INVESTIGATION OF RELATIONS AMONG SCATTERING AMPLITUDES OF OPEN, CLOSED, AND MIXED STRINGS

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ABSTRACT

In this thesis, we explore the relations among open, closed, and mixed string amplitudes. Specifically, we formulate the relationship between closed and mixed string amplitudes at the tree level. By utilizing the analytic continuation of complex variables, we establish a factorization of closed string amplitudes into mixed string amplitudes involving $(n-2)$ open strings and a single closed string. The results demonstrate that closed string amplitudes can be expressed in terms of the product of two mixed string amplitudes with appropriate phase factors. During the Wick rotation, we must take good care of the branch points in the bulk of the complex plane. It turns out that the correction terms arising from the integration along the infinite tube encircle the branch points. The correction terms for the four-point relation are provided. We also provide the alternative way to obtain the relation between closed string amplitudes and mixed string amplitudes and the correction terms for five-point. Nonetheless, the correction term could be neglected in a certain soft limit since it is of subleading order.

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CHAPTER I

INTRODUCTION

1.1 Overview

Historically, Gabriele Veneziano proposed the dual resonance model or string theory, which describes the hadrons and their interactions, in 1968. The scattering amplitude of hadrons can be described by the mathematical function known as the Euler beta function, and this amplitude is called the Veneziano amplitudes [4]. In 1969–1970, Nambu [5], Nielson, and Susskind [6] suggested that the Veneziano amplitude described the interaction of a one-dimensional object called a string, not a point particle, as usual in physics. Different kinds of hadrons can be represented by different modes of vibration of strings. One problem is the existence of massless spin-two particles, which do not exist in the Hadronic world. Another problem is that we need a theory consistent with special relativity. It turns out that the dimension of spacetime needs to be 26 for the bosonic string theory and 10 for the superstring theory. The ground state of a bosonic string theory is tachyon, which has negative mass squared or imaginary mass, where we can get rid of the tachyon state in superstring theory. According to the above problems, string theory is not a good candidate for the theory of strong interaction. As we know, the theory that has been successful for strong interactions is quantum chromodynamics (QCD). In 1974, Scherk and Schwarz [7] took advantage of the existence of a massless spin-two particle, which is one of the string spectra, to reinterpret this spectrum as the graviton. This serves as an indication for the quantum theory of gravity; therefore, string theory is considered one of the candidates for a unified theory.

The fundamental object in string theory is a one-dimensional object, which is called a string. There are two types of strings, namely, closed strings and open strings. In this theory, each mode of vibration of a string is interpreted as a different type of particle. Classically, the motion of a point particle through spacetime creates the trajectory, which is called a worldline, while the motion of a string creates a two-dimensional surface in spacetime, which is called a worldsheet. The action that is proportional to the area of the worldsheet is known as the Nambu-Goto action [8]. This action is difficult to quantize because of the square root of the action. We have an alternative action that is classically equivalent to the Nambu-Goto action. The action is known as the Polyakov action [9, 10]. The classical motion of a string is governed by these actions. When we quantize classical string, the results show that the number of dimensions of spacetime is 26. Combining supersymmetry, the number of dimensions of spacetime is reduced to 10. The extra dimensions are extremely small and compactified. At large distance or low energies, the extra dimensions cannot be detected. One of closed string spectra is the massless spin-two state, which can be identified as graviton. In open string theory, we have another dynamical object, which is called D-brane.

At low energy limits, string theory gives general relativity and the Standard Model. Studying string theory provides new methods to understand aspects of quantum gauge theories, such as the AdS/CFT correspondence [11, 12]. This correspondence gives a connection between strongly coupled quantum field theories and gravity in higher dimensions. This correspondence can be applied in many areas in physics, for example, nuclear physics, condensed matter physics, etc.

Scattering amplitudes are crucial quantities in physics, as they provide probabilities for the outcomes of scattering processes. They serve as a bridge between theoretical and experimental physics. String scattering amplitudes can be captured by worldsheet integral. In the path integral approach, we sum over all fields and worldsheet topologies. The sum over all worldsheet topologies, or genus expansion, can be treated as perturbative expansion in string theory. The genus expansion begins with the lowest order, known as tree-level, followed by one-loop, two-loop, and so forth. There are three kinds of string scattering amplitudes, namely closed string amplitudes, open string amplitudes, and mixed string amplitudes. And string scattering amplitudes can reproduce the field theory amplitudes by taking the field theory limit.

String scattering amplitudes have an interesting relationship among themselves. At the tree-level of amplitudes, we have relations among each type of string scattering amplitude. The relation between closed string amplitudes and open string amplitudes is known as the KLT-relation. This relation shows that closed string amplitudes can be expressed in terms of the summation of the product of open string amplitudes [1, 13]. In low-energy limit, the gravity theory amplitudes in flat space can be written in terms of a sum of the products of gauge theory amplitudes [14]. Moreover, another relation between closed and open superstring amplitudes has been derived. This relation states that the closed superstring amplitude is basically the single-valued version of the open superstring amplitude [15]. The double copy construction establishes an obvious connection between perturbative gauge- and gravity-theories [16, 17, 18]. The open string amplitudes have relations among themselves, which are known as the minimal basis of gauge theory amplitudes. The minimal basis shows that the monodromy relations reduce the number of independent color-ordered amplitudes from $(n-1)!$ to $(n-3)!$ [2]. In field theory limit, $\alpha' \to 0$, the relation monodromy is reduced to the BCJ relation [18] and the Kleiss-Kuijf relation in field theory [19]. The monodromy relations among color-ordered open string amplitudes can be represented by the polygon in the complex plane[20]. The fewer-point physical amplitudes can be treated as building blocks through the on-shell recursion relation (BCFW). The on-shell recursion relation was generalized to string amplitudes [21, 22, 23, 24]. Stieberger and Taylor derived the relation between mixed string amplitudes and open string amplitudes, which expresses mixed string amplitudes in terms of a linear combination of pure open string amplitudes [25, 26]. This relation gives the connection between Einstein-Yang-Mills theory and pure gauge amplitudes at tree-level [27]. In Einstein's gravity, tree-level graviton amplitudes can be expressed as the collinear limit of a linear combination of pure Yang-Mills amplitudes [28]. The mixed string amplitudes have relationships among themselves. The mixed string amplitudes satisfy the monodromy relations with the infinite tube contribution. When we take the field theory limit, it gives the new relations of Einstein Yang-Mills amplitudes, i.e., the gluon part in the amplitudes satisfy the Kleiss-Kuijf relation [3].

1.2 Objectives

- To explore the relations among string scattering amplitudes of different kinds.
- To formulate relations of string amplitudes between closed string amplitudes and mixed string amplitudes at tree-level

1.3 Outline of the Thesis

In Chapter 2, we introduce the fundamental concepts of string theory. We derive the classical equation of motion from the action. The general solution for classical strings is provided. We quantize classical strings by using lightcone quantization. For open strings, they are subject to the Neumann and Dirichlet boundary conditions. The notion of the D-branes is provided. In Chapter 3, we discuss the string interactions which describe by string scattering amplitudes. The scattering process can be captured by worldsheet. We focus on tree-level amplitudes. The expression of closed string, open string, and mixed string amplitudes are provided.

In Chapter 4, we review the relations among string scattering amplitudes, namely, the relations between closed string and open string amplitudes at tree-level, known as KLT-relation [1, 13], the relation among color-ordered open string amplitudes [2], the relations between mixed and open string amplitudes [25], and the relations among mixed string mplitudes [3]. The details of the derivation of these relations are provided.

In Chapter 5, we formulate the relation between closed string amplitudes and mixed string amplitudes at tree-level. By using the analytic continuation of complex variables, we can factorized the closed string amplitudes to the product of mixed string amplitudes with appropriate phase factors. And we discuss about the correction terms which come from avoiding the branch points in the bulk of the complex plane. In Chapter 6, the results are summarized.

In this thesis, we set the Planck's constant and speed of light to be one, $\hbar = c = 1$. The signature of the metric we used is mostly plus, $(-, +, \ldots, +)$. Greek indices are spacetime indices, while Latin indices run are spatial indices. The index a use for Neumann directions, while The index I use for Dirichlet directions.

CHAPTER II

BASIC CONCEPTS OF STRING THEORY

In this chapter, we would like to review basic concepts in string theory including classical and quantum strings. We begin by introducing actions for string theory, namely, Nambu-Goto action and Polyakov action, along with their associated symmetries. By considering the variations of the actions, we derive the equations of motion governing the dynamics of the system. Subsequently, employing mode expansion techniques, we obtain the general solutions for classical strings. Following this, we quantize the classical string, with particular emphasis on the lightcone quantization. This results show that the dimension of spacetime must be 26 in order to preserve the Lorentz symmetry. Additionally, in open string theory, the existence of D-branes is also discussed.

2.1 Classical String

In this section, we discuss the actions of string theory, namely the Nambu-Goto action and the Polyakov action, as well as their symmetries. These two actions are equivalent at the classical level. The general classical solutions of the string are provided.

2.1.1 The Nambu-Goto Action

In string theory, we replace the notion of a point-like particle with a one-dimensional object called a string. There are two types of strings, namely open strings and closed strings. The two-dimensional surface swept out by the string is called the worldsheet. The worldsheet is parameterized by the timelike coordinate τ and the spacelike coordinate σ, as shown in figure. 1, $\sigma \in [0, \pi]$ for open strings, while $\sigma \in [0, 2\pi]$ for closed strings.

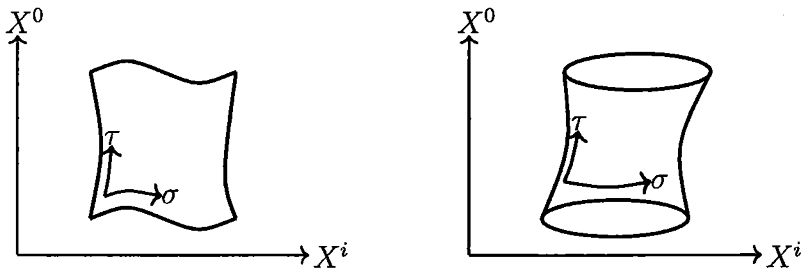

Figure 1 (a) open string worldsheet and (b) closed string worldsheet.

One can construct the the Nambu-Goto action by the proper area of the worldsheet, i.e.

$$S_{NG} = -T \int d^2\sigma \sqrt{-(\dot{X})^2(X')^2 + (\dot{X} \cdot X')^2}, \tag{2.1}$$

where

$$\dot{X} = \frac{\partial X}{\partial \tau} \qquad \text{and} \qquad X' = \frac{\partial X}{\partial \sigma}. \tag{2.2}$$

T is the string tension, which is a dimensionful quantity. We have a parameter which relates to the string tension

$$\alpha' = \frac{1}{2\pi T}, \tag{2.3}$$

which is known as Regge slope. The dimension of α' is (length)^2. We can write the action in terms of an induced metric γ_{ij} which is defined as

$$\gamma_{ij} = \frac{\partial X^\mu}{\partial \sigma^i} \frac{\partial X^\nu}{\partial \sigma^j} \eta_{\mu\nu}. \tag{2.4}$$

Then the action can be written in the form

$$S_{NG} = -T \int d^2\sigma \sqrt{-\gamma}, \tag{2.5}$$

where γ is $\det(\gamma_{ij})$.

Symmetries of the action are as follows:

1. Worldsheet symmetry where the action is invariant under reparametrization.

$$(\tau,\sigma) \to (\tau'(\tau,\sigma), \sigma'(\tau,\sigma)), \tag{2.6}$$

$$X'^{\mu}(\tau',\sigma') = X^{\mu}(\tau,\sigma). \tag{2.7}$$

2. Spacetime symmetry where the action is invariant under Poincaré transformation.

$$X'^{\mu}(\tau,\sigma) = \Lambda^{\mu}{}_{\nu} X^{\nu}(\tau,\sigma) + a^{\mu}, \tag{2.8}$$

where $\Lambda^{\mu}{}_{\nu}$ is the Lorentz transformation and a^{μ} is translation.

To obtain the equations of motion, we vary the action

$$\begin{aligned}\delta S &= -T\int d^2\sigma \frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\delta\left(\frac{\partial X^{\mu}}{\partial\sigma^i}\frac{\partial X_{\mu}}{\partial\sigma^i}\right),\\ &= -T\int d^2\sigma\partial_j(\sqrt{-\gamma}\gamma^{ij}\partial_i X^{\mu}\delta X_{\mu}) + T\int d^2\sigma\partial_j(\sqrt{-\gamma}\gamma^{ij}\partial_i X^{\mu})\delta X_{\mu}.\end{aligned} \tag{2.9}$$

According to the principle of least action (i.e. $\delta S = 0$), the second term on the right hand side gives us the equations of motion which are

$$\partial_j(\sqrt{-\gamma}\gamma^{ij}\partial_i X^{\mu}) = 0. \tag{2.10}$$

The first term in (2.9) is called the boundary terms that is

$$-T\int d^2\sigma\partial_j(\sqrt{-\gamma}\gamma^{ij}\partial_i X^{\mu}\delta X_{\mu}). \tag{2.11}$$

The boundary term can be vanished by imposing the boundary condition which are

1. Dirichlet boundary conditions

$$\delta X^{\mu}(\tau,0) = \delta X^{\mu}(\tau,\pi) = 0. \tag{2.12}$$

2. Neumann boundary conditions

$$\left.\frac{\partial X^{\mu}}{\partial\sigma}\right|_{\sigma=0} = \left.\frac{\partial X^{\mu}}{\partial\sigma}\right|_{\sigma=\pi} = 0. \tag{2.13}$$

3. Periodic boundary conditions

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi). \tag{2.14}$$

Open strings are subject to the conditions (2.12) and (2.13), while closed strings are subject to (2.14)

2.1.2 The Polyakov Action

Because of the square root in the Numbu-Goto action, it is difficult to quantize. The Polyakov action is equivalent to the Numbu-Goto action at the classical level. To obtain the Polyakov action, we introduce a metric $h_{\alpha\beta}(\tau,\sigma)$ on the worldsheet and write

$$S_P[X,h] = -\frac{T}{2}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}, \tag{2.15}$$

where $h = \det h_{\alpha\beta}$.

Symmetries of the action are as follows :

1. Poincaré symmetry

$$\begin{aligned} \delta X^\mu &= a^\mu_\nu X^\nu + b^\mu, \\ \delta h_{\alpha\beta} &= 0. \end{aligned} \tag{2.16}$$

2. Diffeomorphism

$$\begin{aligned} \delta X^\mu &= -\xi^\alpha \partial_\alpha X^\mu, \\ \delta h_{\alpha\beta} &= -(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha), \\ \delta\sqrt{-h} &= -\partial_\alpha(\xi^\alpha\sqrt{-h}). \end{aligned} \tag{2.17}$$

3. Weyl symmetry

$$\begin{aligned} \delta X^\mu &= 0, \\ \delta h_{\alpha\beta} &= 2\Lambda h_{\alpha\beta}. \end{aligned} \tag{2.18}$$

The equations of motion of X^μ are

$$\partial_\alpha(\sqrt{-h}h^{\alpha\beta}\partial_\beta X^\mu) = 0, \tag{2.19}$$

which are subject to the boundary conditions (2.12) -(2.14). The equations of motion of $h_{\alpha\beta}$ are

$$\partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}h_{\alpha\beta}(h^{\sigma\rho}\partial_\sigma X \cdot \partial_\rho X) = 0, \tag{2.20}$$

which are considered as constraint equations. We can solve for $h_{\alpha\beta}$ and substitute it into (2.15). Then, the Polyakov action turns into the Nambu-Goto action. It turns out that these two actions are equivalent at the classical level.

2.1.3 Mode expansions

For simplicity, we will work in worldsheet lightcone coordinates, i.e.

$$(\tau, \sigma) \to (\sigma^+, \sigma^-), \tag{2.21}$$

which are defined as

$$\sigma^\pm = \tau \pm \sigma, \tag{2.22}$$

and their inverse transforms are

$$\tau = \frac{1}{2}(\sigma^+ + \sigma^-) \qquad \text{and} \qquad \sigma = \frac{1}{2}(\sigma^+ - \sigma^-). \tag{2.23}$$

By using the gauge symmetries, we can fix worldsheet metric to be flat. Then the Polyakov action in the worldsheet lightcone coordinates takes the form

$$S_{l.c.} = T\int d\sigma^+ d\sigma^-(\partial_+ X\partial_- X). \tag{2.24}$$

The equations of motion are

$$\partial_+\partial_- X^\mu = 0. \tag{2.25}$$

It is clear that the equations of motion are basically wave equations. Therefore, the general solution for the wave equations is

$$X^\mu(\tau,\sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-). \tag{2.26}$$

The general solution for a closed string can be written as a Fourier expansion giving

$$X^\mu_L(\sigma^+) = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-in\sigma^+}, \tag{2.27}$$

$$X^\mu_R(\sigma^-) = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in\sigma^-}, \tag{2.28}$$

where x^μ and p^μ are the center of mass position and momentum of the string. α^μ_n and $\tilde{\alpha}^\mu_n$ are right and left moving Fourier modes. This solution satisfied the periodic boundary conditions (2.14). We require X^μ to be real, i.e. $(X^\mu)^* = X^\mu$. This implies that

$$\alpha^\mu_{-n} = (\alpha^\mu_n)^\dagger \quad \text{and} \quad \tilde{\alpha}^\mu_{-n} = (\tilde{\alpha}^\mu_n)^\dagger. \tag{2.29}$$

2.1.4 The Energy Momentum Tensor

Similar to general relativity, the energy momentum tensor is defined as

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S}{\delta h^{\alpha\beta}}. \tag{2.30}$$

When we set $h_{\alpha\beta} = \eta_{\alpha\beta}$, the energy momentum tensor is

$$T_{\alpha\beta} = \partial_\alpha X \partial_\beta X - \frac{1}{2}\eta_{\alpha\beta}\eta^{\rho\sigma}\partial_\rho X \partial_\sigma X = 0. \tag{2.31}$$

Note that $T_{\alpha\beta} = 0$ due to the (2.20). This gives the constraints to be

$$\dot{X}\cdot X' = 0 \quad \text{and} \quad \dot{X}^2 + X'^2 = 0. \tag{2.32}$$

In the worldsheet light-cone coordinates, the constraints become

$$(\partial_- X)^2 = 0 \quad \text{and} \quad (\partial_+ X)^2 = 0. \tag{2.33}$$

Substituting (2.27) and (2.28) into the constraints, then we obtain

$$(\partial_- X)^2 = \frac{\alpha'}{2}\sum_{m,p}\alpha_m \cdot \alpha_p e^{-i(m+p)\sigma^-} = 0. \tag{2.34}$$

Let $m + p = n$, the above expression becomes

$$(\partial_- X)^2 = \frac{\alpha'}{2}\sum_{m,n}\alpha_m \cdot \alpha_{n-m} e^{-in\sigma^-} = 0, \tag{2.35}$$

$$(\partial_- X)^2 \equiv \alpha' \sum_{n} L_n e^{-in\sigma^-} = 0, \tag{2.36}$$

where

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}}\alpha_m \cdot \alpha_{n-m} = 0 \qquad \forall n \in \mathbb{Z}. \tag{2.37}$$

Similarly, for the left-moving modes, the Virasoro generators read

$$\tilde{L}_n = \frac{1}{2}\sum_{m\in\mathbb{Z}}\tilde{\alpha}_m \cdot \tilde{\alpha}_{n-m} = 0, \qquad \forall n \in \mathbb{Z}. \tag{2.38}$$

These constraints are valid only at the classical-level.

2.2 Quantization of closed string

There are mainly 4 ways to quantize the string theory which are canonical quantization, lightcone quantization, path integration and BRST quantization. However, in this section we will focus on the lightcone quantization of the closed strings and explore its implications. For lightcone quantization, we solve for the classical constraints by introducing the lightcone coordinates and lightcone gauge. After solving the constraints, we can relate the classical variables. We then quantize these classical variables.

2.2.1 Light-Cone Gauge

Let us introduce the spacetime lightcone coordinates which defined as

$$X^{\pm} = \sqrt{\frac{1}{2}}(X^0 \pm X^{D-1}). \tag{2.39}$$

With this coordinates,

$$dX^+dX^- = \frac{1}{2}((dX^0)^2 - (dX^{D-1})^2). \tag{2.40}$$

The interval is

$$ds^2 = -2dX^+dX^- + \sum_{i=1}^{D-2} dX^i dX^i. \tag{2.41}$$

From the Polyakov action, we obtained the equations of motion as (2.25). When considering $\mu = +$, the equation of motion is

$$\partial_+\partial_- X^+ = 0. \tag{2.42}$$

Then, the solution takes the form

$$X^+ = X_L^+(\sigma^+) + X_R^+(\sigma^-). \tag{2.43}$$

Although we have fixed the worldsheet metric to be flat metric, we still have residual worldsheet symmetry

$$\sigma^\alpha \to \tilde{\sigma}^\alpha = \sigma^\alpha - \epsilon^\alpha(\sigma), \tag{2.44}$$

such that

$$(P \cdot \epsilon)_{\alpha\beta} = 0, \tag{2.45}$$

where

$$(P \cdot \epsilon)_{\alpha\beta} = \partial_\alpha \epsilon_\beta + \partial_\beta \epsilon_\alpha - \eta_{\alpha\beta}(\partial \cdot \epsilon). \tag{2.46}$$

The symmetry are called conformal symmetry. Let us consider $(P \cdot \epsilon)_{++} = 0$ and $(P \cdot \epsilon)_{--} = 0$. These imply that

$$\epsilon^- = \epsilon^-(\sigma^-), \tag{2.47}$$

$$\epsilon^+ = \epsilon^+(\sigma^+). \tag{2.48}$$

Then, we use this gauge to transform

$$\tau \to \tilde{\tau} = \frac{\tilde{\sigma}^+(\sigma^+) + \tilde{\sigma}^-(\sigma^-)}{2} \tag{2.49}$$

where $\tilde{\tau}$ satisfies $\partial_+\partial_-\tilde{\tau} = 0$.

Now, we can choose $\tilde{\tau}$ to relate to one of X^μ, say X^+. Therefore we can write

$$X^+ = x^+ + \alpha' p^+ \tau \tag{2.50}$$

where x^+ is a constant. This can refer to the lightcone gauge. The advantage of the lightcone gauge is that we can fully determine α_n^- and p^- in terms of p^+ and α_n^i. This can be shown by considering

$$(\partial_+ X)^2 = 0, \tag{2.51}$$

$$-2\partial_+ X^+ \partial_+ X^- + \sum_i \partial_+ X^i \partial_+ X^i = 0. \tag{2.52}$$

This gives

$$\partial_+ X^- = \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i \tag{2.53}$$

and

$$\partial_- X^- = \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \partial_- X^i \partial_- X^i. \tag{2.54}$$

Then, by mode expanding the field X^- and X^i, one obtains

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'}} \frac{1}{p^+} \sum_{i=1}^{D-2} ((\alpha_0^i)^2 + \sum_{m \neq 0} \alpha_m^i \alpha_{-m}^i), \tag{2.55}$$

and

$$p^- = \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} (\frac{\alpha'}{2} p^i p^i + \sum_{m \neq 0} \alpha_m^i \alpha_{-m}^i), \tag{2.56}$$

$$= \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} (\frac{\alpha'}{2} p^i p^i + \sum_{m \neq 0} \tilde{\alpha}_m^i \tilde{\alpha}_{-m}^i). \tag{2.57}$$

Mass shell condition of classical string is given by

$$M^2 = 2p^+p^- - \sum_{i=1}^{D-2} p^i p^i = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{m>0} \alpha_{-m}^i \alpha_m^i = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{m>0} \tilde{\alpha}_m^i \tilde{\alpha}_{-m}^i. \tag{2.58}$$

The general classical solution, in light-cone gauge, is fully described by transverse oscillator modes α^i_{-n} and $\tilde{\alpha}^i_m$ together with x^i, p^i, p^+ and x^- where $i = 1, 2, ..., D-2$.

