CONSTRAINT ANALYSIS IN GENERALISED PROCA THEORIES AND CHIRAL BOSON THEORIES

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ABSTRACT

We consider constraint analysis in generalised Proca theories and chiral boson theories. In the first part, we derive conditions which are sufficient for theories consisting of multiple vector fields, which could also couple to non-dynamical external fields, to have the required structure of constraints of diffeomorphism invariant multi-field generalised Proca theories, so that the number of degrees of freedom is correct. The Faddeev–Jackiw constraint analysis is used and is cross-checked by Lagrangian constraint analysis. To ensure the theory is constraint, we impose a special Hessian condition. The sufficient conditions obtained include a refinement of secondary-constraint enforcing relations derived previously in literature, as well as the completion requirements. In the second part of this thesis, we use Dirac constraint analysis in Sen formulation. This formulation for chiral $(2p)$-form in $4p+2$ dimensions describes a system with two separate sectors, one is physical while the other is unphysical. Each contains a chiral form and a metric. We focus on the cases where the self-duality condition for the unphysical sector is linear while for the physical sector can be nonlinear. The decoupling at the Hamiltonian level follows the idea in the literature. Then by an appropriate field redefinition, the separation at the Lagrangian level follows. The Lagrangian for the quadratic theory is separated into two Henneaux–Teitelboim Lagrangians.

LIST OF CONTENTS

LIST OF CONTENTS (CONT.)

Chapter **Page**

CHAPTER I

INTRODUCTION

1.1 Overview

In classical mechanics, a constrained system is the one where the motion of particles is restricted by specific conditions, known as constraints. These constraints reduce degrees of freedom of the system, namely, the number of dynamical variables is less than the total number of generalized coordinates. Important examples of constraint systems are gravity theories and gauge theories.

The methods often used in order to analyses constrained systems are Dirac method [1], [2], [3], Faddeev–Jackiw (FJ) method [4], [5] [6], [7], and Lagrangian constraint analysis [8], [9], [10], [11]. Each offers a different way to identify constraints and determine the number of degrees of freedom of the system. Dirac method and FJ method are based on Hamiltonian formalism but they are technically different. Dirac method begins by identifying primary constraints from the definitions of conjugate momenta. Secondary constraints emerge by ensuring the consistency of primary constraints over time. Then all constraints are reclassified into first-class and second-class constraints. Finally, the number of degree of freedom can be counted. FJ method focuses on the symplectic structure of the system. Primary constraints can be obtained like Dirac method. However, the following steps are carried out in iterations. At the end of each iteration, one obtains new constraints. The process run until no new constraint. The number of degrees of freedom is counted by using the intermediate results from the process [12]. The last one is Lagrangian constraint analysis. It examines behavior of the system directly from its Lagrangian formulation, formulate its equations of motion, identify constraints, and determine the number of physical degrees of freedom.

In this thesis, we mainly follow the process of constrained method in [13], especially the FJ method. We would like to apply the FJ method to cosmology then crosscheck with Lagrangian constraint analysis, and apply Dirac constraint analysis in string theory.

The important equations that describe the behaviour of a physical system are the equations of motion. For example, the well-known equation in classical mechanics; the Newton's second law, Schrödinger equation in quantum mechanics, Einstein field equations in general relativity. All of these equations of motion can be obtained by using variational principle with action. Using principle of least action $\delta S = 0$, we obtain equation of motion. As a concrete example, the equation of motion of metric is Einstein field equation. To obtain it, one starts from Einstein-Hilbert action. Using principle of least action $\delta S_{EH} = 0$, we finally obtain Einstein field equation.

Tests of Einstein field equation at the post-Newtonian level have reached high precision, including the light deflection, the Shapiro time delay, the perihelion advance of Mercury, the Nordtvedt effect in lunar motion, frame-dragging and Gravitational wave. General relativity has been a successful theory giving predictions which accurately agree with observations. However, there are results which cannot be described by general relativity. One of these is the late-time accelerated expansion of the universe which has been confirmed by a wealth of observational evidence from the measured distances of type Ia supernovae measurements of Ω and Λ from 42 high-redshift supernovae [14], [15]. As an attempt towards describing the mechanism behind this, one may consider modifying general relativity.

In modified gravity, there are theories that combine multiple types of fields. Scalar-tensor theories involve both scalar fields and tensor fields, while vector-tensor theories involve both vector fields and tensor fields. These combinations aim to provide a more comprehensive explanation of gravitational phenomena and cosmic evolution. We use constraint analysis to better understand of these theories. Especially, in this thesis we discuss multi-field generalised Proca theory.

In this thesis, we use FJ constraint analysis to derive sufficient conditions for multi-field generalised Proca theories to have the correct number of degrees of freedom. Then, we crosscheck by Lagrangian constraint analysis. Diffeomorphism invariance requirements help to obtain these conditions. Firstly, we impose the special Hessian condition to ensure that the theory has constraints. In the process to obtain secondary constraints, we obtain the correct version of secondary-constraint enforcing relations in [8], [9]. Further conditions should also be imposed to ensure that the symplectic two-form at the second iteration does not have a zero mode so that the constraint analysis terminates [8], [9], [10], [16]. If a theory passes all these requirements, then it is a multi-field generalised Proca theory [17].

String theory is a theory of a one-dimensional object called string. There are 5 types of consistent superstring theories. They are called type I, type IIA, type IIB, heterotic E8 $\times$ E8, and heterotic SO(32). String theory contains, in addition to string, objects called D-branes. There is attempt to unify string theory to obtain the more fundamental theory than string theory called M-theory. This theory does not contain strings but include higher-dimensional entities called M-branes. These branes can have various dimensionalities and play a crucial role in the dynamics of the theory. There are two primary types of branes in M-theory: M2-brane and M5-brane. A comprehensive understanding of the M5-brane remains an open and challenging problem in theoretical physics.

Chiral form fields play a significant role in string theory and M-theory. In a $(4p+2)$-dimensional spacetime with metric g with $p = 0, 1, 2, \ldots$, a chiral $2p$-form A is a type of gauge field. Its field strength $F = dA$ is self-dual or anti-self-dual. In case $p = 0$, a chiral 0-form in 2 dimensions can be used to explain a heterotic string [18]. A chiral 4-form field in 10 dimensions is relevant in type IIB string theory

[19]. Constructing a consistent action for the M5-brane in M-theory is a complex task, primarily due to the presence of a chiral 2-form field on its 6 dimensional world-volume [20]. This field is characterized by a self-dual 3-form field strength, meaning it equals its own Hodge dual.

The challenges associated with formulating consistent actions have been addressed through various approaches. Notably, significant contributions have been made by Henneaux-Teitelboim (HT), Pasti-Sorokin-Tonin (PST), and Ashoke Sen. These approaches are different. HT formulation uses Hamiltonian analysis which splits the spacetime coordinates $x^\mu; \mu \in \{0, 1, \cdots, 4p+1\}$ into time and spatial coordinates as $x^0, x^i; i \in \{1, \cdots, 4p+1\}$. Its action is non-manifestly diffeomorphism invariant [21], [22]. PST approach introduces an auxiliary scalar field, enabling a Lorentz-invariant action that incorporates self-dual fields [23], [24], [25]. Sen proposed a formulation that separates the theory into physical and unphysical sectors, each containing chiral forms and metrics. This method facilitates the analysis of self-duality conditions and their decoupling at Hamiltonian and equation of motion levels [26], [27]. In this thesis, we use constraint analysis for more understanding of Sen formalism. The self-duality condition in the physical sector can be nonlinear. Our goal is to show the separation of the physical and unphysical sectors at Lagrangian level [17], [28], [29], [30].

1.2 Outline of the Dissertation

This thesis is organised as follows. In Chapter 2, we review modified gravity theory. There are scalar-tensor theory and vector-tensor theory. In Chapter 3, we review constraint analysis, which are Dirac constraint analysis, Faddeev-Jackiw constraint analysis, and Lagrangian constraint analysis. The sufficient conditions for degree of freedom counting of multi-field generalised Proca theories are presented in Chapter 4 which is largely based on [17]. In Chapter 5, we review Chiral field theories in M-theory. Constraint analysis in Chiral field theories is discussed in Chapter 6 which is largely based on[31]. We conclude this thesis and discuss possible future work in Chapter 7.

CHAPTER II

MODIFIED GRAVITY THEORIES

2.1 Scalar-tensor theories

A simple way to modified gravity is to introduce a scalar-tensor theory, in which there is a scalar field introduced into the Lagrangian. The most general scalar-tensor theory is Horndeski theory.

2.1.1 Horndeski theory

The Horndeski action is the most general scalar-tensor theory with at most second-order derivatives in the equations of motion [32], [33], [34], [35]. It evades Ostrogradsky instabilities [36], [37] and making it of interest when modifying gravity at large scales. The problem with a higher-derivative theory is that they can contain Ostrogradsky instabilities, which arise from ghost degrees of freedom due to third or higher time derivatives in the equations of motion. To avoid it, one may require the equations of motion to be of at most second order in derivatives. This consideration leads to the Horndeski action. One consider the system of single scalar field ϕ. The Lagrangian of Horndeski theory is given by

$$\mathcal{L} = \sqrt{-g}\sum_{n=2}^{5}\mathcal{L}_n, \tag{2.1}$$

where $\mathcal{L}_n$ are defined by

$$\begin{aligned}
\mathcal{L}_2 &= G_2(\phi, X),\\
\mathcal{L}_3 &= G_3(\phi, X)\Box\phi,\\
\mathcal{L}_4 &= G_4(\phi, X)R - 2G_{4,X}(\phi, X)((\Box\phi)^2 - (\nabla\nabla\phi)^2),\\
\mathcal{L}_5 &= G_5(\phi, X)G^{\mu\nu}\nabla_\mu\nabla_\nu\phi + \frac{1}{3}G_{5,X}(\phi, X)((\Box\phi)^3 - 3\Box\phi(\nabla\nabla\phi)^2\\
&\quad + 2(\nabla\nabla\phi)^3),
\end{aligned}$$

and $X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. Horndeski theories accommodate a wide variety of gravitational theories like Brans-Dicke theory, $f(R)$ gravity, and covariant Galileons. After the discovery of GW170817, the higher order terms in Horndeski theories have been ruled out. It is possible to go beyond Horndeski thoery [38], [39], [40].

2.2 Vector-tensor theories

Vector-Tensor theories are alternative theory to scalar-tensor theories. In these theories, gravity is described by using both the tensor and vector field. The idea of the construction is to first only consider vector theory. One tries to construct the most general vector-tensor theories using the same idea with the scalar-tensor theories. The well-known vector theory is Maxwell theory. One tries to extend this theory. It turns out that there is a no-go theorem forbidding the Horndeski-like vector theory with 2 degrees of freedom [41]. One has to turn to generalising massive vector theory named Proca theory. This generalisation is known as generalised Proca theories. It describes a system of vector field with derivative self interaction.

Original constructions of generalised Proca theories are given in [42], [43]. The extension to the original construction has been constructed by other works, for example, [44], [45], [46]. Further generalisations to generalised Proca theories are possible, for example, references [47], [48], [49] construct beyond generalised Proca theories, references [16], [50], [51], [52] construct Proca-Nuevo[1].

2.2.1 Generalised Proca theory

Generalizing scalar-tensor theories, the introduction of a vector field in place of a scalar field yields vector-tensor theories. Notably, Maxwell theory is a sim-

[1]Note that [52] points out that [16] has obtained incorrect secondary constraint. So the result of [16] are not correct. We thank Claudia de Rham for letting us know this recent development and related discussions.

ple example of a vector theory. It describes the dynamics of the massless spin-1 photon field. This field possesses two transverse polarizations, corresponding to the familiar electric and magnetic fields. It can be related to the vector field A_μ through appropriate gauge transformations. These two independent polarizations translate to two degrees of freedom.

When one introduces a mass term m for the vector field to Maxwell theory, one obtains Proca theory. The Proca Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu. \tag{2.2}$$

This simple modification has significant consequences. It describes a massive vector field and represents an extension of Maxwell theory that incorporates a mass term for the vector potential. The presence of this mass term fundamentally breaks the $U(1)$ gauge symmetry characteristic of Maxwell equations. It leads to a significant alteration in the degrees of freedom of the field.

In the massless case, the vector field possesses only two transverse polarizations corresponding to the physical degrees of freedom of a spin-1 field. However, the introduction of a mass allows for the emergence of a longitudinal polarization mode. Consequently, the Proca theory has a total of three dynamical degrees of freedom for the massive spin-1 field: two associated with the transverse modes and one with the longitudinal mode [53].

The dynamics of the Proca field are shown in the equation of motion which is given by

$$\partial_\nu F^{\nu\mu} - m^2 A^\mu = 0. \tag{2.3}$$

This equation describes how the massive vector field propagates and interacts in the presence of the mass term. The existence of this mass term not only modifies the field equations but also imposes constraints on the gauge invariance that governs the behavior of the vector field. Therefore, the Proca Lagrangian provides a comprehensive framework for understanding the implications of mass in vector fields.

Then let us consider the generalised Proca theory [42], [43]. Its Lagrangian is given by

$$\mathcal{L}_{gen.Proca} = -\sqrt{-g}\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \beta_1\sqrt{-g}G_{\mu\nu}A^{\mu}A^{\nu} + \sqrt{-g}\sum_{n=2}^{6}\beta_n\mathcal{L}_n, \tag{2.4}$$

where $\beta_n, n = 1,2,3,4,5,6$ are arbitrary constants and $\mathcal{L}_n, n = 2,3,4,5,6$ are self-interactions of the vector field given by

$$\begin{aligned}
\mathcal{L}_2 &= G_2(A_\mu, F_{\mu,\nu}, \tilde{F}_{\mu\nu}),\\
\mathcal{L}_3 &= G_3(Y)\nabla\cdot A,\\
\mathcal{L}_4 &= G_4(Y)R + G_4'(Y)[(\nabla\cdot A)^2 - \nabla_\rho A_\sigma\nabla^\sigma A^\rho],\\
\mathcal{L}_5 &= G_5(Y)G_{\mu\nu}\nabla^\mu A^\nu - \frac{1}{6}G_5'(Y)\left[(\nabla\cdot A)^3 - 3(\nabla\cdot A)\nabla_\rho A_\sigma\nabla^\sigma A^\rho\right.\\
&\quad \left.+2\nabla_\rho A_\sigma\nabla^\gamma A^\rho\nabla^\sigma A_\gamma\right] - \tilde{G}_5(Y)\tilde{F}^{\alpha\mu}\tilde{F}^{\beta}{}_{\mu}\nabla_\alpha A_\beta\\
\mathcal{L}_6 &= G_6(Y)L^{\mu\nu\alpha\beta}\nabla_\mu A_\nu\nabla_\alpha A_\beta + \frac{G_6'(Y)}{2}\tilde{F}^{\alpha\beta}\tilde{F}^{\mu\nu}\nabla_\alpha A_\mu\nabla_\beta A_\nu,
\end{aligned} \tag{2.5}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, Y \equiv -A_\mu A^\mu/2, \nabla\cdot A \equiv \nabla_\mu A^\mu, \tilde{F}_{\mu\nu}$ is the Hodge dual of $F_{\mu\nu}$, and

$$L^{\mu\nu\alpha\beta} = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}R_{\rho\sigma\gamma\delta}. \tag{2.6}$$

The Lagrangian (2.4) describes the most general generalised Proca theories. These theories are local and have diffeomorphism invariance A. Furthermore, these theories have additional properties, for example, the vector sector has three propagating degrees of freedom, the equations of motion in the vector sector have at most second order derivative in time, and there is no dynamics in the temporal component of the vector field.

2.3 Ostrogradsky instability

Theories with equations of motion involving derivatives higher than second order are typically prone to Ostrogradsky instability [36], [37]. This issue arises from the presence of ghost degrees of freedom, which are unphysical modes that indicate the system does not behave as expected. The existence of such ghost degrees of freedom suggests that the theory is problematic. Let us consider the Lagrangian of harmonic oscillator with an additional $\epsilon' \ddot{x}^2$ term in the Minkowski space-time

$$L = \frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2}x^2 + \epsilon' \ddot{x}^2, \tag{2.7}$$

where $\epsilon' = -\frac{\epsilon m}{2\omega^2}$ (the unit is $kg \cdot s^2$), ϵ is a dimensionless parameter. The Euler-Lagrange equation is

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}} = 0. \tag{2.8}$$

For $\epsilon' \neq 0$, this equation contains derivatives higher than second order,

$$\ddot{\ddot{x}} = \mathcal{F}(x, \dot{x}, \ddot{x}, \dddot{x}) \rightarrow x(t) = \mathcal{X}(t, x_0, \dot{x}_0, \ddot{x}_0, \dddot{x}_0). \tag{2.9}$$

Using Legendre transformation, the Hamiltonian is given by

$$H = -2\epsilon' \dot{x}\dddot{x} + \epsilon' \ddot{x}^2 + \frac{m}{2}\dot{x}^2 + \frac{m\omega^2}{2}x^2. \tag{2.10}$$

This term $-2\epsilon' \dot{x}\dddot{x}$ leads to an instability associated with the Hamiltonian unbounded from below.

CHAPTER III

CONSTRAINT ANALYSIS

3.1 Constrained system

A constrained system is a system in which the number of generalised coordinates is greater than the number of degrees of freedom. In this system, the conjugate momenta cannot be traded for generalised velocities. Equivalently, some Euler-Lagrange equations are only up to first-order derivative in time. A condition called Hessian condition can be used to determine whether a system is constraint. The methods often made use in order to analyse constrained systems are Dirac method, Faddeev–Jackiw method (FJ), and Lagrangian constraint analysis. We mainly follow the process of constrained method in [13], especially focusing on the Faddeev–Jackiw method. In this chapter, we consider theories in d-dimensional spacetime.

3.1.1 Hessian condition

In this subsection, we will consider Hessian condition of a multi-field Lagrangian theory $\mathcal{L} = \mathcal{L}(\phi^a, \partial_\mu \phi^a)$. Its Euler-Lagrange equation is given by

$$\frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} = 0, \tag{3.1}$$

where ϕ^a are collection of components of fields in the theory and $a = 1, 2, ..., N$, which is equivalent to

$$\frac{\partial \mathcal{L}}{\partial \phi^a} - \frac{\partial^2 \mathcal{L}}{\partial \dot{\phi}^a \partial \phi^b}\dot{\phi}^b - \frac{\partial^2 \mathcal{L}}{\partial \dot{\phi}^a \partial \partial_i \phi^b}\partial_i \dot{\phi}^b - \partial_i \left(\frac{\partial \mathcal{L}}{\partial \partial_i \phi^a}\right) = \frac{\partial^2 \mathcal{L}}{\partial \dot{\phi}^a \partial \dot{\phi}^b}\ddot{\phi}^b. \tag{3.2}$$

Then one find the inverse of $\partial^2 \mathcal{L}/\partial \dot{\phi}^a \partial \dot{\phi}^b$. It satisfies only in the case of its Hessian determinant does not equal to zero, i.e.

$$\det\left(\frac{\partial^2 \mathcal{L}}{\partial \dot{\phi}^a \partial \dot{\phi}^b}\right) \neq 0. \tag{3.3}$$

If the determinant of this matrix is non-zero, the matrix is invertible. It allows for the resolution of generalised velocities $\dot{\phi}^a$ in terms of conjugate momenta π^a. Such a system is referred to as non-degenerate or regular. Conversely, if the determinant of this matrix is zero, the Hessian is singular. It indicates that the Lagrangian is degenerate. In this scenario, the conjugate momenta cannot be uniquely expressed in terms of the generalized velocities, complicating the transition to the Hamiltonian formalism. On the other hand, if some equations relate initial data, namely generalised coordinates and generalised velocities, then $\partial^2\mathcal{L}/\partial\dot{\phi}^a\partial\dot{\phi}^b$ is not invertible. This system is constrained system.

For examples, the construction ideas in [42] consider only the vector theories A_μ and then demand that the theory is constrained. This is by imposing condiion which ensure that Hessian determinant vanish. More explicitly, one considers the Hessian matrix ($\mathcal{H}^{\mu\nu}$) associated with the Lagrangian term as

$$\mathcal{H}^{\mu\nu} = \frac{\partial^2\mathcal{L}}{\partial\dot{A}_\mu\partial\dot{A}_\nu}. \tag{3.4}$$

On the other hand, one expands all components of this matrix

$$\frac{\partial^2\mathcal{L}}{\partial\dot{A}_\mu\partial\dot{A}_\nu} = \begin{bmatrix} \frac{\partial^2\mathcal{L}}{\partial\dot{A}_0\partial\dot{A}_0} & \frac{\partial^2\mathcal{L}}{\partial\dot{A}_0\partial\dot{A}_1} & \frac{\partial^2\mathcal{L}}{\partial\dot{A}_0\partial\dot{A}_2} & \frac{\partial^2\mathcal{L}}{\partial\dot{A}_0\partial\dot{A}_3} \\ \frac{\partial^2\mathcal{L}}{\partial\dot{A}_1\partial\dot{A}_0} & \frac{\partial^2\mathcal{L}}{\partial\dot{A}_1\partial\dot{A}_1} & \frac{\partial^2\mathcal{L}}{\partial\dot{A}_1\partial\dot{A}_2} & \frac{\partial^2\mathcal{L}}{\partial\dot{A}_1\partial\dot{A}_3} \\ \frac{\partial^2\mathcal{L}}{\partial\dot{A}_2\partial\dot{A}_0} & \frac{\partial^2\mathcal{L}}{\partial\dot{A}_2\partial\dot{A}_1} & \frac{\partial^2\mathcal{L}}{\partial\dot{A}_2\partial\dot{A}_2} & \frac{\partial^2\mathcal{L}}{\partial\dot{A}_2\partial\dot{A}_3} \\ \frac{\partial^2\mathcal{L}}{\partial\dot{A}_3\partial\dot{A}_0} & \frac{\partial^2\mathcal{L}}{\partial\dot{A}_3\partial\dot{A}_1} & \frac{\partial^2\mathcal{L}}{\partial\dot{A}_3\partial\dot{A}_2} & \frac{\partial^2\mathcal{L}}{\partial\dot{A}_3\partial\dot{A}_3} \end{bmatrix}. \tag{3.5}$$

In [42], the Hessian matrix is defined the first row components and the first column components equal to zero, such that the Hessian can be written in the matrix form as

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\partial^2\mathcal{L}}{\partial\dot{A}_1\partial\dot{A}_1} & \frac{\partial^2\mathcal{L}}{\partial\dot{A}_1\partial\dot{A}_2} & \frac{\partial^2\mathcal{L}}{\partial\dot{A}_1\partial\dot{A}_3} \\ 0 & \frac{\partial^2\mathcal{L}}{\partial\dot{A}_2\partial\dot{A}_1} & \frac{\partial^2\mathcal{L}}{\partial\dot{A}_2\partial\dot{A}_2} & \frac{\partial^2\mathcal{L}}{\partial\dot{A}_2\partial\dot{A}_3} \\ 0 & \frac{\partial^2\mathcal{L}}{\partial\dot{A}_3\partial\dot{A}_1} & \frac{\partial^2\mathcal{L}}{\partial\dot{A}_3\partial\dot{A}_2} & \frac{\partial^2\mathcal{L}}{\partial\dot{A}_3\partial\dot{A}_3} \end{bmatrix}. \tag{3.6}$$

One can rewrite this Hessian condition as

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{A}_0 \partial \dot{A}_\mu} = 0, \ \det\left(\frac{\partial^2 \mathcal{L}}{\partial \dot{A}_i \partial \dot{A}_j}\right) \neq 0. \tag{3.7}$$

This is the special case of Hessian condition. This make Hessian determinant vanish,

$$\det\left(\frac{\partial^2 \mathcal{L}}{\partial \dot{A}_\mu \partial \dot{A}_\nu}\right) = 0. \tag{3.8}$$

Namely, the requirement eq.(3.8) can be fulfilled by eq.(3.7) showed in Heisenberg [42]. Many references also impose this condition. Once the required action is constructed, one simply assumes that it has three propagating degrees of freedom.

In principle, it is not enough to simply assume that vector theories satisfying Hessian condition has three propagating degrees of freedom. One tries to find whather there are other conditions. The degree of freedom can be counted by Dirac method and the Faddeev–Jackiw method. However, the Lagrangian (2.4) is too complicated. When one tries to follow the process of these two methods, the calculation would be very long. It is not a good idea to attack this action direcly. Instead of that, one tries to consider a class of actions and count the number of degrees of freedom on this class. One considers the class of vector theories which are diffeomorphic invariance A, satisfying the Hessian condition (3.7) and the vector sector being free from Ostrogradski instability.