2.2.2 Lightcone quantization

To quantize, we will promote α^i_n, $\tilde{\alpha}^i_m$, x^i, p^i, p^+ and x^- to be operators with canonical commutation relations which are as follow

$$\begin{aligned} [x^i, p^j] &= i\eta^{ij}, \\ [x^-, p^+] &= -i, \\ [\alpha^i_n, \alpha^j_m] &= [\tilde{\alpha}^i_n, \tilde{\alpha}^j_m] = n\eta^{ij}\delta_{n+m,0}. \end{aligned} \tag{2.59}$$

We define the ground state of strings, $|0; p\rangle$ such that

$$\hat{p}^\mu |0; p\rangle = p^\mu |0; p\rangle \tag{2.60}$$

$$\alpha^i_n |0; p\rangle = \tilde{\alpha}^i_n |0; p\rangle = 0 \qquad \text{for} \quad n > 0 \tag{2.61}$$

The excited states can be obtained by acting the creation operator, i.e. $\alpha^i_{n<0}$ and $\tilde{\alpha}^i_{n<0}$, on the ground state. For example,

$$|\text{excited}\rangle = \alpha^{i_1}_{-1}\alpha^{i_2}_{-1}\dots\alpha^{j_1}_{-2}\dots\tilde{\alpha}^{k_1}_{-1}\tilde{\alpha}^{k_2}_{-1}\dots\tilde{\alpha}^{l_1}_{-2}\dots|0; p\rangle\,. \tag{2.62}$$

Not all states define in this way are physical. Analogy to the classical string, the physical states must satisfy the following conditions

$$(\hat{L}_0 - a)\,|\text{phys}\rangle = 0, \tag{2.63}$$

$$\hat{L}_n\,|\text{phys}\rangle = 0. \qquad \forall n \in \mathbb{Z}^+ \tag{2.64}$$

Note that $\hat{L}_0$ needs to be defined due to the ordering ambiguity, and the constant a we have put in (2.63) arises from the ordering ambiguity of the operator. The constraint (2.63) implies the mass shell condition which is

$$M^2 = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0}\hat{\alpha}^i_{-n}\hat{\alpha}^i_n - a\right) = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0}\hat{\tilde{\alpha}}^i_{-n}\hat{\tilde{\alpha}}^i_n - a\right) \tag{2.65}$$

$$\equiv \frac{4}{\alpha'}(N - a) = \frac{4}{\alpha'}(\tilde{N} - a). \tag{2.66}$$

The last line provides the level matching condition of closed string, i.e.

$$\sum_{i=1}^{D-2}\sum_{n>0}(\hat{\alpha}^i_{-n}\hat{\alpha}^i_n - \hat{\tilde{\alpha}}^i_{-n}\hat{\tilde{\alpha}}^i_n) = 0. \tag{2.67}$$

To fix the value of the constant a, let us consider the first excited states

$$M^2\tilde{\alpha}^i_{-1}\alpha^j_{-1}\left|0;p\right\rangle = \frac{4}{\alpha'}(N-a)(\tilde{\alpha}^i_{-1}\alpha^j_{-1}\left|0;p\right\rangle) \tag{2.68}$$

$$= \frac{4}{\alpha'}(1-a)(\tilde{\alpha}^i_{-1}\alpha^j_{-1}\left|0;p\right\rangle) \tag{2.69}$$

We expect that the first excited states are massless because we need the states $\tilde{\alpha}^i_{-1}\alpha^j_{-1}\left|0;p\right\rangle$ to be Lorentz invariant. This is because we require the states to transform under $SO(D-2)$ which is a little group of the Lorentz group.

$$\frac{4}{\alpha'}(1-a) = 0, \tag{2.70}$$

$$a = 1. \tag{2.71}$$

In fact to make the theory Lorentz invariant, it requires the critical dimension $D = 26$. To see this, let us consider the Lorentz algebra, i.e.

$$[\mathcal{M}^{\rho\sigma}, \mathcal{M}^{\tau\nu}] = \eta^{\sigma\tau}\mathcal{M}^{\rho\nu} - \eta^{\rho\tau}\mathcal{M}^{\sigma\nu} + \eta^{\rho\nu}\mathcal{M}^{\sigma\tau} - \eta^{\sigma\nu}\mathcal{M}^{\rho\tau}. \tag{2.72}$$

For the string theory,

$$\mathcal{M}^{\mu\nu} = \int d\sigma J^{\mu\nu} \tag{2.73}$$

$$J^{\mu\nu} = P^\mu X^\nu - P^\nu X^\mu \tag{2.74}$$

$$P^\mu = \frac{\partial\mathcal{L}}{\partial(\dot{X}_\mu)} = -T\dot{X}^\mu \tag{2.75}$$

After some calculation, we obtain

$$[\mathcal{M}^{-i}, \mathcal{M}^{-j}] = \frac{-1}{(p^+)^2}\sum_{m=1}^{\infty}\left[m\left(\frac{26-D}{12}\right) + \frac{1}{m}\left(\frac{D-26}{12}(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m)\right)\right.$$
$$\left. + 2(1-a)(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m)\right] \tag{2.76}$$

However, to satisfy (2.72), we require that

$$[\mathcal{M}^{-i}, \mathcal{M}^{-j}] = 0. \tag{2.77}$$

This implies that

$$D = 26. \tag{2.78}$$

To summarize, Lorentz invariance of the theory requires that the critical dimension, $D = 26$ and $a = 1$.

2.3 Open strings

Unlike the closed strings, the worldsheet of an open string has boundaries at which we cannot apply periodicity conditions. When we do the variation of the action, the boundary terms do not vanish. We have to impose boundary conditions, (2.12) and (2.13), for open string.

2.3.1 Mode expansions

The mode expansion for open strings are obtained by solving the (2.19) with boundary conditions, (2.12) and (2.13). The general solution for X^μ are given by

$$X^\mu(\tau,\sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-), \tag{2.79}$$

where

$$X^\mu_L(\sigma^+) = \frac{1}{2}x^\mu + \alpha'\tilde{p}^\mu\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-in\sigma^+}, \tag{2.80}$$

$$X^\mu_R(\sigma^-) = \frac{1}{2}x^\mu + \alpha' p^\mu\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in\sigma^-}. \tag{2.81}$$

The modes of the string can be related by imposing the boundary conditions.

- Neumann boundary conditions (NN) : In the direction that we impose Neumann boundary conditions, the end points require that $\tilde{p}^a = p^a$ and $\alpha^a_n = \tilde{\alpha}^a_n$ for a in Neumann directions. The solution in the Neumann directions take the form

$$X^a(\tau,\sigma)\Big|_{NN} = x^a + 2\alpha' p^a + 2i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^a e^{-in\tau}\cos(n\sigma) \tag{2.82}$$

- Dirichlet boundary conditions (DD) : In the direction that we impose the Dirichlet boundary conditions, the end points of the string have to be fixed in a specific positions, as

$$X^I(\tau,\sigma=0) = c^I \qquad \text{and} \qquad X^I(\tau,\sigma=\pi) = d^I \tag{2.83}$$

In Dirichlet direction, with index I, it gives

$$x^I = c^I, \quad \tilde{p}^I = -p^I = \frac{d^I - c^I}{2\pi\alpha'}, \quad \text{and} \quad \tilde{\alpha}_n^I = -\alpha_n^I \tag{2.84}$$

The solution in Dirichlet direction take the form

$$X^I(\tau,\sigma)\Big|_{DD} = c^I + \frac{\sigma}{\pi}(d^I - c^I) - 2\sqrt{\frac{\alpha;}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^I e^{-in\tau}\sin(n\sigma) \tag{2.85}$$

2.3.2 Quantization of open strings

We again apply lightcone quantization to open strings. For spacetime directions with indices $0 \le a \le d_N - 1$, we impose the Neumann boundary conditions, and for $d_N \le I \le D-1$, we impose the Dirichlet boundary conditions. We choose lightcone gauge in Neumann directions

$$X^{\pm} = \sqrt{\frac{1}{2}}(X^0 \pm X^{d_N-1}), \tag{2.86}$$

We do the same procedures as we did for closed strings. However, the mass shell conditions change, and it takes the form

$$\begin{aligned} M^2 &= 2p^+p^- - \sum_{i=1}^{d_N-2} p^i p^i \\ &= \frac{1}{(2\pi\alpha')^2}\sum_{I=d_N}^{D-1}(c^I - d^I)^2 + \frac{1}{\alpha'}\left\{\sum_{n-1}^{\infty}\left(\sum_{i=1}^{d_N-2}\alpha_{-n}^i\alpha_n^i + \sum_{I=d_N}^{D-1}\alpha_{-n}^I\alpha_n^I\right) - a\right\}. \end{aligned} \tag{2.87}$$

Note that the first term refers to the specific position of the end points of an open string, the second term is sum over Neumann directions, and the third term is sum over Dirichlet directions. We notice that there is a sum only over α modes because the $\tilde{\alpha}$ modes can be related by the boundary condition. The constant a and the critical dimensions take the same value as in closed strings, i.e.,

$$a = 1 \quad \text{and} \quad D = 26. \tag{2.88}$$

2.3.3 D-branes

For Dirichlet boundary conditions, the open string endpoints are fixed at $X^I(\tau, \sigma = 0) = c^I$ and $X^I(\tau, \sigma = \pi) = d^I$ which define hypersurfaces in spacetime. We interpret these hypersurfaces as dynamical objects called D-branes, as shown in figure 2.

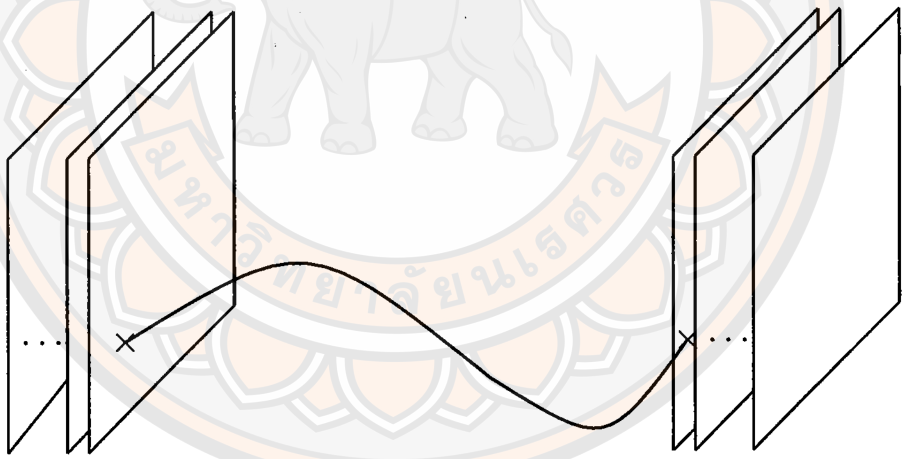

Figure 2 Open string (blue line) and D-branes.

We are considering a stack of N D-branes, i.e., multiple D-branes sitting at the same position. Both endpoints of open strings can be labeled by m, $n = 1, 2, \ldots, N$ to specify the brane they are attached to, where N is the number of branes. This gives us the degree of freedom of non-abelian gauge bosons of $U(N)$. The open-string state is defined as

$$\left|\text{phys}_{\text{op}}; (m, n)\right\rangle = \sum_{a=1}^{N^2} (T^a)_m{}^n \left|\text{phys}_{\text{op}}; a\right\rangle, \tag{2.89}$$

where T^a are hermitian $N \times N$ matrices, called Chan Paton factors.

2.4 The string spectrum

2.4.1 Open string spectrum

The ground state is $|0, p\rangle$. According to the mass shell condition (2.66), the mass of the state is

$$\alpha' m^2 |0, p\rangle = -a |0, p\rangle \tag{2.90}$$

We know that $a = 1$, then the mass square of ground state is negative, i.e. $\alpha' m^2$ = -1. It is a tachyon. The first excited state is $\alpha^i_{-1} |0, p\rangle$. The mass of the first excited state is given by

$$\alpha' m^2 (\alpha^i_{-1} |0, p\rangle) = (1 - a)\alpha^i_{-1} |0, p\rangle = 0 \tag{2.91}$$

Therefore, the first excited states are massless.

2.4.2 Closed string spectrum

Closed string state are tensor products of the left-handed and right-handed modes. The states are subject to the level matching condition, i.e. $N = \tilde{N}$ which were already defined in (2.67). The ground state is again a tachyon with mass $\alpha' m^2 = -4$. The first excited state $\alpha^i_{-1}\tilde{\alpha}^j_{-1} |0\rangle$ has a vanishing mass square, $\alpha' m^2 = 4(1 - a) = 0$. The first excited states of closed string are also massless. One of the interesting closed string spectra is the massless spin-two state. This state is symmetric and traceless which can be identified as massless spin-two particle or graviton.

CHAPTER III

STRING SCATTERING AMPLITUDES

In quantum field theory, a scattering of particles can be represented by Feynmann diagrams. For each process, one needs to sum over all possible intermediate processes of interactions. In contrast, the scattering in string theory can be captured by a worldsheet. A single worldsheet capture all possible processes of Feynmann diagrams at each order. The string scattering amplitude can reproduce the field theory amplitude by taking the low energy limit. In the Polyakov path

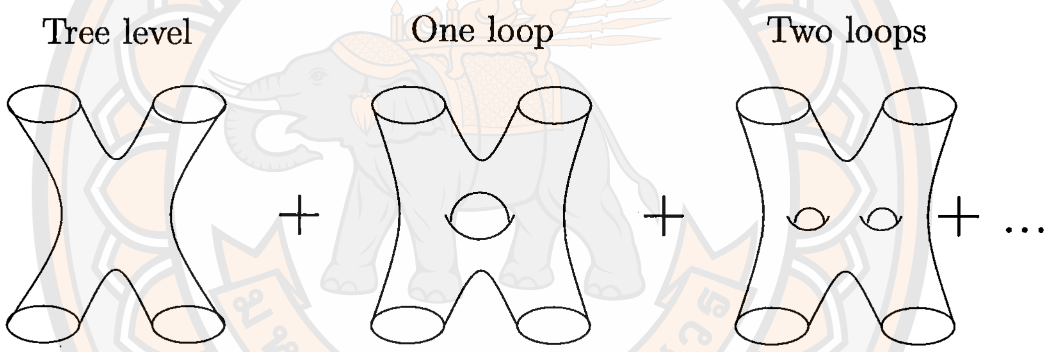

Figure 3 Scattering of four closed strings

integral, we sum over all fields and also sum over all worldsheet topologies as shown in Figure 3. The sum over all worldsheet topologies gives the perturbative expansion of string theory. To see this, let's consider the augmented Polyakov action of the form

$$S_{\text{string}} = S_P + \lambda\chi, \tag{3.1}$$

where λ is a real number. χ is given by an integral over the worldsheet

$$\chi = \frac{1}{4\pi}\int d^2\sigma\sqrt{-h}R \tag{3.2}$$

where R is Ricci scalar of worldsheet metric h. This integral seems like the Einstein-Hilbert action. However, in 2-dimensional case, the term (3.2) does not make the gravity dynamical. This term depends on the topology of the worldsheet. According to the Gauss-Bonnet theorem, the integral (3.2) gives an integer, χ, known as the Euler characteristic of the worldsheet. It is given by

$$\chi = 2 - 2h = 2(1-g), \tag{3.3}$$

where h is the number of handles on the worldsheet and g is called the genus of the surface. For example, the sphere has $g=0$ and $\chi=2$, the torus has $g=2$ and $\chi=0$, for higher $g>1$, the Euler characteristic is negative. The integral over the worldsheet weight by

$$\sum_{\text{topologies metrics}} e^{-S_{\text{string}}} \sim \sum_{\text{topologies}} e^{-2\lambda(1-g)} \int DXDhe^{-S_P[X,h]}. \tag{3.4}$$

We can treat sum over all topologies as perturbative expansion when $e^{\lambda} << 1$.

According to the conformal symmetry, we can reshape the worldsheet and map onto the complex plane. At tree-level, closed string worldsheet maps onto full complex plane, while open string worldsheet maps onto upper-half complex plane. For example, four closed and open string scattering are shown in figure 4 and 5 respectively. The string states can be mapped to local operators called vertex operators. As in figures 4 and 5, the vertex operator is represented by the symbol $\times$, which are inserted in the bulk and along the boundary of the worldsheet for closed strings and open strings, respectively.

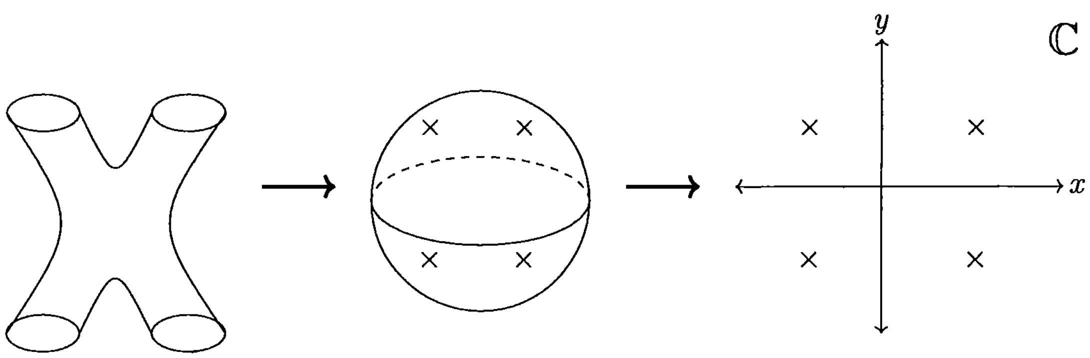

Figure 4 Mapping of closed string worldsheet to full complex plane

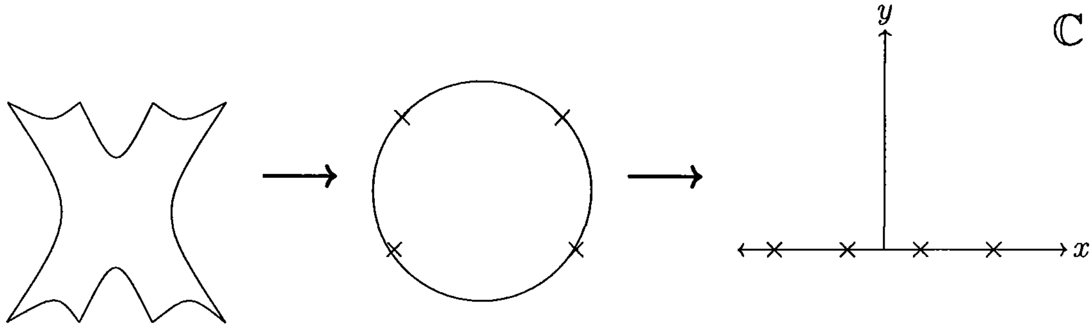

Figure 5 Mapping of open string worldsheet to upper- half complex plane

3.1 The vertex operators and string propagator

3.1.1 Vertex operators

The physical string states have to satisfy (2.63) and (2.64). We can write the physical states as

$$|\text{phys}\rangle = \lim_{z,\bar{z}\to 0} \mathcal{V}(z,\bar{z})\,|0\rangle \tag{3.5}$$

where $\mathcal{V}(z,\bar{z})$ is vertex operators. The vertex operators are primary fields which transform under conformal transformation with conformal weight, $(h,\bar{h})$, is $(1,1)$. Under the conformal transformation, the primary fields transform as follow

$$\phi(z,\bar{z}) \to \phi'(z,\bar{z}) = \left(\frac{\partial z'}{\partial z}\right)^{h}\left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z),\bar{f}(\bar{z})) \tag{3.6}$$

where h and $\bar{h}$ are conformal weights. We will provide some example of vertex operator as follow

1. The closed string vertex operator of tachyon

$$\mathcal{V}(z,\bar{z}) =: e^{ik\cdot X(z,\bar{z})} :, \tag{3.7}$$

with

$$k^2 = -m^2 = \frac{4}{\alpha'}. \tag{3.8}$$

2. The open string vertex operator of tachyon

$$\mathcal{V}(z) =: e^{ik\cdot X(z)} :, \tag{3.9}$$

with

$$k^2 = -m^2 = \frac{1}{\alpha'}. \tag{3.10}$$

3. The closed string vertex operator of first excited state

$$\mathcal{V}(z,\bar{z};\xi) = \chi_{\mu\nu} : \partial X^\mu(z)\bar{\partial}X^\nu(\bar{z})e^{ik\cdot X(z,\bar{z})} :, \tag{3.11}$$

$$k^\mu \chi_{\mu\nu} = 0 \quad \text{and} \quad k^2 = 0, \tag{3.12}$$

where $\chi_{\mu\nu}$ is polarization tensor. If $\chi_{\mu\nu}$ is symmetric and traceless, one can identify the state as graviton.

4. The open string vertex operator of first excited state

$$\mathcal{V}(z;\xi) = \zeta_\mu : \partial X^\mu(z)e^{ik\cdot X(z)} :, \tag{3.13}$$

$$k^\mu \zeta_\mu = 0 \quad \text{and} \quad k^2 = 0, \tag{3.14}$$

where ζ_μ is polarization vector.

3.1.2 String propagator

To compute scattering amplitude, it is useful to find a string propagator beforehand. In conformal gauge, the Polyakov action takes the form

$$S_P\left[X(z,\bar{z})\right] = \frac{1}{2\pi\alpha'}\int d^2z\partial_z X(z,\bar{z})\bar{\partial}_{\bar{z}}X(z,\bar{z}). \tag{3.15}$$

In path integral formalism, the expectation value is given by

$$\langle X(z,\bar{z})\rangle = \frac{1}{Z}\int \mathcal{D}X\, X(z,\bar{z})e^{-S_P[X(z,\bar{z})]} \tag{3.16}$$

where Z is partition function. To obtain the string propagator, we will use Dyson-Schwinger method which is

$$\frac{\delta\langle X(z,\bar{z})\rangle}{\delta X(w,\bar{w})} = 0. \tag{3.17}$$

Substituting (3.16) into (3.17), we then obtain

$$\partial_w \bar{\partial}_{\bar{w}} \langle X(z,\bar{z}) X(w,\bar{w}) \rangle = -\pi\alpha' \delta^{(2)}(z-w). \tag{3.18}$$

We then obtain the string propagator

$$\langle X(z,\bar{z}) X(w,\bar{w}) \rangle = -\frac{\alpha'}{2} \ln |z-w|^2. \tag{3.19}$$

This propagator or two-point correlation function is basically Green's functions.

3.2 String scattering amplitudes at tree-level

The scattering amplitudes can be computed by expectation value of the product of vertex operators. For closed string, we inserted the vertex operators in the bulk of the worldsheet. For open string, the vertex operators were inserted along the boundary of the worldsheet.

3.2.1 Closed string amplitudes

The tree-level n-point closed string amplitude can be computed by

$$\mathcal{A}_n^{\text{cl}} = \frac{g_s^{n-2}}{\text{Vol}(SL(2,\mathbb{C}))} \prod_{i=1}^{n} \int d^2 z_i \langle \prod_j \mathcal{V}_j(z_j, \bar{z}_j) \rangle \tag{3.20}$$

where g_s is string coupling constant. The Vol$(SL(2,\mathbb{C}))$ is given by

$$\text{Vol}(SL(2,\mathbb{C})) = \frac{d^2 z_a d^2 z_b d^2 z_c}{|z_{ab}|^2 |z_{bc}|^2 |z_{ac}|^2} \tag{3.21}$$

The expectation value can be computed by

$$\langle \prod_j^n \mathcal{V}_j(z_j, \bar{z}_j) \rangle = \frac{1}{Z} \int \mathcal{D}[X] \prod_j \mathcal{V}_j(z_j, \bar{z}_j) e^{-S_P[X]}. \tag{3.22}$$

Let's compute the expectation value of n tachyon vertex operator.

$$\begin{aligned} \langle \prod_{j=1} : e^{ik_j \cdot X(z_j,\bar{z}_j)} : \rangle = & \frac{1}{Z} \int \mathcal{D}[X] \exp\left\{ \frac{1}{2\pi\alpha'} \int d^2 z X^\mu(z) \partial\bar{\partial} X_\mu(z) \right. \\ & \left. + i \sum_{j=1}^{n} k_j \cdot X(z_j, \bar{z}_j) \right\}. \end{aligned} \tag{3.23}$$

By utilizing the Gaussian integral formula and we include the contribution of zero mode, we then obtain

$$\langle\prod_{j=1} : e^{ik_j\cdot X(z_j,\bar{z}_j)} :\rangle = (2\pi)^D\delta^D(\sum_i k_i)\exp\Big\{-\frac{1}{2}\int d^2z d^2w \sum_{i,j} k_i\delta^{(2)}(z-z_i) \times G(z,w)k_j\delta^{(2)}(w-z_j)\Big\}, \tag{3.24}$$

where $G(z,w)$ is Green's function or propagator that we obtain in previous section. We integrate z and w out, we then obtain

$$\langle\prod_{j=1} : e^{ik_j\cdot X(z_j,\bar{z}_j)} :\rangle = (2\pi)^D\delta^D(\sum_i k_i)\prod_{1\le i<j}|z_i-z_j|^{\alpha' k_i\cdot k_j}. \tag{3.25}$$

This is known as Koba-Nielsen factor.

The general expression of closed string amplitudes take the form

$$\mathcal{A}_n^{\text{cl}} = C_{S^2}(2\pi)^D\delta^D(\sum_i k_i)\int\frac{|z_{ab}z_{ac}z_{bc}|^2}{dz_a dz_b dz_c}\prod_{i=1}^n d^2z_i\prod_{1\le j<l\le n}|z_j-z_l|^{\alpha' k_j\cdot k_l}F_n(z_i,\bar{z}_i), \tag{3.26}$$

where C_{S^2} is a normalization constant, $z_{ij} = z_i - z_j$ and the function $F_n(z_i, \bar{z}_i)$ contains polarization and kinematic factors of the external states. For tachyon $F_n(z_i, \bar{z}_i)$ is basically one, while for the excited state $F_n(z_i, \bar{z}_i)$ takes the form

$$F_n(z_i,\bar{z}_i) = \exp\Big\{\sum_{i>j}\frac{\zeta_i\cdot\zeta_j}{(z_i-z_j)^2} - \sqrt{\alpha'}\sum_{i\ne j}\frac{k_i\cdot\zeta_j}{(z_i-z_j)} + \sum_{i>j}\frac{\tilde{\zeta}_i\cdot\tilde{\zeta}_j}{(\bar{z}_i-\bar{z}_j)^2} - \sqrt{\alpha'}\sum_{i\ne j}\frac{k_i\cdot\tilde{\zeta}_j}{(\bar{z}_i-\bar{z}_j)}\Big\}\Big|_{\text{linear in }\zeta_i,\tilde{\zeta}_i}, \tag{3.27}$$

where $\zeta_i\otimes\tilde{\zeta}_i = \chi_i$ are polarization of external closed string states. According to conformal symmetry, the points z_a, z_b and z_c are freely fixed to arbitrary points in the complex plane.

3.2.2 Open string amplitudes

The tree-level open string amplitudes are captured by punctured disks, as shown in figure 5. The notion of cyclic ordering comes from the placement of open-string vertex operators on the disk boundary. The choice of cyclic ordering is related to the non-abelian degree of freedom called the Chan Paton factor T^a. And the trace of the Chan-Paton factor, $\text{Tr}\{T^{a_1}\dots T^{a_n}\}$, arises from the contractions from the boundary segment in cyclic ordering. The tree-level n-point open string amplitudes take the form

$$\mathcal{A}_n^{\text{op}} = C_{D_2}(2\pi)^D\delta^D(\sum_i p_i) \sum_{(a_1,\dots,a_n)\in S_n/\mathbb{Z}_n} \text{tr}(T^{a_1}\dots T^{a_n})\mathcal{A}^{\text{op}}(a_1,\dots,a_n) \tag{3.28}$$

where C_{D_2} is a normalization constant and $\mathcal{A}^{\text{op}}(a_1,\dots,a_n)$ is color-ordered open string amplitudes. The n-point color-ordered amplitudes be compute by

$$\mathcal{A}^{\text{op}}(a_1,\dots,a_n) = \frac{1}{\text{Vol}(SL(2,\mathbb{R}))}\prod_{i=1}^{n}\int_{-\infty<x_1<x_2<\dots<x_n<\infty} dx_i\langle\prod_j \mathcal{V}_j(x_j)\rangle. \tag{3.29}$$

The $\text{Vol}(SL(2,\mathbb{R}))$ is given by

$$\text{Vol}(SL(2,\mathbb{R})) = \frac{dx_a dx_b dx_c}{|x_{ab}||x_{bc}||x_{ac}|}. \tag{3.30}$$

By utilizing the same procedure as we did for closed string amplitudes, the color-order amplitudes take the form

$$\begin{aligned}\mathcal{A}^{\text{op}}(a_1,\dots,a_n) = \int\prod_{i=1}^{n} dx_i\frac{|x_{ab}x_{ac}x_{bc}|}{dx_a dx_b dx_c}\prod_{i=1}^{n-1}\Theta(x_{a_i+1}-x_{a_i})\\ \times\prod_{1\le i<j\le n}|x_i-x_j|^{2\alpha' p_i\cdot p_j}P_n(x_i),\end{aligned} \tag{3.31}$$

where $\Theta(x)$ is haviside step function, $x_{ij} = x_j - x_j$ and $P_n(x_i)$ contains polarization and kinematic factors of the external states. Again $P_n(x_i)$ is basically one for tachyon, while P_n of excited states take the form

$$P_n(x_i) = \exp\left\{\sum_{i>j}\frac{\zeta_i\cdot\zeta_j}{(x_i-x_j)^2} - \sqrt{\alpha'}\sum_{i\neq j}\frac{p_i\cdot\zeta_j}{(x_i-x_j)}\right\}\Bigg|_{\text{linear in }\zeta_i}, \tag{3.32}$$

where where ζ_i are polarization of external open string states. According to conformal symmetry, the points x_a, x_b and x_c are freely fixed to arbitrary points in the complex plane.

3.2.3 Mixed string amplitudes

Mixed string amplitudes can be captured by a disk worldsheet where closed string vertex operators are inserted in the bulk of the disk, and open string vertex operators are inserted on the boundary of the disk. The disk can be conformally mapped to the upper half-plane $H_+ = \{z \in \mathbb{C} | \operatorname{Im}\{z\} \geq 0\}$. The general expression of mixed string amplitudes involving N_o open string and N_c closed string takes the form [26]

$$\begin{aligned}\mathcal{M}(1,2,\ldots,N_o;N_c) = C_{D_2}(2\pi)^D\delta^D(\sum_{j=1}^{N_o} p_j + \sum_{i=1}^{N_c} k_i)\int_{\mathcal{I}_{N_o}}\prod_{j=1}^{N_o} dx_j \int_{H_+}\prod_{i=1}^{N_c} d^2z_i \\ \times \prod_{j_1<j_2}^{N_o}|x_{j_1}-x_{j_2}|^{2\alpha' p_{j_1}p_{j_2}}\prod_{i=1}^{N_c}|z_i-\bar{z}_i|^{\alpha' k_i D k_i}\prod_{j=1}^{N_o}\prod_{i=1}^{N_c}|x_j-z_i|^{2\alpha' p_j k_i} \\ \times \prod_{i_1<i_2}^{N_c}|z_{i_1}-z_{i_2}|^{\alpha' k_{i_1}k_{i_2}}|z_{i_1}-\bar{z}_{i_2}|^{\alpha' k_{i_1} D k_{i_2}} K_n(x_i,z_i,\bar{z}_i) \qquad (3.33)\end{aligned}$$

where C_{D_2} is a normalization constant and the function K_n contains polarization and kinematic factor of the external states. Notice that we used letters p_i and k_i to denote the momentum of open strings and a closed string respectively. In flat spacetime, the matrix $D^{\mu\nu}$ is equal to the Minkowski metric $\eta^{\mu\nu}$ in directions that impose Neumann boundary conditions and to $-\eta^{\mu\nu}$ in directions that impose Dirichlet boundary conditions. The open string coordinates x_i obey the ordering of the integration region $\mathcal{I}_{N_o} = \{x_i \in \mathbb{R} | x_1 < x_2 < \ldots < x_{N_o}\}$. The partial amplitudes are associated with a group factor $\operatorname{Tr}\{T^1T^2\ldots T^{N_o}\}$ with T^a being a Chan-Paton factor. To be more specific, we are interested in $n-2$ open strings and one closed string. The expression of mixed string amplitudes, which described scattering of $n-2$ open strings and one closed string, take the form [27]

$$\begin{aligned}\mathcal{M}_n(1,\ldots,n-2;k) =& C_{D_2}(2\pi)^D\delta(\sum_{i=1}^{n-2} p_i + k)\int_{\mathcal{I}_{n-2}}\prod_{i=1}^{n-2} dx_i \prod_{1\leq r<s\leq n-2}|x_r-x_s|^{2\alpha' p_r p_s} \\ &\times \int_{H_+} d^2z|z-\bar{z}|^{\frac{1}{2}\alpha' k\cdot k}\prod_{i=1}^{n-2}|x_i-z|^{2\alpha' p_i k} K_n(x_i,z_i,\bar{z}_i) \qquad (3.34)\end{aligned}$$

Furthermore, we can write the (3.34) in more general integral form as [25]

$$\mathcal{F}_n(1,\ldots,n-2;q_1,q_2) = C_{D_2}\int_{\mathcal{I}_{n-2}}\prod_{i=1}^{n-2}dx_i\prod_{1\le r<s\le n-2}|x_r-x_s|^{2\alpha' p_r p_s+n_{rs}}$$
$$\times\int_{H_+}d^2z(z-\bar{z})^{2\alpha' q_1 q_2+n}\prod_{i=1}^{n-2}(x_i-z)^{2\alpha' p_i q_1+n_i}(x_i-\bar{z})^{2\alpha' p_i q_2+\bar{n}_i}. \qquad (3.35)$$

The integers n_{rs}, n_i, $\bar{n}_i$, and n are determined by external states. Notice that when $q_1 = q_2 = k/2$, the integral (3.35) reproduces the mixed string amplitude (3.34) when the contents of external states are identified.

CHAPTER IV

RELATIONS OF STRING SCATTERING AMPLITUDES

In this chapter, we review the interesting relations among string scattering amplitudes at tree-level, namely, the relations between closed and open string amplitudes, relations among color ordered open string amplitudes, and the relations between mixed string and open string amplitudes.

4.1 Relations between closed string and open string ampliutdes at tree-level

This section is devoted to the review of a paper by Søndergaard [1]. The KLT relation is the relation between closed and open string amplitudes at tree level which gives a description towards gravity and gauge theory. This relation was derived by Kawai, Lewellen and Tye in 1985 [13]. To obtained this relation, They factorized the closed string amplitudes into sum of products of two open string amplitudes.

4.1.1 Factorization

The n-point tree-level closed string amplitudes is given by (3.26). According to the $PSL(2,\mathbb{C})$ symmetry, we can choose three points, i.e., $z_1 = 0$, $z_{n-1} = 1$, and $z_n = \infty$, we then obtain the closed string amplitude of the form

$$\mathcal{A}_n^{\text{cl}} = \left(\frac{i}{2\pi\alpha}\right)^{n-3} \int \prod_{i=2}^{n-2} d^2 z_i |z_i|^{2\alpha' k_1 \cdot k_i} |z_i - 1|^{2\alpha' k_{n-1}\cdot k_i} \prod_{i<j\leq n-2} |z_j - z_i|^{2\alpha' k_i \cdot k_j} f(z_i) g(\bar{z}_i), \tag{4.1}$$

The $f(z_i)g(\bar{z}_i)$ are identified as $F_n(z,\bar{z})$ in (3.26), which depending on the types of external state being considered, but they are not important because they contain no branch cuts. The normalization constant, C_{S^2}, is $(\frac{i}{2\pi\alpha})^{n-3}$

We can write z_i in the form of $z_i = v_i^1 + iv_i^2$, such that

$$|z_i|^{2\alpha' k_1\cdot k_i} = [(v_i^1)^2 + (v_i^2)^2]^{\alpha' k_1\cdot k_i}, \tag{4.2}$$

$$|z_i - 1|^{2\alpha' k_{n-1}\cdot k_i} = [(v_i^1 - 1)^2 + (v_i^2)^2]^{\alpha' k_{n-1}\cdot k_i}, \tag{4.3}$$

$$|z_j - z_i|^{2\alpha' k_i\cdot k_j} = [(v_j^1 - v_i^1)^2 + (v_j^2 - v_i^2)^2]^{\alpha' k_i\cdot k_j}. \tag{4.4}$$

We can rotate the contour of integration of v_i^2 from real axis to the imaginary axis, without changing the value of amplitude.

$$v_i^2 \to ie^{-2i\epsilon} v_i^2 \simeq i(1 - 2i\epsilon)v_i^2, \tag{4.5}$$

where $\epsilon > 0$ is a small number which is there to make sure that we avoid all the branch points. The integrands become (up to linear order of ϵ)

$$[(v_i^1)^2 + (v_i^2)^2]^{\alpha' k_1\cdot k_i} \to [(v_i^1)^2 - (v_i^2)^2 + 4i\epsilon(v_i^2)^2]^{\alpha' k_1\cdot k_i}, \tag{4.6}$$

$$[(v_i^1 - 1)^2 + (v_i^2)^2]^{\alpha' k_{n-1}\cdot k_i} \to [(v_i^1)^2 - (v_i^2)^2 - 2v_i^1 + 1 + 4i\epsilon(v_i^2)]^{\alpha' k_{n-1}\cdot k_i}, \tag{4.7}$$

$$[(v_j^1 - v_i^1)^2 + (v_j^2 - v_i^2)^2]^{\alpha' k_i\cdot k_j} \to [(v_j^1 - v_i^1)^2 - (v_j^2 - v_i^2)^2(1 - 4i\epsilon)]^{\alpha' k_i\cdot k_j}. \tag{4.8}$$

Then, we transform our variables in this way

$$v_i^{\pm} \equiv v_i^1 \pm v_i^2, \tag{4.9}$$

and define

$$\delta_i \equiv v_i^+ - v_i^-. \tag{4.10}$$

Now, the integrand in terms of the new variables are

$$(v_i^+ - i\epsilon\delta_i)^{\alpha' k_1\cdot k_i}(v_i^- + i\epsilon\delta_i)^{\alpha' k_1\cdot k_i}, \tag{4.11}$$

$$(v_i^+ - 1 - i\epsilon\delta_i)^{\alpha' k_{n-1}\cdot k_i}(v_i^- - 1 + i\epsilon\delta_i)^{\alpha' k_{n-1}\cdot k_i}, \tag{4.12}$$

$$(v_i^+ - v_j^+ - i\epsilon(\delta_i - \delta_j))^{\alpha' k_i\cdot k_j}(v_i^- - v_j^- + i\epsilon(\delta_i - \delta_j))^{\alpha' k_i\cdot k_j}. \tag{4.13}$$

Totally, this brings (4.1) into the form

$$\mathcal{A}_n^{\text{cl}} = \left(\frac{i}{2}\right)^{n-3}\left(\frac{i}{2\pi\alpha'}\right)^{n-3}\int_{-\infty}^{+\infty}\prod_{i=2}^{n-2} dv_i^+ dv_i^- f(v_i^-)g(v_i^+)$$
$$\times (v_i^+ - i\epsilon\delta_i)^{\alpha' k_1\cdot k_i}(v_i^- + i\epsilon\delta_i)^{\alpha' k_1\cdot k_i}(v_i^+ - 1 - i\epsilon\delta_i)^{\alpha' k_{n-1}\cdot k_i}(v_i^- - 1 + i\epsilon\delta_i)^{\alpha' k_{n-1}\cdot k_i}$$
$$\times \prod_{i<j\leq n-2}(v_i^+ - v_j^+ - i\epsilon(\delta_i - \delta_j))^{\alpha' k_i\cdot k_j}(v_i^+ - v_j^+ + i\epsilon(\delta_i - \delta_j))^{\alpha' k_i\cdot k_j}, \tag{4.14}$$

where $\left(\frac{i}{2}\right)^{n-3}$ is from the change of variables and rotation of the contours of v_i^2. Assume that at least one $v_i^+ \in]-\infty, 0[$, and consider the contribution of v_i^- that is

$$\int_{-\infty}^{+\infty} dv_i^- f(v_i^-)(v_i^- + i\epsilon\delta_i)^{\alpha' k_1\cdot k_i}(v_i^- - 1 + i\epsilon\delta_i)^{\alpha' k_{n-1}\cdot k_i} \tag{4.15}$$
$$\times \prod_{i<j\leq n-2}(v_i^- - v_j^- - i\epsilon(\delta_i - \delta_j))^{\alpha' k_i\cdot k_j}$$

Let us take a look at the behaviour of the imaginary ϵ terms near the branch points which is

$$v_i^- \sim 0 \;\rightarrow\; \delta_i \sim v_i^+ < 0, \tag{4.16}$$
$$v_i^- \sim 1 \;\rightarrow\; \delta_i \sim v_i^+ - 1 < 0,$$
$$v_i^- \sim v_j^- \;\rightarrow\; \delta_i - \delta_j \sim v_i^+ - v_j^+ < 0 \quad \text{when} \quad v_i^+ < v_j^+.$$

In general, if $v_i^+ < v_j^+$, we will avoid the branch point $v_i^- = v_j^-$ below the real axis. If $v_i^+ > v_j^+$, we will avoid it above the real axis. If $v_i^- \sim 0$, we will avoid the branch points above the real axis. If $v_i^- \sim 1$, we will avoid branch points below the real axis. However, when considering the v_i^- integral corresponding to the smallest v_i^+, which belong to $]-\infty, 0[$. This means that we can closed the contour at lower half-plane of v_i^-, and in this contour there is no pole, the result of the integration is equal to zero. From these arguments, when one of v_i^+ variable is in $]-\infty, 0[$ or $]1, \infty[$, we can closed the contour of v_i^- above or below the real axis which make the integration vanish. Therefore, only the value v_i^+ in $[0, 1]$ can contribute to (4.14).

The n-point closed string amplitudes after splitting up the v_i^+-integration region can be written as

$$\mathcal{A}_n^{\rm cl} = \sum_{\sigma} M_n^{\sigma}(\sigma(2), ..., \sigma(n-2)), \tag{4.17}$$

where $M_n^{\sigma}(\sigma(2), ..., \sigma(n-2))$ is the ordered amplitude in which $v^+_{\sigma(2)} < v^+_{\sigma(3)} < ... < v^+_{\sigma(n-2)}$. At this ordering of v_i^+, the v_i^+-part of M_n^{σ} in (4.17)is given by

$$\int_{0<v^+_{\sigma(2)}<\dots<v^+_{\sigma(n-2)}<1} \prod_{i=2}^{n-2} dv_i^+ g(v_i^+)(v_i^+)^{\alpha' k_1\cdot k_i}(1-v_i^+)^{\alpha' k_{n-1}\cdot k_i} \times \prod_{i<j\leq n-2} (v^+_{\sigma(j)} - v^+_{\sigma(i)})^{\alpha' k_{\sigma(i)}\cdot k_{\sigma(j)}} \tag{4.18}$$

We see that (4.18) exactly corresponds to the color-ordered open string amplitude $\mathcal{A}^{\rm op}(1, \sigma(2, \ldots, n-2), n-1, n)$. Remind that the integrand in (4.18) differs from that of (4.14) by the factor of $(-1)^{\alpha' k_{n-1}\cdot k_i}$, in which we will compensate for the difference by altering the integrand of the v^--section in a similar way. For convenience, after this we specify the ordering as $\{2, 3, \ldots, n-2\}$, and we consider only $M_n^{\sigma}(2, 3, \ldots, n-2)$. Note that the full amplitude can be obtained by permuting the ordering, $\{2, 3, \ldots, n-2\}$.

Now, we see that the v_i^+-part provides $\mathcal{A}_n^{\rm op}(1, 2, \ldots, n-1, n)$. Then let us consider the v_i^--part. The contour v_i^- are deformed to avoid the branch point as follows. In the interval $v_i^+ \in]0, 1[$, the contour of v_i^- is above real axis, when $v_i^- \sim 0$ and it is below the real axis, when $v_i^- \sim 1$. When $v_i^- \sim v_j^-$, the contour of v_i^- is below the contour of v_j^-, for $i < j$. The contours are shown in figure 6. Next, we deform the contour for v_i^- into the form which corresponds to the color-order amplitude, namely, we have freedom to close all contours to the left or to the right, but we have to make sure the integrand is correct. This give us the phase factors. To avoid crossing the branch cut we use the following relation with $\mathrm{Re}(z) < 0$,

$$z^c = \begin{cases} e^{i\pi c}(-z)^c & \mathrm{Im}(z) \geq 0 \\ e^{-i\pi c}(-z)^c & \mathrm{Im}(z) < 0. \end{cases} \tag{4.19}$$

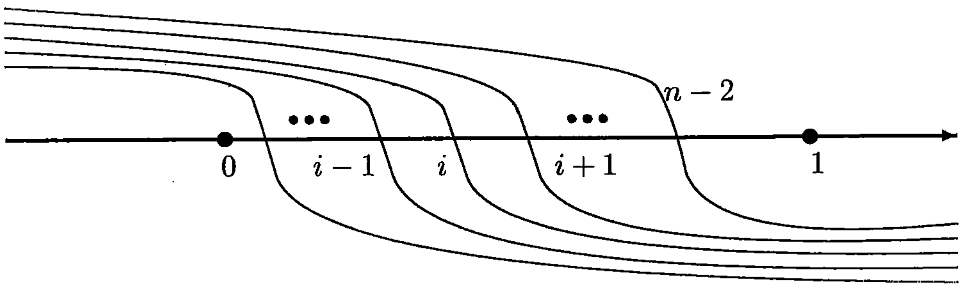

Figure 6 The integrating contour of variables v_i^-. [1]

For $2 \geq j \geq n-1$, we close the contour of 2 to $j-1$ to the left and j to $n-2$ to the right.

4.1.2 Five point KLT Relation

Let us consider the case of $n = 5$, or $\mathcal{A}_5^{\text{cl}}$. Firstly, consider that for $j = 4$ we closed the contour of v_2^- to the left, which is the contour C_2 in figure 7, we obtain

$$\begin{aligned}
&\int_{C_2} dv_2^- (v_2^-)^{\alpha' k_1 \cdot k_2} (1 - v_2^-)^{\alpha' k_4 \cdot k_2} (v_3^- - v_2^-)^{\alpha' k_3 \cdot k_2} f(v_2^-) \\
&\quad = (e^{i\pi\alpha' k_1 \cdot k_2} - e^{-i\pi\alpha' k_1 \cdot k_2}) \left(\int_{-\infty}^{0} dv_2^- (-v_2^-)^{\alpha' k_1 \cdot k_2} (1 - v_2^-)^{\alpha' k_4 \cdot k_2} \right. \\
&\qquad \left. \times (v_3^- - v_2^-)^{\alpha' k_3 \cdot k_2} f(v_2^-) \right) \\
&\quad = 2i \sin(\pi\alpha' k_1 \cdot k_2) \int_{-\infty}^{0} dv_2^- (-v_2^-)^{\alpha' k_1 \cdot k_2} (1 - v_2^-)^{\alpha' k_4 \cdot k_2} (v_3^- - v_2^-)^{\alpha' k_3 \cdot k_2} f(v_2^-).
\end{aligned} \tag{4.20}$$

Similarly, as illustrated in figure 7, the contour of v_3^- is C_3. We then obtain

$$\begin{aligned}
&\int_{C_3} dv_3^- (v_3^-)^{\alpha' k_1 \cdot k_3} (1 - v_3^-)^{\alpha' k_4 \cdot k_3} (v_3^- - v_2^-)^{\alpha' k_3 \cdot k_2} f(v_3^-) \\
&\quad = (e^{i\pi\alpha' (k_1 + k_2) \cdot k_3} - e^{-i\pi\alpha' (k_1 + k_2) \cdot k_3}) \left(\int_{-\infty}^{v_2^-} dv_3^- (v_3^-)^{\alpha' k_1 \cdot k_3} (1 - v_3^-)^{\alpha' k_4 \cdot k_3} \right. \\
&\qquad \left. \times (v_3^- - v_2^-)^{\alpha' k_3 \cdot k_2} f(v_3^-) \right) + (e^{i\pi\alpha' k_1 \cdot k_3} - e^{-i\pi\alpha' k_1 \cdot k_3}) \left(\int_{v_2^-}^{0} dv_3^- (v_3^-)^{\alpha' k_1 \cdot k_3} \right. \\
&\qquad \left. \times (1 - v_3^-)^{\alpha' k_4 \cdot k_3} (v_3^- - v_2^-)^{\alpha' k_3 \cdot k_2} f(v_3^-) \right).
\end{aligned} \tag{4.21}$$

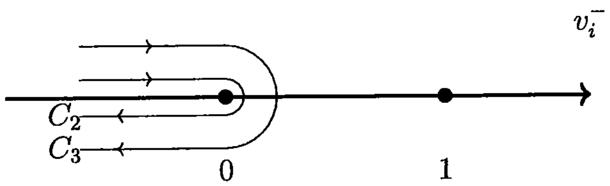

Figure 7 The integration contour of v_2^- and v_3^- for five-point case.