In [13], one considers the case $\mu = 0$ of the first condition of (3.7) as

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{A}_0 \partial \dot{A}_0} = 0. \tag{3.9}$$

One sees that $\partial \mathcal{L}/\partial \dot{A}_0$ shoud be independent from $\dot{A}_0$. However, $\mathcal{L}$ also should be linear in $\dot{A}_0$. Then let us find the exact form of the Lagrangian of this class. Let us first start to consider a general form of Lagrangian

$$\mathcal{L}(A_\mu, \partial_\mu A_\nu, g_{\mu\nu}, \partial_\kappa g_{\mu\nu}, \partial_{\kappa\lambda} g_{\mu\nu}, \cdots, K), \tag{3.10}$$

where $\partial_{\mu\nu\cdots\lambda} \equiv \partial_\mu \partial_\nu \ldots \partial_\lambda$ and K are external fields and their derivatives. Then one substitutes this Lagrangian into the Hessian condition eq.(3.9). One calculates this equation to find the Lagrangian form. One obtains

$$\begin{aligned}\mathcal{L} &= U(A_\mu, \partial_i A_\nu, \dot{A}_j, g_{\mu\nu}, \partial_\kappa g_{\mu\nu}, \partial_{\kappa\lambda} g_{\mu\nu}, \cdots, K)\dot{A}_0 \\ &+ T(A_\mu, \partial_i A_\nu, \dot{A}_j, g_{\mu\nu}, \partial_\kappa g_{\mu\nu}, \partial_{\kappa\lambda} g_{\mu\nu}, \cdots, K).\end{aligned} \tag{3.11}$$

Then one considers the case $\mu = i$ of the first condition of (3.7) as

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{A}_0 \partial \dot{A}_i} = 0. \tag{3.12}$$

One obtains

$$\frac{\partial U}{\partial \dot{A}_i} = 0. \tag{3.13}$$

It shows that U should be independent from $\dot{A}_i$. Therefore, any Lagrangian in this class should take the form

$$\begin{aligned}\mathcal{L} &= T(A_\nu, \partial_k A_\nu, \dot{A}_k, g_{\rho\sigma}, \partial_\kappa g_{\rho\sigma}, \partial_{\kappa\lambda} g_{\rho\sigma}, ..., K) \\ &+ U(A_\nu, \partial_i A_\nu, g_{\rho\sigma}, \partial_\kappa g_{\rho\sigma}, \partial_{\kappa\lambda} g_{\rho\sigma}, ..., K)\dot{A}_0.\end{aligned} \tag{3.14}$$

One demands that the vector field couples to metric and also other external fields as well as their possible derivative of any vector. Namely, Each theory in the class (3.14) describes the dynamics of a vector field A_μ coupled to external metric $g_{\mu\nu}$ and other external fields K and their derivatives, but does not describe the dynamics of the external metric $g_{\mu\nu}$ and other external fields K. Finally, after substituting eq.(3.14) into the second requirement of eq.(3.7), one obtains

$$\det\left(\frac{\partial^2 T}{\partial \dot{A}_i \partial \dot{A}_j}\right) \neq 0. \tag{3.15}$$

It should be evident from the derivation that the condition (3.7) is imposed on the whole Lagrangian instead of separating the Lagrangian into sub-Lagrangians at different orders and impose the Hessian condition separately on each order. In other words, impose the Hessian condition at "mixed order".

Any Lagrangian in the class (3.14) is free of Ostrogradski instability in the vector sector. This is because the equations of motion for A_i are of second order derivatives in time, and there is no dynamics of A_0. To see this clearer, consider the equations of motion for the vector field. One considers the variation of action

$$\begin{aligned}
\delta S &= \int d^{d-1}x\, \delta\mathcal{L}(A_\mu, \partial_\mu A_\nu, g_{\mu\nu}, \partial_\kappa g_{\mu\nu}, \partial_{\kappa\lambda} g_{\mu\nu}, \cdots, K) \\
&= \int d^{d-1}x \left(\frac{\partial\mathcal{L}}{\partial A_\mu}\delta A_\mu + \frac{\partial\mathcal{L}}{\partial\partial_\mu A_\nu}\delta\partial_\mu A_\nu + \cdots \right) \\
&= \int d^{d-1}x \left[\left(\frac{\partial\mathcal{L}}{\partial A_\mu} - \partial_\nu\Big(\frac{\partial\mathcal{L}}{\partial\partial_\nu A_\mu}\Big) \right) \delta A_\mu + \cdots \right] \\
&= 0.
\end{aligned} \tag{3.16}$$

Therefore, one obtains the equation of motion which can be written as

$$\partial_\nu\Big(\frac{\partial\mathcal{L}}{\partial\partial_\nu A_\mu}\Big) - \frac{\partial\mathcal{L}}{\partial A_\mu} + \cdots = 0, \tag{3.17}$$

where $\cdots$ are terms which do not contain $\ddot{A}_\mu$. Then one separately considers space and time indices. One first considers space indices

$$\begin{aligned}
\partial_0\Big(\frac{\partial\mathcal{L}}{\partial\dot{A}_i}\Big) + \partial_j\Big(\frac{\partial\mathcal{L}}{\partial\partial_j A_i}\Big) - \frac{\partial\mathcal{L}}{\partial A_i} + \cdots &= 0 \\
\frac{\partial^2\mathcal{L}}{\partial\dot{A}_i\partial\dot{A}_j}\ddot{A}_j + \partial_j\frac{\partial\mathcal{L}}{\partial\partial_j A_i} - \frac{\partial\mathcal{L}}{\partial A_i} + \cdots &= 0.
\end{aligned} \tag{3.18}$$

Then one considers the time index

$$\begin{aligned}
\partial_0\Big(\frac{\partial\mathcal{L}}{\partial\dot{A}_0}\Big) + \partial_i\Big(\frac{\partial\mathcal{L}}{\partial\partial_i A_0}\Big) - \frac{\partial\mathcal{L}}{\partial A_0} &= 0 \\
\partial_0 U + \partial_i\Big(\frac{\partial\mathcal{L}}{\partial\partial_i A_0}\Big) - \frac{\partial\mathcal{L}}{\partial A_0} &= 0 \\
\dot{U} + \partial_i\frac{\partial\mathcal{L}}{\partial\partial_i A_0} - \frac{\partial\mathcal{L}}{\partial A_0} &= 0,
\end{aligned} \tag{3.19}$$

where $\frac{\partial\mathcal{L}}{\partial\dot{A}_0} = U$. Since the Euler-Lagrange equations do not contain time derivative with order higher than two, the theories are free of Ostrogradski instability [36] in the vector sector. Furthermore, it is clear that the systems are free of Ostrogradski instability and are constrained as Euler-Lagrange equations are of second order derivative in time of A_j while there is only up to first order derivative in time for A_0.

3.2 Dirac constraint analysis

Dirac method is the method which is often used in order to analyse constrained systems. One starts from finding the constraints which are functions of phase space variables. As an example, a Lagrangian density of N scalar fields is considered. It depends on the N scalar fields and their first order derivatives. It is

$$\mathcal{L}(\phi_a(x), \partial_\mu \phi_a(x)), \tag{3.20}$$

where $a = 1, 2, ..., N$. If the determinant of the Hessian is zero:

$$\det\left(\frac{\partial^2 \mathcal{L}}{\partial \dot{\phi}_a \partial \dot{\phi}_b}\right) = 0, \tag{3.21}$$

the system is a constrained system. Then, one defines conjugate momenta by taking the derivative of the Lagrangian density with respect to the generalised velocities $(\dot{\phi}_a)$ as

$$\pi^a = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}. \tag{3.22}$$

If the system is not constrained, eq.(3.22) can be inverted to uniquely express $\dot{\phi}_a$ in terms of π^a. If however the system is constrained, one may extract from eq.(3.22) the constrained equations, which are the equations containing ϕ_a and π^a but without $\dot{\phi}_a$. Suppose there are k constrained equations of the form $\Phi^{\hat{m}} = 0$, for $\hat{m} = 1, 2, \cdots, k$. The quantities $\Phi^{\hat{m}} = 0$ are called constraints. In particular, since they are the initial set of constraints being generated, they are called "primary constraints". One then computes the Hamiltonian density by using Legendre transformation,

$$\mathcal{H}(\phi_a, \pi^a) = \pi^a \dot{\phi}_a - \mathcal{L} - \dot{\gamma}_{\hat{m}} \Phi^{\hat{m}}, \tag{3.23}$$

where $\gamma_{\hat{m}}$ are Lagrange multipliers. The Poisson bracket of a phase space variable and a primary constraint with the Hamiltonian is the time derivative of that variable,

$$\dot{\Phi}^{\hat{m}} = \{\Phi^{\hat{m}}, H\}, \tag{3.24}$$

where the Hamiltonian is obtained by integrating the Hamiltonian density over all of space,

$$H = \int d^{d-1}x\mathcal{H}. \tag{3.25}$$

If this Poisson bracket is independent of $\gamma^{\hat{m}}$ and is not equal to zero after $\Phi^{\hat{m}}$ are imposed, it gives further constraints. These constraints are called "secondary constraints". One requires the secondary constraints to be invariant under time evolution. If the time evolution of secondary constraints is not vanish, one say that the system generates newer constraints called "tertiary constraints". The process should be repeated until there are no further constraints generated. After all the constraints are found, they are reclassified into the first-class and second-class constraints. By definition, a first-class constraint weakly commutes with all other constraints while a second-class constraint does not. The usage that are relevant for us is to use them to count the number of degrees of freedom which is given by

$$\text{number of d.o.f.} = \frac{n_{PS} - 2n_1 - n_2}{2}, \tag{3.26}$$

where n_{PS} is the number of phase space variables, n_1 is the number of first-class constraints, and n_2 is the number of second-class constraints. The counting of degrees of freedom has been useful for example to check whether a proposed theory is free of ghost degrees of freedom.

Each class of constraints also have their important roles. Let us briefly state them. First-class constraints generate gauge transformation. As for second-class constraints, they are used in order to form Dirac bracket, which is the constrained system counterpart of the unconstrained system's Poisson bracket. As part of the canonical quantisation of constrained system, the Dirac bracket is promoted to commutator.

3.2.1 Dirac constraint analysis for Maxwell theory

For the simple example of constraint analysis, we apply it to Maxwell theory. Lagrangian density for Maxwell theory is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{3.27}$$

One calculates the conjugate momenta corresponding to each component of the vector field (A_μ) as

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0}. \tag{3.28}$$

The conjugate momentum for the time component (A_0),

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0. \tag{3.29}$$

This implies that there is no dynamic degree of freedom associated with A_0. It is a constraint, namely eq.(3.29) is a constrained equation. Another case, the conjugate momenta for the spatial component (A_i) are given by

$$\pi^i = F^{i0} = \partial^i A^0 + \dot{A}^i. \tag{3.30}$$

They are not constrained equations. Therefore, one obtains the primary constraint

$$\Omega_1 \equiv \pi^0 = 0. \tag{3.31}$$

Then, one considers Hamiltonian by using Lagendre transformation eq.(3.23)

$$\begin{aligned}
\mathcal{H}(A_\mu, \pi^\mu) &= \pi^\mu \dot{A}_\mu + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \dot{\gamma}_1\pi^0 \\
&= \pi^0\dot{A}_0 + \pi^i\dot{A}_i + \frac{1}{4}(F_{00}F^{00} + F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij}) - \dot{\gamma}_1\pi^0 \\
&= \frac{1}{2}(\pi^i)^2 + \frac{1}{2}F_{ij}F^{ij} + A_0(\partial_i\pi^i) - \dot{\gamma}_1\pi^0,
\end{aligned} \tag{3.32}$$

where γ_1 is a Lagrange multiplier. The Poisson bracket of primary constraint and Hamiltonian is given by

$$\dot{\pi}^0 = \{\pi^0, H\} = \partial_i\pi^i. \tag{3.33}$$

This gives the secondary constraint,

$$\Omega_2 = \partial_i \pi^i = 0. \tag{3.34}$$

To classify these constraints as first-class or second-class constraints, one computes the Poisson brackets between them,

$$\{\Omega_1, \Omega_2\} = 0. \tag{3.35}$$

Since the Poisson bracket vanishes, both Ω_1 and Ω_2 are first-class constraints.

3.2.2 Dirac constraint analysis for Proca theory

Lagrangian density for Proca theory is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu. \tag{3.36}$$

The conjugate momenta corresponding to each component of the vector field is given by

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0}. \tag{3.37}$$

For time component of vector field same as in Maxwell theory, the conjugate momentum vanishes. It implies a primary constraint,

$$\Omega_1 \equiv \pi^0 = 0. \tag{3.38}$$

System (3.36) is a constrained system. Fot spatial components, the conjugate momenta are given by

$$\pi^i = F^{i0} = \partial^i A^0 + \dot{A}^i. \tag{3.39}$$

Eq.(3.39) is not constraint equation. This is the electric field component $E^i = F^{0i}$, as in the Maxwell theory. The Hamiltonian is calculated by using Lagendre transformation

$$\begin{aligned} \mathcal{H}(A_\mu, \pi^\mu) &= \pi^\mu \dot{A}_\mu + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu - \dot{\gamma}_1 \pi^0 \\ &= A_0(\partial_i \pi^i) + \frac{1}{2}(\pi^i)^2 + \frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}m^2 A_\mu A^\mu - \dot{\gamma}_1 \pi^0, \end{aligned} \tag{3.40}$$

where γ_1 is a Lagrange multiplier. The time evolution of a constraint is determined by its Poisson bracket with the Hamiltonian

$$\dot{\pi}^0 = \{\pi^0, H\} = \partial_i \pi^i + m^2 A_0. \tag{3.41}$$

This is a secondary constraint,

$$\Omega_2 \equiv \partial_i \pi^i + m^2 A_0 = 0. \tag{3.42}$$

To classify these constraints, one compute their Poisson brackets.

$$\{\Omega_1(t, \mathbf{x}), \Omega_2(t, \mathbf{y})\} = \{\pi^0(t, \mathbf{x}), m^2 A_0(t, \mathbf{y}) - \partial_i \pi^i(t, \mathbf{y})\} = m^2 \delta^3(\mathbf{x} - \mathbf{y}). \tag{3.43}$$

This does not vanish due to the presence of the mass term. As a result, the constraints do not commute, indicating that both of constraints are second-class constraints.

3.3 Faddeev-Jackiw constraint analysis

In this section, we consider constraint analysis named Faddeev-Jackiw method [4], [5], [6], [7]. We will use the notations and conventions similar to those used in [12], [13]. Let us first consider the Lagrangian density in the form of eq.(3.20) [13]. Then follow the same discussions as in Dirac method until obtaining the Hamiltonian density in eq.(3.23). Then, one defines

$$\mathcal{L}_{FOF} = \pi^a \dot{\phi}_a - \mathcal{H}, \tag{3.44}$$

which is called the first-order form of the Lagrangian. Then one collects the phase space variables namely the generalised coordinates, the conjugate momenta and the Lagrange multipliers into the symplectic variables $\xi^I = (\phi_a, \pi^a, \gamma_{\hat{m}})$ for $I = 1, 2, ..., 2N + k$. Next, one defines the partial derivative with respect to the generalised velocities of the symplectic variables as the form

$$\mathcal{A}_{\xi^I} = \frac{\partial \mathcal{L}_{FOF}}{\partial \dot{\xi}^I}. \tag{3.45}$$

The first-order form of the Lagrangian can be rewritten in the form

$$\mathcal{L}_{FOF} = \mathcal{A}_{\xi^I}\dot{\xi}^I + \mathcal{L}_v, \tag{3.46}$$

where $\mathcal{L}_v$ are the first order form of Lagrangian density without time derivative and constraint terms. Let us define the 1-form corresponding to $\mathcal{A}_{\xi^I}(t, \mathbf{x})$, called the canonical 1-form,

$$\mathcal{A}(t) \equiv \int d^{d-1}\mathbf{x}\mathcal{A}_{\xi^I}(t, \mathbf{x})\delta\xi^I(t, \mathbf{x}). \tag{3.47}$$

Then, one apply to the exterior derivative

$$\delta \equiv \int d^{d-1}\mathbf{x}\delta\xi^I(t, \mathbf{x})\frac{\delta}{\delta\xi^I(t, \mathbf{x})} \tag{3.48}$$

on $\mathcal{A}(t)$ gives

$$\mathcal{F}(t) \equiv \delta\mathcal{A}(t). \tag{3.49}$$

If we take the exterior derivative to the canonical 1-form, we will get the 2-form. So the quantity $\mathcal{F}(t)$ is called the symplectic 2-form. One considers the interior product on the symplectic 2-form in the equation

$$i_{z(t)}\mathcal{F}(t) = 0, \tag{3.50}$$

where $z(t)$ is given by

$$z(t) = \int d^{d-1}\mathbf{x}z^{\xi^I}(t, \mathbf{x})\frac{\delta}{\delta\xi^I(t, \mathbf{x})}, \tag{3.51}$$

where

$$z^{\zeta^I} = i_z\delta\zeta^I. \tag{3.52}$$

One would like to find $z(t)$ which is the non-trivial solution of eq.(3.50). Eq.(3.50) is not a standard matrix multiplication. It is the matrix multiplication also involves integration over $\mathbf{x}$. If a non-trivial solution, $z(t) \neq 0$, exists for equation (3.50), it signifies the presence of a zero mode. In the context of constraint analysis, a zero mode corresponds to an eigenvector of a constraint operator with an eigenvalue of zero. The existence of a zero mode indicates that possess a new constraint. This constraint arises from the geometric structure of the phase space as captured by the symplectic 2-form. Conversely, if no non-trivial solution exists for equation (3.50), it implies that the system has no new constraints. The new constraints are generated from

$$\Omega(t) = i_{z(t)}\delta \int d^{d-1}\mathbf{x}\mathcal{L}_v(t,\mathbf{x}). \tag{3.53}$$

If there are new constrains, one need to repeat the steps to the first-order form of Lagrangian eq.(3.46). Namely, one adds the secondary constraints term to the old L_{FOF}. Then one follows the similar steps to obtain the canonical 1-form, the symplectic 2-form. One follows the steps until there is no more new constrains. In [54] show the method to count the number of degree of freedom as

$$\text{Number of d.o.f.} = \frac{1}{2}\Big(n_{\text{ps}} - n_{\Omega} - n_z\Big), \tag{3.54}$$

where n_{ps} is the number of phase space variables, n_{Ω} is the number of total constraints, and n_z is the number of zero mode at last iteration.

3.3.1 Faddeev-Jackiw constraint analysis for Maxwall theory

Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{3.55}$$

The conjugate momenta are

$$\pi^{\mu} = F^{\mu 0}. \tag{3.56}$$

In the case $\mu = 0$, one obtains $\pi^0 = 0$. It is a constrained equation. Another case $\mu = i$, velocity term is obtained which given by

$$\begin{aligned}\pi^i &= F^{i0}\\ &= \partial^i A^0 - \partial^0 A^i\\ &= \partial^i A^0 + \dot{A}^i.\end{aligned} \tag{3.57}$$

Eq.(3.57) is not constrained equation. Therefore, one obtains the primary constraint

$$\Omega_1 \equiv \pi^0 = 0. \tag{3.58}$$

Then, one considers Hamiltonian by using Lagendre transformation eq.(3.23)

$$\begin{aligned}\mathcal{H}(A_\mu, \pi^\mu) &= \pi^\mu \dot{A}_\mu + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \dot{\gamma}_1\pi^0, \\ &= \frac{1}{2}(\pi^i)^2 + \frac{1}{2}F_{ij}F^{ij} + A_0(\partial_i\pi^i) - \dot{\gamma}_1\pi^0,\end{aligned} \tag{3.59}$$

where γ_1 is a Lagrange multiplier. Then let us start considering first iteration. The first-order form of the Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{FOF} &= \pi^\rho \dot{A}_\rho - \mathcal{H} \\ &= \pi^\rho \dot{A}_\rho - \frac{1}{2}(\pi^i)^2 - \frac{1}{2}F_{ij}F^{ij} - A_0(\partial_i\pi^i) + \dot{\gamma}_1\pi^0.\end{aligned} \tag{3.60}$$

Let us define the symplectic variables as $\xi^I = (A_\rho, \pi^\rho, \gamma_1)$ and the partial derivative of first order form of the Lagrangian with respect to the generalised velocities of the symplectic variables in eq.(3.45). This gives

$$\mathcal{A}_{A_\rho} = \frac{\partial \mathcal{L}_{FOF}}{\partial \dot{A}_\rho} = \pi^\rho, \tag{3.61}$$

$$\mathcal{A}_{\pi^\rho} = \frac{\partial \mathcal{L}_{FOF}}{\partial \dot{\pi}_\rho} = 0, \tag{3.62}$$

$$\mathcal{A}_{\gamma_1} = \frac{\partial \mathcal{L}_{FOF}}{\partial \dot{\gamma}_1} = \pi^0. \tag{3.63}$$

Therefore, the first-order form of the Lagrangian eq.(3.46) can be rewritten in the form

$$\begin{aligned}\mathcal{L}_{FOF} &= \mathcal{A}_{\xi^I}\dot{\xi}^I + \mathcal{L}_v \\ &= \mathcal{A}_{A_\rho}\dot{A}_\rho + \mathcal{A}_{\pi^\rho}\dot{\pi}^\rho + \mathcal{A}_{\gamma_1}\dot{\gamma}_1 + \mathcal{L}_v \\ &= \pi^\rho \dot{A}_\rho + \pi^0\dot{\gamma}_1 + \mathcal{L}_v,\end{aligned} \tag{3.64}$$

where

$$\mathcal{L}_v = -\pi^i(\partial_i A_0) - \frac{1}{2}\pi^i\pi^i - \frac{1}{4}F_{ij}F^{ij}. \tag{3.65}$$

The canonical 1-form is given by

$$\mathcal{A}(t) \equiv \int d^{d-1}\mathbf{x}\Big(\pi^{\mu}(t,\mathbf{x})\delta A_{\mu}(t,\mathbf{x}) + \pi^{0}(t,\mathbf{x})\delta\gamma_1(t,\mathbf{x})\Big). \tag{3.66}$$

Then let us define the symplectic 2-form by taking the exterior derivative

$$\delta \equiv \int d^{d-1}\mathbf{x}\Big(\delta A_{\mu}(t,\mathbf{x})\frac{\delta}{\delta A_{\mu}(t,\mathbf{x})} + \delta\pi^{\mu}(t,\mathbf{x})\frac{\delta}{\delta\pi^{\mu}(t,\mathbf{x})} + \delta\gamma_1(t,\mathbf{x})\frac{\delta}{\delta\gamma_1(t,\mathbf{x})}\Big), \tag{3.67}$$

to the canonical 1-form. This gives

$$\begin{aligned}
\mathcal{F}(t) &\equiv \delta\mathcal{A}(t)\\
&= \int d^{d-1}\mathbf{x}\int d^{d-1}\mathbf{y}\Big(\frac{\delta\pi^{\mu}(t,\mathbf{x})}{\delta\pi^{\nu}(t,\mathbf{y})}\delta\pi^{\nu}(t,\mathbf{y})\wedge\delta A_{\mu}(t,\mathbf{x}) + \frac{\delta\pi^{0}(t,\mathbf{x})}{\delta\pi^{0}(t,\mathbf{y})}\delta\pi^{0}(t,\mathbf{y})\wedge\delta\gamma_1(t,\mathbf{x})\Big)\\
&= \int d^{d-1}\mathbf{x}\int d^{d-1}\mathbf{y}\Big(\delta(\mathbf{x}-\mathbf{y})\delta\pi^{\mu}(t,\mathbf{y})\wedge\delta A_{\mu}(t,\mathbf{x}) + \delta(\mathbf{x}-\mathbf{y})\delta\pi^{0}(t,\mathbf{y})\wedge\delta\gamma_1(t,\mathbf{x})\\
&= \int d^{d-1}\mathbf{x}\Big(\delta\pi^{\mu}(t,\mathbf{x})\wedge\delta A_{\mu}(t,\mathbf{x}) + \delta\pi^{0}(t,\mathbf{x})\wedge\delta\gamma_1(t,\mathbf{x})\Big).
\end{aligned} \tag{3.68}$$

One considers the interior product on the symplectic 2-form, $i_{z(t)}\mathcal{F}(t)$ which is vanish, eq. (3.50), where

$$z(t) = \int d^{d-1}\mathbf{x}\Big(z^{A_{\mu}}(t,\mathbf{x})\frac{\delta}{\delta A_{\mu}(t,\mathbf{x})} + z^{\pi^{\mu}}(t,\mathbf{x})\frac{\delta}{\delta\pi^{\mu}(t,\mathbf{x})} + z^{\gamma_1}(t,\mathbf{x})\frac{\delta}{\delta\gamma_1(t,\mathbf{x})}\Big). \tag{3.69}$$