We see that the total integration of v_3^- and v_2^- is corresponding to color-ordered open string amplitude, denoted by $\tilde{\mathcal{A}}^{\mathrm{op}}$. We combine the v_i^+ and v_i^-, we obtain the relation between five-point closed string amplitude $\mathcal{A}_5^{\mathrm{cl}}$ and the color-ordered open string amplitude $\mathcal{A}^{\mathrm{op}}$,$\tilde{\mathcal{A}}^{\mathrm{op}}$

$$\begin{aligned}\mathcal{A}_5^{\mathrm{cl}} = \frac{-1}{4(\pi\alpha')^2}\Big[&\sin(\pi\alpha' k_1\cdot k_2)\sin(\pi\alpha'(k_1+k_2)\cdot k_3)\\ &\times \mathcal{A}^{\mathrm{op}}(1,2,3,4,5)\tilde{\mathcal{A}}^{\mathrm{op}}(3,2,1,4,5)+\sin(\pi\alpha' k_1\cdot k_2)\sin(\pi\alpha' k_1\cdot k_3)\\ &\times \mathcal{A}^{\mathrm{op}}(1,2,3,4,5)\tilde{\mathcal{A}}^{\mathrm{op}}(2,3,1,4,5)\Big]+(2\leftrightarrow 3)\end{aligned} \tag{4.22}$$

If we close all contour to the right, we will obtain

$$\begin{aligned}\mathcal{A}_5^{\mathrm{cl}} = \frac{-1}{4(\pi\alpha')^2}\Big[&\sin(\pi\alpha' k_4\cdot k_3)\sin(\pi\alpha' k_4\cdot k_2)\mathcal{A}^{\mathrm{op}}(1,2,3,4,5)\tilde{\mathcal{A}}^{\mathrm{op}}(1,4,2,3,5)\\ &+\sin(\pi\alpha' k_4\cdot k_3)\sin(\pi\alpha'(k_4+k_3)\cdot k_2)\mathcal{A}^{\mathrm{op}}(1,2,3,4,5)\tilde{\mathcal{A}}^{\mathrm{op}}(1,4,2,3,5)\Big]\\ &+(2\leftrightarrow 3).\end{aligned} \tag{4.23}$$

If we closed v_2^- to the left and v_3^- to the right, we will obtain

$$\begin{aligned}\mathcal{A}_5^{\mathrm{cl}} =&\frac{-1}{4(\pi\alpha')^2}\Big[\sin(\pi\alpha' k_1\cdot k_2)\sin(\pi\alpha' k_4\cdot k_3)\mathcal{A}^{\mathrm{op}}(1,2,3,4,5)\tilde{\mathcal{A}}^{\mathrm{op}}(2,1,3,4,5)\Big]\\ &+(2\leftrightarrow 3).\end{aligned} \tag{4.24}$$

We can write all different form into one compact form by introducing the momentum kernel [29]

$$\mathcal{S}_{\alpha'}[i_1,...,i_k|j_1,...,j_k]_p \equiv (\pi\alpha'/2)^{-k}\prod_{t=1}^{k}\sin\Big(\pi\alpha'(p\cdot k_{i_t}+\sum_{q>t}^{k}\theta(i_t,i_q)k_{i_t}\cdot k_{i_q})\Big) \tag{4.25}$$

where $\theta(i_t, i_q)$ equal to 1 if ordering of i_t and i_q is opposite in $\{i_1, ..., i_k\}$ and $\{j_1, ..., j_k\}$ and 0 if the ordering is the same. For example,

$$\begin{aligned}
\mathcal{S}_{\alpha'}[2|2]_{k_1} &= (\pi\alpha'/2)^{-1}\sin(\pi\alpha' k_1 \cdot k_2),\\
\mathcal{S}_{\alpha'}[23|23]_{k_1} &= (\pi\alpha'/2)^{-2}\sin(\pi\alpha' k_1 \cdot k_2)\sin(\pi\alpha' k_1 \cdot k_3),\\
\mathcal{S}_{\alpha'}[23|32]_{k_1} &= (\pi\alpha'/2)^{-2}\sin(\pi\alpha'(k_1+k_3) \cdot k_2)\sin(\pi\alpha' k_1 \cdot k_3),
\end{aligned} \tag{4.26}$$

and also define $\mathcal{S}_{\alpha'}[\emptyset|\emptyset]_p = 1$. Finally, we obtain the relation in term of momentum kernel as

$$\begin{aligned}
\mathcal{A}_5^{\text{cl}} = (\frac{-i}{4})^2 \sum_{\sigma}\sum_{\gamma,\beta} & \mathcal{S}_{\alpha'}[\gamma(\sigma(2),\ldots,\sigma(j-1))|\sigma(2),\ldots,\sigma(j-1))]_{k_1}\\
& \times \mathcal{S}'_{\alpha}[\sigma(j),\cdot,\sigma(3)|\beta(\sigma(j),\ldots,\sigma(3))]_{k_4} \times \mathcal{A}^{\text{op}}(1,\sigma(2,3),4,5)\\
& \times \tilde{\mathcal{A}}^{\text{op}}(\gamma(\sigma(2),\ldots,\sigma(j-1)),1,4,\beta(\sigma(j),\ldots,\sigma(3)),5),
\end{aligned} \tag{4.27}$$

with j ={2,3,4}.

4.1.3 General n-point KLT Relations

We can generalize the relation to the n-point case, the relation take the form

$$\begin{aligned}
\mathcal{A}_n^{\text{cl}} = (\frac{-i}{4})^{n-3} \Big[& \sum_{\sigma}\sum_{\gamma,\beta} \mathcal{S}_{\alpha'}[\gamma(\sigma(2),\ldots,\sigma(j-1))|\sigma(2),\ldots,\sigma(j-1))]_{k_1}\\
& \times \mathcal{S}'_{\alpha}[\sigma(j),\cdot,\sigma(n-2)|\beta(\sigma(j),\ldots,\sigma(n-2))]_{k_{n-1}} \mathcal{A}^{\text{op}}(1,\sigma(2,\ldots,n-2),n-1,n)\\
& \times \tilde{\mathcal{A}}^{\text{op}}(\gamma(\sigma(2),\ldots,\sigma(j-1)),1,n-1,\beta(\sigma(j),\ldots,\sigma(n-2)),n)\Big],
\end{aligned} \tag{4.28}$$

with $2 \leq j \leq n-1$. The expression (4.28) shows that the n-point closed string amplitudes $\mathcal{A}_n^{\text{cl}}$ can be factorized into the product of n-point color-ordered open string amplitudes $\mathcal{A}^{\text{op}}$ and $\tilde{\mathcal{A}}^{\text{op}}$ with the $\mathcal{S}_{\alpha'}$ which contain the kinematic factors.

4.2 Relations among tree-level color-ordered open string amplitudes

The relation among color-ordered open string amplitudes were covered by N.E.J. Bjerrum-Bohr, Poul H. Damgaard and Pierre Vanhove [2]. They formulated relation among tree-level string theory amplitudes. The relations among different color ordered tree-level amplitudes were derived via monodromy relations. It turns out that the number of independent n-point amplitudes is $(n-3)!$ which is the minimal number of basis for amplitudes in gauge theory.

4.2.1 The four-point amplitudes

For a four-point amplitude, we can expand the amplitude as

$$\mathcal{A}_4^{\mathrm{op}} \sim g_{YM}^2 \,\mathrm{tr}\big(T^1T^2T^3T^4\big)\mathcal{A}^{\mathrm{op}}(1,2,3,4) + \text{permutations} \tag{4.29}$$

For simplicity, we will consider the tachyon amplitude, i.e. $F_n = 1$. Since we use the choice x_1, x_3 and x_4 are equal to 0,1 and ∞ respectively, all three different color-ordered amplitudes $\mathcal{A}^{\mathrm{op}}(i,j,k,l)$ are given by the same integrands but different domains of integration

$$\mathcal{A}^{\mathrm{op}}(1,2,3,4) = \int_0^1 dx\; x^{2\alpha' p_1\cdot p_2}(1-x)^{2\alpha' p_2\cdot p_3}, \tag{4.30}$$

$$\mathcal{A}^{\mathrm{op}}(1,3,2,4) = \int_1^{\infty} dx\; x^{2\alpha' p_1\cdot p_2}(x-1)^{2\alpha' p_2\cdot p_3}, \tag{4.31}$$

$$\mathcal{A}^{\mathrm{op}}(2,1,3,4) = \int_{-\infty}^{0} dx\; (-x)^{2\alpha' p_1\cdot p_2}(1-x)^{2\alpha' p_2\cdot p_3}. \tag{4.32}$$

Let us consider the $\mathcal{A}^{\mathrm{op}}(1,3,2,4)$, we can see that we integrate from 1 to $+\infty$. Assuming that the $\alpha' p_i \cdot p_j$ are complex number with negative real part, we can deform the integration region, so we can close contour slightly above the real axis. By deforming the contour, we can change the expression into the integration from $-\infty$ to 1. But we have to put the appropriate the phase factor when rotating the contour.

$$(x-y)^{\alpha} = (y-x)^{\alpha} \times \begin{cases} e^{+i\pi\alpha} & \text{for clockwise rotation} \\ e^{-i\pi\alpha} & \text{for counterclockwise rotation.} \end{cases} \tag{4.33}$$

The amplitude $\mathcal{A}^{\mathrm{op}}(1,3,2,4)$ becomes

$$\mathcal{A}^{\mathrm{op}}(1,3,2,4) = -e^{-2i\alpha'\pi p_2\cdot p_3}\mathcal{A}^{\mathrm{op}}(1,2,3,4) - e^{-2i\alpha'\pi p_2\cdot(p_1+p_3)}\mathcal{A}^{\mathrm{op}}(2,1,3,4). \tag{4.34}$$

We know that the original amplitude is real, we have

$$\mathcal{A}^{\text{op}}(1,3,2,4) = -\operatorname{Re}\Big\{e^{-2i\alpha'\pi p_2\cdot p_3}\mathcal{A}^{\text{op}}(1,2,3,4) + e^{-2i\alpha'\pi p_2\cdot(p_1+p_3)}\mathcal{A}^{\text{op}}(2,1,3,4)\Big\}, \tag{4.35}$$

$$0 = -\operatorname{Im}\Big\{e^{-2i\alpha'\pi p_2\cdot p_3}\mathcal{A}^{\text{op}}(1,2,3,4) + e^{-2i\alpha'\pi p_2\cdot(p_1+p_3)}\mathcal{A}^{\text{op}}(2,1,3,4)\Big\}. \tag{4.36}$$

According to above equations, we can relate all amplitudes to the $\mathcal{A}^{\text{op}}(1,2,3,4)$, which are

$$\mathcal{A}^{\text{op}}(1,3,2,4) = \frac{\sin(2\alpha'\pi p_1\cdot p_2)}{\sin(2\alpha'\pi p_2\cdot p_4)}\mathcal{A}^{\text{op}}(1,2,3,4), \tag{4.37}$$

$$\mathcal{A}^{\text{op}}(2,1,3,4) = \frac{\sin(2\alpha'\pi p_2\cdot p_3)}{\sin(2\alpha'\pi p_2\cdot p_4)}\mathcal{A}^{\text{op}}(1,2,3,4). \tag{4.38}$$

By taking filed theory limit, $\alpha' \to 0$, we obtain the relations between field theory amplitudes :

$$\begin{aligned} A(1,3,2,4) &= \frac{p_1\cdot p_2}{p_2\cdot p_4}A(1,2,3,4), \\ A(2,1,3,4) &= \frac{p_2\cdot p_3}{k_2\cdot p_4}A(1,2,3,4). \end{aligned} \tag{4.39}$$

These relations agree with [18].

4.2.2 The n-point amplitude

In this part we will see the derivation that any color-ordered n-point amplitude can be written in terms of a minimal basis of $(n-3)!$ amplitude. For an n-point ordered amplitude, there are three fixed points, i.e. $x_1 = 0$, $x_{\alpha_k} = 1$ and $x_n = \infty$. Let say they contain r points, $\{\beta_1,\ldots,\beta_r\}$, in $]-\infty,0[$, $k-1$ points ,$\{\alpha_1,\ldots,\alpha_{k-1}\}$,in $]1,0[$, and $s-k$ points, $\{\alpha_{k+1},\ldots,\alpha_s\}$, in $]1,\infty[$ illustrated in figure 8.

Let us consider the integrations of the $\{\beta_1,\ldots,\beta_r\}$, we can flip them from the region $]-\infty,0[$ to $]0,+\infty[$.

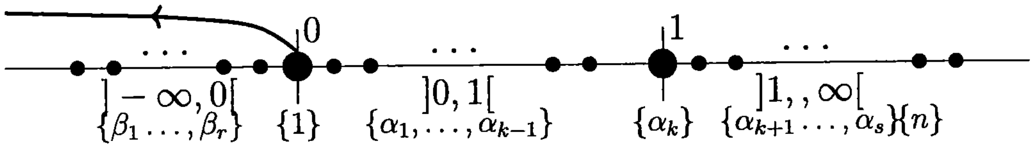

Figure 8 Contour for amplitude $\mathcal{A}^{\rm op}(\beta_1,\ldots,\beta_r,1,\alpha_1,\ldots,\alpha_s,n)$[2]

By taking the real part of the amplitude, we then obtain the following relation

$$\mathcal{A}^{\rm op}(\beta_1,\ldots,\beta_r,1,\alpha_1,\ldots,\alpha_s,n)=(-1)^r\times$$
$$\mathrm{Re}\left[\prod_{1\le i<j\le r} e^{2i\pi\alpha'(p_{\beta_i}\cdot p_{\beta_j})}\sum_{\sigma\subset \mathrm{OP}\{\alpha\}\cup\{\beta^T\}}\prod_{i=0}^{s}\prod_{j=1}^{r} e^{(\alpha_i,\beta_j)}\mathcal{A}^{\rm op}(1,\sigma,n)\right] \tag{4.40}$$

where $e^{(\alpha,\beta)}\equiv e^{2i\pi\alpha'(p_\alpha\cdot p_\beta)}$. The relation (4.40) reduces the set of independent amplitudes from $(n-1)!$ to $(n-2)!$.

Since the amplitudes are real, then the imaginary part vanishes, i.e.

$$0=\mathrm{Im}\left[\prod_{1\le i<j\le r} e^{2i\pi\alpha'(p_{\beta_i}\cdot p_{\beta_j})}\sum_{\sigma\subset \mathrm{OP}\{\alpha\}\cup\{\beta^T\}}\prod_{i=0}^{s}\prod_{j=1}^{r} e^{(\alpha_i,\beta_j)}\mathcal{A}^{\rm op}(1,\sigma,n)\right] \tag{4.41}$$

By using(4.40) we eliminate all amplitudes with point in the $]-\infty,0[$ in terms of amplitudes having the points in $]0,+\infty[$. Then, by using (4.41), we can write amplitudes in $]1,+\infty[$ in terms of amplitude in $[0,1]$ which has $n-3$ points. We can say that color-ordered amplitudes can be written in terms of $(n-3)!$ amplitudes whose unfixed points are in the interval $[0,1]$.

4.3 Relations between mixed and open string amplitudes

Stephen Stieberger and Tomasz R. Taylor formulated the relation between mixed string amplitude and open string amplitudes [25]. They demonstrate that the mixed string amplitude can be decomposed as linear combination of pure open string amplitudes.

The integral form of the disk amplitude given by (3.35). We write $z = z_1 + iz_2$, the integrands become analytic function of z_2 with $2(N-2)$ branch points at $\pm i(x_i - z_1)$. We deform the integration of z_2 from real axis to pure imaginary axis, $z_2 \to iz_2$. In this way, the variables

$$\xi = z_1 + iz_2 \equiv z \quad , \quad \eta = z_1 - iz_2 \equiv \bar{z}, \tag{4.42}$$

become real and are subject to

$$\eta > \xi. \tag{4.43}$$

After changing the variables, (3.35) takes the form

$$\begin{aligned}\mathcal{F}_N = C_{D_2}\delta(\sum_{i=1}^{N} k_i) \int \prod_{i=1}^{N-2} dx_i \int_{-\infty}^{\infty} d\xi \int_{\xi}^{\infty} d\eta \prod_{1\leq r<s\leq N-2} |x_r - x_s|^{2\alpha' p_r p_s} (x_r - x_s)^{n_{rs}} \\ \times \frac{i}{2}(\xi - \eta)^n |\xi - \eta|^{2\alpha' k_{N-1} k_N} \Omega(\xi, \eta) \\ \times \prod_{i=1}^{N-2} \Pi(x_i, \xi, \eta) |x_i - \xi|^{2\alpha' k_i k_{N-1}} |x_i - \eta|^{2\alpha' k_i k_N} (x_i - \xi)^{n_i} (x_i - \eta)^{\bar{n}_i}.\end{aligned} \tag{4.44}$$

The phase factors in (4.44) are defined as

$$\begin{aligned}\Omega(\xi, \eta) &= e^{2\pi i\alpha' k_{N-1} k_N \Theta(\eta - \xi)}, \\ \Pi(x_i, \xi, \eta) &= e^{-2\pi i\alpha' k_i k_{N-1} \Theta(\xi - x_i)} e^{2\pi i\alpha' k_i k_N \Theta(\eta - x_i)},\end{aligned} \tag{4.45}$$

where $\Theta(x)$ is the Heaviside step function.

After we analyze the phase factors in (4.44) and fixed $x_1 = -\infty$. We then obtain

$$\begin{aligned}\mathcal{F}_N =& \frac{i}{2} \sum_{l=1}^{N-2} \sum_{i=l}^{N-2} \exp\left\{\pi i(S_{N,N-1} + \sum_{j=1}^{i} S_{j,N} - \sum_{j=1}^{l} S_{j,N-1})\right\} \\ & \times \mathcal{A}^{\text{op}}(1, \ldots, l, N-1, l+1, \ldots, i, N, \ldots, N-2),\end{aligned} \tag{4.46}$$

where $S_{ij} = 2\alpha' k_i k_j$. We define $\mathcal{F}_N$ as

$$\mathcal{F}_N \equiv \mathcal{M}(1, 2, \ldots, N-2; q_1, q_2). \tag{4.47}$$

We obtain the relation between mixed string amplitude and open strings as

$$\mathcal{M}(1,2,\ldots,N-2;q_1,q_2) = \frac{i}{2}\sum_{l=1}^{N-2}\sum_{i=l}^{N-2}\mathcal{A}^{\rm op}(1,\ldots,l,N-1,l+1,\ldots,i,N,\ldots,N-2)$$
$$\times \exp\left\{\pi i(S_{1,N-1} - S_{1,N} + S_{N,N-1} + \sum_{j=1}^{i} S_{j,N} - \sum_{j=1}^{l} S_{j,N-1})\right\}. \quad (4.48)$$

We can use monodromy relations, we then obtain

$$\mathcal{M}(1,2,\ldots,N-2;q_1,q_2) = -\frac{i}{2}\sum_{l=1}^{\lceil\frac{N}{2}\rceil-1}\sum_{i=2}^{l+1}\exp\left\{\pi i\left(\sum_{j=2}^{i-1} S_{j,N} - \sum_{j=2}^{l+1} S_{j,N-1}\right)\right\}$$
$$\times \mathcal{A}^{\rm op}(1,\ldots,i-1,N,i,\ldots,l,N-1,l+1,\ldots,N-2)$$
$$+\frac{i}{2}\sum_{l=\lceil\frac{N}{2}\rceil}^{N-2}\sum_{i=l}^{N-2}\exp\left\{\pi i\left(S_{1,N-1} - S_{1,N} + S_{N,N-1} + \sum_{j=l+1}^{N-2} S_{j,N} - \sum_{j=i+1}^{N-2} S_{j,N-1}\right)\right\}$$
$$\times \mathcal{A}^{\rm op}(1,\ldots,l,N-1,l+1,\ldots,i,N,i+1,\ldots,N-2). \quad (4.49)$$

The above expression shows that mixed amplitudes can be decomposed as linear combinations of pure open string amplitudes.

By utilizing the string monodromy relations [2], the expression (4.49) can be expressed in terms of the $(N-3)!$ open string amplitudes:

$$\mathcal{M}(1,2,\ldots,N-2;q_1,q_2) = (-1)^N e^{-\pi i(S_{1,N}+S_{2,N-1})}\sum_{l=2}^{N-2}(-1)^l \sin\left(\pi S_{l,N-1}\right)$$
$$e^{\pi i(-1)^l S_{l,N-1}}\sum_{\rho\in\{OP(\alpha,\beta^t),l\}} e^{\pi i\sum_{k=1}^{\lfloor\frac{N-3}{2}\rfloor}\tau_{2k+1}(\rho)}\mathcal{S}(\rho)\mathcal{A}^{\rm op}(1,\rho,N-1,N). \quad (4.50)$$

The $\mathcal{S}(\rho)$ is given by

$$\mathcal{S}(\rho) \equiv \mathcal{S}\left[\rho(2,\ldots,N-2)\right] = \prod_{i=2}^{N-2}\prod_{j=i+1}^{N-2}\exp\{\pi i\Theta(\rho^{-1}(i) - \rho^{-1}(j))S_{i,j}\}, \quad (4.51)$$

where $\rho^{-1}(i)$ is the position of i in ρ.

The $\tau_i(\rho)$ takes the form

$$\tau_i(\rho) = \begin{cases} \mathrm{sign}(\rho^{-1}(i) - \rho^{-1}(i+1))(S_{i,N-1} + S_{i+1,N-1}), & 3 \leq i \leq N-3, \\ S_{N-2,N-1}, & i = N-2. \end{cases} \quad (4.52)$$

The set $OP(\alpha, \beta^t)$ is the merged sets of $\alpha(2, \dots, l-1)$ and $\beta(l+1, \dots, N-2)$ and the β^t denotes reversal of elements in set β. The second summation in (4.50) sums over all permutations ρ, which are composed of the element l and the ordered set of permutations $OP(\alpha, \beta^t)$.

In the field theory limit, $\alpha' \to 0$, the string amplitudes are reduced to the field theory amplitudes A_{FT}. If all open string states are specified to be gauge bosons, the field theory amplitudes are basically Yang-Mills (YM) amplitudes, which are denoted by A_{YM} [30]. Similarly, the mixed string amplitudes can be reduced to field theory amplitudes in this limit. If the closed string state is graviton and the open string states are gauge bosons, the field theory amplitudes are Einstein-Yang-Mills (EYM) amplitudes, which are denoted by A_{EYM}. The relations between EYM and YM are discussed in [25, 27, 28].

In the field theory limit, the leading order of the expression in (4.50) is the order of α', while the leading order in (4.49) is the order of α'^0. This gives us relations similar to Kleiss-Kuijf relations [19],

$$\begin{aligned} 0 = &- \sum_{l=1}^{\lceil \frac{N}{2} \rceil - 1} \sum_{i=2}^{l+1} A_{YM}(1, \dots, i-1, N, i, \dots, l, N-1, l+1, \dots, N-2) \\ &+ \sum_{l=\lceil \frac{N}{2} \rceil}^{N-2} \sum_{i=l}^{N-2} A_{YM}(1, \dots, l, N-1, l+1, \dots, i, N, i+1, \dots, N-2). \end{aligned} \tag{4.53}$$

Let's consider the case of four-dimensional spacetime filled by a D-brane and the closed string momentum is purely four-dimensional momentum which is on-shell, P^2=0. We split the closed string momentum as

$$q_1 = k_{N-1} = xP, \qquad q_2 = k_N = (1-x)P. \tag{4.54}$$

Then, we have

$$\begin{aligned} &S_{N-1,N} = 0, \\ &S_{i,N-1} = xS_{iP}, \qquad S_{i,N} = (1-x)S_{iP}, \qquad i = 1, \dots, N-2 \end{aligned} \tag{4.55}$$

In the field theory limit, the α'^0 order vanishes as result (4.53), we will consider the α' order of (4.49) and (4.50). This yields

$$
\begin{aligned}
&A_{EYM}(1,2,\ldots,N-2;P) = \\
&\frac{\pi}{2}x\Big\{\sum_{l=1}^{\lceil\frac{N}{2}\rceil-1}\sum_{i=2}^{l}(\sum_{j=i}^{l}S_{jP})A_{YM}(1,\ldots,i-1,N,i,\ldots,l,N-1,l+1,\ldots,N-2) \\
&+\sum_{l=\lceil\frac{N}{2}\rceil}^{N-3}\sum_{i=l+1}^{N-2}(\sum_{j=l+1}^{i}S_{jP})A_{YM}(1,\ldots,l,N-1,l+1,\ldots,i,N,i+1,\ldots,N-2).\Big\} \\
&+\frac{\pi}{2}(2x-1)\Big\{\sum_{l=1}^{\lceil\frac{N}{2}\rceil-1}\sum_{i=2}^{l+1}(\sum_{j=2}^{i-1}S_{jP})A_{YM}(1,\ldots,i-1,N,i,\ldots,l,N-1,l+1,\ldots,N-2) \\
&-\sum_{l=\lceil\frac{N}{2}\rceil}^{N-2}\sum_{i=l}^{N-2}(\sum_{j=2}^{i}S_{jP})A_{YM}(1,\ldots,l,N-1,l+1,\ldots,i,N,i+1,\ldots,N-2)\Big\}.
\end{aligned}
\tag{4.56}
$$

By applying the BCJ [18] and Kleiss-Kuijf relations [19], we then obtain

$$
\begin{aligned}
&A_{EYM}(1,2,\ldots,N-2;P) = \pi x(1-x) \\
&\times\Big\{\sum_{l=2}^{\lceil\frac{N}{2}\rceil-1}\sum_{i=2}^{l}(\sum_{j=i}^{l}S_{jP})A_{YM}(1,\ldots,i-1,N,i,\ldots,l,N-1,l+1,\ldots,N-2) \\
&+\sum_{l=\lceil\frac{N}{2}\rceil}^{N-3}\sum_{i=l+1}^{N-2}(\sum_{j=l+1}^{i}S_{jP})A_{YM}(1,\ldots,l,N-1,l+1,\ldots,i,N,i+1,\ldots,N-2)\Big\}.
\end{aligned}
\tag{4.57}
$$

To summarize, the mixed string amplitudes can be expressed as a linear combination of open string amplitudes. By considering the low-energy limit, we found that at the order of α'^0, the relations yield the Kleiss-Kuijf relations [19], while at the order of α', they yield the relations (4.57).

4.4 Relations among mixed string amplitudes

The relation among mixed string amplitudes were formulated by Stephen Stieberger and Tomasz R. Taylor [3]. They obtained new relations between Einstein-Yang-Mills amplitudes associated with N gauge bosons and a single graviton and pure Yang-Mills amplitudes associated with N gauge bosons and one vector boson. Especially the worldsheet monodromy relation for mixed string amplitudes, which gives the novel string tube contributions.

4.4.1 Worldsheet monodromy relations for mixed string amplitudes

Mixed string amplitudes involving $N-2$ open strings and one closed string describe by (3.34) or the more general integral form (3.35). According to conformal symmetry, we choose $x_1 = -\infty$, $z = i$ and $\bar{z} = -i$. Then we consider the real integration with respect to x_2 and by utilizing the analytic continuation of x_2 along the contour integral depicted in Fig. 9 give us the following relation

$$\begin{aligned}
&\mathcal{M}(1,2,\ldots,N-2;q_1,q_2) + e^{-i\pi s_{23}}\mathcal{M}(1,3,2,\ldots,N-2;q_1,q_2) \\
&\quad + e^{-i\pi(s_{23}+s_{24})}\mathcal{M}(1,3,4,2,\ldots,N-2;q_1,q_2) \\
&\quad + \cdots + e^{-i\pi(s_{23}+s_{24}+\cdots+s_{2,N-2})}\mathcal{M}(1,3,\ldots,N-2,2;q_1,q_2) = T(3,\ldots,N-2),
\end{aligned} \tag{4.58}$$

where $s_{ij} \equiv 2\alpha' p_i p_j$. The tube contribution, $T(3,\ldots,N-2)$, takes the form

$$\begin{aligned}
T(3,\ldots,N-2) &= \delta(\sum_{i=1}^{N} k_i)\sin(\pi s_{2,N-1})e^{-i\pi s_{1,N}} \\
&\times \int_1^\infty dy|y-1|^{s_{2,N-1}}|y+1|^{s_{2,N}} \int_{x_3<\cdots<x_{N-2}} (\prod_{i=3}^{N-2} dx_i) \prod_{3\le r<s\le N-2} |x_r - x_s|^{s_{rs}} \\
&\times \prod_{j=3}^{N-2} |x_j - iy|^{s_{2,j}} |x_j - i|^{s_{j,N-1}} |x_j + i|^{s_{j,N}}.
\end{aligned} \tag{4.59}$$

For simplicity, we consider $N = 5$, the infinite tube takes the form

$$\begin{aligned}
T(3) = \sin(\pi s_{24})e^{-i\pi s_{51}} &\int_1^\infty dy|y-1|^{s_{24}}|y+1|^{s_{25}} \\
&\times \int_{-\infty}^{\infty} dx_3 |x_3 - iy|^{s_{23}}|x_3 - i|^{s_{34}}|x_3 + i|^{s_{35}}.
\end{aligned} \tag{4.60}$$

We deform the contour of integration of x_3 to imaginary axis along contour C depicted in Fig. 10.

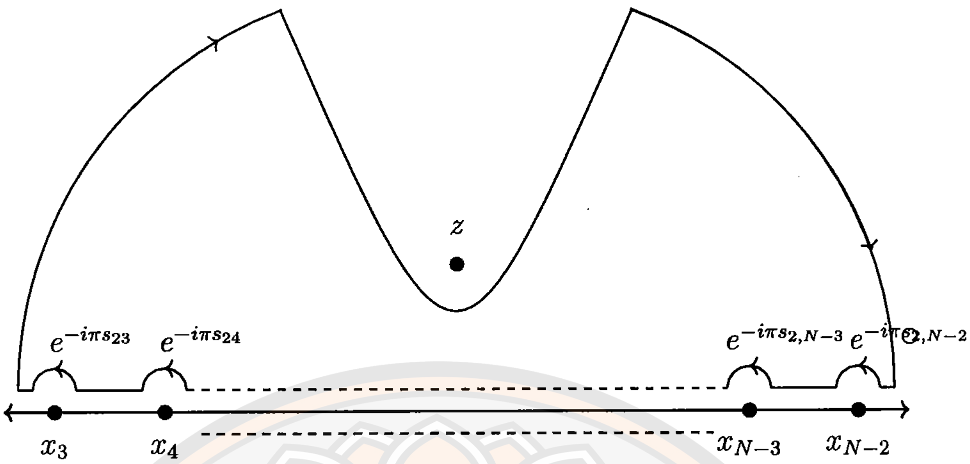

Figure 9 Contour integral in the complex x_2-plane [3].

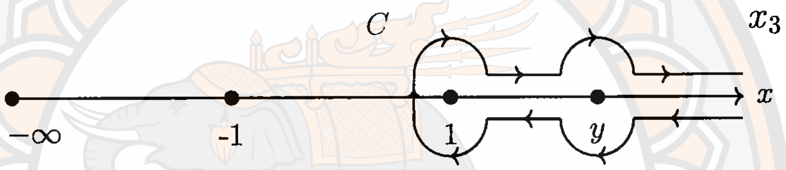

Figure 10 Contour integral in the complex x_3-plane [3].

The (4.60) becomes

$$T(3) = \sin(\pi s_{24}) e^{-i\pi s_{51}} \int_1^{\infty} dy |y-1|^{s_{24}} |y+1|^{s_{25}} \int_C dx |x-y|^{s_{23}} |x-1|^{s_{34}} |x+1|^{s_{35}}. \tag{4.61}$$

This integral can be decomposed into the sum of five open strings amplitudes

$$\begin{aligned} T(3) = 2ie^{-i\pi s_{25}} \sin(\pi s_{24}) \Big\{ & \sin(\pi s_{34}) \mathcal{A}^{\text{op}}(1,5,4,3,2) \\ & + \sin[(\pi(s_{24}+s_{34})] \mathcal{A}^{\text{op}}(1,5,4,2,3) \Big\}. \end{aligned} \tag{4.62}$$

Let us add one more point: $N = 6$. There are two real integrations with respect to x_3 and x_4. We will deform the contour of the integral of two real variables to the imaginary axis as we did in the case of $N = 5$ in (4.59). The important thing is that the tube contribution also satisfies the monodromy relations. We found that

$$T(3,4)+e^{-i\pi s_{34}}T(4,3)\equiv R(3,4)=\sin(\pi s_{25})e^{-i\pi s_{16}}\int_1^\infty dy|y-1|^{s_{25}}|y+1|^{s_{26}}$$
$$\times\int_{C_x}dx\int_{C_z}dz|z-x|^{s_{34}}|x-y|^{s_{23}}|x-1|^{s_{35}}|x+1|^{s_{36}}|z-y|^{s_{24}}|z-1|^{s_{45}}|z+1|^{s_{46}} \tag{4.63}$$

with the contours C_x and C_z depicted in Fig 11.

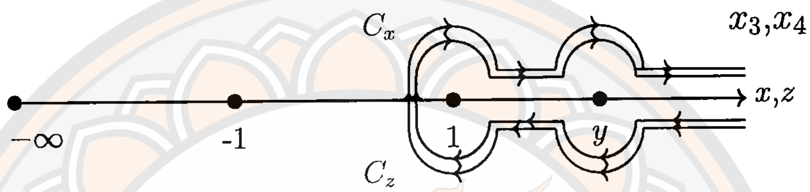

Figure 11 Contour integral in the complex x_3- and x_4-planes [3].

The contour integrals can be expressed as the sum over six-point open string amplitudes yielding the following expression:

$$R(3,4)=-4\sin(\pi s_{25})e^{-i\pi s_{26}}$$
$$\times\sum_{\sigma\in\{2,3,4\}}\prod_{j=3}^{4}\sin\left\{\pi\left(s_{j,5}+\sum_{l=2}^{j-1}s_{jl}\Theta\left[\sigma^{-1}(j)-\sigma^{-1}(l)\right]\right)\right\}\mathcal{A}^{\text{op}}(1,6,5,\sigma(2,3,4)). \tag{4.64}$$

By utilizing the monodromy relation with (4.63), we can express the tube $T(3,4)$ in terms of $R(3,4)$ and $R(4,3)$ as

$$T(3,4)=-\frac{i}{2}\frac{1}{\sin(\pi s_{34})}\left[e^{i\pi s_{34}}R(3,4)-R(4,3)\right] \tag{4.65}$$

We can generalize this to N-point, this yields

$$\sum_{\sigma\in\{3,\ldots,N-2\}}\prod_{2\le j<l\le N-2}\exp\{-i\pi\Theta\left[\sigma^{-1}(j)-\sigma^{-1}(l)\right]s_{jl}\}T(\sigma(2,\ldots,N-2))$$
$$=R(3,\ldots,N-2), \tag{4.66}$$

with

$$R(3,\dots,N-2) = (2i)^{N-4}e^{-i\pi s_{1,N}}\sin\left(\pi s_{2,N-1}\right)$$
$$\times \sum_{\sigma\in\{2,\dots,N-2\}}\prod_{j=3}^{N-2}\sin\left\{\pi\left(\left|s_{j,N-1}+\sum_{l=2}^{j-1}s_{jl}\Theta\left[\sigma^{-1}(j)-\sigma^{-1}(l)\right]\right)\right\}$$
$$\times \mathcal{A}^{\mathrm{op}}(1,N,N-1,\sigma(2,\dots,N-2)). \tag{4.67}$$

4.4.2 Relations for EYM subamplitudes from string worldsheet monodromies

In the field theory limit, as $\alpha'\to 0$, we only consider the lowest order in α' of (4.58). This yields:

$$A_{FT}(1,2,\dots,N-2;q_1,q_2)+A_{FT}(1,3,2,\dots,N-2;q_1,q_2)$$
$$+A_{FT}(1,3,4,2,\dots,N-2;q_1,q_2)+A_{FT}(1,3,\dots,N-2,2;q_1,q_2)=0. \tag{4.68}$$

Notice that the right-hand side of (4.68) equals zero because the contribution of the infinite tube (4.59) is a higher-order string effect. To be more specific, its real part is in the order of α'^3, while its imaginary part is in the order of α'^2. For $q_1 = q_2 = \frac{k}{2}$, the real part of A_{FT} correspond to EYM amplitude:

$$\mathrm{Re}\{A_{FT}(1,\dots,N-2;q_1,q_2)\}\Big|_{q_1=q_2=\frac{k}{2}} = -\frac{1}{2}A_{EYM}(1,\dots,N-2;k). \tag{4.69}$$

Note that the amplitude (4.69) satisfied the reflection property, i.e.,

$$A(1,\dots,N-2;k) = (-1)^N A(1,N-2,\dots,2;k)^*. \tag{4.70}$$

We then obtain the relation

$$A_{EYM}(1,2,\dots,N-2;k)+A_{EYM}(1,3,2,\dots,N-2;k)$$
$$+A_{EYM}(1,3,4,2,\dots,N-2;k)+\cdots+A_{EYM}(1,3,\dots,N-2,2;k)=0, \tag{4.71}$$

which is the gluon sector of Kleiss-Kuijf [19] relations.

CHAPTER V

RELATION BETWEEN CLOSED AND MIXED STRING AMPLITUDES

In this chapter, we derive the relation between closed and mixed string amplitudes by utilizing analytic continuation of complex variables to factorized the closed string amplitudes into the product of mixed string amplitudes. We show only 3 non-trivial cases namely, four- , five- and six-point relation. Finally, we generalize to n-point relation

5.1 Factorization of closed string amplitudes

In this section, we factorize closed string amplitudes using the analytic continuation of complex variables. We begin with four-point amplitudes and proceed up to six-point amplitudes.

5.1.1 The four-points amplitudes

According to the $PSL(2,\mathbb{C})$ symmetry, we choose $z_2 = i$, $z_3 = -i$ and $z_4 = \infty$. The closed string amplitude takes the form

$$\mathcal{A}_4^{\text{cl}} = 4C_{S^2} \int d^2z |z-i|^{2s_{12}} |z+i|^{2s_{13}} |2i|^{2s_{23}} F_4(z,\bar{z}) \tag{5.1}$$

where $s_{ij} = \frac{\alpha'}{2} k_i \cdot k_j$. Note that the factor $(2\pi)^D \delta^D(\sum_i k_i)$ has been dropped for convenience. We can decompose F_4 as $f_4(z)\bar{f}_4(\bar{z})$ without losing generality. By writing $z = x + iy$, the closed string amplitudes take the form

$$\begin{aligned}\mathcal{A}_4^{\text{cl}} =& 4C_{S^2} \int dxdy (x+iy-i)^{s_{12}} (x-iy+i)^{s_{12}} (x+iy+i)^{s_{13}} \\ & (x-iy-i)^{s_{13}} (2i)^{s_{23}} (-2i)^{s_{23}} f_4(z) \bar{f}_4(\bar{z}).\end{aligned} \tag{5.2}$$

Then rotating the contour integral of the variable y from the real axis to the imaginary axis

$$y \to iy, \tag{5.3}$$

the integrands transform as follows:

$$(x+iy-i)^{s_{12}}(x-iy+i)^{s_{12}} \to (x-y-i)^{s_{12}}(x+y+i)^{s_{12}}, \tag{5.4}$$

$$(x+iy+i)^{s_{13}}(x-iy-i)^{s_{13}} \to (x-y+i)^{s_{13}}(x+y-i)^{s_{13}}. \tag{5.5}$$

We then define new variables

$$\xi = x+y \qquad \text{and} \qquad \eta = x-y. \tag{5.6}$$

Accordingly, the amplitude becomes

$$\begin{aligned} \mathcal{A}_4^{\text{cl}} = 4C_{S^2} \int d\xi (\xi+i)^{s_{12}}(\xi-i)^{s_{13}}(-2i)^{s_{23}}\bar{f}_4(\xi) \\ \times \int d\eta(\eta-i)^{s_{12}}(\eta_1+i)^{s_{13}}(2i)^{s_{23}} f_4(\eta). \end{aligned} \tag{5.7}$$

The integrals are exactly those of mixed string amplitudes $\mathcal{F}_4$. Upon setting $C_{S^2} = (C_{D_2})^2$ and identifying the functions f_4 and $\bar{f}_4$ as the external-state dependent function K_4, we obtain

$$\mathcal{A}_4^{\text{cl}} = \mathcal{F}_4(1,4;2,3)\widetilde{\mathcal{F}}_4(1,4;3,2). \tag{5.8}$$

In the limit $k_2 = k_3$, the four-point closed string amplitudes can be expressed as a product of mixed string amplitudes

$$\mathcal{A}_4^{\text{cl}}\Big|_{k_2=k_3=k} = \mathcal{M}_4(1,4;k)\widetilde{\mathcal{M}}_4(1,4;k). \tag{5.9}$$

5.1.2 The five-points amplitudes

For five-points string scattering amplitudes, we fix the points z_3, z_4, and z_5 to be the points i, $-i$, and ∞ respectively in the complex plane, the five-points string amplitude reads

$$\begin{aligned} \mathcal{A}_5^{\text{cl}} = 4C_{S^2} \int d^2z_1 d^2z_2 |z_1-i|^{2s_{13}}|z_1+i|^{2s_{14}}|z_2-i|^{2s_{23}}|z_2+i|^{2s_{24}}|2i|^{2s_{34}} \\ \times |z_1-z_2|^{2s_{12}} F_5(z_1,\bar{z}_1,z_2,\bar{z}_2). \end{aligned} \tag{5.10}$$

We follow the same procedures as used in the four-point scenario, where we rewriting $z_j = x_j + iy_j$ for $j = 1, 2$. Subsequently, the variables y_j are analytically continued to complex variables, which transform as

$$y_j \to ie^{-i\epsilon} y_j \approx iy_j + \epsilon y_j. \tag{5.11}$$

Notice that we avoid the possible branch points of $|z_1 - z_2|^{2s_{12}}$ by rotating the variable y_j from the real axis into almost the imaginary axis. The integrand becomes

$$\begin{aligned}
|z_j - i|^{2s_{j3}} &\to (x_j - y_j - i + i\epsilon y_j)^{s_{j3}}(x_j + y_j + i - i\epsilon y_j)^{s_{j3}} \\
|z_j + i|^{2s_{j4}} &\to (x_j - y_j + i + i\epsilon y_j)^{s_{j4}}(x_j + y_j - i - i\epsilon y_j)^{s_{j4}} \\
|z_1 - z_2|^{2s_{12}} &\to ((x_1 - y_1) - (x_2 - y_2) + i\epsilon(y_1 - y_2))^{s_{12}} \\
&\quad \times ((x_1 + y_1) - (x_2 + y_2) - i\epsilon(y_1 - y_2))^{s_{12}}
\end{aligned} \tag{5.12}$$

for $j = 1, 2$. If we introduce new variables

$$\xi_j = x_j + y_j \qquad \text{and} \qquad \eta_j = x_j - y_j, \tag{5.13}$$

the integral (5.10) can be written as

$$\begin{aligned}
\mathcal{A}_5^{\text{cl}} =& 4C_{S^2} \int d\xi_1 d\eta_1 d\xi_2 d\eta_2 (\eta_1 - i)^{s_{13}}(\eta_1 + i)^{s_{14}}(\eta_2 - i)^{s_{23}}(\eta_2 + i)^{s_{24}} \\
&\times (\xi_1 + i)^{s_{13}}(\xi_1 - i)^{s_{14}}(\xi_2 + i)^{s_{23}}(\xi_2 - i)^{s_{24}}|2i|^{2s_{34}} \\
&\times (\eta_1 - \eta_2 + i\epsilon\delta)^{s_{12}}(\xi_1 - \xi_2 - i\epsilon\delta)^{s_{12}} f_5(\eta_1, \eta_2)\bar{f}_5(\xi_1, \xi_2)
\end{aligned} \tag{5.14}$$

where $\delta = y_1 - y_2$ and $F_5(z_1, \bar{z}_1, z_2, \bar{z}_2) = f_5(z_1, z_2)\bar{f}_5(\bar{z}_1, \bar{z}_2)$. The integral of above expression resembles the integral form of mixed string amplitudes $\mathcal{F}_5$, although it has to include the possible phase factors because of the intertwining of integrating variables ξ_j and η_j within the integrand.

The deformation of contours around branch points, i.e. $\eta_1 \sim \eta_2$ and $\xi_1 \sim \xi_2$, can be determined by the factors $i\epsilon\delta$. When approaching the branch points we need to be careful. Let's consider the behavior of $i\epsilon\delta$ near the branch points. When η_1 approaches η_2, the position of the branch point depends on the sign of $\xi_1 - \xi_2$

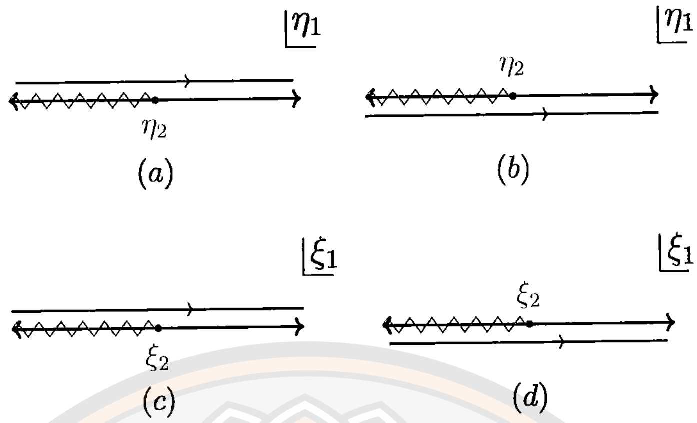

Figure 12 Contours of integration for the variable η_1 in the cases when (a) $\xi_1 > \xi_2$ and (b) $\xi_1 < \xi_2$, and contours of integration for the variable ξ_1 for the cases where (c) $\eta_1 > \eta_2$ and (d) $\eta_1 < \eta_2$.

because $i\epsilon\delta \sim i\frac{\epsilon}{2}(\xi_1 - \xi_2)$. In this case, the branch point is located slightly below the real axis when $\xi_1 > \xi_2$. On the other hand, the branch point is slightly above real line when $\xi_1 < \xi_2$. Therefore, to avoid the branch point, we can slightly shift the contour of η_1 to above the real axis when $\xi_1 > \xi_2$ and slightly shift the contour down in the case of $\xi_1 < \xi_2$. The contours of η_1 are illustrated in figure 12(a) and 12(b).

Similar for the term $(\xi_1 - \xi_4 - i\epsilon\delta)^{s_{12}}$ when ξ_1 approaches ξ_2. When $\eta_1 < \eta_2$, the factor $i\epsilon\delta$ is positive, while it is negative when $\eta_1 > \eta_2$. Consequently, the contours of ξ_1 are above and below the real axis for the case $\eta_1 > \eta_2$ and $\eta_1 < \eta_2$ repectively. The contours of ξ_1 are depicted in figure 12(c) and 12(d)

The integral (5.14) is separated into four terms by inserting the identity

$$1 = (\Theta(\xi_1 - \xi_2) + \Theta(\xi_2 - \xi_1))(\Theta(\eta_1 - \eta_2) + \Theta(\eta_2 - \eta_1)) \tag{5.15}$$

where $\Theta(x - y)$ is a Heaviside step function. This gives four different integral regions among variables. As a result, the integral in each term can be related to the product of $\mathcal{F}_5$. Nonetheless, we need to make sure the integrand matches with the expression (3.35). Therefore, we utilize the following relation to correct the integrand:

$$(\eta_1 - \eta_2)^c = \begin{cases} e^{i\pi c}(\eta_2 - \eta_1)^c, & \xi_1 > \xi_2 \\ e^{-i\pi c}(\eta_2 - \eta_1)^c, & \xi_1 < \xi_2 \end{cases} \tag{5.16}$$

when $\mathrm{Re}(\eta_1) < \mathrm{Re}(\eta_2)$ and

$$(\xi_1 - \xi_2)^c = \begin{cases} e^{i\pi c}(\xi_2 - \xi_1)^c, & \eta_1 > \eta_2. \\ e^{-i\pi c}(\xi_2 - \xi_1)^c, & \eta_1 < \eta_2 \end{cases} \tag{5.17}$$

when $\mathrm{Re}(\xi_1) < \mathrm{Re}(\xi_2)$.

We then obtain

$$\begin{aligned} \mathcal{A}_5^{\mathrm{cl}} =& e^{-i2\pi s_{12}}\mathcal{F}_5(1,2,5;3,4)\widetilde{\mathcal{F}}_5(1,2,5;4,3) + e^{i\pi s_{12}}\mathcal{F}_5(1,2,5;3,4)\widetilde{\mathcal{F}}_5(2,1,5;4,3) \\ &+ e^{i\pi s_{12}}\mathcal{F}_5(2,1,5;3,4)\widetilde{\mathcal{F}}_5(1,2,5;4,3) + \mathcal{F}_5(2,1,5;3,4)\widetilde{\mathcal{F}}_5(2,1,5;4,3), \end{aligned} \tag{5.18}$$

if the constant $C_{S^2} = (C_{D_2})^2$ is fixed. In the case of $k_3 = k_4 = k/2$, the integral $\mathcal{F}_5$ reduces to the corresponding mixed string amplitude $\mathcal{M}_5$ with the closed string momentum k.

5.1.3 The six-points amplitudes

To observe more patterns in the relations, we have to consider another non-trivial case regarding six-point string amplitudes. The six-point closed string amplitude is given by

$$\mathcal{A}_6^{cl} = 4C_{S^2}\int\prod_{j=1}^{3} d^2z_j|z_j - i|^{2s_{j4}}|z_j + i|^{2s_{j5}}|2i|^{2s_{45}}F_6(z_j,\bar{z}_j)\prod_{1\le l<m\le 3}|z_l - z_m|^{2s_{lm}} \tag{5.19}$$

where z_4, z_5, and z_6 are set to i, $-i$, and ∞, respectively, to fix the $PSL(2,\mathbb{C})$ symmetry. We follow the same method as the previous case by writing $z_j = x_j + iy_j$ then the variables y_j transform as (5.11). We define the new variables as

$$\xi_j = x_j + y_j \qquad \text{and} \qquad \eta_j = x_j - y_j \tag{5.20}$$

while now j runs from 1 to 3, the amplitude takes the form

$$\begin{aligned}\mathcal{A}_6^{\text{cl}} =& 4C_{S^2}\int\prod_{j=1}^{3} d\eta_j d\xi_j (\eta_j - i)^{s_{j4}}(\eta_j + i)^{s_{j5}}(\xi_j + i)^{s_{j4}}(\xi_j - i)^{s_{j5}}|2i|^{2s_{45}} \\ &\times f_6(\xi_j)\bar{f}_6(\eta_j)\prod_{1\le l<m\le 3}(\eta_l - \eta_m + i\epsilon\delta_{lm})^{s_{lm}}(\xi_l - \xi_m - i\epsilon\delta_{lm})^{s_{lm}} \end{aligned} \tag{5.21}$$

where $\delta_{ij} = y_i - y_j$.

Similar to four- and five- point cases, the (5.21) can be expressed in terms of the product of $\mathcal{F}_6$ with appropriate phase factors. The presence of $i\epsilon\delta_{lm}$ in the integrand determines the deformation of the integration contour. Specifically, considering the term $(\eta_l - \eta_m + i\epsilon\delta_{lm})^{s_{lm}}$, the contour of η_l is shifted slightly above real axis when $\xi_l > \xi_m$ and it moves slightly downward from the real axis when $\xi_l < \xi_m$. Similarly, regarding the term $(\xi_l - \xi_m + i\epsilon\delta_{lm})^{s_{lm}}$, the contour of ξ_l is translated upward from the real axis when $\eta_l > \eta_m$ and is shifted downward from the real axis when $\eta_l < \eta_m$. The deformation of integration contour depending on how the values of the integrating variables are relative to each other, we will separate the integration region into 36 pieces by inserting

$$1 = \sum_{i\neq j\neq k}^{3}\Theta(\xi_i - \xi_j)\Theta(\xi_j - \xi_k)\sum_{l\neq m\neq n}^{3}\Theta(\eta_l - \eta_m)\Theta(\eta_m - \eta_n) \tag{5.22}$$

to the expression (5.21).