This gives

$$\begin{aligned}
i_{z(t)}\mathcal{F}(t) &= \int d^{d-1}\mathbf{x}\Big(i_{z(t)}\big(\delta\pi^{\mu}(t,\mathbf{x})\wedge\delta A_{\mu}(t,\mathbf{x})\big) + i_{z(t)}\big(\delta\pi^{0}(t,\mathbf{x})\wedge\delta\gamma_1(t,\mathbf{x})\big)\\
&= \int d^{d-1}\mathbf{x}\Big(\big(i_{z(t)}\delta\pi^{\mu}(t,\mathbf{x})\big)\wedge\delta A_{\mu}(t,\mathbf{x}) - \delta\pi^{\mu}(t,\mathbf{x})\wedge\big(i_{z(t)}\delta A_{\mu}(t,\mathbf{x})\big)\\
&\qquad + \big(i_{z(t)}\delta\pi^{0}(t,\mathbf{x})\big)\wedge\delta\gamma_1(t,\mathbf{x}) - \delta\pi^{0}(t,\mathbf{x})\wedge\big(i_{z(t)}\delta\gamma_1(t,\mathbf{x})\big)\Big)\\
&= \int d^{d-1}\mathbf{x}\Big(z^{\pi^{\mu}}\delta A_{\mu} - z^{A_{\mu}}\delta\pi^{\mu} + z^{\pi^{0}}\delta\gamma_1 - z^{\gamma_1}\delta\pi^{0}\Big),
\end{aligned} \tag{3.70}$$

where the interior product in each term are given by

$$z^{A_{\mu}}(\mathbf{x}) = i_{z(t)}\delta A_{\mu}(\mathbf{x}),\ z^{\pi_{\mu}}(\mathbf{x}) = i_{z(t)}\delta\pi_{\mu}(\mathbf{x}),\ z^{\gamma_1}(\mathbf{x}) = i_{z(t)}\delta\gamma_1(\mathbf{x}). \tag{3.71}$$

Then one obtains $z^{\pi^\mu} = 0, z^{A_i} = 0$ and $z^{A_0} = z^{\gamma_1}$. Therefore, the solution to eq.(3.50) is

$$z(t) = \int d^{d-1}\mathbf{x}\, z^{A_0}\Big(\frac{\delta}{\delta A_0} - \frac{\delta}{\delta \gamma_1}\Big). \tag{3.72}$$

Since there exists zero mode, there might be new constraints. They are generated from

$$\begin{aligned}\Omega_2(t) &= i_{z(t)}\delta\int d^{d-1}\mathbf{x}\mathcal{L}_v(t,\mathbf{x})\\ &= i_{z(t)}\delta\int d^{d-1}\mathbf{x}\Big(-\pi^i(\partial A_0) - \frac{1}{2}\pi^i\pi^i - \frac{1}{4}F_{ij}F^{ij}\Big)\\ &= \int d^{d-1}\mathbf{x}\Big(-(\partial_i A_0)(i_{z(t)}\delta\pi^i) - \partial_i\pi^i(i_{z(t)}\delta A_0) - \pi^i(i_{z(t)}\delta\pi^i) + \partial_i F_{ij}(i_{z(t)}\delta A_j)\Big)\\ &= \int d^{d-1}\mathbf{x}\Big(-\partial_i\pi^i z^{A_0}\Big)\\ &= 0.\end{aligned} \tag{3.73}$$

Therefore, a new constraint is

$$\Omega_2(t) \equiv \partial_i\pi^i = 0. \tag{3.74}$$

Since there are new constrains, one needs to repeat the steps from the first-order form of Lagrangian eq.(3.60). Then let us start considering second iteration. One adds the secondary constraints term to the first-order form of Lagrangian eq.(3.60). It is given by

$$\mathcal{L}_{FOF} = \pi^\mu\dot{A}_\mu - \pi^i(\partial_i A_0) - \frac{1}{2}\pi^i\pi^i - \frac{1}{4}F_{ij}F^{ij} + \dot{\gamma}_1\pi^0 + \dot{\gamma}_2\partial_i\pi^i. \tag{3.75}$$

One obtains

$$\mathcal{A}_{A_\mu} = \pi^\mu, \mathcal{A}_{\pi^\mu} = 0, \mathcal{A}_{\gamma_1} = \pi^0, \mathcal{A}_{\gamma_2} = \partial_i\pi^i, \tag{3.76}$$

where $\xi^I = (A_\mu, \pi^\mu, \gamma_1, \gamma_2)$. Therefore, the first-order form of the Lagrangian can be rewritten in the form

$$\begin{aligned}\mathcal{L}_{FOF} &= \mathcal{A}_{\xi^I}\dot{\xi}^I + \mathcal{L}_v\\ &= \pi^\mu\dot{A}_\mu + \pi^0\dot{\gamma}_1 + \partial_i\pi^i\dot{\gamma}_2 + \mathcal{L}_v.\end{aligned} \tag{3.77}$$

Then one follows the following steps to obtain the canonical 1-form

$$\mathcal{A}(t) \equiv \int d^{d-1}\mathbf{x}\Big(\pi^\mu(t,\mathbf{x})\delta A_\mu(t,\mathbf{x}) + \pi^0(t,\mathbf{x})\delta\gamma_1(t,\mathbf{x}) + \partial_i\pi^i(t,\mathbf{x})\delta\gamma_2(t,\mathbf{x})\Big). \quad (3.78)$$

Let us define the symplectic 2-form by taking the exterior derivative

$$\begin{aligned}\delta \equiv \int d^{d-1}\mathbf{x}\Big(&\delta A_\mu(t,\mathbf{x})\frac{\delta}{\delta A_\mu(t,\mathbf{x})} + \delta\pi^\mu(t,\mathbf{x})\frac{\delta}{\delta\pi^\mu(t,\mathbf{x})} \\ &+ \delta\gamma_1(t,\mathbf{x})\frac{\delta}{\delta\gamma_1(t,\mathbf{x})} + \delta\gamma_2(t,\mathbf{x})\frac{\delta}{\delta\gamma_2(t,\mathbf{x})}\Big)\end{aligned} \quad (3.79)$$

to the canonical 1-form. This gives

$$\begin{aligned}\mathcal{F}(t) &\equiv \delta\mathcal{A}(t) \\ &= \int d^{d-1}\mathbf{x}\int d^{d-1}\mathbf{y}\Big(\frac{\delta\pi^\mu(t,\mathbf{x})}{\delta\pi^\nu(t,\mathbf{y})}\delta\pi^\nu(t,\mathbf{y})\wedge\delta A_\mu(t,\mathbf{x}) + \frac{\delta\pi^0(t,\mathbf{x})}{\delta\pi^0(t,\mathbf{y})}\delta\pi^0(t,\mathbf{y})\wedge\delta\gamma_1(t,\mathbf{x}) \\ &\quad + \frac{\delta(\partial_i\pi^i)(t,\mathbf{x})}{\delta\pi^j(t,\mathbf{y})}\delta\pi^j(t,\mathbf{y})\wedge\delta\gamma_2(t,\mathbf{x})\Big) \\ &= \int d^{d-1}\mathbf{x}\int d^{d-1}\mathbf{y}\Big(\delta(\mathbf{x}-\mathbf{y})\delta\pi^\mu(t,\mathbf{y})\wedge\delta A_\mu(t,\mathbf{x}) + \delta(\mathbf{x}-\mathbf{y})\delta\pi^0(t,\mathbf{y})\wedge\delta\gamma_1(t,\mathbf{x}) \\ &\quad + \delta(\mathbf{x}-\mathbf{y})\partial_i\delta\pi^i(t,\mathbf{y})\wedge\delta\gamma_2(t,\mathbf{x})\Big) \\ &= \int d^{d-1}\mathbf{x}\Big(\delta\pi^\mu(t,\mathbf{x})\wedge\delta A_\mu(t,\mathbf{x}) + \delta\pi^0(t,\mathbf{x})\wedge\delta\gamma_1(t,\mathbf{x}) + \partial_i\delta\pi^i(t,\mathbf{x})\wedge\delta\gamma_2(t,\mathbf{x})\Big).\end{aligned} \quad (3.80)$$

The interior product on the symplectic 2-form is vanish eq.(3.50). This gives

$$\begin{aligned}i_{z(t)}\mathcal{F}(t) = \int d^{d-1}\mathbf{x}\Big(&\big(i_{z(t)}\delta\pi^\mu(t,\mathbf{x})\big)\wedge\delta A_\mu(t,\mathbf{x}) - \delta\pi^\mu(t,\mathbf{x})\wedge\big(i_{z(t)}\delta A_\mu(t,\mathbf{x})\big) \\ &+ \big(i_{z(t)}\delta\pi^0(t,\mathbf{x})\big)\wedge\delta\gamma_1(t,\mathbf{x}) - \delta\pi^0(t,\mathbf{x})\wedge\big(i_{z(t)}\delta\gamma_1(t,\mathbf{x})\big) \\ &+ \partial_i\big(i_{z(t)}\delta\pi^i(t,\mathbf{x})\big)\wedge\delta\gamma_2(t,\mathbf{x}) - \partial_i\delta\pi^i(t,\mathbf{x})\wedge\big(i_{z(t)}\delta\gamma_2(t,\mathbf{x})\big)\Big) \\ = \int d^{d-1}\mathbf{x}\Big(&z^{\pi^\mu}\delta A_\mu - z^{A_\mu}\delta\pi^\mu + z^{\pi^0}\delta\gamma_1 - z^{\gamma_1}\delta\pi^0 + \partial_i(z^{\pi^i})\delta\gamma_2 - z^{\gamma_2}\partial_i\delta\pi^i\Big).\end{aligned} \quad (3.81)$$

Considering the coefficients, one obtains $z^{\pi^\mu} = 0$, $z^{A_i} = -\partial_i z^{\gamma_2}$ and $z^{A_0} = -z^{\gamma_1}$. One can rewrite $z(t)$ as

$$z(t) = \int d^{d-1}\mathbf{x}\Big(z^{\gamma_1}\Big(\frac{\delta}{\delta\gamma_1} - \frac{\delta}{\delta A_0}\Big) + z^{\gamma_2}\frac{\delta}{\delta\gamma_2} - \partial_i z^{\gamma_2}\frac{\delta}{\delta A_i}\Big). \quad (3.82)$$

The identification of a non-trivial solution, zero mode to eq.(3.50) suggests the potential existence of new constraints within the dynamic system. These newly identified constraints would arise from

$$
\begin{aligned}
\Omega_3(t) &= i_{z(t)}\delta \int d^{d-1}\mathbf{x}\mathcal{L}_v(t,\mathbf{x}) \\
&= i_{z(t)}\delta \int d^{d-1}\mathbf{x}\Big(-\pi^i(\partial A_0) - \frac{1}{2}\pi^i\pi^i - \frac{1}{4}F_{ij}F^{ij}\Big) \\
&= \int d^{d-1}\mathbf{x}\Big(-(\partial_i A_0)(i_{z(t)}\delta\pi^i) - \partial_i\pi^i(i_{z(t)}\delta A_0) - \pi^i(i_{z(t)}\delta\pi^i) - \partial_i F_{ij}(i_{z(t)}\delta A_j)\Big) \\
&= \int d^{d-1}\mathbf{x}\Big(\partial_i\pi^i z^{\gamma_1} + \partial_i F_{ij}\partial_j z^{\gamma_2}\Big) \\
&= 0.
\end{aligned}
\tag{3.83}
$$

Due to eq.(3.74), the new constraint $\Omega_3(t)$ is identically zero. There is no further constraint. Therefore, Maxwell theory has 2 constraints, namely $\Omega_1 \equiv \pi^0$ and $\Omega_2 \equiv \partial_i\pi_i$. The iterative application of eq. (3.50) to identify constraints within Maxwell theory concludes here. This signifies the absence of any further zero modes in the symplectic 2-form derived from the previously identified constraints.

3.3.2 Faddeev-Jackiw constraint analysis for Proca theory

Then we apply constraint analysis to Proca theory. Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu. \tag{3.84}$$

The conjugate momenta are

$$\pi^\mu = F^{\mu 0}. \tag{3.85}$$

One obtains the primary constraint

$$\Omega_1 \equiv \pi^0 = 0. \tag{3.86}$$

Velocity term is also obtained

$$\pi^i = \partial^i A^0 + \dot{A}^i. \tag{3.87}$$

The Hamiltonian is obtained by using Lagendre transformation

$$\begin{aligned}\mathcal{H}(A_\mu,\pi^\mu) &= \pi^\mu \dot{A}_\mu - \mathcal{L} - \dot{\gamma}_1\pi^0 \\ &= (\partial_i A_0)\pi^i + \frac{1}{2}\pi^i\pi^i + \frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}m^2A_\mu A^\mu - \dot{\gamma}_1\pi^0,\end{aligned} \tag{3.88}$$

where γ_1 is a Lagrange multiplier. Then let us start considering first iteration. The first-order form of the Lagrangian is given by

$$\mathcal{L}_{FOF} = \pi^\rho \dot{A}_\rho - (\partial_i A_0)\pi^i - \frac{1}{2}\pi^i\pi^i - \frac{1}{4}F_{ij}F^{ij} - \frac{1}{2}m^2A_\mu A^\mu + \dot{\gamma}_1\pi^0. \tag{3.89}$$

Let us define the symplectic variables as $\xi^I = (A_\rho, \pi^\rho, \gamma_1)$ and the patial derivative of first order form of Lagrangian with respect to the generalised velocities of the symplectic variables. This gives

$$\mathcal{A}_{A_\rho} = \pi^\rho, \mathcal{A}_{\pi^\rho} = 0, \mathcal{A}_{\gamma_1} = \pi^0. \tag{3.90}$$

Therefore, the first-order form of Lagrangian can be rewritten in the form

$$\begin{aligned}\mathcal{L}_{FOF} &= \mathcal{A}_{\xi^I}\dot{\xi}^I + \mathcal{L}_v \\ &= \mathcal{A}_{A_\rho}\dot{A}_\rho + \mathcal{A}_{\pi^\rho}\dot{\pi}^\rho + \mathcal{A}_{\gamma_1}\dot{\gamma}_1 + \mathcal{L}_v \\ &= \pi^\rho\dot{A}_\rho + \pi^0\dot{\gamma}_1 + \mathcal{L}_v,\end{aligned} \tag{3.91}$$

where

$$\mathcal{L}_v = -\pi^i(\partial_i A_0) - \frac{1}{2}\pi^i\pi^i - \frac{1}{2}m^2A_\mu A^\mu - \frac{1}{4}F_{ij}F^{ij}. \tag{3.92}$$

Then the process is similar with Maxwell theory. The canonical 1-form can be written as eq.(3.66). Taking the exterior derivative eq.(3.67) to the canonical 1-form is obtained the symplectic 2-form eq.(3.68). Then one consider the interior product on the symplectic 2-form, $i_{z(t)}\mathcal{F}(t)$ which is vanish eq.(3.50). Its solution is given by eq.(3.72).

There exists zero mode, there might be new constraints which generated from

$$
\begin{aligned}
\Omega_2(t) &= i_{z(t)}\delta \int d^{d-1}\mathbf{x}\mathcal{L}_v(t,\mathbf{x}) \\
&= i_{z(t)}\delta \int d^{d-1}\mathbf{x}\Big(-\pi^i(\partial A_0) - \frac{1}{2}\pi^i\pi^i - \frac{1}{2}m^2 A_\mu A^\mu - \frac{1}{4}F_{ij}F^{ij}\Big) \\
&= \int d^{d-1}\mathbf{x}\Big(-(\partial_i A_0)(i_{z(t)}\delta\pi^i) - \partial_i\pi^i(i_{z(t)}\delta A_0) - \pi^i(i_{z(t)}\delta\pi^i) - m^2 A_\mu(i_{z(t)}\delta A^\mu) \\
&\quad - \partial_i F_{ij}(i_{z(t)}\delta A_j)\Big) \\
&= \int d^{d-1}\mathbf{x}\, z^{A_0}\Big(-\partial_i\pi^i - m^2 A_0\Big) \\
&= 0.
\end{aligned}
\tag{3.93}
$$

Therefore, a new constraint is

$$\Omega_2(t) \equiv \partial_i\pi^i + m^2 A_0. \tag{3.94}$$

Since there are new constrains, one need to repeat the steps to the first-order form of Lagrangian eq.(3.89). Then let us start considering second iteration. One adds the secondary constraints term to the first-order form of Lagrangian,

$$\mathcal{L}_{FOF} = \pi^\rho \dot{A}_\rho - \pi^i(\partial_i A_0) - \frac{1}{2}\pi^i\pi^i - \frac{1}{4}F_{ij}F^{ij} - \frac{1}{2}m^2 A_\mu A^\mu + \dot{\gamma}_1\pi^0 + \dot{\gamma}_2(\partial_i\pi^i + m^2 A_0). \tag{3.95}$$

One obtains

$$\mathcal{A}_{A_\mu} = \pi^\mu, \mathcal{A}_{\pi^\mu} = 0, \mathcal{A}_{\gamma_1} = \pi^0, \mathcal{A}_{\gamma_2} = \partial_i\pi^i + m^2 A_0, \tag{3.96}$$

where $\xi^I = (A_\mu, \pi^\mu, \gamma_1, \gamma_2)$. Therefore, the first-order form of the Lagrangian can be rewritten in the form

$$
\begin{aligned}
\mathcal{L}_{FOF} &= \mathcal{A}_{\xi^I}\dot{\xi}^I + \mathcal{L}_v \\
&= \mathcal{A}_{A_0}\dot{A}_0 + \mathcal{A}_{A_i}\dot{A}_i + \mathcal{A}_{\pi^0}\dot{\pi}^0 + \mathcal{A}_{\pi^i}\dot{\pi}^i + \mathcal{A}_{\gamma_1}\dot{\gamma}_1 + \mathcal{A}_{\gamma_2}\dot{\gamma}_2 + \mathcal{L}_v \\
&= \pi^0\dot{A}_0 + \pi^i\dot{A}_i + \pi^0\dot{\gamma}_1 + (\partial_i\pi^i + m^2 A_0)\dot{\gamma}_2 + \mathcal{L}_v,
\end{aligned}
\tag{3.97}
$$

where $\mathcal{L}_v$ is in eq.(3.92). Then we follow the following steps to obtain the canonical 1-form

$$\begin{aligned} \mathcal{A}(t) \equiv & \int d^{d-1}\mathbf{x}\Big(\pi^{\mu}(t,\mathbf{x})\delta A_{\mu}(t,\mathbf{x}) + \pi^{0}(t,\mathbf{x})\delta\gamma_1(t,\mathbf{x}) \\ & +(\partial_i\pi^i(t,\mathbf{x}) + m^2 A_0(t,\mathbf{x}))\delta\gamma_2(t,\mathbf{x})\Big). \end{aligned} \tag{3.98}$$

Let us define the symplectic 2-form by taking the exterior derivative eq.(3.79) to the canonical 1-form. This gives

$$\begin{aligned} \mathcal{F}(t) = \int d^{d-1}\mathbf{x}\Big(& \delta\pi^{\mu}(t,\mathbf{x}) \wedge \delta A_{\mu}(t,\mathbf{x}) + \delta\pi^{0}(t,\mathbf{x}) \wedge \delta\gamma_1(t,\mathbf{x}) \\ & + (\partial_i\delta\pi^i(t,\mathbf{x}) + m^2\delta A_0(t,\mathbf{x})) \wedge \delta\gamma_2(t,\mathbf{x})\Big). \end{aligned} \tag{3.99}$$

One considers the interior product on the symplectic 2-form which is vanish as eq.(3.50). This gives

$$\begin{aligned} i_{z(t)}\mathcal{F}(t) = & \int d^{d-1}\mathbf{x}\Big(\big(i_{z(t)}\delta\pi^{\mu}(t,\mathbf{x})\big) \wedge \delta A_{\mu}(t,\mathbf{x}) - \delta\pi^{\mu}(t,\mathbf{x}) \wedge \big(i_{z(t)}\delta A_{\mu}(t,\mathbf{x})\big) \\ & + \big(i_{z(t)}\delta\pi^{0}(t,\mathbf{x})\big) \wedge \delta\gamma_1(t,\mathbf{x}) - \delta\pi^{0}(t,\mathbf{x}) \wedge \big(i_{z(t)}\delta\gamma_1(t,\mathbf{x})\big) \\ & + \big(\partial_i i_{z(t)}\delta\pi^i(t,\mathbf{x}) + m^2 i_{z(t)}\delta A_0(t,\mathbf{x})\big) \wedge \delta\gamma_2(t,\mathbf{x}) \\ & - (\partial_i\delta\pi^i(t,\mathbf{x}) + m^2\delta A_0(t,\mathbf{x})) \wedge \big(i_{z(t)}\delta\gamma_2(t,\mathbf{x})\big)\Big) \\ = & \int d^{d-1}\mathbf{x}\Big(z^{\pi^{\mu}}\delta A_{\mu} - z^{A_{\mu}}\delta\pi^{\mu} + z^{\pi^{0}}\delta\gamma_1 - z^{\gamma_1}\delta\pi^{0} \\ & + (\partial_i z^{\pi^i} + m^2 z^{A_0})\delta\gamma_2 - z^{\gamma_2}\partial_i\delta\pi^i - m^2 z^{\gamma_2}\delta A_0\Big). \end{aligned} \tag{3.100}$$

One considers the interior product in each term to obtain the above equation. Then one obtains $z^{\pi^0} = 0, z^{\pi^i} = 0, z^{A_0} = 0, z^{A_i} = 0, z^{\gamma_1} = 0, z^{\gamma_2} = 0$. The iterative constraint analysis finishes with the identification of two constraints within the system. These constraints are expressed as the primary constraint $\Omega_1 = \pi^0$ and the secondary constraints $\Omega_2 = \partial_i\pi^i + m^2 A_0$. The absence of new zero modes in the symplectic 2-form derived from these constraints indicates that this analysis has successfully captured all the relevant constraints, and no further iterations are necessary.

3.4 Lagrangian constraint analysis

In Lagrangian constraint analysis [8], [9], [10], [11], one analyses Euler–Lagrange equation to obtain constraints. One separate Euler–Lagrange equation to equation of motion (containing generalised acceleration $\ddot{\phi}$) and primary constraints (not containing generalised acceleration $\ddot{\phi}$). Time evolution of primary constraints may give secondary constraints. Then, one repeats these steps until no new constraints.

3.4.1 Lagrangian constraint analysis for Maxwell theory

Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{3.101}$$

Equation of motion is

$$\partial_\nu F^{\mu\nu} = 0. \tag{3.102}$$

Separate to space indices, it gives equation of motion as

$$\ddot{A}_i = \partial_i \dot{A}_0 + \partial_j \partial_j A_i - \partial_j \partial_i A_j, \tag{3.103}$$

and time index, it gives primary constraint as

$$\partial_i \dot{A}_i = \partial_i \partial_i A_0. \tag{3.104}$$

Time evolution of primary constraints give

$$\partial_i \partial_j F_{ij} = 0. \tag{3.105}$$

It is identically vanish. Therefore, there is no new constraint. The steps stop here.

3.4.2 Lagrangian constraint analysis for Proca theory

Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu. \tag{3.106}$$

Equation of motion is

$$\partial_\nu F^{\mu\nu} + m^2 A^\nu = 0. \tag{3.107}$$

Separate to space indices give equation of motion, which is given by

$$\ddot{A}_i = \partial_i \dot{A}_0 + \partial_j \partial_j A_i - \partial_j \partial_i A_j + m^2 A_i, \tag{3.108}$$

and time index which is primary constraints as

$$\partial_i \dot{A}_i = \partial_i \partial_i A_0 - m^2 A_0. \tag{3.109}$$

Time evolution of primary constraints is given by

$$\begin{aligned} \partial_i \partial_j F_{ij} + m^2(\partial_i A_i + \dot{A}_0) &= 0 \\ \partial_i A_i + \dot{A}_0 &= 0 \\ \partial_\mu A^\mu &= 0. \end{aligned} \tag{3.110}$$

This is the secondary constraints. The time evolution of the secondary constraints give equation of motion for A_0 which is given by

$$\ddot{A}_0 + \partial_i \dot{A}_i = 0. \tag{3.111}$$

There is no further constraint.

CHAPTER IV

CONSTRAINT ANALYSIS IN GENERALISED PROCA THEORIES

4.1 Single-field generalised Proca theories

This section reviews constraint analysis of single-field generalised Proca theories which is shown in [13]. The calculations and insights from this work forms parts of basis of the calcution in our work.