In each term, we have to make sure that the integrand is in the correct form of $\mathcal{F}_6$, one can utilize the relations

$$(\eta_i - \eta_j)^c = \begin{cases} e^{i\pi c}(\eta_j - \eta_i)^c, & \xi_i > \xi_j \\ e^{-i\pi c}(\eta_j - \eta_i)^c, & \xi_i < \xi_j \end{cases} \tag{5.23}$$

when $\text{Re}(\eta_i) < \text{Re}(\eta_j)$, and

$$(\xi_i - \xi_j)^c = \begin{cases} e^{i\pi c}(\xi_j - \xi_i)^c, & \eta_i > \eta_j. \\ e^{-i\pi c}(\xi_j - \xi_i)^c, & \eta_i < \eta_j \end{cases} \tag{5.24}$$

when $\mathrm{Re}(\xi_i) < \mathrm{Re}(\xi_j)$. After a careful examination, we acquire the expression

$$\mathcal{A}_6^{\rm cl} = \sum_{\sigma,\sigma'} e^{i\pi(\Lambda(\sigma,\sigma')+\Omega(\sigma,\sigma'))}\mathcal{F}_6(\sigma(1,2,3),6;4,5)\widetilde{\mathcal{F}}_6(\sigma'(1,2,3),6;5,4). \tag{5.25}$$

The expression is summed over the ordering σ and σ' which refer to the order of open string vertex operators. Note that the object inside the bracket are to be permuted. The kinematics variables are contained in the functions $\Lambda(\sigma,\sigma')$ and $\Omega(\sigma,\sigma')$ which depend on the ordering σ and σ' respectively. The values of both functions are shown in table 1

5.2 Relations between closed and mixed string amplitudes

In this section, the results formulated in previous section are generalized to an arbitrary number of strings. To achieve this, we repeat the same procedure as we used in the last section to factorize the expression for n-point closed string amplitudes,

$$\begin{aligned}\mathcal{A}_n^{\rm cl} =& 4C_{S^2}\int\prod_{j=1}^{n-3} d^2z_j|z_j-i|^{2s_{j,n-2}}|z_j+i|^{2s_{j,n-1}}|2i|^{2s_{n-2,n-1}}F_n(z_j,\bar{z}_j)\\ &\prod_{1\leq l<m\leq n-3}|z_l-z_m|^{2s_{lm}}\end{aligned} \tag{5.26}$$

with $z_{n-2}=i, z_{n-1}=-i$ and $z_n=\infty$, into products of mixed string amplitudes. By rewriting $z_j = x_j + iy_j$ and then transforming the variable y_j to $ie^{-i\epsilon}y_j$, it yields

$$\begin{aligned}\mathcal{A}_n^{\rm cl} =& 4C_{S^2}\int\prod_{j=1}^{n-3} d\eta_j d\xi_j(\eta_j-i)^{s_{j,n-2}}(\eta_j+i)^{s_{j,n-1}}(\xi_j+i)^{s_{j,n-2}}(\xi_j-i)^{s_{j,n-1}}\\ & f_n(\xi_j)\bar{f}_n(\eta_j)\prod_{1\leq l<m\leq n-3}(\eta_l-\eta_m+i\epsilon\delta_{lm})^{s_{lm}}(\xi_l-\xi_m-i\epsilon\delta_{lm})^{s_{lm}}|2i|^{2s_{n-2,n-1}}\end{aligned} \tag{5.27}$$

where the variables ξ_j and η_j are defined in (5.20). By careful analysis of branch cuts and contour deformations, the general expression of closed string amplitudes

σ	σ'	$\Lambda(\sigma, \sigma')$	$\Omega(\sigma, \sigma')$
1,2,3	1,2,3	$-s_{12} - s_{13} - s_{23}$	$-s_{12} - s_{13} - s_{23}$
1,2,3	2,1,3	$s_{12} - s_{13} - s_{23}$	$-s_{13} - s_{23}$
1,2,3	2,3,1	$s_{12} + s_{13} - s_{23}$	$-s_{23}$
1,2,3	3,2,1	$s_{12} + s_{13} + s_{23}$	0
1,2,3	3,1,2	$-s_{12} + s_{13} + s_{23}$	$-s_{12}$
1,2,3	1,3,2	$-s_{12} - s_{13} + s_{23}$	$-s_{12} - s_{13}$
2,1,3	1,2,3	$-s_{13} - s_{23}$	$s_{12} - s_{13} - s_{23}$
2,1,3	2,1,3	$-s_{13} - s_{23}$	$-s_{13} - s_{23}$
2,1,3	2,3,1	$s_{13} - s_{23}$	$-s_{23}$
2,1,3	3,2,1	$s_{13} + s_{23}$	0
2,1,3	3,1,2	$s_{13} + s_{23}$	s_{12}
2,1,3	1,3,2	$-s_{13} + s_{23}$	$s_{12} - s_{13}$
2,3,1	1,2,3	$-s_{23}$	$s_{12} + s_{13} - s_{23}$
2,3,1	2,1,3	$-s_{23}$	$s_{13} - s_{23}$
2,3,1	2,3,1	$-s_{23}$	$-s_{23}$
2,3,1	3,2,1	s_{23}	0
2,3,1	3,1,2	s_{23}	s_{12}
2,3,1	1,3,2	s_{23}	$s_{12} + s_{13}$
3,2,1	1,2,3	0	$s_{12} + s_{13} + s_{23}$
3,2,1	2,1,3	0	$s_{13} + s_{23}$
3,2,1	2,3,1	0	s_{23}
3,2,1	3,2,1	0	0
3,2,1	3,1,2	0	s_{12}
3,2,1	1,3,2	0	$s_{12} + s_{13}$
3,1,2	1,2,3	$-s_{12}$	$-s_{12} + s_{13} + s_{23}$
3,1,2	2,1,3	s_{12}	$s_{13} + s_{23}$
3,1,2	2,3,1	s_{12}	s_{23}
3,1,2	3,2,1	s_{12}	0
3,1,2	3,1,2	$-s_{12}$	$-s_{12}$
3,1,2	1,3,2	$-s_{12}$	$-s_{12} + s_{13}$
1,3,2	1,2,3	$-s_{12} - s_{13}$	$-s_{12} - s_{13} + s_{23}$
1,3,2	2,1,3	$s_{12} - s_{13}$	$-s_{13} + s_{23}$
1,3,2	2,3,1	$s_{12} + s_{13}$	s_{23}
1,3,2	3,2,1	$s_{12} + s_{13}$	0
1,3,2	3,1,2	$-s_{12} + s_{13}$	$-s_{12}$
1,3,2	1,3,2	$-s_{12} - s_{13}$	$-s_{12} - s_{13}$

Table 1 Values of functions $\Lambda(\sigma, \sigma')$ and $\Omega(\sigma, \sigma')$ corresponding to the ordering σ and σ'.

can be written as

$$\mathcal{A}_n^{\text{cl}} = \sum_{\sigma,\sigma'} \mathcal{P}(\sigma(1,2,3,\ldots,n-3)|\sigma'(1,2,3,\ldots,n-3))$$
$$\times \mathcal{F}_n(\sigma(1,2,3,\ldots,n-3),n;n-2,n-1)$$
$$\times \widetilde{\mathcal{F}}_n(\sigma'(1,2,3,\ldots,n-3),n;n-1,n-2), \tag{5.28}$$

with the help of relations (5.23) and (5.24).The above expression resembles the relation (5.25), where all the phase factors that appear when correcting the integrand of (5.27) are encapsulated in the function $\mathcal{P}(\sigma|\sigma')$. To defining the function $\mathcal{P}(\sigma|\sigma')$, we introduce the two new functions $\theta(i_j,i_k)$ and $\beta(i_j,i_k)$. The function $\theta(i_j,i_k)$ is defined as

$$\theta(i_p,i_q) = \begin{cases} 1 \quad ; \quad (i_p,i_q) \text{ has the same ordering as set } I_n \\ 0 \quad ; \quad (i_p,i_q) \text{ has the opposite ordering as set } I_n \end{cases} \tag{5.29}$$

where the set I_n are given by

$$I_n = \{1,2,3,\ldots,n-3\}. \tag{5.30}$$

For example, $\theta(1,2) = 1$, $\theta(2,1) = 0$, $\theta(4,7) = 0$, etc. The function $\beta(i_j,i_k)$ is defined as

$$\beta(i_p,i_q|j_r,j_s) = \begin{cases} -1 \quad ; \quad (i_p,i_q) \text{ and } (j_r,j_s) \text{ have the same ordering,} \\ +1 \quad ; \quad (i_p,i_q) \text{ and } (j_r,j_s) \text{ have the opposite ordering ,} \\ 0 \quad ; \quad (i_p,i_q) \text{ and } (j_r,j_s) \text{ are different.} \end{cases} \tag{5.31}$$

For example, $\beta(1,2|1,2)$ = -1, $\beta(1,2|2,1) = +1$, $\beta(1,2|1,3) = 0$, etc. We then construct the function $\mathcal{P}(\sigma|\sigma')$ to be

$$\mathcal{P}(i_1,i_2,\ldots,i_k|j_1,j_2,\ldots,j_k) = \exp\Big\{ i\pi\Big(\sum_{p<q}^{k} \sum_{r<s}^{k} \theta(i_p,i_q)\beta(i_p,i_q|j_r,j_s)s_{i_pi_q}$$
$$+ \theta(j_r,j_s)\beta(j_r,j_s|i_p,i_q)s_{j_rj_s}\Big)\Big\}. \tag{5.32}$$

The expression (5.28) involves a summation of $(n-3)! \times (n-3)!$ terms, accounting for the permutations of $n-3$ objects within the orderings σ and σ'. In the case where $k_{n-2} = k_{n-1} = k/2$, the relation reduces to

$$\begin{aligned} \mathcal{A}_n^{\text{cl}}\Big|_{k_{n-2}=k_{n-1}=k/2} = & \sum_{\sigma,\sigma'} \mathcal{P}(\sigma(1,2,3,\ldots,n-3)|\sigma'(1,2,3,\ldots,n-3)) \\ & \times \mathcal{M}_n(\sigma(1,2,3,\ldots,n-3),n;k) \\ & \times \widetilde{\mathcal{M}}_n(\sigma'(1,2,3,\ldots,n-3),n;k). \end{aligned} \tag{5.33}$$

Note that the closed string polarizations χ_i, corresponding the the physical states, are decomposed into the open string polarizations ζ_i and $\tilde{\zeta}_i$ for the amplitudes $\mathcal{M}_n$ and $\widetilde{\mathcal{M}}_n$, respectively, via $\chi_i = \zeta_i \otimes \tilde{\zeta}_i$. However, in the mixed amplitudes $\mathcal{M}n$ and $\widetilde{\mathcal{M}}_n$, the polarizations of the closed strings are $\zeta_{n-2} \otimes \zeta_{n-1}$ and $\tilde{\zeta}_{n-2} \otimes \tilde{\zeta}_{n-1}$, respectively. Additional details are available in Appendix A.

5.3 Correction terms

Although the relations between closed string amplitudes and those of mixed string amplitudes, as presented (5.28) and (5.33), have been formulated, the calculation require Wick rotations of the variables y_j via (5.11). When implementing the contour integral, we assume that no poles or branch points. In this section, we will carefully consider possible branch points that exist in the bulk of the complex plane, which could lead to correction terms of the proposed relations. For simplicity, we will confine ourselves to consider only the four-point relation and the comments on higher-point cases are provided at the end of the section

The terms $|z-i|^{2s_{12}}$ and $|z+i|^{2s_{13}}$ in the four-point closed string amplitude (5.1) imply the presence of branch points at $y = \pm(1 \pm ix)$ inside the worldsheet. To transform y into an imaginary axis, the y-contour has to avoid intersecting any cuts in the complex plane. To illustrate this, the y-contour is defined as shown in figure 13. Following this contour, additional terms arise form the integration along

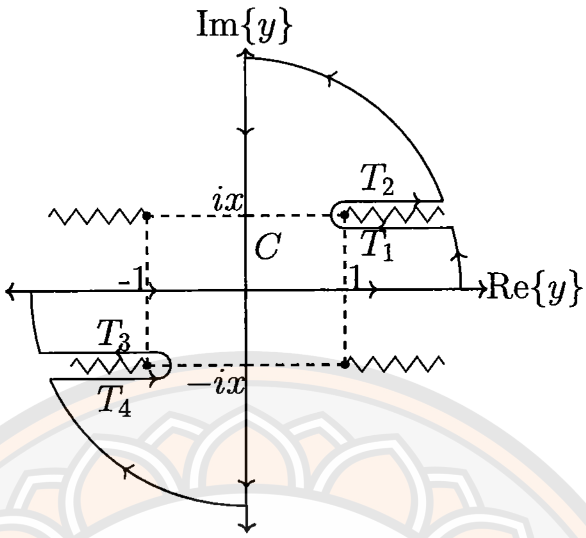

Figure 13 Contour integral in the complex y plane.

the infinite tubes, namely paths T_1, T_2, T_3, and T_4. Since there are no poles enclosed by the contour, the variable y undergoes transformation through the application of the Cauchy theorem:

$$\left(\int_{-\infty}^{\infty}+\int_{T_1+T_2}+\int_{T_3+T_4}+\int_{C}\right)dy\ I(y)=0, \tag{5.34}$$

where $I(y)$ is an arbitrary analytic function. Note that the contribution from the integral along the infinite arcs are disregarded since we assume that the function $I(y)\to 0$ when $|y|\to\infty$ which holds true for our integrand. Thus, the tree-level four-point closed string amplitudes take the form

$$\mathcal{A}_4^{\text{cl}}=\mathcal{F}_4(1,4;2,3)\widetilde{\mathcal{F}}_4(1,4;3,2)+8iC_{S^2}\int_{-\infty}^{\infty}dx\left(\int_{T_1+T_2}+\int_{T_3+T_4}\right)dy\ \mathcal{I}(x,y) \tag{5.35}$$

where

$$\mathcal{I}(x,y)=(x+iy-i)^{s_{12}}(x-iy+i)^{s_{12}}(x+iy+i)^{s_{13}}(x-iy-i)^{s_{13}}|2i|^{2s_{23}}F_4(x,y). \tag{5.36}$$

The first term of (5.35) has already been discussed, which arises from the contour integration along the imaginary axis (path C).

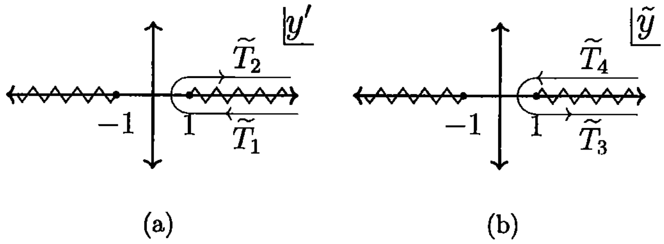

Figure 14 Contour along the infinite tubes of (a) the variable y' and (b) the variable $\tilde{y}$.

Now let's investigate the additional terms resulting from the integration along the infinite tubes by considering the following integral:

$$A_1 \equiv \int_{T_1+T_2} dy\ \mathcal{I}(x,y) \qquad \text{and} \qquad A_2 \equiv \int_{T_3+T_4} dy\ \mathcal{I}(x,y) \tag{5.37}$$

For convenience, F_4 is set to one. To compute, the additional integral is divided into two cases: $x > 0$ and $x < 0$. For the first case $x > 0$, we change the variable y to y' where

$$y = y' + ix, \tag{5.38}$$

the integral A_1 becomes

$$\begin{aligned} A_1\Big|_{x>0} =& i^{(s_{12}+s_{13})} 4^{s_{23}} \left(\int_{\widetilde{T}_1} + \int_{\widetilde{T}_2} \right) dy' (y'-1)^{s_{12}} (x - iy' + x + i)^{s_{12}} \\ & \times (y'+1)^{s_{13}} (x - iy' + x - i)^{s_{13}}. \end{aligned} \tag{5.39}$$

The contour paths $\widetilde{T}_1$ and $\widetilde{T}_2$ of the contour y' are depicted in figure 14(a). Utilizing the relation

$$(y'-1)^c = \begin{cases} |y'-1|^c e^{2\pi i c}; & \text{for path } \widetilde{T}_1 \\ |y'-1|^c; & \text{for path } \widetilde{T}_2, \end{cases} \tag{5.40}$$

we can write

$$\begin{aligned} A_1\Big|_{x>0} =& -(2i) i^{(s_{12}+s_{13})} 4^{s_{23}} \sin(\pi s_{12}) e^{i\pi s_{12}} \int_1^\infty dy' |y'-1|^{s_{12}} |y'+1|^{s_{13}} \\ & \times (x - iy' + x + i)^{s_{12}} (x - iy' + x - i)^{s_{13}}. \end{aligned} \tag{5.41}$$

Similarly, we can compute A_2 for the case $x > 0$ to be

$$A_2\Big|_{x>0} = i^{(s_{12}+s_{13})}4^{s_{23}}\Big(\int_{\widetilde{T}_3} + \int_{\widetilde{T}_4}\Big)d\tilde{y}(\tilde{y}-1)^{s_{13}}(x - i\tilde{y} + x + i)^{s_{13}} \times (\tilde{y}+1)^{s_{12}}(x - i\tilde{y} + x - i)^{s_{12}}, \tag{5.42}$$

where we define

$$y = -\tilde{y} - ix. \tag{5.43}$$

The figure 14(b) depicts paths $\widetilde{T}_3$ and $\widetilde{T}_4$ of the $\tilde{y}$-contour along the infinite tube. By applying the relation

$$(\tilde{y}-1)^c = \begin{cases} |\tilde{y}-1|^c e^{2\pi i c}; & \text{for path } \widetilde{T}_3, \\ |\tilde{y}-1|^c; & \text{for path } \widetilde{T}_4, \end{cases} \tag{5.44}$$

the integral A_2 takes the form

$$A_2\Big|_{x>0} = -(2i)i^{(s_{12}+s_{13})}4^{s_{23}}\sin(\pi s_{13})e^{i\pi s_{13}}\int_1^\infty d\tilde{y}|\tilde{y}-1|^{s_{13}}|\tilde{y}+1|^{s_{12}} \times (x - i\tilde{y} + x + i)^{s_{13}}(x - i\tilde{y} + x - i)^{s_{12}}. \tag{5.45}$$

Similarly, we can compute the expressions for A_1 and A_2 in the case of $x < 0$ and find out that

$$\int_0^\infty dx\Big(A_1\Big|_{x>0} + A_2\Big|_{x>0}\Big) = \int_{-\infty}^0 dx\Big(A_1\Big|_{x<0} + A_2\Big|_{x<0}\Big). \tag{5.46}$$

Finally, the four-point closed string amlitude (5.35) takes the form

$$\mathcal{A}_4^{\text{cl}} = \mathcal{F}_4(1,4;2,3)\widetilde{\mathcal{F}}_4(1,4;3,2) + 32C_{S^2}i^{(s_{12}+s_{13})}(2^{s_{23}})\Big(\mathcal{B}(s_{12},s_{13}) + \mathcal{B}(s_{13},s_{12}),\Big) \tag{5.47}$$

where

$$\mathcal{B}(s_{12},s_{13}) = \sin(\pi s_{12})e^{i\pi s_{12}}\int_0^\infty dx\int_1^\infty dy'|y-1|^{s_{12}}|y+1|^{s_{13}} \times (x - iy + x + i)^{s_{12}}(x - iy + x - i)^{s_{13}}2^{s_{23}}. \tag{5.48}$$

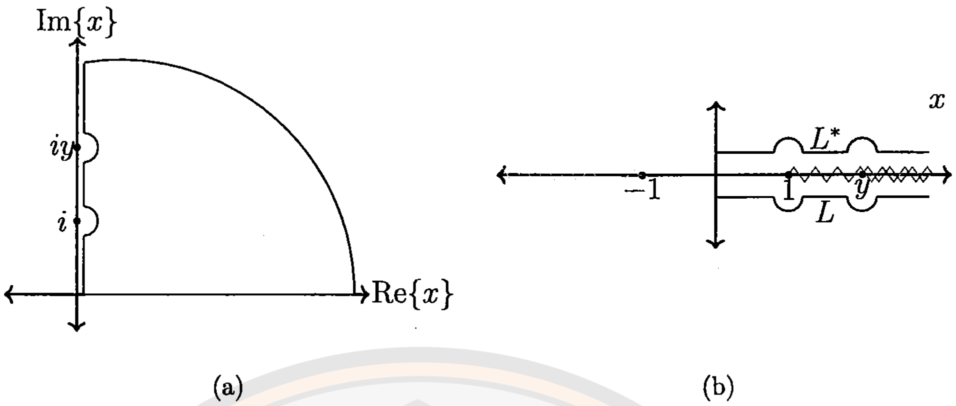

Figure 15 (a) Contour of integration providing a Wick rotation of variable x and (b) contour path of the variable x after the rotation

Remind that the order of the arguments of $\mathcal{B}(x,y)$ matters, i.e. $\mathcal{B}(x,y) \neq \mathcal{B}(y,x)$. Moreover, the integral $\mathcal{B}(x,y)$ can be related to the color-ordered open string amplitudes by applying the binomial expansion to the expression (5.48). It yields

$$\begin{aligned}\mathcal{B}(s_{12},s_{13}) =& \sin(\pi s_{12}) e^{i\pi s_{12}} \sum_{a,b=0}^{\infty} \binom{s_{12}}{a}\binom{s_{13}}{b} \int_1^{\infty} dy\ |y-1|^{s_{12}} |y+1|^{s_{13}} \\ &\times \int_0^{\infty} dx\ (x+i)^{s_{12}-a}(x-iy)^{a+b}(x-i)^{s_{13}-b} 2^{s_{23}}.\end{aligned} \tag{5.49}$$

If we transform $x \to ie^{-i\epsilon}x$, $\mathcal{B}(s_{12},s_{13})$ becomes

$$\begin{aligned}\mathcal{B}(s_{12},s_{13}) =& i \sin(\pi s_{12}) e^{i\frac{\pi}{2}(3s_{12}+s_{13})} \sum_{a,b=0}^{\infty} \binom{s_{12}}{a}\binom{s_{13}}{b} \int_1^{\infty} dy |y-1|^{s_{12}} |y+1|^{s_{13}} \\ &\times \int_L dx (x+1)^{s_{12}-a}(x-y)^{a+b}(x-1)^{s_{13}-b} 2^{s_{23}}.\end{aligned} \tag{5.50}$$

The parameter ϵ was used to avoid intersecting branch points on the imaginary axis. The contour of integration of the variable x is shown in figure 15(a), which the variable x is rotated to the imaginary axis. After the rotation, the contour of the new variable x, denoted by L, is depicted in figure 15(b). To be more specific, we will write $\mathcal{B}(s_{12},s_{13})$ as $\mathcal{B}_L(s_{12},s_{13})$ to indicate the path of integration. The branch cuts are assigned to be along the real axis shown in figure 15(b),we can write

$$\mathcal{B}_L(s_{12}, s_{13}) = i \sin(\pi s_{12}) e^{i\frac{\pi}{2}(3s_{12}+s_{13})} \sum_{a,b=0}^{\infty} \binom{s_{12}}{a} \binom{s_{13}}{b} \int_1^{\infty} dy |y-1|^{s_{12}} |y+1|^{s_{13}}$$
$$\times e^{\pi i(s_{13}+a)} \left(\int_0^1 dx + e^{\pi i(s_{13}-b)} \int_1^y dx + e^{\pi i(s_{13}+a)} \int_y^{\infty} dx \right)$$
$$\times |x+1|^{s_{12}-a} |x-y|^{a+b} |x-1|^{s_{13}-b} 2^{s_{23}}. \tag{5.51}$$

We notice that the last two terms in the second line of (5.51) are the integral form of color-ordered open string amplitudes considering that the position of open string vertex operators are fixed to be at $-1, 1$ and ∞. According to [31], the integral of tree-level five-point partial string amplitude is given by

$$\mathcal{I}_5^{\text{op}}(2,3,1,4,5;\{n_{ij}\}) = \int_1^{\infty} dx_4 \int_1^{x_4} dx_1 (x_1+1)^{s_{12}+n_{12}} (x_1-1)^{s_{13}+n_{13}}$$
$$\times (x_4+1)^{s_{24}+n_{24}} (x_4-1)^{s_{34}+n_{34}} (x_4-x_1)^{s_{14}+n_{14}} 2^{s_{23}} \tag{5.52}$$

with a set of integers n_{ij}.The three points of open string vertices are fixed to be $x_2 = -1$, $x_3 = 1$, and $x_5 = \infty$. To identify the integral (5.52) to the open string amplitude, the numbers n_{ij} must be set to specific values associated with external string states. To eliminate the terms that involve the integration of x from 0 to 1 of (5.51), we consider the function $\mathcal{B}_{L^*}(s_{12}, s_{13})$ where the contour of integration L^* is defined as illustrated in figure 15(b). The function $\mathcal{B}_{L^*}(s_{12}, s_{13})$ takes the form

$$\mathcal{B}_{L^*}(s_{12}, s_{13}) = i \sin(\pi s_{12}) e^{i\frac{\pi}{2}(3s_{12}+s_{13})} \sum_{a,b=0}^{\infty} \binom{s_{12}}{a} \binom{s_{13}}{b} \int_1^{\infty} dy |y-1|^{s_{12}} |y+1|^{s_{13}}$$
$$\times e^{\pi i(s_{13}+a)} \left(\int_0^1 dx + e^{-\pi i(s_{13}-b)} \int_1^y dx + e^{-\pi i(s_{13}+a)} \int_y^{\infty} dx \right)$$
$$\times |x+1|^{s_{12}-a} |x-y|^{a+b} |x-1|^{s_{13}-b} 2^{s_{23}}. \tag{5.53}$$

If we subtract $\mathcal{B}_L(s_{12}, s_{13})$ by $\mathcal{B}_{L^*}(s_{12}, s_{13})$, one yields

$$i \sin(\pi s_{12}) e^{i\frac{\pi}{2}(3s_{12}+s_{13})} \sum_{a,b=0}^{\infty} \binom{s_{12}}{a}\binom{s_{13}}{b} \int_1^{\infty} dy |y-1|^{s_{12}} |y+1|^{s_{13}}$$
$$\times\, 2ie^{\pi i(s_{13}+a)} \left(\sin(\pi(s_{13}-b)) \int_1^y dx + \sin(\pi(s_{13}+a)) \int_y^{\infty} dx \right)$$
$$\times\, |x+1|^{s_{12}-a} |x-y|^{a+b} |x-1|^{s_{13}-b}, \tag{5.54}$$

which can be written in terms of $\mathcal{I}_5^{\text{op}}$ as

$$\mathcal{B}_L(s_{12}, s_{13}) - \mathcal{B}_{L^*}(s_{12}, s_{13}) = -2\sin(\pi s_{12}) e^{i\frac{3}{2}\pi(s_{12}+s_{13})} \sum_{a,b=0}^{\infty} \binom{s_{12}}{a}\binom{s_{13}}{b}(-1)^a$$
$$\times \left(\sin(\pi(s_{13}-b)) \mathcal{I}_5^{\text{op}}(2,3,1,4,5;\{n_{ij}\})\Big|_{p_5=s_{14}=0} \right.$$
$$\left. + \sin(\pi(s_{13}+a)) \mathcal{I}_5^{\text{op}}(2,3,4,1,5;\{n_{ij}\})\Big|_{p_5=s_{14}=0} \right). \tag{5.55}$$

For all strings are in the same level of string mass spectrum, the above expression is evaluated at $s_{14} = 0$ and $p_5 = 0$ providing that $s_{12} = s_{24}$, $s_{13} = s_{34}$. The set $\{n_{ij}\}$ is $\{n_{12} = -a, n_{13} = -b, n_{14} = a + b\}$.