For definiteness, we consider theories in 4-dimensional spacetime. However, the analysis of this chapter can easily be extended to spacetime with other number of dimensions. We define Lagrangian density $\mathcal{L}$ via

$$S = \int d^4x \; \mathcal{L}. \tag{4.1}$$

We denote spacetime coordinates by x^μ with $\mu = 0, 1, 2, 3$. We also use other middle lower-case Greek indices $\mu, \nu, \rho \in \{0, 1, 2, 3\}$ to denote spacetime indices. We will denote spatial indices by using middle lower-case Latin indices $i, j, k, l \in \{1, 2, 3\}$. When expressing field we will omit the dependence on time coordinate t. We will also often drop the dependence on space variables $\mathbf{x}$ (but keep explicit other space variables e.g. $\mathbf{x}', \mathbf{y}, \mathbf{z}$). So for example φ stands for $\varphi(t, \mathbf{x})$, whereas $\varphi(\mathbf{y})$ stands for $\varphi(t, \mathbf{y})$.

Then, let us review conditions from the Faddev-Jackiw constrained analysis. For both the Dirac method and the Faddeev–Jackiw method, firstly, one need to transform to phase space. One is interested in vector sector. So the fields of interests are A_μ. One considers the Faddeev-Jackiw constrained analysis of the Lagrangian in the whole class (3.14) in [13]. All theories in this class have constraints which the Hessian determinant vanishs, i.e. eq.(3.8) and eq.(3.7). The conjugate momenta are

$$
\begin{aligned}
\pi^{\mu} &= \frac{\partial \mathcal{L}}{\partial \dot{A}_{\mu}} \\
&= \frac{\partial T}{\partial \dot{A}_i}\delta_i^{\mu} + U\delta_0^{\mu}.
\end{aligned} \tag{4.2}
$$

The zeroth component of these conjugate momenta do not contain generalised velocities $\dot{A}_{\mu}$. Therefore, it gives the constraint:

$$\pi^0 = U. \tag{4.3}$$

This is the first constraint of the system. It is called primary constraint Ω_1 which is written as

$$\Omega_1 \equiv \pi^0 - U = 0. \tag{4.4}$$

Its dependence on phase space variables and external fields is then of the form

$$\Omega_1(A_{\mu}, \partial_i A_{\mu}, \pi^0, g_{\mu\nu}, \partial_{\sigma} g_{\mu\nu}, ..., K). \tag{4.5}$$

It can be confirmed that this is the constrained system. The spatial components are

$$\pi^i = \frac{\partial T}{\partial \dot{A}_i}. \tag{4.6}$$

In this equation, one can find the term $\dot{A}_i$ to use for the next calculation. The inverse of this equation is of the form

$$\dot{A}_i = \Lambda_i(A_{\nu}, \partial_k A_{\nu}, \pi^k, g_{\rho\sigma}, \partial_{\kappa} g_{\rho\sigma}, \partial_{\kappa\lambda} g_{\rho\sigma}, ..., K). \tag{4.7}$$

One then computes the Hamiltonian density by using Legendre transformation. This gives

$$
\begin{aligned}
\mathcal{H} &= \pi^{\mu}\dot{A}_{\mu} - \mathcal{L} - \dot{\gamma}_1\Omega_1 \\
&= \pi^0\dot{A}_0 + \pi^i\dot{A}_i - T - U\dot{A}_0 - \dot{\gamma}_1\Omega_1 \\
&= \pi^i\Lambda_i - \mathcal{T} - \dot{\gamma}_1\Omega_1,
\end{aligned} \tag{4.8}
$$

where $\mathcal{T}$ is obtained by replacing $\dot{A}_i$ in T by Λ_i and using eq.(4.3) and eq.(4.7). However, $\mathcal{T}$ depends on phase space variables and external fields which can be expressed as

$$\mathcal{T}(A_\nu, \partial_j A_\nu, \pi^j, g_{\rho\sigma}, \partial_\kappa g_{\rho\sigma}, ..., K). \tag{4.9}$$

Therefore, the first-order form of Lagrangian density is given by

$$\begin{aligned}\mathcal{L}_{FOF} &= \pi^\mu \dot{A}_\rho - \mathcal{H} \\ &= \pi^\mu \dot{A}_\mu - \pi^i \Lambda_i + \mathcal{T} + \dot{\gamma}_1 \Omega_1.\end{aligned} \tag{4.10}$$

Then one collects the phase space variables and the constraints into the symplectic variables

$$\xi^I = (A_\mu, \pi^\mu, \gamma_1). \tag{4.11}$$

Then, one obtains $\mathcal{A}_{\xi^I} = \frac{\partial \mathcal{L}_{FOF}}{\partial \dot{\xi}^I}$ as

$$\mathcal{A}_{A_\mu} = \pi^\mu, \quad \mathcal{A}_{\pi^\mu} = 0, \quad \mathcal{A}_{\gamma_1} = \Omega_1. \tag{4.12}$$

Therefore, first-order form of Lagrangian can be rewritten in the form

$$\begin{aligned}\mathcal{L}_{FOF} &= \mathcal{A}_{\xi^I} \dot{\xi}^I + \mathcal{L}_v \\ &= \mathcal{A}_{A_\mu} \dot{A}_\mu + \mathcal{A}_{\pi_\mu} \dot{\pi}_\mu + \mathcal{A}_{\gamma_1} \dot{\gamma}_1 \\ &= \pi^\mu \dot{A}_\mu + \Omega_1 \dot{\gamma}_1 + \mathcal{L}_v.\end{aligned} \tag{4.13}$$

Then one compares eq.(4.10) and eq.(4.13), it gives

$$\mathcal{L}_v = -\pi^i \Lambda_i + \mathcal{T}. \tag{4.14}$$

The canonical 1-form is given by

$$\mathcal{A} = \int d^3\mathbf{x} (\pi^\mu \delta A_\mu + \Omega_1 \delta \gamma_1). \tag{4.15}$$

Applying the exterior derivative

$$\delta = \int d^3\mathbf{x} \left(\delta A_\mu \frac{\delta}{\delta A_\mu} + \delta \pi^\mu \frac{\delta}{\delta \pi^\mu} + \delta \gamma_1 \frac{\delta}{\delta \gamma_1} \right), \tag{4.16}$$

on $\mathcal{A}$ gives the symplectic 2-form $\mathcal{F}$

$$\mathcal{F} = \int d^3\mathbf{x}\Bigg[\left(\frac{\partial\Omega_1}{\partial A_\mu} - \partial_i\left(\frac{\partial\Omega_1}{\partial\partial_i A_\mu}\right)\right)\delta A_\mu \wedge \delta\gamma_1 - \frac{\partial\Omega_1}{\partial\partial_i A_\mu}\delta A_\mu \wedge \partial_i\delta\gamma_1 \\ + \delta\pi^\mu \wedge \delta A_\mu + \delta\pi^0 \wedge \delta\gamma_1\Bigg]. \tag{4.17}$$

Demanding

$$i_z\mathcal{F} = 0, \tag{4.18}$$

where

$$z = \int d^3\mathbf{x}\left(z^{A_\mu}\frac{\delta}{\delta A_\mu} + z^{\pi^\mu}\frac{\delta}{\delta\pi^\mu} + z^{\gamma_1}\frac{\delta}{\delta\gamma_1}\right), \tag{4.19}$$

the interior product $i_z\mathcal{F}$ can be straighforwardly be computed as

$$i_z\mathcal{F} = \int d^3\mathbf{x}\Bigg[\left(\frac{\partial\Omega_1}{\partial A_\mu} - \partial_i\Big(\frac{\partial\Omega_1}{\partial\partial_i A_\mu}\Big)\right)\Big(z^{A_\mu}\delta\gamma_1 - \delta A_\mu z^{\gamma_1}\Big) \\ + \left(\partial_i\left(\frac{\partial\Omega_1}{\partial\partial_i A_\mu}\right)z^{A_\mu} + \frac{\partial\Omega_1}{\partial\partial_i A_\mu}\partial_i z^{A_\mu}\right)\delta\gamma_1 + \frac{\partial\Omega_1}{\partial\partial_i A_\mu}\partial_i z^{\gamma_1}\delta A_\mu \\ + z^{\pi^\mu}\delta A_\mu - \delta\pi^\mu z^{A_\mu} + z^{\pi^0}\delta\gamma_1 - \delta\pi^0 z^{\gamma_1}\Bigg]. \tag{4.20}$$

Then let us consider the coefficients of $\delta A_\mu, \delta\pi^0, \delta\pi^i, \delta\gamma_1$ which vanish. It gives

$$z^{\pi^\mu} = \left(\frac{\partial\Omega_1}{\partial A_\mu} - \partial_i\left(\frac{\partial\Omega_1}{\partial\partial_i A_\mu}\right) - \frac{\partial\Omega_1}{\partial\partial_i A_\mu}\partial_i\right)z^{\gamma_1}, \tag{4.21}$$

$$z^{\gamma_1} = -z^{A_0}, \tag{4.22}$$

$$z^{A_i} = 0, \tag{4.23}$$

$$z^{\pi^0} = -\left(\frac{\partial\Omega_1}{\partial A_\mu} - \partial_i\Big(\frac{\partial\Omega_1}{\partial\partial_i A_\mu}\Big)\right)z^{A_\mu} - \partial_i\left(\frac{\partial\Omega_1}{\partial\partial_i A_\mu}\right)z^{A_\mu} - \frac{\partial\Omega_1}{\partial\partial_i A_\mu}\partial_i z^{A_\mu}. \tag{4.24}$$

From some calculations, it gives

$$2\frac{\partial\Omega_1}{\partial\partial_i A_0}\partial_i z^{A_0} + \partial_i\left(\frac{\partial\Omega_1}{\partial\partial_i A_0}\right)z^{A^0} = 0. \tag{4.25}$$

Then one imposes eq.(A.21), which is one of the diffeomorphism invariance requirement. Therefore, eq.(4.25) now identically vanishes. So there is no restriction placed on z^{A_0}. It follows that the condition $i_z\mathcal{F} = 0$ can be consistently solved. Using eq.(4.21) - eq.(4.25), the zero mode of $\mathcal{F}$ is found to be

$$\begin{aligned} z &= \int d^3\mathbf{x}\Big(z^{A_0}\frac{\delta}{\delta A_0} + z^{A_i}\frac{\delta}{\delta A_i} + z^{\pi^0}\frac{\delta}{\delta \pi^0} + z^{\pi^i}\frac{\delta}{\delta \pi^i} + z^{\gamma_1}\frac{\delta}{\delta \gamma_1}\Big) \\ &= \int d^3\mathbf{x}\bigg(\Big(-\frac{\partial\Omega_1}{\partial A_0}\Big)z^{A_0}\frac{\delta}{\delta\pi^0} - \frac{\partial\Omega_1}{\partial A_i}z^{A_0}\frac{\delta}{\delta\pi^i} + \partial_j\Big(\frac{\partial\Omega_1}{\partial\partial_j A_i}z^{A_0}\Big)\frac{\delta}{\delta\pi^i} \\ &\quad + z^{A_0}\Big(\frac{\delta}{\delta A_0} - \frac{\delta}{\delta\gamma_1}\Big)\bigg). \end{aligned} \tag{4.26}$$

Since the zero mode depends only on one arbitary function z^{A_0}, so there is at most one new constraint. The new constraints are generated form eq.(3.53). It gives

$$\begin{aligned} \Omega_2 &= i_z\delta\int d^3\mathbf{x}\big(-\pi^i\Lambda_i + \mathcal{T}\big) \\ &= \int d^3\mathbf{x}\big(-i_z\delta\pi^i\Lambda_i - \pi^i i_z\delta\Lambda_i + i_z\delta\mathcal{T}\big). \end{aligned} \tag{4.27}$$

Then one considers the terms $i_z\delta\pi^i$, $i_z\delta\Lambda_i$ and $i_z\delta\mathcal{T}$ in eq.(4.27) separately. Therefore, it gives

$$\Omega_2 = -\Lambda_i\frac{\partial U}{\partial A_i} - \partial_j\Lambda_i\frac{\partial U}{\partial\partial_j A_i} + \frac{\partial\mathcal{T}}{\partial A_0} - \partial_i\Big(\frac{\partial\mathcal{T}}{\partial\partial_i A_0}\Big). \tag{4.28}$$

The dependence of Ω_2 on phase space variables and external fields can be expressed as

$$\Omega_2(A_\mu, \partial_i A_\mu, \partial_i\partial_j A_\mu, \pi^i, \partial_j\pi^i, g_{\mu\nu}, \partial_\sigma g_{\mu\nu}, \cdots, K). \tag{4.29}$$

It is easy to see that the constraint (4.29) is indeed a new constraint. This is because it is independent from π^0, which appears in Ω_1. With introduction of an extra constraint, one need to start the second iteration by first-order form of Lagrangian from eq.(4.13) to

$$\mathcal{L}_{FOF} = \pi^\mu\dot{A}_\mu + \mathcal{L}_v + \dot{\gamma}_1\Omega_1 + \dot{\gamma}_2\Omega_2. \tag{4.30}$$

Then the canonical 1-form for the Lagrangian (4.30) is given by

$$\mathcal{A} = \int d^3\mathbf{x}\big(\pi^\mu\delta A_\mu + \Omega_1\delta\gamma_1 + \Omega_2\delta\gamma_2\big). \tag{4.31}$$

Applying the exterior derivative

$$\delta = \int d^3\mathbf{x}\bigg(\delta A_\mu \frac{\delta}{\delta A_\mu} + \delta\pi^\mu \frac{\delta}{\delta\pi^\mu} + \delta\gamma_1 \frac{\delta}{\delta\gamma_1} + \delta\gamma_2 \frac{\delta}{\delta\gamma_2}\bigg) \tag{4.32}$$

to $\mathcal{A}$ gives the symplectic 2-form $\mathcal{F}$

$$\begin{aligned}\mathcal{F} = \int d^3\mathbf{x}\bigg(&\frac{\delta\Omega_1}{\delta A_\mu}\delta A_\mu \wedge \delta\gamma_1 + \frac{\delta\Omega_2}{\delta A_\mu}\delta A_\mu \wedge \delta\gamma_2 + \frac{\delta\pi^\mu}{\delta\pi^\nu}\delta\pi^\nu \wedge \delta A_\mu \\ &+ \frac{\delta\Omega_1}{\delta\pi^0}\delta\pi^0 \wedge \delta\gamma_1 + \frac{\delta\Omega_2}{\delta\pi^i}\delta\pi^i \wedge \delta\gamma_2\bigg).\end{aligned} \tag{4.33}$$

Demanding $i_z\mathcal{F} = 0$, where

$$z = \int d^3\mathbf{x}\bigg(z^{A_\mu}\frac{\delta}{\delta A_\mu} + z^{\pi^\mu}\frac{\delta}{\delta\pi^\mu} + z^{\gamma_1}\frac{\delta}{\delta\gamma_1} + z^{\gamma_2}\frac{\delta}{\delta\gamma_2}\bigg). \tag{4.34}$$

One obtains

$$\begin{aligned} i_z\mathcal{F} = \int d^3\mathbf{x}\bigg[&\bigg(\frac{\partial\Omega_1}{\partial A_\mu} - \partial_i\bigg(\frac{\partial\Omega_1}{\partial\partial_i A_\mu}\bigg)\bigg)\big(z^{A_\mu}\delta\gamma_1 - \delta A_\mu z^{\gamma_1}\big) \\ &- \frac{\partial\Omega_1}{\partial\partial_i A_\mu}\big(z^{A_\mu}\partial_i\delta\gamma_1 - \delta A_\mu\partial_i z^{\gamma_1}\big) \\ &+ \bigg(\frac{\partial\Omega_2}{\partial A_\mu} - \partial_i\bigg(\frac{\partial\Omega_2}{\partial\partial_i A_\mu}\bigg) + \partial_i\partial_j\bigg(\frac{\partial\Omega_2}{\partial\partial_i\partial_j A_\mu}\bigg)\bigg)\big(z^{A_\mu}\delta\gamma_2 - \delta A_\mu z^{\gamma_2}\big) \\ &- \bigg(\frac{\partial\Omega_2}{\partial\partial_i A_\mu} - 2\partial_j\bigg(\frac{\partial\Omega_2}{\partial\partial_i\partial_j A_\mu}\bigg)\bigg)\big(z^{A_\mu}\partial_i\gamma_2 - \delta A_\mu\partial_i z^{\gamma_2}\big) \\ &+ \frac{\partial\Omega_2}{\partial\partial_i\partial_j A_\mu}\big(z^{A_\mu}\partial_i\partial_j\delta\gamma_2 - \delta A_\mu\partial_i\partial_j z^{\gamma_2}\big) + \big(z^{\pi^\mu}\delta A_\mu - \delta\pi^\mu z^{A_\mu}\big) \\ &+ \big(z^{\pi^0}\delta\gamma_1 - \delta\pi^0 z^{\gamma_1}\big) + \bigg(\frac{\partial\Omega_2}{\partial\pi^i} - \partial_j\bigg(\frac{\partial\Omega_2}{\partial\partial_j\pi^i}\bigg)\bigg)\big(z^{\pi^i}\delta\gamma_2 - \delta\pi^i z^{\gamma_2}\big) \\ &- \frac{\partial\Omega_2}{\partial\partial_j\pi^i}\big(z^{\pi^i}\partial_j\delta\gamma_2 - \delta\pi^i\partial_j z^{\gamma_2}\big)\bigg].\end{aligned} \tag{4.35}$$

Having computed $i_z\mathcal{F}$, one now demands that the vector sector has 3 propagating degrees of freedom. So there should be no more than 2 constraints in total. Since there are already 2 constraints, let us demand that none of the zero modes of $i_z\mathcal{F} = 0$ produce extra constraints. One considers $i_z\mathcal{F} = 0$, by eliminating $z^{\pi^0}, z^{\gamma_1}, z^{A_i}$ from coefficient of $\delta\gamma_1, \delta\pi^0, \delta\pi^i$, which is given by

$$z^{\pi^0} = -\bigg(\frac{\partial\Omega_1}{\partial A_\mu} + \bigg(\frac{\partial\Omega_1}{\partial\partial_i A_\mu}\bigg)\partial_i\bigg)z^{A_\mu}, \tag{4.36}$$

$$z^{\gamma_1} = -z^{A_0}, \tag{4.37}$$

$$z^{A_i} = -\left(\frac{\partial\Omega_2}{\partial\pi^i} - \partial_j\left(\frac{\partial\Omega_2}{\partial\partial_j\pi^i}\right)z^{\gamma_2} + \frac{\partial\Omega_2}{\partial\partial_j\pi^i}\partial_j z^{\gamma_2}, \tag{4.38}$$

and substituting into the coefficient of δA_0, it gives

$$0 = \mathcal{C}_2^{jk}\partial_j\partial_k z^{\gamma_2} + \mathcal{C}_1^j\partial_j z^{\gamma_2} + \mathcal{C}_0 z^{\gamma_2}, \tag{4.39}$$

where

$$\mathcal{C}_2^{jk} \equiv -\frac{\partial\Omega_2}{\partial\partial_j\partial_k A_0} - \frac{\partial\Omega_1}{\partial\partial_{(j|}A_i}\frac{\partial\Omega_2}{\partial\partial_{|k)}\pi^i}, \tag{4.40}$$

$$\begin{aligned}\mathcal{C}_1^j \equiv & -\frac{\partial\Omega_1}{\partial A_i}\frac{\partial\Omega_2}{\partial\partial_j\pi^i} + \frac{\partial\Omega_1}{\partial\partial_j A_i}\frac{\partial\Omega_2}{\partial\pi^i} - \frac{\partial\Omega_1}{\partial\partial_i A_k}\partial_i\left(\frac{\partial\Omega_2}{\partial\partial_j\pi^k}\right) \\ & - \frac{\partial\Omega_1}{\partial\partial_j A_k}\partial_i\left(\frac{\partial\Omega_2}{\partial\partial_i\pi^k}\right) + \frac{\partial\Omega_2}{\partial\partial_j A_0} - 2\partial_k\left(\frac{\partial\Omega_2}{\partial\partial_j\partial_k A_0}\right),\end{aligned} \tag{4.41}$$

$$\begin{aligned}\mathcal{C}_0 \equiv & \frac{\partial\Omega_1}{\partial A_i}\frac{\partial\Omega_2}{\partial\pi^i} - \frac{\partial\Omega_1}{\partial A_i}\partial_j\left(\frac{\partial\Omega_2}{\partial\partial_j\pi^i}\right) + \frac{\partial\Omega_1}{\partial\partial_i A_j}\partial_i\left(\frac{\partial\Omega_2}{\partial\pi^j}\right) - \frac{\partial\Omega_1}{\partial\partial_i A_j}\partial_i\partial_k\left(\frac{\partial\Omega_2}{\partial\partial_k\pi^j}\right) \\ & - \frac{\partial\Omega_2}{\partial A_0} + \partial_j\left(\frac{\partial\Omega_2}{\partial\partial_j A_0}\right) - \partial_j\partial_k\left(\frac{\partial\Omega_2}{\partial\partial_j\partial_k A_0}\right).\end{aligned} \tag{4.42}$$

It can be shown that diffeomorphism invariance requirements demand that $\mathcal{C}_2^{ij} = 0$ and $\mathcal{C}_1^i = 0$. See Appendix B. If one impose these requirements, then eq.(4.39) reduce to

$$0 = \mathcal{C}_0 z^{\gamma_2}. \tag{4.43}$$

If $\mathcal{C}_0 = 0$, then either there are extra constraints or there are only two constraints in total, but both of them are of first-class. Both of these cases give degree of freedom less than three. Therefore one should demand $\mathcal{C}_0 \neq 0$.

Let us summarise the result of this subsection. One imposes (A.21) which is one of the diffeomorphism invariance requirements, one see that eq.(4.25) is trivially satisfied. This eventually results in the existence of the second constraint as shown in eq.(4.27). Furthermore, if any given theory in the class (3.14) satisfies

$$\mathcal{C}_2^{ij} = 0, \tag{4.44}$$

$$\mathcal{C}_1^i = 0, \tag{4.45}$$

$$\mathcal{C}_0 \neq 0, \tag{4.46}$$

then the vector sector of that theory has 3 degrees of freedom as required. It can be shown that the conditions (4.44) and (4.45) follow from diffeomorphism invariance requirements. For consistency, it can be shown by using the Dirac method that the result agree with the Faddeev–Jackiw method.

4.2 Multi-field generalised Proca theories

This section is largely based on [31]. We are interested in the class of multi-field generalised Proca theories which is a system of n vector fields A_μ^α possibly coupled to external fields, which might also include the metric $g_{\mu\nu}$, and their derivatives. The external fields can be thought of as being dependent explicitly on time and space. For example, the system of multiple massive vector fields might be put in a flat or curved backgrounds and might also couple to other external fields. As for the notations, we use beginning lower-case Greek indices $\alpha, \beta, \gamma \in \{1, 2, \cdots, n\}$ to denote internal indices for vector fields. We call the external fields and their derivative collectively as K.

We consider theories whose Lagrangians are local, diffeomorphism invariance, free of Ostrogradski instability and depend up to first order derivatives of the vector fields. For definiteness, we call the space of the vector fields and their first order time derivatives as the tangent bundle.

In order for the vector sector to be constraint, the Hessian condition[2]

$$\det\left(\frac{\partial^2 \mathcal{L}}{\partial \dot{A}_\mu^\alpha \partial \dot{A}_\nu^\beta}\right) = 0 \tag{4.47}$$

[2]The determinants in eq.(4.47), eq.(4.48), and eq.(4.50) are defined as follows. We combine the two indices of each vector field into one collective index. The matrices appearing within the determinants then have two collective indices. Standard definition for determinant then applies.

should be satisfied. However, in this thesis, we will restrict the study to theories satisfying condition

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{A}_0^\alpha \partial \dot{A}_\mu^\beta} = 0, \quad \det\left(\frac{\partial^2 \mathcal{L}}{\partial \dot{A}_i^\alpha \partial \dot{A}_j^\beta}\right) \neq 0, \tag{4.48}$$

which would imply the Hessian condition (4.47). For definiteness, let us call eq.(4.48) as "the special Hessian condition". This condition has also been imposed by many references for example [8], [9], [55], [56], [57], in order to construct multi-field generalised Proca theories.

By requiring $\partial^2\mathcal{L}/\partial\dot{A}_0^\alpha\partial\dot{A}_0^\beta = 0$, we see that $\mathcal{L}$ should be at most linear in $\dot{A}_0^\alpha$. Then by using the condition $\partial^2\mathcal{L}/\partial\dot{A}_0^\alpha\partial\dot{A}_i^\beta = 0$, we see that the coefficient of the linear term does not depend on $\dot{A}_i^\alpha$. Then imposing $\det(\partial^2\mathcal{L}/\partial\dot{A}_i^\alpha\partial\dot{A}_j^\beta) \neq 0$ exhausts all the requirements of eq.(4.48).

Therefore, theories we consider have Lagrangians of the form

$$\mathcal{L} = T(A_\mu^\alpha, \partial_i A_\mu^\alpha, \dot{A}_i^\alpha, K) + U_\beta(A_\mu^\alpha, \partial_i A_\mu^\alpha, K)\dot{A}_0^\beta, \tag{4.49}$$

subject to

$$\det\left(\frac{\partial^2 T}{\partial \dot{A}_i^\alpha \partial \dot{A}_j^\beta}\right) \neq 0. \tag{4.50}$$

Since these theories are diffeomorphism invariance, they satisfy conditions on T, U_β as given in Appendix C. Further requirements will be imposed in order for the theory to possess the correct number of degrees of freedom. These requirements are known in the literature to allow secondary constraints and to terminate the process of constraint analysis [8], [9], [10], [57]. The conditions which we will present are slightly differed from their counterparts in the literature. These differences, however, are important. Later in this section, we will comment on how and why they differ.

Euler-Lagrange equations for the vector fields are of the form

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{A}_i^\alpha \partial \dot{A}_j^\beta}\ddot{A}_j^\beta + \partial_j \frac{\partial \mathcal{L}}{\partial \partial_j A_i^\alpha} - \frac{\partial \mathcal{L}}{\partial A_i^\alpha} + \cdots = 0, \tag{4.51}$$

$$\dot{U}_\alpha + \partial_i \frac{\partial \mathcal{L}}{\partial \partial_i A_0^\alpha} - \frac{\partial \mathcal{L}}{\partial A_0^\alpha} = 0, \tag{4.52}$$

where $\cdots$ are terms which do not contain $\ddot{A}_\mu^\alpha$. Since the Euler-Lagrange equations do not contain time derivative with order higher than two, the theories are free of Ostrogradski instability [36] in the vector sector. Furthermore, it is clear that the systems are constrained as Euler-Lagrange equations describe only the dynamics of A_j^β but there is no dynamics of A_0^β. In section 4.3, we will start from these Euler-Lagrange equations and rederive, as a cross-check to the analysis of the present section, secondary-constraint enforcing relations [8], [9]. As to be seen in the analysis, the relations given in [8], [9] miss one term, which would invalidate some of their justifications on behaviour of example theories. Then. let us consider Faddeev-Jackiw Constraint analysis. We require that theories presented above should have the correct number of degrees of freedom. For this, we are going to make use of constraint analysis using the Faddeev-Jackiw method [4], [5], [6], [7]. The analysis will give further conditions that the theories should satisfy. We will use the notations and conventions similar to those used in [12], [13].

4.2.1 First iteration

In order to transform from the tangent bundle to phase space, one considers conjugate momenta. Conjugate momenta for the Lagrangian eq.(4.49) are

$$\pi_\beta^\mu = \delta_0^\mu U_\beta + \delta_i^\mu \frac{\partial T}{\partial \dot{A}_i^\beta}. \tag{4.53}$$

These equations allow us to identify primary constraints

$$\Omega_\beta = \pi_\beta^0 - U_\beta. \tag{4.54}$$

The spatial components of conjugate momenta are given by

$$\pi_\beta^i = \frac{\partial T}{\partial \dot{A}_i^\beta}. \tag{4.55}$$

Because of the condition (4.50), these equations can be inverted to give

$$\dot{A}_i^\beta = \Lambda_i^\beta (A_\mu^\alpha, \partial_i A_\mu^\alpha, \pi_\alpha^i, K). \tag{4.56}$$

Since we work in phase space, it would be convenient to define

$$\mathcal{T}(A^\alpha_\mu, \partial_i A^\alpha_\mu, \Lambda^\alpha_i, K) = T(A^\alpha_\mu, \partial_i A^\alpha_\mu, \dot{A}^\alpha_i, K)\Big|_{\dot{A}^\alpha_i \to \Lambda^\alpha_i}. \tag{4.57}$$

Hamiltonian is given by

$$\begin{aligned}\mathcal{H} &= \pi^\mu_\alpha \dot{A}^\alpha_\mu - \mathcal{L} - \dot{\gamma}^\alpha \Omega_\alpha \\ &\approx \pi^i_\alpha \Lambda^\alpha_i - \mathcal{T} - \dot{\gamma}^\alpha \Omega_\alpha,\end{aligned} \tag{4.58}$$

where γ^α are Lagrange multipliers. Note that the time derivatives of external fields is allowed in the Hamiltonian (through $\mathcal{T}$) since for the system of interest, the external fields are predetermined functions of time and space. So their time derivatives are also predetermined functions. The presence of time-dependent external fields in the Hamiltonian simply means that the Hamiltonian depends explicitly on time. It is also not possible and not relevant to work out the conjugate momenta of the external fields as, apart from the fact that the external fields are predetermined functions, $\mathcal{L}$ does not contain terms describing dynamics of the external fields.

Let us start considering first iteration. First order form of the Lagrangian is given by

$$\mathcal{L}_{FOF} = \pi^\mu_\alpha \dot{A}^\alpha_\mu + \mathcal{L}_v + \dot{\gamma}^\alpha \Omega_\alpha, \tag{4.59}$$

where

$$\mathcal{L}_v \equiv \mathcal{T} - \pi^i_\alpha \Lambda^\alpha_i. \tag{4.60}$$

Symplectic variables are

$$\xi^I = (A^\alpha_\mu, \pi^\mu_\alpha, \gamma^\alpha). \tag{4.61}$$

Note that since the system of interest only describes the dynamics of A^α_μ, the phase space only contain variables relevant to A^α_μ. On the other hand, the external fields (K) are simply predetermined functions of time and space and are not treated as variables. Each of them is a real valued indexed object at each given spacetime position.

Canonical one-form is given by

$$\mathcal{A} = \int d^3\mathbf{x}(\pi^\mu_\alpha \delta A^\alpha_\mu + \Omega_\alpha \delta\gamma^\alpha). \tag{4.62}$$

So symplectic two-form is

$$\mathcal{F} = \int d^3\mathbf{x}\left(\delta\pi^\mu_\alpha \wedge \delta A^\alpha_\mu + \delta\pi^0_\alpha \wedge \delta\gamma^\alpha - \frac{\partial U_\alpha}{\partial A^\beta_\mu}\delta A^\beta_\mu \wedge \delta\gamma^\alpha - \frac{\partial U_\alpha}{\partial\partial_i A^\beta_\mu}\delta\partial_i A^\beta_\mu \wedge \delta\gamma^\alpha\right). \tag{4.63}$$

Demanding $i_z\mathcal{F} = 0$ gives

$$z^{\pi^\mu_\alpha} + \frac{\partial U_\beta}{\partial A^\alpha_\mu} z^{\gamma^\beta} - \partial_i\left(\frac{\partial U_\beta}{\partial\partial_i A^\alpha_\mu} z^{\gamma^\beta}\right) = 0, \tag{4.64}$$

$$z^{A^\alpha_\mu} + \delta^0_\mu z^{\gamma^\alpha} = 0, \tag{4.65}$$

$$z^{\pi^0_\alpha} - \frac{\partial U_\alpha}{\partial A^\beta_\mu} z^{A^\beta_\mu} - \frac{\partial U_\alpha}{\partial\partial_i A^\beta_\mu}\partial_i z^{A^\beta_\mu} = 0. \tag{4.66}$$

In order for these equations to be consistent, the equation

$$\begin{aligned} &\left(\frac{\partial U_\alpha}{\partial A^\beta_0} - \frac{\partial U_\beta}{\partial A^\alpha_0} + \partial_i \frac{\partial U_\beta}{\partial\partial_i A^\alpha_0}\right) z^{\gamma^\beta} \\ &\quad + \left(\frac{\partial U_\beta}{\partial\partial_i A^\alpha_0} + \frac{\partial U_\alpha}{\partial\partial_i A^\beta_0}\right)\partial_i z^{\gamma^\beta} = 0 \end{aligned} \tag{4.67}$$

has to be satisfied. In fact as analysed in Appendix ?? diffeomorphism invariance requires, among others, eq.(C.9). So we are left with

$$\left(\frac{\partial U_\alpha}{\partial A^\beta_0} - \frac{\partial U_\beta}{\partial A^\alpha_0} + \partial_i \frac{\partial U_\beta}{\partial\partial_i A^\alpha_0}\right) z^{\gamma^\beta} = 0. \tag{4.68}$$

Let us denote

$$q_{\alpha\beta} \equiv \frac{\partial U_\alpha}{\partial A^\beta_0} - \frac{\partial U_\beta}{\partial A^\alpha_0} + \partial_i \frac{\partial U_\beta}{\partial\partial_i A^\alpha_0}. \tag{4.69}$$

We are particularly interested in the case where $\operatorname{rank}(q_{\alpha\beta}) = 0$, that is

$$\frac{\partial U_\alpha}{\partial A^\beta_0} - \frac{\partial U_\beta}{\partial A^\alpha_0} + \partial_i \frac{\partial U_\beta}{\partial\partial_i A^\alpha_0} = 0. \tag{4.70}$$

As will be seen later, enforcing these conditions would lead to n secondary constraints. We are only interested in the class of theories with this constraint structure. This class include, for example, a theory of n uncoupled generalised Proca fields (an analysis will be given in subsection 4.4.1). On the other hand, if $\text{rank}(q_{\alpha\beta}) \neq 0$, and we want the procedure not to terminate after the second iteration, the theory would either have undesired number of degrees of freedom or have first class constraints. Either of these cases are not what we are interested in.

As a cross-check, one may note that after imposing diffeomorphism invariance requirement,

$$[\Omega_\alpha, \Omega_\beta(\mathbf{x}')] \approx q_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{x}'). \tag{4.71}$$

Therefore, the condition eq.(4.70) is equivalent to the vanishing of the Poisson's brackets of the primary constraints among themselves. That is

$$[\Omega_\alpha, \Omega_\beta(\mathbf{x}')] \approx 0. \tag{4.72}$$

In Dirac constraint analysis [2], [1], if we demand the primary constraints to be preserved in time we would have, with Hamiltonian density being $\mathcal{H} = \mathcal{H}_0 + u^\beta\Omega_\beta$ where $\mathcal{H}_0 = \pi^i_\alpha\Lambda^\alpha_i - \mathcal{T}$,

$$\int d^3\mathbf{x}'[\Omega_\alpha, \mathcal{H}_0(\mathbf{x}')] + \int d^3\mathbf{x}' u^\beta(\mathbf{x}')[\Omega_\alpha, \Omega_\beta(\mathbf{x}')] + \frac{\partial\Omega_\alpha}{\partial t} \approx 0. \tag{4.73}$$

Then since the explicit dependence on time of Ω_α appears in U_α due to the presence of K we may simply use the chain rule to obtain

$$\int d^3\mathbf{x}'[\Omega_\alpha, \mathcal{H}_0(\mathbf{x}')] + \int d^3\mathbf{x}' u^\beta(\mathbf{x}')[\Omega_\alpha, \Omega_\beta(\mathbf{x}')] - \frac{\partial U_\alpha}{\partial K}\dot{K} \approx 0, \tag{4.74}$$

where it is understood that in the third term on LHS of eq.(4.74) there is a sum over the collection of the external fields and their derivatives. If the conditions (4.72) are not fulfilled, i.e. $\text{rank}(q_{\alpha\beta}) \neq 0$, eq.(4.74) would determine some components of u^β. So there will be less than n secondary constraints. In the extreme case where $\text{rank}(q_{\alpha\beta}) = n$, i.e. $\det(q_{\alpha\beta}) \neq 0$, there is no secondary constraint. Furthermore, after classification, it is easy to see that all of these constraints are of second class. So the number of degrees of freedom is less than $3n$, which is not desirable.

Note that in the tangent bundle, eq.(4.70) can also be expressed as

$$\frac{\partial^2\mathcal{L}}{\partial\dot{A}_0^\alpha\partial A_0^\beta}-\frac{\partial^2\mathcal{L}}{\partial A_0^\alpha\partial\dot{A}_0^\beta}+\partial_i\left(\frac{\partial^2\mathcal{L}}{\partial\partial_i A_0^\alpha\partial\dot{A}_0^\beta}\right)=0, \tag{4.75}$$

which is a correction to the secondary-constraint enforcing relations derived in [8], [9]. Only the last term on the LHS of eq.(4.75) is not present in these references. This could be due to the fact that their analysis discards the dependence on spatial derivatives of vector fields. While this is sufficient for the main purpose of counting the number of degrees of freedom, one should be careful with the conditions derived in the process. In order to make use of such conditions, one should appropriately restore the dependence on spatial derivatives of vector fields. It turns out that the restoration in this case is given by the inclusion of the third term on LHS of eq.(4.75). As a consequence of the missing term in the secondary-constraint enforcing relations, behaviours of some theories receive incorrect interpretations. For example, a special case of theory presented in [56] is interpreted by [8] to contain extra degrees of freedom. In fact, however, by a careful analysis to be discussed in subsection 4.4.1, the theory is a legitimate multi-field generalised Proca theory since it has the desirable number of degrees of freedom.

It would be helpful to first demonstrate that eq.(4.75) is indeed satisfied by some simple cases. In particular, it can be shown that eq.(4.75) is satisfied by single field generalised Proca theories. In this case, eq.(4.75) reduces to

$$\partial_i\left(\frac{\partial^2\mathcal{L}}{\partial\partial_i A_0\partial\dot{A}_0}\right)=0, \tag{4.76}$$

which is in fact trivially satisfied. The systems of interest as described at the start of section 4.2 automatically satisfies the diffeomorphism invariant requirement. In particular, consider a diffeomorphism condition, eq.(C.9), which reduces to

$$\frac{\partial^2\mathcal{L}}{\partial\partial_i A_0\partial\dot{A}_0}=0, \tag{4.77}$$

where we recall that $U\equiv\partial\mathcal{L}/\partial\dot{A}_0$. So by imposing the condition (4.77), it can be seen that eq.(4.76) is trivially satisfied. Having shown that eq.(4.75) is satisfied by single field generalised Proca theories, it can immediately be seen that it is also satisfied by separable multi-field generalised Proca theories. See section 4.4.1.1 for more details.

Let us continue the Faddeev-Jackiw analysis. The zero mode of $\mathcal{F}$ is

$$\begin{aligned} z_1 = z^{\gamma^\alpha}\left(\frac{\delta}{\delta\gamma^\alpha} - \frac{\delta}{\delta A_0^\alpha}\right) + \left(-\frac{\partial U_\alpha}{\partial A_0^\beta} z^{\gamma^\beta} - \frac{\partial U_\alpha}{\partial\partial_i A_0^\beta}\partial_i z^{\gamma^\beta}\right)\frac{\delta}{\delta\pi_\alpha^0} \\ + \left(-\frac{\partial U_\beta}{\partial A_i^\alpha} z^{\gamma^\beta} + \partial_j\left(\frac{\partial U_\beta}{\partial\partial_j A_i^\alpha} z^{\gamma^\beta}\right)\right)\frac{\delta}{\delta\pi_\alpha^i}, \end{aligned} \tag{4.78}$$

subject to secondary-constraint enforcing relations (4.70). Having obtained the zero mode, let us check whether there are further constraints in the system by considering

$$i_{z_1}\int d^3\mathbf{x}\,\delta\mathcal{L}_v = \int d^3\mathbf{x}\Bigg(-\frac{\partial\mathcal{T}}{\partial A_0^\beta} + \partial_i\frac{\partial\mathcal{T}}{\partial\partial_i A_0^\beta} + \left(\frac{\partial U_\beta}{\partial A_i^\alpha} + \frac{\partial U_\beta}{\partial\partial_j A_i^\alpha}\partial_j\right)\Lambda_i^\alpha\Bigg) z^{\gamma^\beta}, \tag{4.79}$$

where we have used the identity

$$\pi_\alpha^i = \frac{\partial\mathcal{T}}{\partial\Lambda_i^\alpha}, \tag{4.80}$$

which is equivalent to eq.(4.55). The result from eq.(4.79) gives secondary constraints

$$\tilde{\Omega}_\beta = \frac{\partial\mathcal{T}}{\partial A_0^\beta} - \partial_i\frac{\partial\mathcal{T}}{\partial\partial_i A_0^\beta} - \left(\frac{\partial U_\beta}{\partial A_i^\alpha} + \frac{\partial U_\beta}{\partial\partial_j A_i^\alpha}\partial_j\right)\Lambda_i^\alpha - \frac{\partial U_\beta}{\partial K}\dot{K} \tag{4.81}$$

which, written as functions,

$$\tilde{\Omega}_\beta = \tilde{\Omega}_\beta(A_\mu^\alpha, \partial_i A_\mu^\alpha, \partial_i\partial_j A_\mu^\alpha, \pi_i^\alpha, \partial_i\pi_j^\alpha, K). \tag{4.82}$$

Note that when reading off the constraint (4.81), there is also the contribution from external fields as presented in the last term on RHS. This is because the external fields are considered to be functions with explicit dependence on time. So when working out secondary constraints which essentially involves taking derivative of primary constraints with respect to time, the explicit time derivative of the external field should also be taken into account.

4.2.2 Second iteration

Having obtained new constraints from the first iteration, let us start the second iteration by including Lagrange multipliers corresponding to the new constraints. Symplectic variables are

$$\xi^I = (A_\mu^\alpha, \pi_\alpha^\mu, \gamma^\alpha, \tilde{\gamma}^\alpha). \tag{4.83}$$

Canonical one-form is given by

$$\mathcal{A} = \int d^3\mathbf{x}(\pi_\alpha^\mu \delta A_\mu^\alpha + \Omega_\alpha \delta\gamma^\alpha + \tilde{\Omega}_\alpha \delta\tilde{\gamma}^\alpha). \tag{4.84}$$

So symplectic two-form is

$$\mathcal{F} = \int d^3\mathbf{x}\left(\delta\pi_\alpha^\mu \wedge \delta A_\mu^\alpha + \delta\Omega_\alpha \wedge \delta\gamma^\alpha + \delta\tilde{\Omega}_\alpha \wedge \delta\tilde{\gamma}^\alpha\right). \tag{4.85}$$

We may also denote the constraints and Lagrange multipliers as $\Omega_\alpha^{(1)} \equiv \Omega_\alpha, \Omega_\alpha^{(2)} \equiv \tilde{\Omega}_\alpha, \gamma_{(1)}^\alpha \equiv \gamma^\alpha, \gamma_{(2)}^\alpha \equiv \tilde{\gamma}^\alpha$.

When solving for zero mode of the symplectic two-form $\mathcal{F}$, equations involving Poisson's brackets would arise. In order to easily see this, it will be useful to define the notation for generalised derivatives $\partial_\mathcal{I}$ as follows. Suppose that f and g are functions of $A_\mu^\alpha, \partial_i A_\mu^\alpha, \partial_i\partial_j A_\mu^\alpha, \cdots, \pi_\alpha^\mu, \partial_i \pi_\alpha^\mu, \partial_i\partial_j\pi_\alpha^\mu, \cdots, K$. So

$$\begin{aligned}\frac{\delta f}{\delta A_\mu^\alpha(\mathbf{z})} &= \frac{\partial f}{\partial A_\mu^\alpha}\delta^{(3)}(\mathbf{x}-\mathbf{z}) + \frac{\partial f}{\partial\partial_i A_\mu^\alpha}\partial_i\delta^{(3)}(\mathbf{x}-\mathbf{z}) + \frac{\partial f}{\partial\partial_i\partial_j A_\mu^\alpha}\partial_i\partial_j\delta^{(3)}(\mathbf{x}-\mathbf{z}) + \cdots \\ &\equiv \frac{\partial f}{\partial\partial_\mathcal{I} A_\mu^\alpha}\partial_\mathcal{I}\delta^{(3)}(\mathbf{x}-\mathbf{z}),\end{aligned} \tag{4.86}$$

where summation over $\mathcal{I}$ is understood. Similarly,

$$\frac{\delta f}{\delta\pi_\alpha^\mu(\mathbf{z})} = \frac{\partial f}{\partial\partial_\mathcal{I}\pi_\alpha^\mu}\partial_\mathcal{I}\delta^{(3)}(\mathbf{x}-\mathbf{z}). \tag{4.87}$$

Then in this notation Poisson's bracket can be written as

$$\begin{aligned}[f, g(\mathbf{y})] =&(-1)^{|\mathcal{J}|}\frac{\partial f}{\partial\partial_{\mathcal{I}}A^{\alpha}_{\mu}}\partial_{\mathcal{I}}\partial_{\mathcal{J}}\left(\frac{\partial g}{\partial\partial_{\mathcal{J}}\pi^{\mu}_{\alpha}}\delta^{(3)}(\mathbf{x}-\mathbf{y})\right)\\ &-(-1)^{|\mathcal{J}|}\frac{\partial f}{\partial\partial_{\mathcal{I}}\pi^{\mu}_{\alpha}}\partial_{\mathcal{I}}\partial_{\mathcal{J}}\left(\frac{\partial g}{\partial\partial_{\mathcal{J}}A^{\alpha}_{\mu}}\delta^{(3)}(\mathbf{x}-\mathbf{y})\right),\end{aligned} \tag{4.88}$$

where $|\mathcal{J}|$ is the order of partial derivatives of $\mathcal{J}$, and summation over $\mathcal{I}$ and $\mathcal{J}$ is understood.

Let us then find zero mode of $\mathcal{F}$. Demanding $i_z\mathcal{F} = 0$ gives

$$z^{\pi^{\mu}_{\beta}} - \sum_{\mathfrak{s}=1}^{2}(-1)^{|\mathcal{I}|}\partial_{\mathcal{I}}\left(z^{\gamma^{\alpha}_{(\mathfrak{s})}}\frac{\partial\Omega^{(\mathfrak{s})}_{\alpha}}{\partial\partial_{\mathcal{I}}A^{\beta}_{\mu}}\right) = 0, \tag{4.89}$$

$$-z^{A^{\beta}_{\mu}} - \sum_{\mathfrak{s}=1}^{2}(-1)^{|\mathcal{I}|}\partial_{\mathcal{I}}\left(z^{\gamma^{\alpha}_{(\mathfrak{s})}}\frac{\partial\Omega^{(\mathfrak{s})}_{\alpha}}{\partial\partial_{\mathcal{I}}\pi^{\mu}_{\beta}}\right) = 0, \tag{4.90}$$

$$\partial_{\mathcal{I}}z^{A^{\alpha}_{\mu}}\frac{\partial\Omega^{(\mathfrak{s})}_{\beta}}{\partial\partial_{\mathcal{I}}A^{\alpha}_{\mu}} + \partial_{\mathcal{I}}z^{\pi^{\mu}_{\alpha}}\frac{\partial\Omega^{(\mathfrak{s})}_{\beta}}{\partial\partial_{\mathcal{I}}\pi^{\mu}_{\alpha}} = 0, \quad \text{for } \mathfrak{s} = 1, 2. \tag{4.91}$$

Eliminating $z^{A^{\alpha}_{\mu}}$ and $z^{\pi^{\mu}_{\alpha}}$ and using the identity eq.(4.88), we obtain

$$\sum_{\mathfrak{s}=1}^{2}\int d^3\mathbf{y}[\Omega^{(1)}_{\alpha}, \Omega^{(\mathfrak{s})}_{\beta}(\mathbf{y})]z^{\gamma^{\beta}_{(\mathfrak{s})}}(\mathbf{y}) = 0, \tag{4.92}$$

$$\sum_{\mathfrak{s}=1}^{2}\int d^3\mathbf{y}[\Omega^{(2)}_{\alpha}, \Omega^{(\mathfrak{s})}_{\beta}(\mathbf{y})]z^{\gamma^{\beta}_{(\mathfrak{s})}}(\mathbf{y}) = 0. \tag{4.93}$$

Note that

$$[\Omega_{\alpha}, \Omega_{\gamma}(\mathbf{y})] = \left(-q_{\alpha\gamma} + \left(\frac{\partial\Omega_{\alpha}}{\partial\partial_i A^{\gamma}_0} + \frac{\partial\Omega_{\gamma}}{\partial\partial_i A^{\alpha}_0}\right)\partial_i\right)\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{4.94}$$