Besides, we notice that the expression of correction terms (5.54) is similar to the infinite tube amplitudes presented in [3]albeit an infinite sum. For the higher-point string amplitudes, we would obtain the correction terms as in the four-point case. However, the computation is more complicated because we have to deal with the multiple contours of integration together with multiple branch points and cuts in the bulk of the complex plane. This issue can be overlooked by considering it within a certain limit. The natural choice is the field theory limit, $\alpha' \to 0$, to which it relates Einstein's theory with Einstein-Yang-Mills theory. In this limit, the correction terms in (5.47) are at the order of α' assuming that the open string amplitudes are at the leading order, while the product of $\mathcal{F}_4$ is at the order of α'^2. This means that the correction terms are at the leading order. Unfortunately, we cannot ignore them within this limit.

Now let's consider another limit where the correction terms can be ignored, i.e., the soft limit. In this limit, we can choose one of momenta, denoted as k_s, approaches zero. We write $k_s = \delta \hat{k}_s$ with $\hat{k}_s$ being some fixed momentum and limit the parameter δ. In the four-point relations, the mixed string integral $\mathcal{F}_4$ is at the order of δ^0 since the open string amplitudes are at the order of δ^{-1} in this limit [32, 33, 34] together with the relations between mixed and open string amplitudes [25]. For the correction terms, they are at the order of δ, which is subleading order, due to the sine function. This argument holds for higher-point string amplitudes as well, because the factor $\sin(\pi s_{ij})$ always appears when the integrating contour encircling around the branch cuts. In principle, we can choose some momentum k_s to be vanished such that the correction terms of higher-point string amplitudes are in the subleading order. By considering the soft limit or some certain limit of the relations, it may provide new interesting structures among string scattering amplitudes.

5.4 Relations between tree-level closed and mixed string amplitudes at five-point

In this section, we show the alternative way to formulate the five-point relations between closed and mixed string amplitudes. To obtain this, we choose different points to fix, i.e., $z_2 = \frac{i}{2}$, $z_3 = \frac{i}{2}$, $z_5 = x_5(y_5 = 0)$, and $y_1 = y_4 = y$. Notice that we choose y_1 and y_4 to be the same because we would like to avoid dealing with multiple contours of integration and branch cuts. By writing $z_j = x_j + iy_j$, we then obtain the amplitudes of the form

$$\begin{aligned}\mathcal{A}_5^{\text{cl}} =&(-2i)^2 C_{S^2} \int dx_1 dx_4 dx_5 dy \left|x_1 + iy - \frac{i}{2}\right|^{2s_{12}} \left|x_1 + iy + \frac{i}{2}\right|^{2s_{13}} \\ &|x_1 - x_4|^{2s_{14}} |x_1 + iy - x_5|^{2s_{15}} \left|x_4 + iy - \frac{i}{2}\right|^{2s_{24}+2} \left|x_5 - \frac{i}{2}\right|^{2s_{25}+2} \\ &\left|x_4 + iy + \frac{i}{2}\right|^{2s_{34}+2} \left|x_5 + \frac{i}{2}\right|^{2s_{35}+2} |x_4 + iy - x_5|^{2s_{45}} F_5(x_1, x_4, x_5, y) \end{aligned} \tag{5.56}$$

As in four-point, there are terms in (5.56) that contain the branch points in the bulk of the complex plane. The branch points are at $y = \pm\frac{1}{2} \pm ix_1$ and $y =$

$\pm\frac{1}{2} \pm ix_4$. During the Wick rotations of the variable y, we need to make sure that the y-contour does not intersect with any brach cuts. We define the contour of y shown in figure 16.

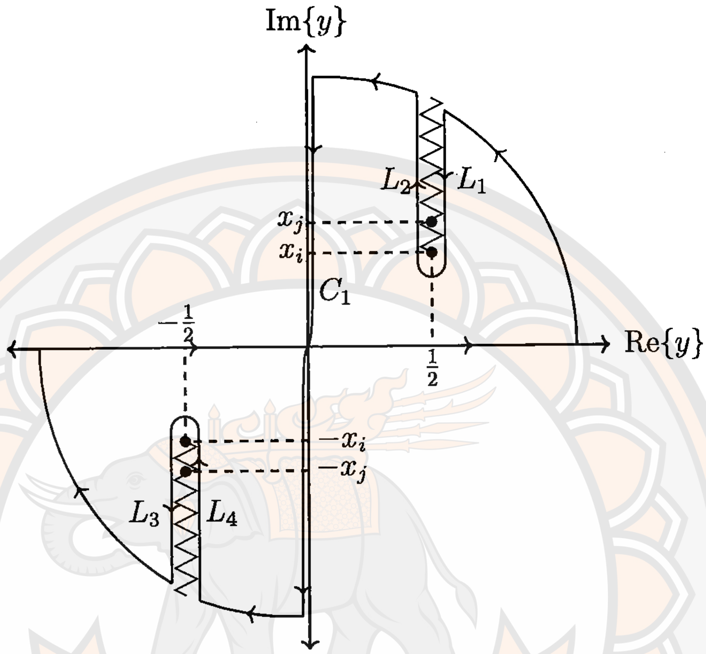

Figure 16 The contour of integration of y.

Again, we apply the Cauchy theorem as we did in previous section. The five-point tree-level closed string amplitudes take the form

$$\mathcal{A}_5^{\rm cl} = 4C_{S^2} \int dx_1 dx_4 dx_5 \mathcal{I}_1(x_5)\Big(\int_{C_1} + \int_{L_1+L_2} + \int_{L_3+L_4}\Big) dy \mathcal{I}_2(x_1, x_4, x_5, y) \quad (5.57)$$

where

$$\mathcal{I}_1(x_5) \equiv (x_5 - \frac{i}{2})^{s_{25}+1}(x_5 + \frac{i}{2})^{s_{25}+1}(x_5 + \frac{i}{2})^{s_{35}+1}(x_5 - \frac{i}{2})^{s_{35}+1}, \quad (5.58)$$

and

$$\begin{aligned}\mathcal{I}_2(x_1,x_4,x_5,y) \equiv &(x_1+iy-\frac{i}{2})^{s_{12}}(x_1-iy+\frac{i}{2})^{s_{12}}(x_1+iy+\frac{i}{2})^{s_{13}}\\ &(x_1-iy-\frac{i}{2})^{s_{13}}(x_1+iy-x_5)^{s_{15}}(x_1-iy-x_5)^{s_{15}}\\ &(x_4+iy-x_5)^{s_{45}}(x_4-iy-x_5)^{s_{45}}(x_4+iy-\frac{i}{2})^{s_{24}+1}\\ &(x_4-iy+\frac{i}{2})^{s_{24}+1}(x_4+iy+\frac{i}{2})^{s_{34}+1}(x_4-iy-\frac{i}{2})^{s_{34}+1}\\ &(x_1-x_4)^{2s_{14}}F_5(x_1,x_4,x_5,y). \end{aligned} \tag{5.59}$$

Let us consider the following integral :

$$B_1 \equiv \int dx_1 dx_4 dx_5 \mathcal{I}_1(x_5) \int_{C_1} dy \mathcal{I}_2(x_1,x_4,x_5,y) \tag{5.60}$$

Along the contour C_1, we transform y as

$$y \to ie^{-i\epsilon}\tilde{y} \approx i\tilde{y} + \epsilon\tilde{y}. \tag{5.61}$$

The integrand $\mathcal{I}_2(x_1,x_4,x_5,y)$ becomes

$$\begin{aligned}\mathcal{I}_2(x_1,x_4,x_5,y) \to &(x_1-\tilde{y}-\frac{i}{2}+i\epsilon\tilde{y})^{s_{12}}(x_1+\tilde{y}+\frac{i}{2}-i\epsilon\tilde{y})^{s_{12}}(x_1-\tilde{y}+\frac{i}{2}+i\epsilon\tilde{y})^{s_{13}}\\ &(x_1+\tilde{y}-\frac{i}{2}-i\epsilon\tilde{y})^{s_{13}}(x_1-\tilde{y}-x_5+i\epsilon\tilde{y})^{s_{15}}(x_1+\tilde{y}-x_5-i\epsilon\tilde{y})^{s_{15}}\\ &(x_4-\tilde{y}-x_5+i\epsilon\tilde{y})^{s_{45}}(x_4+\tilde{y}-x_5-i\epsilon\tilde{y})^{s_{45}}(x_1-x_4)^{2s_{14}}\\ &(x_4-\tilde{y}-\frac{i}{2}+i\epsilon\tilde{y})^{s_{24}+1}(x_4+\tilde{y}+\frac{i}{2}-i\epsilon\tilde{y})^{s_{24}+1}\\ &(x_4-\tilde{y}+\frac{i}{2}+i\epsilon\tilde{y})^{s_{34}+1}(x_4+\tilde{y}-\frac{i}{2}-i\epsilon\tilde{y})^{s_{34}+1}. \end{aligned} \tag{5.62}$$

By defining the new variables as

$$\zeta_j = x_j - \tilde{y} \qquad \text{and} \qquad \omega_j = x_j + \tilde{y}, \tag{5.63}$$

where $j = \{1,4\}$. We then obtain integral B_1 of the form

$$B_1 = \int dx_5 \mathcal{I}_1(x_5) \int d\omega_1 d\omega_4 d\zeta_1 d\zeta_4 \delta(\omega_1-\omega_4+\zeta_4-\zeta_1)\tilde{\mathcal{I}}_2(\zeta_1,\zeta_4,\omega_1,\omega_4,x_5), \tag{5.64}$$

where

$$\tilde{\mathcal{I}}_2(\zeta_1,\zeta_4,\omega_1,\omega_4,x_5) \equiv (\zeta_1 - \frac{i}{2} + \frac{i\epsilon\gamma}{2})^{s_{12}}(\zeta_1 + \frac{i}{2} + \frac{i\epsilon\gamma}{2})^{s_{13}}(\zeta_1 - x_5 + \frac{i\epsilon\gamma}{2})^{s_{15}}$$
$$(\zeta_1 - \zeta_4)^{s_{14}}(\zeta_4 - x_5 + \frac{i\epsilon\gamma}{2})^{s_{45}}(\zeta_4 - \frac{i}{2} + \frac{i\epsilon\gamma}{2})^{s_{24}+1}$$
$$(\zeta_4 + \frac{i}{2} + \frac{i\epsilon\gamma}{2})^{s_{34}+1}(\omega_1 + \frac{i}{2} - \frac{i\epsilon\gamma}{2})^{s_{12}}(\omega_1 - \frac{i}{2} - \frac{i\epsilon\gamma}{2})^{s_{13}}$$
$$(\omega_1 - x_5 - \frac{i\epsilon\gamma}{2})^{s_{15}}(\omega_4 - x_5 - \frac{i\epsilon\gamma}{2})^{s_{45}}(\omega_4 + \frac{i}{2} - \frac{i\epsilon\gamma}{2})^{s_{24}+1}$$
$$(\omega_4 - \frac{i}{2} - \frac{i\epsilon\gamma}{2})^{s_{34}+1}(\omega_1 - \omega_4)^{s_{14}} \tag{5.65}$$

and $\gamma \equiv \omega_j - \zeta_j$. Note that the Dirac delta function has been inserted because the new variables are not independent. By utilizing the binomial expansions to separate the variable x_5 from $\tilde{\mathcal{I}}_2(\zeta_1,\zeta_4,\omega_1,\omega_4,x_5)$. The integral B_1 becomes

$$B_1 = \sum_{a,b,c,d=0}^{\infty}\binom{s_{15}}{a}\binom{s_{45}}{b}\binom{s_{15}}{c}\binom{s_{45}}{d}\int dx_5\left|x_5 - \frac{i}{2}\right|^{2s_{25}+2}\left|x_5 + \frac{i}{2}\right|^{2s_{35}+2}(-x_5)^{a+b+c+d}$$
$$\times\int d\omega_1 d\omega_4 d\zeta_1 d\zeta_4 \delta(\omega_1 - \omega_4 + \zeta_4 - \zeta_1)\mathcal{I}_3(\zeta_1,\zeta_4,\omega_1,\omega_4), \tag{5.66}$$

where

$$\mathcal{I}_3(\zeta_1,\zeta_4,\omega_1,\omega_4) \equiv (\zeta_1 - \frac{i}{2} + \frac{i\epsilon\gamma}{2})^{s_{12}}(\zeta_1 + \frac{i}{2} + \frac{i\epsilon\gamma}{2})^{s_{13}}(\zeta_1 + \frac{i\epsilon\gamma}{2})^{s_{15}-a}$$
$$(\zeta_1 - \zeta_4)^{s_{14}}(\zeta_4 + \frac{i\epsilon\gamma}{2})^{s_{45}-b}(\zeta_4 - \frac{i}{2} + \frac{i\epsilon\gamma}{2})^{s_{24}+1}$$
$$(\zeta_4 + \frac{i}{2} + \frac{i\epsilon\gamma}{2})^{s_{34}+1}(\omega_1 + \frac{i}{2} - \frac{i\epsilon\gamma}{2})^{s_{12}}(\omega_1 - \frac{i}{2} - \frac{i\epsilon\gamma}{2})^{s_{13}}$$
$$(\omega_1 - \frac{i\epsilon\gamma}{2})^{s_{15}-c}(\omega_4 - \frac{i\epsilon\gamma}{2})^{s_{45}-d}(\omega_4 + \frac{i}{2} - \frac{i\epsilon\gamma}{2})^{s_{24}+1}$$
$$(\omega_4 - \frac{i}{2} - \frac{i\epsilon\gamma}{2})^{s_{34}+1}(\omega_1 - \omega_4)^{s_{14}}. \tag{5.67}$$

By using the integral form the Dirac delta function

$$\delta(\omega_1 - \omega_4 + \zeta_4 - \zeta_1) = \frac{1}{2\pi}\int dl \exp\{il(\omega_1 - \omega_4 + \zeta_4 - \zeta_1)\}, \tag{5.68}$$

we then write

$$B_1 = \sum_{a,b,c,d=0}^{\infty} \binom{s_{15}}{a}\binom{s_{45}}{b}\binom{s_{15}}{c}\binom{s_{45}}{d} \int dx_5 \left|x_5 - \frac{i}{2}\right|^{2s_{25}+2} \left|x_5 + \frac{i}{2}\right|^{2s_{35}+2} (-x_5)^{a+b+c+d}$$
$$\int \frac{dl}{2\pi} \Big[\int d\omega_1 d\omega_4 e^{il(\omega_1-\omega_4)} (\omega_1 + \frac{i}{2} - \frac{i\epsilon\gamma}{2})^{s_{12}} (\omega_1 - \frac{i}{2} - \frac{i\epsilon\gamma}{2})^{s_{13}} (\omega_1 - \omega_4)^{s_{14}}$$
$$(\omega_1 - \frac{i\epsilon\gamma}{2})^{s_{15}-c} (\omega_4 - \frac{i\epsilon\gamma}{2})^{s_{45}-d} (\omega_4 + \frac{i}{2} - \frac{i\epsilon\gamma}{2})^{s_{24}+1} (\omega_4 - \frac{i}{2} - \frac{i\epsilon\gamma}{2})^{s_{34}+1} \Big]$$
$$\Big[\int d\zeta_1 d\zeta_4 e^{il(\zeta_4-\zeta_1)} (\zeta_1 - \frac{i}{2} + \frac{i\epsilon\gamma}{2})^{s_{12}} (\zeta_1 + \frac{i}{2} + \frac{i\epsilon\gamma}{2})^{s_{13}} (\zeta_1 + \frac{i\epsilon\gamma}{2})^{s_{15}-a}$$
$$(\zeta_1 - \zeta_4)^{s_{14}} (\zeta_4 + \frac{i\epsilon\gamma}{2})^{s_{45}-b} (\zeta_4 - \frac{i}{2} + \frac{i\epsilon\gamma}{2})^{s_{24}+1} (\zeta_4 + \frac{i}{2} + \frac{i\epsilon\gamma}{2})^{s_{34}+1} \Big]. \qquad (5.69)$$

The integral in the square parentheses can be defined as the one-parameter dependent amplitudes, i.e.

$$M_5(k_1, k_4, k_5; p_2, p_3 : l, \{n_{ij}\}) \equiv \int d\omega_1 d\omega_4 e^{il(\omega_1-\omega_4)} (\omega_1 + \frac{i}{2} - \frac{i\epsilon\gamma}{2})^{s_{12}+n_{12}}$$
$$(\omega_1 - \frac{i}{2} - \frac{i\epsilon\gamma}{2})^{s_{13}+n_{13}} (\omega_1 - \omega_4)^{s_{14}+n_{14}}$$
$$(\omega_1 - \frac{i\epsilon\gamma}{2})^{s_{15}-n_{15}} (\omega_4 - \frac{i\epsilon\gamma}{2})^{s_{45}-n_{45}}$$
$$(\omega_4 + \frac{i}{2} - \frac{i\epsilon\gamma}{2})^{s_{24}+n_{24}} (\omega_4 - \frac{i}{2} - \frac{i\epsilon\gamma}{2})^{s_{34}+n_{34}}, \qquad (5.70)$$

where n_{ij} are integer number. The expression (5.70) is considered as one-parameter amplitudes, where three vertex operators are fixed at $\omega_5 = 0$, $\omega_2 = -\frac{i}{2}$, and $\omega_3 = \frac{i}{2}$. We need to consider all contour of ω_j and ζ_j shown in figure 17 This can be done

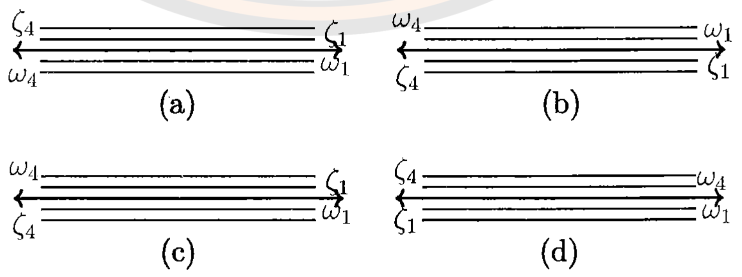

Figure 17 Contours of integration of ζ_1, ζ_4, ω_1, and ω_4 (a) $y_1 > 0$ and $y_4 > 0$, (b) $y_1 < 0$ and $y_4 < 0$, (c) $y_1 > 0$ and $y_4 < 0$, and (d) $y_1 < 0$ and $y_4 > 0$

by inserting an identity,

$$1 = \Theta(y_1)\Theta(y_4) + \Theta(-y_1)\Theta(-y_4) + \Theta(-y_1)\Theta(y_4) + \Theta(y_1)\Theta(-y_4). \tag{5.71}$$

Nontheless, all cases give us the same expression for $M_5(k_1, k_2, k_5; p_2, p_3; l, \{n_{ij}\})$ which is

$$\begin{aligned} M_5(k_1, k_4, k_5; p_2, p_3; l, \{n_{ij}\}) \equiv & e^{i\pi s_{14}} M_5(5,1,4;2,3;l,\{n_{ij}\}) + M_5(5,4,1;2,3;l,\{n_{ij}\}) \\ & + e^{-i\pi(s_{15}+s_{45}-s_{14})} M_5(1,4,5;2,3;l,\{n_{ij}\}) \\ & + e^{-i\pi(s_{15}+s_{45})} M_5(4,1,5;2,3;l,\{n_{ij}\}) \\ & + e^{-i\pi(s_{15}-s_{14})} M_5(1,5,4;2,3;l,\{n_{ij}\}) \\ & + e^{-i\pi s_{45}} M_5(4,5,1;2,3;l,\{n_{ij}\}). \end{aligned} \tag{5.72}$$

where

$$\begin{aligned} M_5(1,4,5;2,3;l,\{n_{ij}\}) \equiv \int_{\omega_1<\omega_4<0} & d\omega_1 d\omega_4 e^{il(\omega_1-\omega_4)} (\omega_1 + \frac{i}{2})^{s_{12}+n_{12}} \\ & \times (\omega_1 - \frac{i}{2})^{s_{13}+n_{13}} (\omega_4 - \omega_1)^{s_{14}+n_{14}} (\omega_1)^{s_{15}-n_{15}} \\ & \times (\omega_4)^{s_{45}-n_{45}} (\omega_4 + \frac{i}{2})^{s_{24}+n_{24}} (\omega_4 - \frac{i}{2})^{s_{34}+n_{34}}. \end{aligned} \tag{5.73}$$

The integration regions correspond to the color order of the amplitudes. The set $\{n_{ij}\}$ is $\{\ n_{12} = n_{13} = n_{14} = 0,\ n_{24} = n_{34} = 1,\ n_{15} = a,\ c,\ n_{45} = b,\ d\ \}$. $M_5(1,4,5;2,3;l,n_{ij})$ correspond to the five-point mixed string amplitudes when $l = 0$. We then write

$$\begin{aligned} B_1 = \sum_{a,b,c,d=0}^{\infty} \binom{s_{15}}{a}\binom{s_{45}}{b}\binom{s_{15}}{c}\binom{s_{45}}{d} \int dx_5 \left|x_5 - \frac{i}{2}\right|^{2s_{25}+2} \left|x_5 + \frac{i}{2}\right|^{2s_{35}+2} (-x_5)^{a+b+c+d} \\ \int \frac{dl}{2\pi} M_5(k_1, k_2, k_5; p_2, p_3; l, \{n_{ij}\}) \times \tilde{M}_5(k_1, k_2, k_5; p_2, p_3; l, \{n_{ij}\}). \end{aligned} \tag{5.74}$$

Furthermore, we analyze the integration of x_5 by using the analytic continuations. Let us consider the following integral:

$$C_1 \equiv \int dx_5 \left|x_5 - \frac{i}{2}\right|^{2s_{25}+2} \left|x_5 + \frac{i}{2}\right|^{2s_{35}+2} (-x_5)^{a+b+c+d}. \tag{5.75}$$

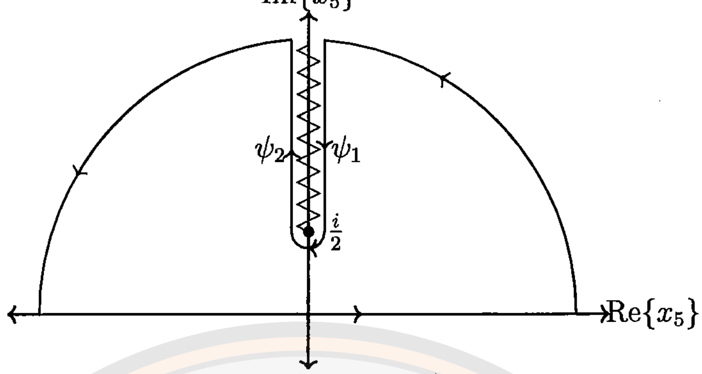

Figure 18 Contour of integration of x_5

Note that the branch points are at $\pm\frac{i}{2}$. By defining the contour of x_5, as shown in figure 18, and applying the Cauchy's theorem. The integration C_1 takes the form

$$C_1 = -\left(\int_{\psi_1} + \int_{\psi_2}\right) dx_5 \left|x_5 - \frac{i}{2}\right|^{2s_{25}+2} \left|x_5 + \frac{i}{2}\right|^{2s_{35}+2} (-x_5)^{a+b+c+d}. \tag{5.76}$$

Along the contour path ψ_1 and ψ_2, the x_5 transform as

$$x_5 \to \frac{ix_5}{2}. \tag{5.77}$$

Then the integration of x_5 yields

$$\begin{aligned} C_1 =& (\frac{i}{2})^{1+2s_{25}+2s_{35}+4}(-1)^{a+b+c+d} e^{i\pi(s_{25}+s_{35})}(-2i\sin(\pi(s_{25}+s_{35}+2)) \\ &\times \frac{B(3+s_{25}+s_{35}, -\frac{5}{2}-s_{25}-s_{35}-\frac{a+b+c+d}{2})}{2}, \end{aligned} \tag{5.78}$$

where $B(x,y)$ is beta functions. $\mathrm{Re}\{2(s_{25}+s_{35})+(a+b+c+d)\}$ is less than -1, $\mathrm{Re}\{a+b+c+d\}$ is greater than -1 and $\mathrm{Re}\{s_{25}+s_{35}\}$ is greater than -3. Finally, the expression of B_1 is given by

$$\begin{aligned} B_1 = \sum_{a,b,c,d=0}^{\infty} & \binom{s_{15}}{a}\binom{s_{45}}{b}\binom{s_{15}}{c}\binom{s_{45}}{d}(\frac{i}{2})^{1+2s_{25}+2s_{35}}(-1)^{a+b+c+d} e^{i\pi(s_{25}+s_{35})} \\ & (-2i\sin(\pi(s_{25}+s_{35})) \times \frac{B(3+s_{25}+s_{35}, -\frac{5}{2}-s_{25}-s_{35}-\frac{a+b+c+d}{2})}{2} \\ & \int \frac{dl}{2\pi} M_5(k_1,k_2,k_5;p_2,p_3;l) \times \tilde{M}_5(k_1,k_2,k_5;p_2,p_3;l). \end{aligned} \tag{5.79}$$

We also examine the correction terms which arise from the integration of y along infinite tube namely, L_1, L_2, L_3, and L_4. And we need to specify the region of integration of x_1 and x_4. There are eight cases which are i.) $0< x_1<x_4$, ii.) $0< x_4<x_1$, iii.)$x_4<x_1<0$, iv.) $x_1<x_4<0$, v.) $x_1 > 0$, $x_4 <0$ with $|x_4| > |x_1|$, vi.) $x_1 > 0$, $x_4 <0$ with $|x_1| > |x_4|$, vii.) $x_1 < 0$, $x_4 >0$ with $|x_4| > |x_1|$, and viii.) $x_1 < 0$, $x_4 >0$ with $|x_1| > |x_4|$.