Imposing diffeomorphism conditions eq.(C.9) and secondary-constraint enforcing relations eq.(4.70), we obtain

$$[\Omega_{\alpha}, \Omega_{\gamma}(\mathbf{y})] = 0. \tag{4.95}$$

Next, after expressing the Poisson's brackets between primary and secondary constraints and substituting this along with eq.(4.95) into eq.(4.92), one obtains

$$\mathcal{C}_{0\alpha\gamma}z^{\tilde{\gamma}^{\gamma}} + \mathcal{C}^{i}_{1\alpha\gamma}\partial_i z^{\tilde{\gamma}^{\gamma}} + \mathcal{C}^{ij}_{2\alpha\gamma}\partial_i\partial_j z^{\tilde{\gamma}^{\gamma}} = 0, \tag{4.96}$$

where

$$\begin{aligned}\mathcal{C}_{0\alpha\gamma} \equiv & \frac{\partial\tilde{\Omega}_\gamma}{\partial A_0^\alpha} - \partial_i\left(\frac{\partial\tilde{\Omega}_\gamma}{\partial\partial_i A_0^\alpha}\right) + \partial_i\partial_j\left(\frac{\partial\tilde{\Omega}_\gamma}{\partial\partial_i\partial_j A_0^\alpha}\right) \\ & - \left(\frac{\partial\Omega_\alpha}{\partial A_k^\beta} + \frac{\partial\Omega_\alpha}{\partial\partial_i A_k^\beta}\partial_i\right)\left(\frac{\partial\tilde{\Omega}_\gamma}{\partial\pi_\beta^k} - \partial_j\left(\frac{\partial\tilde{\Omega}_\gamma}{\partial\partial_j\pi_\beta^k}\right)\right),\end{aligned} \tag{4.97}$$

$$\begin{aligned}\mathcal{C}_{1\alpha\gamma}^i \equiv & -\frac{\partial\tilde{\Omega}_\gamma}{\partial\partial_i A_0^\alpha} + 2\partial_j\left(\frac{\partial\tilde{\Omega}_\gamma}{\partial\partial_i\partial_j A_0^\alpha}\right) + \frac{\partial\Omega_\alpha}{\partial A_k^\beta}\frac{\partial\tilde{\Omega}_\gamma}{\partial\partial_i\pi_\beta^k} \\ & - \frac{\partial\Omega_\alpha}{\partial\partial_i A_k^\beta}\left(\frac{\partial\tilde{\Omega}_\gamma}{\partial\pi_\beta^k} - \partial_j\left(\frac{\partial\tilde{\Omega}_\gamma}{\partial\partial_j\pi_\beta^k}\right)\right) + \frac{\partial\Omega_\alpha}{\partial\partial_j A_k^\beta}\partial_j\left(\frac{\partial\tilde{\Omega}_\gamma}{\partial\partial_i\pi_\beta^k}\right),\end{aligned} \tag{4.98}$$

$$\mathcal{C}_{2\alpha\gamma}^{ij} \equiv \frac{\partial\tilde{\Omega}_\gamma}{\partial\partial_i\partial_j A_0^\alpha} + \frac{\partial\Omega_\alpha}{\partial\partial_{(i|}A_k^\beta}\frac{\partial\tilde{\Omega}_\gamma}{\partial\partial_{|j)}\pi_\beta^k}. \tag{4.99}$$

It would be helpful to rewrite eqs.(4.97)-(4.99) in the forms which are easier to use. In particular, one may express $\mathcal{C}_{0\alpha\gamma}, \mathcal{C}_{1\alpha\gamma}^i, \mathcal{C}_{2\alpha\gamma}^{ij}$ in terms of $\mathcal{T}$ and U_β. However, even with the help of diffeomorphism invariance requirements, the expressions are still not simple to use. It is in fact even better to express these quantities in tangent bundle. We will postpone the presentation of these forms to section 4.3, where the relevant expressions are given in eqs.(4.145)-(4.146). Nevertheless, we may readily note here that by working in phase space and using diffeomorphism invariance requirements, it can be seen explicitly that

$$\mathcal{C}_{1\alpha\gamma}^i = -\mathcal{C}_{1\gamma\alpha}^i, \qquad \mathcal{C}_{2\alpha\gamma}^{ij} = 0. \tag{4.100}$$

After using eq.(4.100), it can be seen that eq.(4.96) becomes

$$\mathcal{C}_{0\alpha\gamma}z^{\tilde{\gamma}^\gamma} + \mathcal{C}_{1\alpha\gamma}^i\partial_i z^{\tilde{\gamma}^\gamma} = 0. \tag{4.101}$$

It is clear that $z^{\tilde{\gamma}^\gamma} = 0$ is a solution to eq.(4.101). However, the question is whether this solution is unique. If $z^{\tilde{\gamma}^\gamma} = 0$ is the unique solution to eq.(4.101), then after substituting into eq.(4.93), we obtain

$$(\mathcal{C}_{0\gamma\alpha} - \partial_i\mathcal{C}_{1\gamma\alpha}^i)z^{\gamma^\gamma} + \mathcal{C}_{1\alpha\gamma}^i\partial_i z^{\gamma^\gamma} = 0. \tag{4.102}$$

As to be discussed in section 4.3, it can be shown by using diffeomorphism conditions that

$$\mathcal{C}_{0\alpha\gamma} - \mathcal{C}_{0\gamma\alpha} = \partial_i \mathcal{C}^i_{1\alpha\gamma}. \tag{4.103}$$

So eq.(4.102) is equivalent to eq.(4.101). If eq.(4.101) has the unique solution $z^{\bar{\gamma}^\gamma} = 0$, then $z^{\gamma^\gamma} = 0$ should also be the unique solution to eq.(4.102). Then by using eqs.(4.89)-(4.90) we obtain $z^{A^\alpha_\mu} = z^{\pi^\mu_\alpha} = 0$. So there is no zero mode, and the procedure terminates. Note that the requirement that the constraint analysis should terminate is previously suggested and emphasised in [8], [9], [10], [16]. By using the criteria presented by [54], it can be concluded that the number of degrees of freedom is $3n$ as required.

For definiteness, let us call the condition

$$\mathcal{C}_{0\alpha\gamma} z^{\bar{\gamma}^\gamma} + \mathcal{C}^i_{1\alpha\gamma}\partial_i z^{\bar{\gamma}^\gamma} = 0 \implies \text{unique solution } z^{\bar{\gamma}^\gamma} = 0 \tag{4.104}$$

as the "completion requirement" since it signals the end of the second iteration. There are two main cases which would satisfy the completion requirement (4.104):

- Case 1: $\mathcal{C}^i_{1\alpha\gamma} \neq 0$, and the boundary condition that fields should vanish fast enough near spatial infinity (this is the boundary condition which is required in the whole analysis to make integrals of total derivatives vanish) is sufficient to fix the solution to the equation in (4.104) to be unique.
- Case 2: $\mathcal{C}^i_{1\alpha\gamma} = 0$ and $\det(\mathcal{C}_{0\alpha\gamma}) \neq 0$.

In the case where $\mathcal{C}^i_{1\alpha\gamma} \neq 0$, it is not clear whether the boundary condition would be sufficient to fix the solution to the equation in (4.104) to be unique. We expect that the analysis should be done separately for each given specific theory. Even then, it would still be quite difficult, if at all possible, to show that the solution is unique. This means that it would not be simple to show whether a given theory with $\mathcal{C}^i_{1\alpha\gamma} \neq 0$ is within the case 1. As for the case where a theory has $\mathcal{C}^i_{1\alpha\gamma} = 0$, it could be very likely that $\det(\mathcal{C}_{0\alpha\gamma}) \neq 0$. This is because the form of $\mathcal{C}_{0\alpha\gamma}$ contains many terms in the expression, which make it difficult for $\mathcal{C}_{0\alpha\gamma}$ to be singular. On the other hand, the requirement $\mathcal{C}^i_{1\alpha\gamma} = 0$ itself would look quite restrictive, which might bring an immediate question as to whether it is possible to find theories within case 2. In fact, as to be explicitly discussed in subsection 4.4.1, theories passing this requirement have already appeared in the literature. However, some of them might have been mistakenly ruled out due to the usage of the incorrect version of secondary-constraint enforcing relations [8], [9]. We will only provide one such example.

4.2.3 Matrix form of $\mathcal{F}$

In Faddeev-Jackiw constraint analysis, it is often convenient to consider the matrix form of $\mathcal{F}$. This would allow us to cross-check the analysis at the second iteration and at the same time further justify the completion requirement (4.104). In order to obtain the components of $\mathcal{F}$, it is convenient to first denote

$$f_{\xi^I} \equiv i_{\frac{\delta}{\delta\xi^I}}\mathcal{F}. \tag{4.105}$$

From direct calculation, we obtain

$$f_{A^\alpha_\mu} = -\delta\pi^\mu_\alpha + \sum_{s=1}^{2}\int d^3\mathbf{y}\frac{\delta\Omega^{(s)}_\beta(\mathbf{y})}{\delta A^\alpha_\mu}\delta\gamma^\beta_{(s)}(\mathbf{y}), \tag{4.106}$$

$$f_{\pi^\mu_\alpha} = \delta A^\alpha_\mu + \sum_{s=1}^{2}\int d^3\mathbf{y}\frac{\delta\Omega^{(s)}_\beta(\mathbf{y})}{\delta\pi^\mu_\alpha}\delta\gamma^\beta_{(s)}(\mathbf{y}), \tag{4.107}$$

$$f_{\gamma^\alpha} = -\int d^3\mathbf{y}\frac{\delta\Omega_\alpha}{\delta A^\beta_\mu(\mathbf{y})}\delta A^\beta_\mu(\mathbf{y}) - \int d^3\mathbf{y}\frac{\delta\Omega_\alpha}{\delta\pi^\mu_\beta(\mathbf{y})}\delta\pi^\mu_\beta(\mathbf{y}), \tag{4.108}$$

$$f_{\tilde{\gamma}^\alpha} = -\int d^3\mathbf{y}\frac{\delta\tilde{\Omega}_\alpha}{\delta A^\beta_\mu(\mathbf{y})}\delta A^\beta_\mu(\mathbf{y}) - \int d^3\mathbf{y}\frac{\delta\tilde{\Omega}_\alpha}{\delta\pi^\mu_\beta(\mathbf{y})}\delta\pi^\mu_\beta(\mathbf{y}). \tag{4.109}$$

The matrix element of $\mathcal{F}$ can then be obtained by taking interior product of eqs.(4.106)-(4.109) with respect to phase space coordinate basis as follows

$$\mathcal{F}_{IJ}(\mathbf{x},\mathbf{x}') = i_{\frac{\delta}{\delta\xi^J(\mathbf{x}')}}f_{\xi^I}(\mathbf{x}). \tag{4.110}$$

The matrix form of $\mathcal{F}$ is given by

$$\mathcal{F}(\mathbf{x},\mathbf{x}') = \begin{pmatrix} A(\mathbf{x},\mathbf{x}') & B(\mathbf{x},\mathbf{x}') \\ C(\mathbf{x},\mathbf{x}') & D(\mathbf{x},\mathbf{x}') \end{pmatrix}, \tag{4.111}$$

where

$$A(\mathbf{x},\mathbf{x}') = \begin{pmatrix} 0 & -\delta^{\beta}_{\alpha}\delta^{\mu}_{\nu} \\ \delta^{\alpha}_{\beta}\delta^{\nu}_{\mu} & 0 \end{pmatrix} \delta^{(3)}(\mathbf{x}-\mathbf{x}'), \tag{4.112}$$

$$B(\mathbf{x},\mathbf{x}') = \begin{pmatrix} \frac{\partial\Omega_{\beta}}{\partial\partial_{\mathcal{I}}A^{\alpha}_{\mu}}(\mathbf{x}') & \frac{\partial\tilde{\Omega}_{\beta}}{\partial\partial_{\mathcal{I}}A^{\alpha}_{\mu}}(\mathbf{x}') \\ \frac{\partial\Omega_{\beta}}{\partial\partial_{\mathcal{I}}\pi^{\mu}_{\alpha}}(\mathbf{x}') & \frac{\partial\tilde{\Omega}_{\beta}}{\partial\partial_{\mathcal{I}}\pi^{\mu}_{\alpha}}(\mathbf{x}') \end{pmatrix} \partial'_{\mathcal{I}}\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \tag{4.113}$$

$$C(\mathbf{x},\mathbf{x}') = -\begin{pmatrix} \frac{\partial\Omega_{\alpha}}{\partial\partial_{\mathcal{I}}A^{\beta}_{\nu}} & \frac{\partial\Omega_{\alpha}}{\partial\partial_{\mathcal{I}}\pi^{\nu}_{\beta}} \\ \frac{\partial\tilde{\Omega}_{\alpha}}{\partial\partial_{\mathcal{I}}A^{\beta}_{\nu}} & \frac{\partial\tilde{\Omega}_{\alpha}}{\partial\partial_{\mathcal{I}}\pi^{\nu}_{\beta}} \end{pmatrix} \partial_{\mathcal{I}}\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \tag{4.114}$$

$$D(\mathbf{x},\mathbf{x}') = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.115}$$

where $\partial'_{\mathcal{I}}$ are generalised derivatives with respect to $\mathbf{x}'$. One important steps of Faddeev-Jackiw constraint analysis is to find the determinant of $\mathcal{F}$. This determinant would also be useful when working out path integral quantisation as its square root would appear in the path integration measure. By the standard formula of determinant of block matrix, we have

$$\det\mathcal{F} = \det(A)\det(D - CA^{-1}B). \tag{4.116}$$

By direct calculation, it can be shown that $\det(A) = 1$. So in order to evaluate $\det\mathcal{F}$, one needs to first compute $(D - CA^{-1}B)$. Direct computation gives, after applying eq.(4.100) and eq.(4.103),

$$\begin{aligned}(D - CA^{-1}B)(\mathbf{x},\mathbf{x}') &= \begin{pmatrix} [\Omega_{\alpha},\Omega_{\beta}(\mathbf{x}')] & [\Omega_{\alpha},\tilde{\Omega}_{\beta}(\mathbf{x}')] \\ [\tilde{\Omega}_{\alpha},\Omega_{\beta}(\mathbf{x}')] & [\tilde{\Omega}_{\alpha},\tilde{\Omega}_{\beta}(\mathbf{x}')] \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\mathcal{C}_{0\alpha\beta} - \mathcal{C}^{i}_{1\alpha\beta}\partial_i \\ \mathcal{C}_{0\beta\alpha} - \mathcal{C}^{i}_{1\beta\alpha}(\mathbf{x}')\partial_i & \mathcal{D}^{\mathcal{I}}_{\alpha\beta}\partial_{\mathcal{I}} \end{pmatrix} \delta^{(3)}(\mathbf{x}-\mathbf{x}'),\end{aligned} \tag{4.117}$$

where $\mathcal{D}^{\mathcal{I}}_{\alpha\beta}$ are functions whose form are not relevant to the analysis of this thesis, so we do not provide its explicit form.

In order for $(D-CA^{-1}B)$ to be invertible, the solution w of $(D-CA^{-1}B)w = \psi$ should be unique. Let us denote $w(\mathbf{x}') \equiv (u^\beta(\mathbf{x}'), v^\beta(\mathbf{x}'))^T$, and $\psi(\mathbf{x}) \equiv (\chi_\alpha(\mathbf{x}), \lambda_\alpha(\mathbf{x}))^T$. So

$$-\left(\mathcal{C}_{0\alpha\beta} + \mathcal{C}^i_{1\alpha\beta}\partial_i\right) v^\beta = \chi_\alpha, \tag{4.118}$$

$$(\mathcal{C}_{0\alpha\beta} + \mathcal{C}^i_{1\alpha\beta}\partial_i)u^\beta + \mathcal{D}^{\mathcal{I}}_{\alpha\beta}\partial_{\mathcal{I}} v^\beta = \lambda_\alpha. \tag{4.119}$$

Solution to eq.(4.118) is

$$v^\beta = \int d^3\mathbf{x}' G^{\beta\gamma}(\mathbf{x},\mathbf{x}')\chi_\gamma(\mathbf{x}') + v^\beta_0, \tag{4.120}$$

where $G^{\beta\gamma}(\mathbf{x},\mathbf{x}')$ and v^β_0 satisfy

$$-\left(\mathcal{C}_{0\alpha\beta} + \mathcal{C}^i_{1\alpha\beta}\partial_i\right) G^{\beta\gamma}(\mathbf{x},\mathbf{x}') = \delta^\gamma_\alpha \delta^{(3)}(\mathbf{x}-\mathbf{x}'), \tag{4.121}$$

and

$$-\left(\mathcal{C}_{0\alpha\beta} + \mathcal{C}^i_{1\alpha\beta}\partial_i\right) v^\beta_0 = 0. \tag{4.122}$$

In order for v^β to be the unique solution to eq.(4.118), we demand that v^β_0 is unique. This is precisely the completion requirement (4.104).

In the case where $D - CA^{-1}B$ is invertible, the determinant of $\mathcal{F}$ can be determined. In this case, by direct calculation using the standard formula of determinant of block matrix and using the property of determinant of product of square matrices, one obtains

$$\det \mathcal{F} = \{\det[(\mathcal{C}_{0\alpha\beta} + \mathcal{C}^i_{1\alpha\beta}\partial_i)\delta^{(3)}(\mathbf{x}-\mathbf{x}')]\}^2. \tag{4.123}$$

Demanding that there is no zero mode of $\mathcal{F}$ at the second iteration is equivalent to demanding that $\det \mathcal{F} \neq 0$. So by using eq.(4.123), it can be seen that one should demand the differential operator $\mathcal{C}_{0\alpha\beta} + \mathcal{C}^i_{1\alpha\beta}\partial_i$ to have no zero mode. This also implies the completion requirement.

The class of theories we consider indeed include the particular theories investigated in [10], in which the conditions called "quantum consistency condition" are derived. The result of our thesis suggests that these conditions can indeed be generalised to a larger class of theories. The generalisation is simply the condition we called "completion requirement". The idea is that our differential operator $\mathcal{C}_{0\alpha\beta} + \mathcal{C}^i_{1\alpha\beta}\partial_i$ could be thought of as a generalisation to their differential operator $Z_{\alpha\beta}$. We have provided in eqs.(4.145)-(4.146) the formula to directly compute the coefficients $\mathcal{C}_{0\alpha\beta}$ and $\mathcal{C}^i_{1\alpha\beta}$, which in turn give rise the required differential operator. The quantum consistency condition derived in [10] is $Z_{\alpha\beta} \neq 0$. This seems to demand a differential operator to be non-zero. We suppose that it would be useful to give a slightly clearer interpretation. In particular, one should interpret it as being that the differential operator $Z_{\alpha\beta}$ has no zero mode. This is exactly generalised to our requirement.

Furthermore, by using diffeomorphism invariance requirement, we have shown that $\mathcal{C}_{0\beta\alpha} = \mathcal{C}_{0\alpha\beta} - \partial_i\mathcal{C}^i_{1\alpha\beta}$ and $\mathcal{C}^i_{1\alpha\beta} = -\mathcal{C}^i_{1\beta\alpha}$. This implies that $\mathcal{C}_{0\beta\alpha} - \mathcal{C}^i_{1\beta\alpha}(\mathbf{x}')\partial_i = \mathcal{C}_{0\alpha\beta} + \mathcal{C}^i_{1\alpha\beta}\partial_i$, which should be the generalisation to $-Z'_{\beta\alpha} = Z_{\alpha\beta}$ of the theories in [10]. This provides an explanation why the determinant of the symplectic two-form factorises as eq.(4.123). For example, in the particular theories of [10], the determinant reduces as $\det\mathcal{F} = (\det(Z\delta^{(3)}(\mathbf{x}-\mathbf{x}')))^2 = \det(Z\cdot Z\delta^{(3)}(\mathbf{x}-\mathbf{x}')) = \det(-Z'\cdot Z\delta^{(3)}(\mathbf{x}-\mathbf{x}'))$, in agreement, modulo a possible minor typographical error, with [10].

An immediate application is that if the theory passes the completion requirement, path integral quantisation can be carried out [12]. In particular, it is possible to read off

$$\sqrt{\det\mathcal{F}} = \det[(\mathcal{C}_{0\alpha\beta} + \mathcal{C}^i_{1\alpha\beta}\partial_i)\delta^{(3)}(\mathbf{x}-\mathbf{x}')], \tag{4.124}$$

which is an expression that appears in the measure of the generating functional in path integral quantisation.

4.3 Consistency check using Lagrangian constraint analysis

In the previous subsection, we have presented the criteria for which the theories of n vector fields with Lagrangian of the form eq.(4.49) would have $3n$ degrees of freedom, which corresponds to theories of multi-field generalised Proca. In short, the criteria is that the theory should transform in a standard way under diffeomorphism transformation and should satisfy eqs.(4.48), (4.70), and (4.104).

In this subsection, we present a consistency check of our result by using Lagrangian constraint analysis developed in [8], [9], [10], [11], and work out the equivalence between the conditions to be obtained in this section with those from the previous section.

In this analysis, it is convenient to define collective coordinates as follows. Let Q^M, Q^α, Q^A be collective for $A^\alpha_\mu, A^\alpha_0, A^\alpha_i$, respectively. The Lagrangian we are interested in is given by

$$\mathcal{L} = \mathcal{L}(Q^M, \dot{Q}^M, \partial_i Q^M, K), \tag{4.125}$$

Euler-Lagrange equations for vector fields are

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{Q}^M}\right) + \partial_i \left(\frac{\partial \mathcal{L}}{\partial \partial_i Q^M}\right) - \frac{\partial \mathcal{L}}{\partial Q^M} \\ &= W_{MN}\ddot{Q}^N + \alpha_M, \end{aligned} \tag{4.126}$$

where

$$W_{MN} \equiv \frac{\partial^2 \mathcal{L}}{\partial \dot{Q}^M \partial \dot{Q}^N}, \tag{4.127}$$

$$\begin{aligned} \alpha_M = &\frac{\partial^2 \mathcal{L}}{\partial \dot{Q}^M \partial Q^N}\dot{Q}^N + \frac{\partial^2 \mathcal{L}}{\partial \dot{Q}^M \partial \partial_i Q^N}\partial_i \dot{Q}^N + \frac{\partial^2 \mathcal{L}}{\partial \dot{Q}^M \partial K}\dot{K} \\ &+ \partial_i \left(\frac{\partial \mathcal{L}}{\partial \partial_i Q^M}\right) - \frac{\partial \mathcal{L}}{\partial Q^M}. \end{aligned} \tag{4.128}$$

Note that eqs.(4.126)-(4.128) suggest that higher time derivatives on the external fields in the Euler-Lagrange equations for vector fields can be present. This, however, is not problematic since the external fields are non-dynamical since they appear in these equations as predetermined functions. So their time derivatives are also predetermined functions. The special Hessian conditions eq.(4.48) give the following conditions on W_{MN} :

$$W_{\alpha N} = 0, \qquad \det(W_{AB}) \neq 0. \tag{4.129}$$

So Euler Lagrange equations (4.126) can be separated into equations of motion:

$$W_{AB}\ddot{Q}^B + \alpha_A = 0, \tag{4.130}$$

and primary constraints

$$\alpha_\alpha = 0. \tag{4.131}$$

Let M^{AB} be the inverse of W_{AB}. So the equations of motion imply

$$\ddot{Q}^A + M^{AB}\alpha_B = 0. \tag{4.132}$$

Time evolution of constraints is given by, after making use of eq.(4.132),

$$\dot{\alpha}_\alpha = \frac{\partial \alpha_\alpha}{\partial \dot{Q}^\beta}\ddot{Q}^\beta + \frac{\partial \alpha_\alpha}{\partial \partial_i \dot{Q}^\beta}\partial_i \ddot{Q}^\beta - \frac{\partial \alpha_\alpha}{\partial \dot{Q}^B}M^{BC}\alpha_C - \frac{\partial \alpha_\alpha}{\partial \partial_i \dot{Q}^B}\partial_i(M^{BC}\alpha_C) + \frac{\partial \alpha_\alpha}{\partial Q^M}\dot{Q}^M + \frac{\partial \alpha_\alpha}{\partial \partial_i Q^M}\partial_i \dot{Q}^M + \frac{\partial \alpha_\alpha}{\partial \partial_i \partial_j Q^M}\partial_i \partial_j \dot{Q}^M + \frac{\partial \alpha_\alpha}{\partial K}\dot{K}. \tag{4.133}$$

We demand that the process should not terminate at this stage. So the conditions $\dot{\alpha}_\alpha = 0$ should not introduce further dynamics on the vector fields. This means that the expressions with second order derivative in time of Q^β should not appear in eq.(4.133). These expressions are $\ddot{Q}^\beta$ and $\partial_i \ddot{Q}^\beta$. From direct calculation, their coefficients are

$$\frac{\partial \alpha_\alpha}{\partial \dot{Q}^\beta} \equiv \frac{\partial^2 \mathcal{L}}{\partial \dot{Q}^\alpha \partial Q^\beta} - \frac{\partial^2 \mathcal{L}}{\partial Q^\alpha \partial \dot{Q}^\beta} + \partial_i\left(\frac{\partial^2 \mathcal{L}}{\partial \partial_i Q^\alpha \partial \dot{Q}^\beta}\right), \tag{4.134}$$

and

$$\frac{\partial \alpha_\alpha}{\partial \partial_i \dot{Q}^\beta} = \frac{\partial^2 \mathcal{L}}{\partial \dot{Q}^\alpha \partial \partial_i Q^\beta} + \frac{\partial^2 \mathcal{L}}{\partial \dot{Q}^\beta \partial \partial_i Q^\alpha}. \tag{4.135}$$

By using a diffeomorphism condition (C.9), the coefficient of $\partial_i \ddot{Q}^\beta$ vanishes. So we are left with the terms with $\ddot{Q}^\beta$. In order for the coefficients of these terms to vanish, we should set

$$\frac{\partial \alpha_\alpha}{\partial \dot{Q}^\beta} = 0, \tag{4.136}$$

which turns out to be equivalent to eq.(4.70).

Two remarks are in order. The first is that the analysis in [8] does not show explicit dependence on spatial derivatives of fields. While this might be sufficient for the purpose of counting the number of degrees of freedom, the conditions derived in the process are not readily correct until time dependence on spatial derivatives of fields are re-introduced. From their analysis, the last term on RHS of eq.(4.134) is missing. This term could be considered as restoring spatial derivatives of fields. The second remark is that the reference [10] does not seem to mention the dependence of $\dot{\alpha}_\alpha$ on $\partial_i \ddot{Q}^\beta$ nor on whether their coefficients disappear. We have learned from the analysis above that diffeomorphism invariance requirement is crucial, at least in the case of multi-field generalised Proca theories that we are analysing, to make the the coefficients disappear. It would be interesting to see whether this behaviour is also the case in the analysis of more general theories given in [10].

Although Lagrangian constraint analysis is more advantageous than Hamiltonian constraint analysis in that it treats time and space on a more equal footing, the nature of constraint analysis still requires that time and space should be treated differently. For example, to see whether there are further constraints, only the time evolution is required. Some information on manifest covariance would then be lost. In order to recover them, one needs to make use of the fact that theories are diffeomorphism invariance (or, in the case of flat spacetime, Lorentz invariance).

Let us continue the analysis. By imposing eq.(4.136), we then have n secondary constraints $\phi_\alpha = \dot{\alpha}_\alpha \approx 0$. The next step is to consider the time evolution of ϕ_α. We demand that the condition $\dot{\phi}_\alpha \approx 0$ should not lead to further constraints.

For this, $\dot{\phi}_\alpha$ should contain terms with second order derivative in time on Q^β. These terms are

$$\frac{\partial \phi_\alpha}{\partial \dot{Q}^\beta}\ddot{Q}^\beta + \frac{\partial \phi_\alpha}{\partial \partial_i \dot{Q}^\beta}\partial_i \ddot{Q}^\beta + \frac{\partial \phi_\alpha}{\partial \partial_i \partial_j \dot{Q}^\beta}\partial_i \partial_j \ddot{Q}^\beta \in \dot{\phi}_\alpha. \tag{4.137}$$

The analysis in [10] does not mention terms with $\partial_i \ddot{Q}^\beta$ and $\partial_i \partial_j \ddot{Q}^\beta$. In principle, these terms are also crucial in determining whether the procedure should be terminated. Analysis of a particular case, for example in [57], also show the dependence of constraints on these terms, especially $\partial_i \ddot{Q}^\beta$.

Let us connect the result in this subsection with the analysis in phase space given in section 4.2. For this, we first show that by transforming to tangent bundle, $\tilde{\Omega}_\alpha = -\alpha_\alpha$. We start from eq.(4.81). Then by using $T = \mathcal{L} - U_\gamma \dot{Q}^\gamma$, and realising that U_α is independent of $\dot{Q}^M$, we obtain[3]

$$\tilde{\Omega}_\alpha = -\alpha_\alpha + \left(\frac{\partial U_\alpha}{\partial Q^\beta} - \frac{\partial U_\beta}{\partial Q^\alpha} + \partial_i \left(\frac{\partial U_\beta}{\partial \partial_i Q^\alpha}\right)\right)\dot{Q}^\beta + \left(\frac{\partial U_\alpha}{\partial \partial_i Q^\beta} + \frac{\partial U_\beta}{\partial \partial_i Q^\alpha}\right)\partial_i \dot{Q}^\beta. \tag{4.138}$$

The second and the third term on RHS vanish due to secondary-constraint enforcing relations (4.70) and diffeomorphism invariance requirement (C.9). This finally gives

$$\tilde{\Omega}_\alpha = -\alpha_\alpha, \tag{4.139}$$

as required. Then by following the calculations outlined in Appendix D, we obtain

$$\frac{\partial \phi_\alpha}{\partial \dot{Q}^\beta} = -\mathcal{C}_{0\alpha\beta}, \quad \frac{\partial \phi_\alpha}{\partial \partial_i \dot{Q}^\beta} = -\mathcal{C}^i_{1\alpha\beta}, \quad \frac{\partial \phi_\alpha}{\partial \partial_i \partial_j \dot{Q}^\beta} = 0. \tag{4.140}$$

Note in passing that the condition

$$\mathcal{C}_{0\alpha\beta} = \mathcal{C}_{0\beta\alpha} - \partial_i \mathcal{C}^i_{1\beta\alpha}, \tag{4.141}$$

which is also proven in Appendix D is crucial in the derivation of eq.(4.140).

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[3]It is understood that LHS of eq.(4.138) is actually the pullback of $\tilde{\Omega}_\alpha$ to tangent bundle. Throughout this chapter, we do not use different notations to distinguish the functions from their pullbacks as it should be clear from the context.

Therefore, time evolution of $\dot{\phi}_\alpha$ is of the form

$$\dot{\phi}_\alpha = -\left(\mathcal{C}_{0\alpha\beta} + \mathcal{C}^i_{1\alpha\beta}\partial_i\right)\ddot{Q}^\beta + \cdots , \tag{4.142}$$

where $\cdots$ are terms with up to first order in time derivative in Q^M. In order for $\dot{\phi} \approx 0$ not to lead to further constraints, we should demand that it is equivalent to

$$\ddot{Q}^\beta + \cdots = 0. \tag{4.143}$$

This would be possible only when the differential operator

$$\left(\mathcal{C}_{0\alpha\beta} + \mathcal{C}^i_{1\alpha\beta}\partial_i\right) \tag{4.144}$$

is invertible. Equivalently, this differential operator should have no zero mode. This would lead exactly to the completion requirements (4.104) given at the end of subsection 4.2.2.

We have seen that the analysis of Lagrangian constraint analysis agree with the Faddeev-Jackiw constraint analysis. In particular, the functions $\mathcal{C}_{0\alpha\beta}$ and $\mathcal{C}^i_{1\alpha\beta}$ appear in ones of the important conditions. Having worked with Lagrangian analysis, we are now in a position to express them in a more useful form. They are

$$\mathcal{C}_{0\alpha\beta} = -\frac{\partial\alpha_\alpha}{\partial A^\beta_0} - \partial_j\left(\frac{\partial\alpha^j_\alpha}{\partial\dot{A}^\beta_0}\right) + \frac{\partial\alpha^k_\gamma}{\partial\dot{A}^\alpha_0}M^{\gamma\delta}_{kl}\frac{\partial\alpha^l_\delta}{\partial\dot{A}^\beta_0}, \tag{4.145}$$

$$\mathcal{C}^i_{1\alpha\beta} = -\frac{\partial\alpha_\alpha}{\partial\partial_i A^\beta_0} - \frac{\partial\alpha^i_\alpha}{\partial\dot{A}^\beta_0} - \frac{\partial\alpha_\alpha}{\partial\dot{A}^\beta_i}, \tag{4.146}$$

where

$$\frac{\partial\alpha_\alpha}{\partial A^\beta_0} = \partial_\mu\left(\frac{\partial^2\mathcal{L}}{\partial\partial_\mu A^\alpha_0\partial A^\beta_0}\right) - \frac{\partial^2\mathcal{L}}{\partial A^\alpha_0\partial A^\beta_0}, \tag{4.147}$$

$$\frac{\partial\alpha^k_\gamma}{\partial\dot{A}^\beta_0} = \frac{\partial^2\mathcal{L}}{\partial A^\beta_0\partial\dot{A}^\gamma_k} + \partial_i\left(\frac{\partial^2\mathcal{L}}{\partial\dot{A}^\beta_0\partial\partial_i A^\gamma_k}\right) - \frac{\partial^2\mathcal{L}}{\partial\dot{A}^\beta_0\partial A^\gamma_k}, \tag{4.148}$$

$$\frac{\partial\alpha_\alpha}{\partial\partial_i A^\beta_0} = \partial_\mu\left(\frac{\partial^2\mathcal{L}}{\partial\partial_\mu A^\alpha_0\partial\partial_i A^\beta_0}\right) + 2\frac{\partial^2\mathcal{L}}{\partial\partial_i A^{[\alpha}_0\partial A^{\beta]}_0}. \tag{4.149}$$

4.4 Application of the sufficient conditions

In the previous sections, we have studied a class of theories of n vector fields, with a possiblity to couple to external fields. A theory in this class describes $n-$field generalised Proca system coupled to external fields if it passes the special Hessian condition (4.48), secondary-constraint enforcing relation (4.75), as well as the completion requirement which demands that eq.(4.101) contains no zero mode. The completion requirement is the most involved. In order to consider them, one needs to write down the expression of $\mathcal{C}_{0\alpha\beta}$ and $\mathcal{C}^{i}_{1\alpha\beta}$. Their explicit forms can be computed by using eqs.(4.145)-(4.149).

In this section, we will demonstrate the use of the criteria presented in sections 4.2-4.3. We provide a few examples of theories which pass these requirements, as well as an example theory which does not pass, but is previously incorrectly identified in the literature as being legitimate. These examples should be sufficient to serve the purpose. They are, however, far from exhaustive. We expect that many other theories passing these requirements are already presented in the literature, but some of them may have been previously misinterpreted.

4.4.1 Examples

4.4.1.1 Separable multi-field generalised Proca theories.

One of simple examples is the case where each of the n vector fields in the system does not couple to one another. The system is considered to be separated into n sub-systems of single vector field, possibly coupled to external fields. It could then be expected that one can simply separately apply the constraint analysis on on each sub-system. For example, an analysis of [13] confirms that as long as each sub-system describes a generalised Proca field, possibly coupled to external fields, then the vector sector has 3 degrees of freedom.

Direct use of the results presented in sections 4.2-4.3 can also easily be done. The Lagrangian of the example system takes the form

$$\mathcal{L} = \sum_{\alpha=1}^{n} \mathcal{L}_{(\alpha)}, \tag{4.150}$$

where for each $\alpha \in \{1, 2, \cdots, n\}$, the sub-Lagrangian $\mathcal{L}_{(\alpha)}$ is a function of only the αth vector field A^α_μ, its first order derivative $\partial_\mu A^\alpha_\nu$, and possibly external fields; but $\mathcal{L}_{(\alpha)}$ does not depend on the βth vector fields nor their derivatives if $\beta \neq \alpha$. After demanding that it satisfies the special Hessian condition (4.48), we obtain

$$\mathcal{L}_{(\alpha)} = T_{(\alpha)} + U_\alpha \dot{A}^\alpha_0 \quad \text{(no summation over } \alpha). \tag{4.151}$$

So we have

$$\frac{\partial U_\alpha}{\partial A^\beta_0} = \delta_{\alpha\beta} \frac{\partial U_\alpha}{\partial A^\alpha_0} \quad \text{(no summation over } \alpha), \tag{4.152}$$

and from eq.(C.9), we have

$$\frac{\partial U_\alpha}{\partial \partial_i A^\beta_0} = 0. \tag{4.153}$$

Therefore, the secondary-constraint enforcing relations are automatically satisfied. Next, since the derivative of $\mathcal{L}_{(\alpha)}$ with respect to A^β_μ or $\partial_\mu A^\beta_\nu$ vanish if $\alpha \neq \beta$, then $\mathcal{C}_{0\alpha\beta}$ and $\mathcal{C}^i_{1\alpha\beta}$ are diagonal matrices. In fact, since $\mathcal{C}^i_{1\alpha\beta} = -\mathcal{C}^i_{1\beta\alpha}$, we can conclude that $\mathcal{C}^i_{1\alpha\beta} = 0$. So we have

$$\mathcal{C}_{0\alpha\beta} = \mathcal{C}_{0\alpha\alpha}\delta_{\alpha\beta}, \quad \mathcal{C}^i_{1\alpha\beta} = 0, \quad \text{(no sum over } \alpha). \tag{4.154}$$

Then in order for eq.(4.101) to have no zero mode, we should require

$$\det(\mathcal{C}_{0\alpha\beta}) = \prod_{\alpha=1}^{n} \mathcal{C}_{0\alpha\alpha} \neq 0, \tag{4.155}$$

which is possible if $\mathcal{C}_{0\alpha\alpha} \neq 0$ for each $\alpha \in \{1, 2, \cdots, n\}$. This means that each sub-system has to be described by a generalised Proca field, possibly coupled to external fields.

4.4.1.2 A less trivial example.

Let us consider an example theory whose Lagrangian is of the form

$$\mathcal{L} = \mathcal{L}_2(A^\alpha_\mu, A^\alpha{}_{\mu\nu}, K), \tag{4.156}$$

where $A^\alpha{}_{\mu\nu} \equiv \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu$. It is one of the simplest forms of multi-field generalised Proca theories being presented in the literature, see for example [8], [9], [55], [56]. We confirm that the theory is indeed legitimate. For this theory,

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu A^\alpha_\nu} = -\frac{\partial \mathcal{L}}{\partial \partial_\nu A^\alpha_\mu} = 2\frac{\partial \mathcal{L}_2}{\partial A^\alpha{}_{\mu\nu}}. \tag{4.157}$$

This immediately gives $U_\alpha = 0$. So the secondary-constraint enforcing relations (4.75) is trivially satisfied. Furthermore, $\mathcal{C}_{0\alpha\beta}$ and $\mathcal{C}^i_{1\alpha\beta}$ are simplified to

$$\mathcal{C}_{0\alpha\beta} = \frac{\partial^2 \mathcal{L}_2}{\partial A^\alpha_0 \partial A^\beta_0} - 4\frac{\partial^2 \mathcal{L}_2}{\partial A^\gamma{}_{0j} \partial A^\alpha_0} M^{\gamma\delta}_{jk} \frac{\partial^2 \mathcal{L}_2}{\partial A^\delta{}_{0k} \partial A^\beta_0}, \quad \mathcal{C}^i_{1\alpha\beta} = 0. \tag{4.158}$$

It can be seen that, apart from some exceptions, $\det(\mathcal{C}_{0\alpha\beta}) \neq 0$. So the theory has the required number of degrees of freedom, and hence is an $n-$field generalised Proca theory.

A notable exception is when $\mathcal{L}$ is independent from $A^{\alpha_1}_0$ for $\alpha_1 \in \{1, 2, \cdots, r\}$, where $1 < r \leq n$. While the criteria provided in sections 4.2-4.3 can only be used to state that this exception is not an $n-$field generalised Proca theory, it should nevertheless intuitively be expected that it describes $(n - r)$ generalised Proca fields while the other r fields might be, provided that it passes some further criteria, generalised Maxwell fields. These criteria, if any, should arise when one considers multi-field generalised Maxwell-Proca theories. While [8], [9], [10] might have already provided the criteria for identifying multi-field generalised Maxwell-Proca theories, we have found in this work that even when restricted to purely (multi-field) Proca theories, their analysis seems to require some non-trivial refinements. So we expect that the refinements to the criteria of multi-field generalised Maxwell-Proca theories are needed. We leave this for future works.

Nevertheless, suppose that we have considered a Lagrangian $\mathcal{L}^{(1)}$ whose $\mathcal{C}^i_{1\alpha\beta}$, denoted $\mathcal{C}^i_{1\alpha\beta}(\mathcal{L}^{(1)})$, is zero while its $\mathcal{C}_{0\alpha\beta}$, denoted $\mathcal{C}_{0\alpha\beta}(\mathcal{L}^{(1)})$, is singular. It could still be possible to add to it another Lagrangian $\mathcal{L}^{(2)}$ with $\mathcal{C}^i_{1\alpha\beta}(\mathcal{L}^{(2)}) = 0$ so that the resulting Lagrangian $\mathcal{L}^{(1)}+\mathcal{L}^{(2)}$ might describe an $n-$field generalised Proca theory. This is because, due to eq.(4.146), $\mathcal{C}^i_{1\alpha\beta}$ is linear. So $\mathcal{C}^i_{1\alpha\beta}(\mathcal{L}^{(1)}+\mathcal{L}^{(2)}) = \mathcal{C}^i_{1\alpha\beta}(\mathcal{L}^{(1)})+\mathcal{C}^i_{1\alpha\beta}(\mathcal{L}^{(2)}) = 0$. On the other hand, due to the last term on RHS of eq.(4.145), $\mathcal{C}_{0\alpha\beta}$ is non-linear. So $\mathcal{C}_{0\alpha\beta}(\mathcal{L}^{(1)}+\mathcal{L}^{(2)}) = \mathcal{C}_{0\alpha\beta}(\mathcal{L}^{(1)})+\mathcal{C}_{0\alpha\beta}(\mathcal{L}^{(2)})+\text{non-linear}(\mathcal{L}^{(1)},\mathcal{L}^{(2)})$. Due to non-linearity of $\mathcal{C}_{0\alpha\beta}$ and of its determinant, it is likely that $\mathcal{C}_{0\alpha\beta}(\mathcal{L}^{(1)}+\mathcal{L}^{(2)})$ is not singular even if both $\mathcal{C}_{0\alpha\beta}(\mathcal{L}^{(1)})$ and $\mathcal{C}_{0\alpha\beta}(\mathcal{L}^{(2)})$ are singular. Of course, although highly likely to be the case, direct calculations are required in each case to confirm whether this is truly the case.

4.4.1.3 A legitimate theory previously misinterpreted.

In [56], actions for multiple vector fields are constructed by using a systematic approach which demands that the special Hessian condition is satisfied. In principle, this is not sufficient to give legitimate theories as further conditions, for example secondary-constraint enforcing relations, are required. The reference [8] points out that one of theories proposed in [56], does not pass secondary-constraint enforcing relations and hence contains extra degrees of freedom. The Lagrangian of this theory is

$$\mathcal{L} = -\frac{1}{4}A^{\alpha}{}_{\mu\nu}A_{\alpha}{}^{\mu\nu} - 4\lambda\left(A^{\alpha\sigma}A^{\beta}_{\sigma}\partial^{\mu}A^{\alpha}_{[\mu}\partial^{\nu}A^{\beta}_{\nu]} + A^{\alpha}_{[\mu}A^{\beta}_{\nu]}\partial^{\mu}A^{\alpha}_{\rho}\partial^{\nu}A^{\beta\rho}\right), \tag{4.159}$$

where λ is a non-zero constant. Actually, since secondary-constraint enforcing relations presented in [8] miss some terms in the expression, in principle, the interpretation being drawn should be revised.

Let us argue that in fact the theory (4.159) is legitimate. By direct calculation, one obtains

$$\frac{\partial^2\mathcal{L}}{\partial\dot{A}^{\alpha}_0\partial A^{\beta}_0} - \frac{\partial^2\mathcal{L}}{\partial\dot{A}^{\beta}_0\partial A^{\alpha}_0} = -8\lambda\partial_i(A^{\alpha}_{[0}A^{\beta}_{i]}) = -\partial_i\left(\frac{\partial^2\mathcal{L}}{\partial\dot{A}^{\beta}_0\partial\partial_i A^{\alpha}_0}\right), \tag{4.160}$$

which means that the secondary-constraint enforcing relation (4.75) is satisfied. Therefore, contrary to the interpretation given in [8], the theory eq.(4.159) has secondary constraints. Furthermore, this theory is in fact an $n-$field generalised Proca theory. To see this, one notes that by making direct computation one obtains

$$\mathcal{C}^i_{1\alpha\beta} = 0. \tag{4.161}$$

It can then be checked that if $\lambda \neq 0$, then $\det(\mathcal{C}_{0\alpha\beta}) \neq 0$. Therefore, the completion requirement (4.104) is satisfied.

Of course, the same conclusion can also be reached if one directly starts from the Lagrangian (4.159) and performs either Hamiltonian or Lagrangian constraint analysis.

We expect that there are also other theories presented in [56] which are legitimate but is previously incorrectly ruled out. A common feature for these theories is that

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{A}^\alpha_0 \partial A^\beta_0} - \frac{\partial^2 \mathcal{L}}{\partial \dot{A}^\beta_0 \partial A^\alpha_0} \neq 0, \tag{4.162}$$

which makes them incorrectly ruled out. So if $\partial^2\mathcal{L}/(\partial \dot{A}^\beta_0 \partial\partial_i A^\alpha_0) \neq 0$, then one might try to see if $-\partial_i(\partial^2\mathcal{L}/(\partial \dot{A}^\beta_0 \partial\partial_i A^\alpha_0))$ would cancel out with LHS of (4.162). If this is the case, then one can proceed to check the completion requirement.

4.4.1.4 An undesired theory previously misinterpreted.

After the reference [8] suggests that the special Hessian conditions are not sufficient, and that the secondary-constraint enforcing relations should be satisfied, theories are being proposed in the literature in order to satisfy the required relations. Notable examples are [8], [9], [55].

Let us argue that, by using a refined version of secondary-constraint enforcing relations, some of the theories in fact are undesired, i.e. they contain extra degrees of freedom. In particular, we explicitly show one example from [55]. This particular example has the Lagrangian of the form

$$\mathcal{L} = -2A^{\alpha}{}_{\mu\nu}S^{\beta\mu}{}_{\sigma}A_{\alpha\rho}A_{\beta\lambda}\epsilon^{\nu\sigma\rho\lambda} + S^{\alpha}{}_{\mu\nu}S^{\beta\nu}{}_{\sigma}A_{\alpha\rho}A_{\beta\lambda}\epsilon^{\mu\sigma\rho\lambda}, \tag{4.163}$$

where $S^{\alpha}{}_{\mu\nu} \equiv \partial_\mu A^\alpha_\nu + \partial_\nu A^\alpha_\mu$. By direct calculation, one obtains

$$\frac{\partial^2\mathcal{L}}{\partial\dot{A}^\alpha_0\partial A^\beta_0} - \frac{\partial^2\mathcal{L}}{\partial\dot{A}^\beta_0\partial A^\alpha_0} = 0 \neq -\partial_i\left(\frac{\partial^2\mathcal{L}}{\partial\dot{A}^\beta_0\partial\partial_i A^\alpha_0}\right). \tag{4.164}$$

Therefore, this theory is in fact undesired.

We expect that there are also other theories presented in the literature which contain extra degrees of freedom but is previously interpreted as being well-behaved. For these theories, $\partial^2\mathcal{L}/(\partial\dot{A}^\alpha_0\partial A^\beta_0) - \partial^2\mathcal{L}/(\partial\dot{A}^\beta_0\partial A^\alpha_0) = 0$. So if they are truly undesired, one should find that $-\partial_i(\partial^2\mathcal{L}/(\partial\dot{A}^\beta_0\partial\partial_i A^\alpha_0)) \neq 0$, which would violate the secondary-constraint enforcing relations (4.75).

4.4.2 Cosmological implications

Multi-field generalised Proca theories have been applied for example in [58], [59], [60] to explain cosmological phenomena. In some of these studies, the conditions presented by [8], [9] are taken into consideration. However, as we have been discussing, these conditions are incorrect and should be replaced by eq.(4.75). In principle, one should then investigate the validation of the cosmological implications presented in [58], [59], [60]. In this subsection, we discuss a direction for further investigations on these works.

In [58], a Lagrangian involving Einstein-Hilbert term, $SU(2)$ Yang-Mills term $\mathcal{L}_{YM}$, and a term called $\alpha\mathcal{L}^1_4$ where α is a constant is considered. Autonomous dynamical system analysis of this model in a homogeneous and isotropic background is studied which allows dark energy and primordial inflation to be discussed. While the dark energy case leads to an interesting result, the primordial inflation case is problematic as the model is strongly sensitive to initial conditions and the value of α. It is then suggest that one should also include a term $\kappa\mathcal{L}^2_4$, where κ is a constant, into the Lagrangian and see if the problem can be evaded.

Let us discuss whether the Lagrangian presented in [58] would pass the sufficient conditions in section 4.2. Note that for the theory in [58], gravity is dynamical whereas the sufficient conditions we have presented is useful when the gravity is non-dynamical. Nevertheless, a simple check can still be performed in the case of flat spacetime, in which case $\mathcal{L}_{YM}$ is a function of $A^{\alpha}_{\mu}, A^{\alpha}{}_{\mu\nu}$, whereas $\mathcal{L}^1_4$ is a function of $A^{\alpha}_{\mu}, \partial_{\mu}A^{\alpha}_{\nu}$ in such a way that $\partial^2\mathcal{L}^1_4/\partial\dot{A}^{\alpha}_0\partial A^{\beta}_0 = \partial^2\mathcal{L}^1_4/\partial\dot{A}^{\beta}_0\partial A^{\alpha}_0$, $\partial^2\mathcal{L}^1_4/(\partial\dot{A}^{\alpha}_0\partial\partial_i A^{\beta}_0) = 0$. So it can easily be seen from the discussion of subsection 4.4.1 that the theory in [58] pass the sufficient conditions.