In the thesis, we show only one of the cases, which is $0< x_1<x_4$. Let's consider the following integrals:

$$B_2 \equiv \int_{L_1+L_2} dy\mathcal{I}_2(x_1,x_4,x_5,y) \quad \text{and} \quad B_3 \equiv \int_{L_3+L_4} dy\mathcal{I}_2(x_1,x_4,x_5,y). \tag{5.80}$$

For convenience, we set $F_5 = 1$. By transforming the variable y to y' where

$$y = \frac{1}{2} + iy', \tag{5.81}$$

the integral B_2 becomes

$$\begin{aligned} B_2 =& i\Big(\int_{\tilde{L}_1} + \int_{\tilde{L}_2}\Big)dy'(x_1-y')^{s_{12}}(x_1+y')^{s_{12}}(x_1-y'+i)^{s_{13}}(x_1+y'-i)^{s_{13}}(x_4+y')^{s_{24}} \\ & (x_1-y'-x_5+\frac{i}{2})^{s_{15}}(x_1+y'-x_5-\frac{i}{2})^{s_{15}}(x_4-y'+i)^{s_{34}+1}(x_4+y'-i)^{s_{34}+1} \\ & (x_4-y'-x_5+\frac{i}{2})^{s_{45}}(x_1-x_4)^{2s_{14}}(x_4+y'-x_5-\frac{i}{2})^{s_{45}+1}(x_4-y')^{s_{24}+1}. \end{aligned} \tag{5.82}$$

The contour paths $\tilde{L}_1$ and $\tilde{L}_2$ of the variable y' and $-y'$ are illustrated in figure

Figure 19 Contour of integration of (a) y' and (b) $-y'$.

19(a) and 19(b) respectively. By using the following relations:

$$(x_1-y')^c = \begin{cases} |x_1-y'|^c e^{ic\pi}; & \text{for path } \tilde{L}_1 \\ |x_1-y'|^c e^{-ic\pi}; & \text{for path } \tilde{L}_2, \end{cases} \tag{5.83}$$

for $y' \in [x_1, x_4]$

$$(x_4 - y')^c = |x_4 - y'|^c \qquad ;\text{for path} \quad \tilde{L}_1, \tilde{L}_2, \tag{5.84}$$

for $y' \in [x_4, +\infty)$

$$(x_4 - y')^c = \begin{cases} |x_4 - y'|^c e^{ic\pi}; & \text{for path} \quad \tilde{L}_1, \\ |x_4 - y'|^c e^{-ic\pi}; & \text{for path} \quad \tilde{L}_2, \end{cases} \tag{5.85}$$

one can write

$$B_2 = 2\sin\left(\pi(s_{12} + s_{24} + 1)\right) \int_{x_4}^{\infty} dy' \mathcal{I}_3 + 2\sin\left(\pi s_{12}\right) \int_{x_1}^{x_4} dy' \mathcal{I}_3, \tag{5.86}$$

where

$$\begin{aligned} \mathcal{I}_3 =& |x_1 - y'|^{s_{12}} |x_4 - y'|^{s_{24}+1} (x_1 + y')^{s_{12}} (x_1 - y' + i)^{s_{13}} (x_1 + y' - i)^{s_{13}} \\ & (x_1 - y' - x_5 + \frac{i}{2})^{s_{15}} (x_1 + y' - x_5 - \frac{i}{2})^{s_{15}} (x_4 - y' + i)^{s_{34}+1} (x_4 + y' - i)^{s_{34}+1} \\ & (x_4 - y' - x_5 + \frac{i}{2})^{s_{45}} (x_1 - x_4)^{2s_{14}} (x_4 + y' - x_5 - \frac{i}{2})^{s_{45}} (x_4 + y')^{s_{24}+1}. \end{aligned} \tag{5.87}$$

Likewise, B_3 can be computed by the same as we did for B_2 but we define y as

$$y = -\frac{1}{2} - i\tilde{y}. \tag{5.88}$$

The path $\tilde{L}_3$ and $\tilde{L}_4$ are illustrated in figure 20. We also apply the following

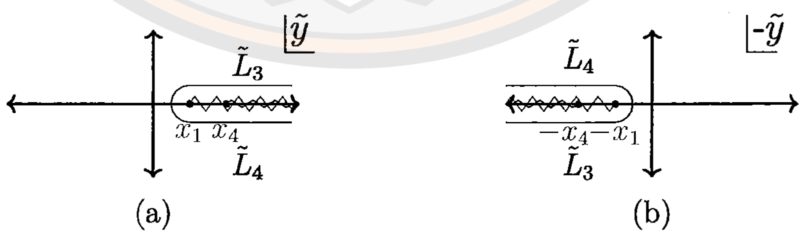

Figure 20 Contour of integration of (a) $\tilde{y}$ and (b) $-\tilde{y}$.

relations:

$$(x_1 - \tilde{y})^c = \begin{cases} |x_1 - \tilde{y}|^c e^{ic\pi}; & \text{for path} \quad \tilde{L}_4 \\ |x_1 - \tilde{y}|^c e^{-ic\pi}; & \text{for path} \quad \tilde{L}_3 \end{cases} \tag{5.89}$$

for $\tilde{y} \in [x_1, x_4]$

$$(x_4 - \tilde{y})^c = |x_4 - \tilde{y}|^c; \quad \text{for path } \tilde{L}_3 \text{ and } \tilde{L}_4 \tag{5.90}$$

for $\tilde{y} \in [x_4, +\infty)$

$$(x_4 - \tilde{y})^c = \begin{cases} |x_4 - \tilde{y}|^c e^{ic\pi}; & \text{for path } \tilde{L}_4 \\ |x_4 - \tilde{y}|^c e^{-ic\pi}; & \text{for path } \tilde{L}_3 \end{cases} \tag{5.91}$$

the integral B_3 takes the form

$$B_3 = 2\sin\left(\pi(s_{13} + s_{34} + 1)\right) \int_{x_4}^{\infty} d\tilde{y}\mathcal{I}_4 + 2\sin\left(\pi s_{13}\right) \int_{x_1}^{x_4} d\tilde{y}\mathcal{I}_4, \tag{5.92}$$

where

$$\begin{aligned} \mathcal{I}_4 =& |x_1 - \tilde{y}|^{s_{13}} |x_4 - \tilde{y}|^{s_{34}+1} (x_1 - \tilde{y} + i)^{s_{12}} (x_1 + \tilde{y} - i)^{s_{12}} (x_1 + \tilde{y})^{s_{13}} \\ & (x_1 - \tilde{y} - x_5 + \frac{i}{2})^{s_{15}} (x_1 + \tilde{y} - x_5 - \frac{i}{2})^{s_{15}} (x_4 + \tilde{y})^{s_{34}+1} (x_4 - \tilde{y} - x_5 + \frac{i}{2})^{s_{45}} \\ & (x_1 - x_4)^{2s_{14}} (x_4 + \tilde{y} - x_5 - \frac{i}{2})^{s_{45}} (x_4 - \tilde{y} + i)^{s_{24}+1} (x_4 + \tilde{y} - i)^{s_{24}+1}. \end{aligned} \tag{5.93}$$

We can compute all possible cases for the correction terms by using the same procedures and combining all the results. Finally, we then obtain the following expression:

$$\begin{aligned}
\mathcal{A}_5^{\text{cl}} =& 8C_{S^2}\int dx_5\mathcal{I}_1\Big[\int_0^\infty dx_1\int_{x_1}^\infty dx_4\Big(\sin(\pi(s_{12}+s_{24}+1))\int_{x_4}^\infty dy'\mathcal{I}_3 \\
&+\sin(\pi s_{12})\int_{x_1}^{x_4} dy'\mathcal{I}_3+\sin(\pi(s_{13}+s_{34}+1))\int_{x_4}^\infty d\tilde{y}\mathcal{I}_4+\sin(\pi s_{13})\int_{x_1}^{x_4} d\tilde{y}\mathcal{I}_4\Big) \\
&+\int_0^\infty dx_1\int_0^{x_4} dx_4\Big(\sin(\pi(s_{12}+s_{24}+1))\int_{x_4}^\infty dy'\mathcal{I}_3+\sin(\pi s_{24})\int_{x_4}^{x_1} dy'\mathcal{I}_3 \\
&+\sin(\pi(s_{13}+s_{34}+1))\int_{x_4}^\infty d\tilde{y}\mathcal{I}_4+\sin(\pi s_{34})\int_{x_4}^{x_1} d\tilde{y}\mathcal{I}_4\Big) \\
&+\int_{-\infty}^0 dx_1\int_{-\infty}^{x_1} dx_4\Big(\sin(\pi(s_{12}+s_{24}+1))\int_{-x_4}^\infty dy'\mathcal{I}_3+\sin(\pi s_{12})\int_{-x_1}^{-x_4} dy'\mathcal{I}_3 \\
&+\sin(\pi(s_{13}+s_{34}+1))\int_{-x_4}^\infty d\tilde{y}\mathcal{I}_4+\sin(\pi s_{34})\int_{-x_4}^{-x_1} d\tilde{y}\mathcal{I}_4\Big) \\
&+\int_{-\infty}^0 dx_4\int_{-\infty}^{x_4} dx_1\Big(\sin(\pi(s_{12}+s_{24}+1))\int_{-x_1}^\infty dy'\mathcal{I}_3+\sin(\pi s_{24})\int_{-x_4}^{-x_1} dy'\mathcal{I}_3 \\
&+\sin(\pi(s_{13}+s_{34}+1))\int_{-x_1}^\infty d\tilde{y}\mathcal{I}_4+\sin(\pi s_{34})\int_{-x_4}^{-x_1} d\tilde{y}\mathcal{I}_4\Big) \\
&+\int_{-\infty}^0 dx_4\int_0^{x_4} dx_1\Big(\sin(\pi(s_{12}+s_{24}+1))\int_{-x_4}^\infty dy'\mathcal{I}_3+\sin(\pi s_{12})\int_{x_1}^{-x_4} dy'\mathcal{I}_3 \\
&+\sin(\pi(s_{13}+s_{34}+1))\int_{-x_4}^\infty d\tilde{y}\mathcal{I}_4+\sin(\pi s_{13})\int_{x_1}^{-x_4} d\tilde{y}\mathcal{I}_4\Big) \\
&+\int_{-\infty}^0 dx_4\int_{-x_4}^\infty dx_1\Big(\sin(\pi(s_{12}+s_{24}+1))\int_{x_1}^\infty dy'\mathcal{I}_3+\sin(\pi s_{24})\int_{-x_4}^{x_1} dy'\mathcal{I}_3 \\
&+\sin(\pi(s_{13}+s_{34}+1))\int_{x_1}^\infty d\tilde{y}\mathcal{I}_4+\sin(\pi s_{34})\int_{-x_4}^{x_1} d\tilde{y}\mathcal{I}_4\Big) \\
&+\int_0^\infty dx_4\int_\infty^{-x_4} dx_1\Big(\sin(\pi(s_{12}+s_{24}+1))\int_{x_4}^\infty dy'\mathcal{I}_3+\sin(\pi s_{12})\int_{-x_1}^{x_4} dy'\mathcal{I}_3 \\
&+\sin(\pi(s_{13}+s_{34}+1))\int_{x_4}^\infty d\tilde{y}\mathcal{I}_4+\sin(\pi s_{13})\int_{-x_1}^{x_4} d\tilde{y}\mathcal{I}_4\Big) \\
&+\int_0^\infty dx_4\int_{-x_4}^0 dx_1\Big(sin(\pi(s_{12}+s_{24}+1))\int_{-x_1}^\infty dy'\mathcal{I}_3+\sin(\pi s_{24})\int_{x_4}^{-x_1} dy'\mathcal{I}_3 \\
&+\sin(\pi(s_{13}+s_{34}+1))\int_{-x_1}^\infty d\tilde{y}\mathcal{I}_4+\sin(\pi s_{34})\int_{x_4}^{-x_1} d\tilde{y}\mathcal{I}_4\Big)\Big] \\
&+\sum_{a,b,c,d=0}^\infty\binom{s_{15}}{a}\binom{s_{45}}{b}\binom{s_{15}}{c}\binom{s_{45}}{d}(\frac{i}{2})^{1+2s_{25}+2s_{35}}(-1)^{a+b+c+d}e^{i\pi(s_{25}+s_{35})} \\
&(-2i\sin(\pi(s_{25}+s_{35}))\times\frac{B(3+s_{25}+s_{35},-\frac{5}{2}-s_{25}-s_{35}-\frac{a+b+c+d}{2})}{2} \\
&\int\frac{dl}{2\pi}M_5(k_1,k_2,k_5;p_2,p_3;l,\{n_{ij}\})\times\tilde{M}_5(k_1,k_2,k_5;p_2,p_3;l,\{n_{ij}\}). \qquad (5.94)
\end{aligned}$$

CHAPTER VI

CONCLUSIONS

In this thesis, we investigate the relations among string scattering amplitudes at tree-level. In Chapter 2, we provide basic notions of string theory both classical and quantum string theory. In Chapter 3, the basic ideas of string scattering amplitudes were provided.

In Chapter 4, we review the interesting relations among string scattering amplitudes, which are 1.) relations between closed and open string amplitudes at tree-level, 2.) relations among color-ordered open string scattering amplitudes, 3.) relations between mixed and open string amplitudes, and 4.) relations among mixed string amplitudes. The relations between closed string and open string show that closed string amplitudes can be expressed as products of open string amplitudes at tree-level, which are known as KLT-relations[13, 1]. The factors in this relation can be captured by the momentum kernel [29]. In low energy limits, this gives the connection between perturbative gravity and Yang-Mills amplitudes. The relations among color-ordered open string amplitudes show that color-ordered open string amplitudes can be related by monodromy relations. The number of independent color-ordered open string amplitudes can be reduced from $(n-1)!$ to $(n-3)!$. We can write the color-ordered open string amplitudes in terms of $(n-3)!$ independent color-ordered amplitudes. The set of independent amplitudes can be treated as the basis, known as minimal basis [2]. At the low energy limit, the relations reproduce the Kleiss-Kuijf relations [19]. The relations between mixed and open string amplitudes show that the mixed string amplitudes can be decomposed as linear combinations of pure open string amplitudes with appropriate phase factors. At the field theory limit, this gives the relation between Einstein Yang-Mills amplitudes and Yang-Mills amplitudes[25]. Mixed string amplitudes also have relations among themselves. The amplitudes can be connected by the monodromy relations and give a novel infinite tube contribution. By taking the field theory limit, this gives a new relation for Einstein Yang-Mills amplitudes, and the gluon part satisfies the Kleiss-Kuijf relation [19]. The infinite tube contribution does not play an important role in this limit; we could say that this is string correction terms [3].

In Chapter 5, we construct the relations between closed string and mixed string amplitudes, which involve $(n-2)$ open strings and a single closed string, at tree level. We start the derivation with the fewer-point relation, namely, the four-, five-, and six-point relation, which are presented in (5.9), (5.18), and (5.25), respectively. We then construct the n-point relation, and the result shows that the n-point closed string amplitudes can be expressed in terms of the product of mixed string amplitudes with the phase factors, as in (5.33). The phase factors correspond to the color-ordered mixed string amplitudes, which can be captured by $\mathcal{P}(\sigma|\sigma')$ (5.32). The function $\mathcal{P}(\sigma|\sigma')$ is defined by two new functions $\theta(i_j, i_k)$ and $\beta(i_j, i_k)$, which are defined as (5.29) and (5.31) respectively. However, there are correction terms that come from avoiding the branch cut in the bulk of the complex plane when we do the Wick rotations. We have shown the correction terms for the 4-point relation as in (5.51). Furthermore, we analyze the correction terms by utilizing the binomial expansion, analytic continuation of complex variables, and the contour in Fig. 15(b). The result shows that the correction terms are similar to the infinite tube amplitudes presented in [3]. Again, these correction terms can be neglected within a certain limit. For the five-point relation, we also provide an alternative way to formulate the relations between closed and mixed string amplitudes. The results show that the closed string amplitudes can be expressed as the product of two one-parameter mixed string amplitudes (5.72), and the correction terms also exist for the same reason.

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APPENDIX

APPENDIX A POLARIZATIONS AND KINEMATIC FACTORS

As mentioned in Section 5.2, the polarizations and kinematic factors of the external closed string state are contained in the function $F_n(z, \bar{z})$. Through out the factorization process, the functions F_n are decomposed to those open and closed strings in the mixed amplitudes captured by K_n and $\widetilde{K}_n$

For tachyons, $F_n = K_n = \widetilde{K}_n = 1$. For the first excited states, $F_n(z_i, \bar{z}_i)$ is given by

$$F_n(z_i, \bar{z}_i) = \exp\left\{ \sum_{i>j} \frac{\zeta_i \cdot \zeta_j}{(z_i - z_j)^2} - \sqrt{\alpha'} \sum_{i \neq j} \frac{k_i \cdot \zeta_j}{(z_i - z_j)} + \sum_{i>j} \frac{\tilde{\zeta}_i \cdot \tilde{\zeta}_j}{(\bar{z}_i - \bar{z}_j)^2} - \sqrt{\alpha'} \sum_{i \neq j} \frac{k_i \cdot \tilde{\zeta}_j}{(\bar{z}_i - \bar{z}_j)} \right\}\Bigg|_{\text{linear in } \zeta_i, \tilde{\zeta}_i}, \tag{A.1}$$

where $\zeta_i \otimes \tilde{\zeta}_i = \chi_i$ are polarization of external closed string states. The function $K_n(x_i)$, involving $n-2$ open strings and a single closed string, takes the form

$$\begin{aligned} \exp\Bigg\{ &\sum_{i>j}^{n-2} \frac{\zeta_i \cdot \zeta_j}{(x_i - x_j)^2} - \sqrt{\alpha'} \sum_{i \neq j}^{n-2} \frac{k_i \cdot \zeta_j}{(x_i - x_j)} + \sum_{j=1}^{n-3} \Bigg[\frac{\vartheta \cdot \zeta_j}{(x_j - z)^2} + \frac{\tilde{\vartheta} \cdot \zeta_j}{(x_j - \bar{z})^2} \\ &- \sqrt{\alpha'} \left(\frac{k_{n-1} \cdot \zeta_j}{(\bar{z} - \eta_j)} + \frac{k_{n-2} \cdot \zeta_j}{(z - \eta_j)} + \frac{k_j \cdot \vartheta}{(x_j - z)} + \frac{k_j \cdot \tilde{\vartheta}}{(x_j - \bar{z})} \right) \Bigg] \\ &+ \frac{\vartheta \cdot \tilde{\vartheta}}{(z - \bar{z})^2} - \sqrt{\alpha'} \left(\frac{k_{n-1} \cdot \vartheta}{(\bar{z} - z)} + \frac{k_{n-2} \cdot \tilde{\vartheta}}{(z - \bar{z})} \right) \Bigg\}\Bigg|_{\text{linear in } \zeta_i, \vartheta, \tilde{\vartheta}} \end{aligned} \tag{A.2}$$

where ζ_i is open string polarization for $i \in \{1, \ldots, n-2\}$, $\chi = \vartheta \otimes \tilde{\vartheta}$ is closed string polarization, and $k_{n-2} + k_{n-1}$ is closed string momentum.

The function $F_n(z_i, \bar{z}_i)$ (A.1) can be factorized into the product of two functions as $f_n(z_i) \times \tilde{f}_n(\bar{z})$. Next, we proceed the similar procedure as in section 5.1 applying the analytic continuation of complex variables and introducing the new variables as defined in (5.20. We then obtain

$$
\begin{aligned}
f_n(\eta_i) =& \exp\Bigg\{ \sum_{i>j}^{n-3} \frac{\zeta_i \cdot \zeta_j}{(\eta_i - \eta_j)^2} - \sqrt{\alpha'} \sum_{i \neq j}^{n-3} \frac{k_i \cdot \zeta_j}{(\eta_i - \eta_j)} + \sum_{j=1}^{n-3} \Bigg[\frac{\zeta_{n-1} \cdot \zeta_j}{(\eta_j + i)^2} + \frac{\zeta_{n-2} \cdot \zeta_j}{(\eta_j - i)^2} \\
& - \sqrt{\alpha'} \left(\frac{k_{n-1} \cdot \zeta_j}{(-i - \eta_j)} + \frac{k_{n-2} \cdot \zeta_j}{(i - \eta_j)} + \frac{k_j \cdot \zeta_{n-1}}{(\eta_j + i)} + \frac{k_j \cdot \zeta_{n-2}}{(\eta_j - i)} \right) \Bigg] \\
& + \frac{\zeta_{n-1} \cdot \zeta_{n-2}}{(-2i)^2} - \sqrt{\alpha'} \left(\frac{k_{n-1} \cdot \zeta_{n-2}}{(-2i)} + \frac{k_{n-2} \cdot \zeta_{n-1}}{(2i)} \right) \Bigg\} \Bigg|_{\text{linear in } \zeta_i} \qquad \text{(A.3)}
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{f}_n(\xi_i) =& \exp\Bigg\{ \sum_{i>j}^{n-3} \frac{\tilde{\zeta}_i \cdot \tilde{\zeta}_j}{(\xi_i - \xi_j)^2} - \sqrt{\alpha'} \sum_{i \neq j}^{n-3} \frac{k_i \cdot \tilde{\zeta}_j}{(\xi_i - \xi_j)} + \sum_{j=1}^{n-3} \Bigg[\frac{\tilde{\zeta}_{n-1} \cdot \tilde{\zeta}_j}{(\xi_j - i)^2} + \frac{\tilde{\zeta}_{n-2} \cdot \tilde{\zeta}_j}{(\xi_j + i)^2} \\
& - \sqrt{\alpha'} \left(\frac{k_{n-1} \cdot \tilde{\zeta}_j}{(i - \xi_j)} + \frac{k_{n-2} \cdot \tilde{\zeta}_j}{(-i - \xi_j)} + \frac{k_j \cdot \tilde{\zeta}_{n-1}}{(\xi_j - i)} + \frac{k_j \cdot \tilde{\zeta}_{n-2}}{(\xi_j + i)} \right) \Bigg] \\
& + \frac{\tilde{\zeta}_{n-1} \cdot \tilde{\zeta}_{n-2}}{(2i)^2} - \sqrt{\alpha'} \left(\frac{k_{n-1} \cdot \tilde{\zeta}_{n-2}}{(2i)} + \frac{k_{n-2} \cdot \tilde{\zeta}_{n-1}}{(-2i)} \right) \Bigg\} \Bigg|_{\text{linear in } \tilde{\zeta}_i} . \qquad \text{(A.4)}
\end{aligned}
$$

Note that the positions of the three vertex operators z_{n-2}, z_{n-1}, and z_n are fixed to i, $-i$, and ∞, respectively. The functions f_n and $\tilde{f}_n$ can be identified as K_n and $\tilde{K}_n$ (A.2). In the mixed string amplitudes, closed string polarization regrading K_n and $\tilde{K}_n$ are $\zeta_{n-1} \otimes \zeta_{n-2}$ and $\tilde{\zeta}_{n-1} \otimes \tilde{\zeta}_{n-2}$,respectively.