It would also be interesting to investigate whether the suggestion to include the term $\kappa\mathcal{L}^2_4$ still valid, as far as our sufficient conditions are concerned. So let us also consider the case of flat spacetime. In this case, it can easily be seen that $\mathcal{L}_{YM} + \kappa\mathcal{L}^2_4$ is simply expressible as a summation of the Lagrangians (4.156) and (4.159). So indeed the term $\kappa\mathcal{L}^2_4$ can be included to extend the model of [58]. Note on the other hand that if one had used the criteria of [8], [9], the term $\kappa\mathcal{L}^2_4$ would have been incorrectly ruled out.

In [59], [60], cosmological implications of multi-field generalised Proca theories are also investigated. It turns out however that some terms of the Lagrangian, for example $\mathcal{L}^2_4$ presented in [58], has been incorrectly ruled out according to the criteria of [8], [9]. But as discussed in the previous paragraph, such a term in fact passes the criteria presented in section 4.2, so there is no problem with the number of degrees of freedom. It would be interesting to see for example the cosmological implication of the inclusion of $\mathcal{L}^2_4$ to the models of [59], [60].

CHAPTER V

CHIRAL FIELD THEORIES

In this chapter, we review Floreanini-Jackiw theory, Henneaux-Teitelboim formulation, $2p$-form Maxwell theory, and Sen formulation for chiral fields.

5.1 Conventions

In order to translate differential form to index notation, the followings are defined for definiteness. A differential $q-$form is expressed as

$$\omega_{(q)} = \frac{1}{q!} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_q} \omega_{\mu_q \cdots \mu_1}. \tag{5.1}$$

Interior products and exterior derivatives act from the right. The wedge product of all coordinate basis 1-form is

$$dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{4p+1} = d^{4p+2}x. \tag{5.2}$$

There are two Hodge star operators $*, \bar{*}$ corresponding to the two metrics $g, \bar{g}$. In this thesis, we only require the expression for the Hodge star operators acting on $(2p+1)$-forms. They are defined as

$$\begin{aligned} *(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{2p+1}}) = & \frac{dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{2p+1}}}{(2p+1)! \sqrt{-\det(g)}} \\ & \times g_{\nu_1 \rho_1} \cdots g_{\nu_{2p+1} \rho_{2p+1}} \\ & \times \epsilon^{\rho_1 \cdots \rho_{2p+1} \mu_1 \cdots \mu_{2p+1}}, \end{aligned} \tag{5.3}$$

$$\begin{aligned} \bar{*}(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{2p+1}}) = & \frac{dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{2p+1}}}{(2p+1)! \sqrt{-\det(\bar{g})}} \\ & \times \bar{g}_{\nu_1 \rho_1} \cdots \bar{g}_{\nu_{2p+1} \rho_{2p+1}} \\ & \times \epsilon^{\rho_1 \cdots \rho_{2p+1} \mu_1 \cdots \mu_{2p+1}}, \end{aligned} \tag{5.4}$$

where $\epsilon^{\mu_1 \cdots \mu_{4p+2}}$ is Levi-Civita symbol with $\epsilon^{01 \cdots (4p+1)} = 1$.

5.2 Floreanini-Jackiw theory

This subsection, we review Floreanini-Jackiw theory [21] which $p = 0$ case of Henneaux-Teitelboim formulation. Lagrangian is given by

$$\mathcal{L} = \frac{1}{4}\partial A\widetilde{\partial A} - \frac{1}{4}\widetilde{\partial A}^2, \tag{5.5}$$

where

$$\begin{aligned} \partial A &\equiv \partial_0 A = \dot{A}, \\ \widetilde{\partial A} &\equiv -\partial_1 A = -A'. \end{aligned} \tag{5.6}$$

Conjugate momentum is

$$\begin{aligned} \pi &= \frac{\partial\mathcal{L}}{\partial\partial A} \\ &= \frac{1}{4}\widetilde{\partial A} \\ &= -\frac{1}{4}A'. \end{aligned} \tag{5.7}$$

So one obtains primary constraint as

$$\Omega_1 \equiv \pi - \frac{1}{4}\widetilde{\partial A}. \tag{5.8}$$

Hamiltonian is given by

$$\begin{aligned} \mathcal{H} &= \pi\partial A - \mathcal{L} - \dot{\gamma}_1\Omega_1 \\ &= \pi\partial A - (\frac{1}{4}\partial A\widetilde{\partial A} - \frac{1}{4}\widetilde{\partial A}^2) - \dot{\gamma}_1(\pi - \frac{1}{4}\widetilde{\partial A}) \\ &= \frac{1}{4}\widetilde{\partial A}^2 - \dot{\gamma}_1(\pi - \frac{1}{4}\widetilde{\partial A}). \end{aligned} \tag{5.9}$$

First-order form of Lagrangian is

$$\begin{aligned} \mathcal{L}_{FOF} &= \pi\dot{A} - \mathcal{H} \\ &= \pi\dot{A} + \mathcal{L}_v + \dot{\gamma}_1(\pi - \frac{1}{4}\widetilde{\partial A}), \end{aligned} \tag{5.10}$$

where

$$\mathcal{L}_v = -\frac{1}{4}\widetilde{\partial A}^2. \tag{5.11}$$

Symplectic variables are

$$\zeta \equiv (A, \pi, \gamma_1). \tag{5.12}$$

Canonical one-form is given by

$$\mathcal{A} = \int dx(\pi \delta A + \Omega_1 \delta \gamma_1). \tag{5.13}$$

So symplectic two-form is

$$\mathcal{F} = \int dx(\delta \pi \wedge \delta A + \delta \Omega_1 \wedge \gamma_1). \tag{5.14}$$

Let us consider

$$i_z \mathcal{F} = \int dx\Big(z^{\pi} \delta A - z^{A} \delta \pi + (z^{\pi} - \frac{1}{4}\partial_1 z^{A})\delta\gamma_1 - z^{\gamma_1}\delta(\pi + \frac{1}{4}A')\Big). \tag{5.15}$$

Demanding $i_z \mathcal{F} = 0$ gives

$$z^{\pi} = 0, \tag{5.16}$$

$$\partial_1 z^{\gamma_1} = 0, \tag{5.17}$$

and

$$z^{A} = -z^{\gamma_1}. \tag{5.18}$$

Therefore, one obtains zero mode which is given by

$$z = \int dx z^{\gamma_1}\Big(\frac{\delta}{\delta\gamma_1} - \frac{\delta}{\delta A}\Big), \quad \text{and} \quad \partial_1 z^{\gamma_1} = 0. \tag{5.19}$$

It gives

$$i_z \delta \int dx \mathcal{L}_v = -\frac{1}{2}\int dx \partial_1 z^{\gamma_1} \partial_1 A = 0. \tag{5.20}$$

There is no new constraint.

5.3 Henneaux-Teitelboim formulation

This section we review Henneaux-Teitelboim formulation which its equation of motion leads to self-duality condition [22]. Let us consider $p = 1$ case. Lagrangian is given by

$$\mathcal{L} = \frac{3}{8}\partial_{[0}A_{ij]}\partial_m A_{np}\epsilon^{ijmnp} - \frac{3}{4}\partial_{[i}A_{jk]}\partial_i A_{jk}. \tag{5.21}$$

One obtains equation of motion which satisfy self-duality condition

$$F_{ijk} + \frac{1}{2}\epsilon^{ijkmn}F_{0mn} = 0. \tag{5.22}$$

Then, we consider any p in $4p+2$ dimensions. Lagrangian is given by

$$\begin{aligned}\mathcal{L} = &-\frac{(2p+1)}{4(2p)!}\partial_{[0}A_{i_1\cdots i_{2p}]}\partial_{j_1}A_{j_2\cdots j_{2p+1}}\epsilon^{i_1\cdots i_{2p}j_1\cdots j_{2p+1}} \\ &-\frac{1}{4(2p)!(2p)!}\partial_{j_1}A_{j_2\cdots j_{2p+1}}\epsilon^{i_1\cdots i_{2p}j_1\cdots j_{2p+1}}\partial_{k_1}A_{k_2\cdots k_{2p+1}}\epsilon^{i_1\cdots i_{2p}k_1\cdots k_{2p+1}},\end{aligned} \tag{5.23}$$

where

$$\begin{aligned}(\partial A)_{0i_1\cdots i_{2p}} &\equiv (2p+1)\partial_{[0}A_{i_1\cdots i_{2p}]} \\ &= \partial_0 A_{i_1\cdots i_{2p}} + (2p)\partial_{[i_1}A_{i_2\cdots i_{2p}]0}, \\ (\widetilde{\partial A})^{i_1\cdots i_{2p}} &\equiv -\frac{1}{(2p)!}\partial_{j_1}A_{j_2\cdots j_{2p+1}}\epsilon^{i_1\cdots i_{2p}j_1\cdots j_{2p+1}}.\end{aligned} \tag{5.24}$$

So

$$\begin{aligned}\mathcal{L} = &-\frac{(2p+1)}{4(2p)!}\partial_{[0}A_{i_1\cdots i_{2p}]}\partial_{j_1}A_{j_2\cdots j_{2p+1}}\epsilon^{i_1\cdots i_{2p}j_1\cdots j_{2p+1}} \\ &-\frac{(2p+1)}{4}\partial_{[k_1}A_{k_2\cdots k_{2p+1}]}\partial_{k_1}A_{k_2\cdots k_{2p+1}},\end{aligned} \tag{5.25}$$

where we use

$$\epsilon^{i_1\cdots i_{2p}j_1\cdots j_{2p+1}}\epsilon_{i_1\cdots i_{2p}k_1\cdots k_{2p+1}} = (2p)!(2p+1)!\delta^{j_1\cdots j_{2p+1}}_{k_1\cdots k_{2p+1}}, \tag{5.26}$$

and

$$\delta^{i_1\cdots i_{2p+1}}_{j_1\cdots j_{2p+1}} = \delta^{i_1}_{[j_1}\cdots\delta^{i_{2p+1}}_{j_{2p+1}]}. \tag{5.27}$$

Let us consider the terms that contain $A_{i_2\cdots i_{2p}0}$ by using product rule, one obtains

$$\begin{aligned}&\partial_{i_1}A_{i_2\cdots i_{2p}0}\partial_{j_1}A_{j_2\cdots j_{2p+1}}\epsilon^{i_1\cdots i_{2p}j_1\cdots j_{2p+1}} \\ &\quad= \partial_{i_1}(A_{i_2\cdots i_{2p}0}\partial_{j_1}A_{j_2\cdots j_{2p+1}}\epsilon^{i_1\cdots i_{2p}j_1\cdots j_{2p+1}}) - A_{i_2\cdots i_{2p}0}(\partial_{i_1}\partial_{j_1}A_{j_2\cdots j_{2p+1}})\epsilon^{i_1\cdots i_{2p}j_1\cdots j_{2p+1}} \\ &\quad= \partial_{i_1}(A_{i_2\cdots i_{2p}0}\partial_{j_1}A_{j_2\cdots j_{2p+1}}\epsilon^{i_1\cdots i_{2p}j_1\cdots j_{2p+1}}),\end{aligned} \tag{5.28}$$

where $A_{i_2\cdots i_{2p}0}$ appears in Lagrangian only though total derivative. Therefore

$$\mathcal{L} = -\frac{1}{4(2p)!}\partial_0 A_{i_1\cdots i_{2p}}\partial_{j_1}A_{j_2\cdots j_{2p+1}}\epsilon^{i_1\cdots i_{2p}j_1\cdots j_{2p+1}} - \frac{(2p+1)}{4}\partial_{[k_1}A_{k_2\cdots k_{2p+1}]}\partial_{k_1}A_{k_2\cdots k_{2p+1}}. \tag{5.29}$$

Conjugate momentum is

$$\pi^{i_1\cdots i_{2p}} = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_{i_1\cdots i_{2p}})} = -\frac{1}{4(2p)!}\partial_{j_1} A_{j_2\cdots j_{2p+1}}\epsilon^{i_1\cdots i_{2p}j_1\cdots j_{2p+1}}. \tag{5.30}$$

So one obtains primary constraint as

$$\Omega_1^{i_1\cdots i_{2p}} \equiv \pi^{i_1\cdots i_{2p}} + \frac{1}{4(2p)!}\partial_{j_1} A_{j_2\cdots j_{2p+1}}\epsilon^{i_1\cdots i_{2p}j_1\cdots j_{2p+1}}. \tag{5.31}$$

Hamiltonian is given by

$$\mathcal{H} = \pi^{i_1\cdots i_{2p}}\partial_0 A_{i_1 i_2} - \mathcal{L} - \dot{\gamma}_1\Omega_1 = \frac{(2p+1)}{4}\partial_{[k_1}A_{k_2\cdots k_{2p+1}]}\partial_{k_1}A_{k_2\cdots k_{2p+1}} - \dot{\gamma}_1\Omega_1, \tag{5.32}$$

where $\dot{\gamma}_1\Omega_1 = (\dot{\gamma}_1)_{i_1\cdots i_{2p}}\Omega_1^{i_1\cdots i_{2p}}$. First-order form of Lagrangian is

$$\mathcal{L}_{FOF} = \pi^{i_1\cdots i_{2p}}\partial_0 A_{i_1\cdots i_{2p}} - \mathcal{H} = \pi^{i_1\cdots i_{2p}}\partial_0 A_{i_1\cdots i_{2p}} + \mathcal{L}_v + \dot{\gamma}_1\Omega_1, \tag{5.33}$$

where

$$\mathcal{L}_v = -\frac{(2p+1)}{4}\partial_{[k_1}A_{k_2\cdots k_{2p+1}]}\partial_{k_1}A_{k_2\cdots k_{2p+1}}. \tag{5.34}$$

Symplectic variables are

$$\zeta \equiv (A_{i_1\cdots i_{2p}}, \pi^{i_1\cdots i_{2p}}, \gamma_1). \tag{5.35}$$

Canonical one-form is given by

$$\mathcal{A} = \int d^{4p+1}x(\pi^{i_1\cdots i_{2p}}\delta A_{i_1\cdots i_{2p}} + \Omega_1\delta\gamma_1). \tag{5.36}$$

So symplectic two-form is

$$\mathcal{F} = \int d^{4p+1}x(\delta\pi^{i_1\cdots i_{2p}}\wedge\delta A_{i_1\cdots i_{2p}} + \delta\Omega_1\wedge\gamma_1). \tag{5.37}$$

Let us consider

$$\begin{aligned} i_z\mathcal{F} = \int d^{4p+1}x(&z^{\pi^{i_1\cdots i_{2p}}}\delta A_{i_1\cdots i_{2p}} - \delta\pi^{i_1\cdots i_{2p}}z^{A_{i_1\cdots i_{2p}}} \\ &+ (z^{\pi^{i_1\cdots i_{2p}}} + \frac{1}{4(2p)!}\partial_{j_1}z^{A_{j_2\cdots j_{2p+1}}}\epsilon^{i_1\cdots i_{2p}j_1\cdots j_{2p+1}})\delta\gamma_1^{i_1\cdots i_{2p}} \\ &- (\delta\pi_{i_1\cdots i_{2p}} - \frac{1}{4(2p)!}\partial_{j_1}\delta A_{j_2\cdots j_{2p+1}}\epsilon^{i_1\cdots i_{2p}j_1\cdots j_{2p+1}})z^{\gamma_1^{i_1\cdots i_{2p}}}). \end{aligned} \tag{5.38}$$

Demanding $i_z\mathcal{F}=0$, gives

$$z^{\pi^{i_1\cdots i_{2p}}}+\frac{1}{4(2p)!}\partial_{j_1}z^{\gamma_1^{j_2\cdots j_{2p+1}}}\epsilon^{j_2\cdots j_{2p+1}j_1i_1\cdots i_{2p}}=0, \tag{5.39}$$

$$z^{\gamma_1^{i_1\cdots i_{2p}}}=-z^{A_{i_1\cdots i_{2p}}}, \tag{5.40}$$

and

$$z^{\pi^{i_1\cdots i_{2p}}}+\frac{1}{4(2p)!}\partial_{j_1}z^{A^{j_2\cdots j_{2p+1}}}\epsilon^{i_1\cdots i_{2p}j_1\cdots j_{2p+1}}=0. \tag{5.41}$$

Then one obtains

$$z^{\pi^{i_1\cdots i_{2p}}}=0, \tag{5.42}$$

and

$$\begin{aligned}\partial_{[i_1}z^{A_{i_2\cdots i_{2p+1}]}}&=0\\ z^{A_{i_2\cdots i_{2p+1}}}&=\partial_{[i_1}\zeta_{i_2\cdots i_{2p}]}.\end{aligned} \tag{5.43}$$

Zero mode is given by

$$\int d^{4p+1}xz^{A_{i_1\cdots i_{2p}}}\Big(\frac{\delta}{\delta A_{i_1\cdots i_{2p}}}+\frac{1}{4(2p)!}\epsilon^{j_2\cdots j_{2p+1}j_1i_1\cdots i_{2p}}\partial_{j_1}\frac{\delta}{\delta\pi_{j_2\cdots j_{2p+1}}}-\frac{\delta}{\delta\gamma_1^{i_1\cdots i_{2p}}}\Big). \tag{5.44}$$

Therefore, it gives

$$\begin{aligned}\Omega_2&=i_z\delta\int d^{4p+1}x\mathcal{L}_v\\ &=-\frac{(2p+1)}{2}i_z\int d^{4p+1}x\partial_{[k_1}A_{k_2\cdots k_{2p+1}]}\partial_{k_1}\delta A_{k_2\cdots k_{2p+1}}\\ &=-\frac{(2p+1)}{2}\int d^{4p+1}x\partial_{[k_1}A_{k_2\cdots k_{2p+1}]}\partial_{k_1}z^{A_{k_2\cdots k_{2p+1}}}\\ &=0.\end{aligned} \tag{5.45}$$

There is no new constraint.

5.3.1 Number of degree of freedom of Henneaux-Teitelboim theory in $4p+2$ dimemsions

As in subsection 3.3, one obtains the number of phase space variable $A_{i_1\cdots i_{2p}}$ equal to $\binom{4p+1}{2p}$ and $\pi^{i_1\cdots i_{2p}}$ also equal to $\binom{4p+1}{2p}$. Therefore, the number of phase space variables equal to $2\binom{4p+1}{2p}$. The number of constraints (5.31) is $\binom{4p+1}{2p}$. The number of zero mode at last iteration (5.43) is given by

$$\binom{4p+1}{2p-1} - \binom{4p+1}{2p-2} + \cdots - \binom{4p+1}{0} = \binom{4p}{2p-1}, \tag{5.46}$$

where using

$$\sum_{i=0}^{D}(-1)^i\binom{n}{i} = (-1)^D\binom{n-1}{D}, \tag{5.47}$$

where

$$n = 4p+1, \quad \text{and} \quad D = 2p-1. \tag{5.48}$$

The first term of eq.(5.46) is the number of components of ζ in eq.(5.43). Actually, ζ is not independent. If we consider ζ and $\zeta + d\kappa$, we also obtain the same z. So the number of ζ must be subtracted by the number of κ which is $\binom{4p+1}{2p-2}$ in eq.(5.46). However, κ is not all independent. If we consider κ and $\kappa + d\phi$, we also obtain the same $\zeta + d\kappa$. However, ϕ is not all independent, we must follow the above steps and repeat untill we get zero-form, namely the last term of left-hand side of eq.(5.46) $\binom{4p+1}{0}$. Therefore, the number of degree of freedom is given by

$$\begin{aligned} \text{Number of d.o.f.} &= \frac{1}{2}\left(n_{\text{ps}} - n_{\Omega} - n_z\right), \\ &= \frac{1}{2}\left(2\binom{4p+1}{2p} - \binom{4p+1}{2p} - \binom{4p}{2p-1}\right) \\ &= \frac{1}{2}\left(\binom{4p+1}{2p} - \binom{4p}{2p-1}\right) \\ &= \frac{1}{2}\binom{4p}{2p}, \end{aligned} \tag{5.49}$$

where using

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \tag{5.50}$$

where

$$n = 4p+1, \quad \text{and} \quad k = 2p. \tag{5.51}$$

5.4 2p-form Maxwell theory

This section, we review $2p$-form Maxwell theory in d dimensions. Lagrangian density is given by

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2(2p+1)!}F_{\mu_1\cdots\mu_{2p+1}}F^{\mu_1\cdots\mu_{2p+1}} \\ &= -\frac{1}{2(2p)!}F_{0\mu_2\cdots\mu_{2p+1}}F^{0\mu_2\cdots\mu_{2p+1}} - \frac{1}{2(2p+1)!}F_{i_1\cdots i_{2p+1}}F^{i_1\cdots i_{2p+1}},\end{aligned} \tag{5.52}$$

where

$$F_{\mu_1\cdots\mu_{2p+1}} = (2p+1)\partial_{[\mu_1}A_{\mu_2\cdots\mu_{2p+1}]}. \tag{5.53}$$

Let us consider

$$\begin{aligned}\delta\mathcal{L} &= -\frac{1}{(2p+1)!}F_{\mu_1\cdots\mu_{2p+1}}\delta F^{\mu_1\cdots\mu_{2p+1}} \\ &= -\frac{1}{(2p)!}F^{\mu_1\cdots\mu_{2p+1}}\partial_{[\mu_1}\delta A_{\mu_2\cdots\mu_{2p+1}]} \\ &= -\frac{1}{(2p)!}\Big((2p+1)F^{0\mu_2\cdots\mu_{2p+1}}\partial_{[0}\delta A_{\mu_2\cdots\mu_{2p+1}]} + \cdots\Big).\end{aligned} \tag{5.54}$$

Conjugate momentum is

$$\pi^{\mu_1\cdots\mu_{2p}} = \frac{\delta\mathcal{L}}{\delta\partial_0 A_{\mu_1\cdots\mu_{2p}}} = \frac{1}{(2p)!}F_{[0\mu_1\cdots\mu_{2p}]} \tag{5.55}$$

So one ontains primary constraint as

$$\Omega_1^{i_2\cdots i_{2p}} \equiv \pi^{0i_2\cdots i_{2p}}, \tag{5.56}$$

and generalised velocity as

$$\begin{aligned}\pi^{i_1\cdots i_{2p}} &= -\frac{1}{(2p)!}F^{0i_1\cdots i_{2p}} \\ &= \frac{1}{(2p)!}\Big(\partial_0 A_{i_1\cdots i_{2p}} - (2p)\partial_{[i_1}A_{|0|i_2\cdots i_{2p}]}\Big),\end{aligned} \tag{5.57}$$

$$\partial_0 A_{i_1\cdots i_{2p}} = (2p)!\pi^{i_1\cdots i_{2p}} + (2p)\partial_{[i_1}A_{|0|i_2\cdots i_{2p}]}. \tag{5.58}$$

Hamiltonian is given by

$$
\begin{aligned}
\mathcal{H} &= \pi^{\mu_1\cdots\mu_{2p}}\dot{A}_{\mu_1\cdots\mu_{2p}} - \mathcal{L} - \dot{\gamma}_1\Omega_1 \\
&= \pi^{0\mu_2\cdots\mu_{2p}}\dot{A}_{0\mu_2\cdots\mu_{2p}} + \pi^{i_1\cdots i_{2p}}\dot{A}_{i_1\cdots i_{2p}} \\
&\quad - \Big(- \frac{1}{2(2p)!}F_{0\mu_2\cdots\mu_{2p+1}}F^{0\mu_2\cdots\mu_{2p+1}} - \frac{1}{2(2p+1)!}F_{i_1\cdots i_{2p+1}}F^{i_1\cdots i_{2p+1}}\Big) - \dot{\gamma}_1\Omega_1 \\
&= \pi^{i_1\cdots i_{2p}}\Big((2p)!\pi^{i_1\cdots i_{2p}} + (2p)\partial_{i_1}A_{0i_2\cdots i_{2p}}\Big) \\
&\quad - \Big(\frac{(2p)!}{2}(\pi^{i_1\cdots i_{2p}})^2 - \frac{1}{2(2p+1)!}F_{i_1\cdots i_{2p+1}}F^{i_1\cdots i_{2p+1}}\Big) - \dot{\gamma}_1\Omega_1 \\
&= \frac{(2p)!}{2}(\pi^{i_1\cdots i_{2p}})^2 + (2p)\pi^{i_1\cdots i_{2p}}\partial_{i_1}A_{0i_2\cdots i_{2p}} + \frac{1}{2(2p+1)!}F_{i_1\cdots i_{2p+1}}F^{i_1\cdots i_{2p+1}} - \dot{\gamma}_1\Omega_1.
\end{aligned}
\tag{5.59}
$$

First-order form of Lagrangian is

$$
\begin{aligned}
\mathcal{L}_{FOF} &= \pi^{\mu_1\cdots\mu_{2p}}\dot{A}_{\mu_1\cdots\mu_{2p}} - \mathcal{H} \\
&= \pi^{\mu_1\cdots\mu_{2p}}\dot{A}_{\mu_1\cdots\mu_{2p}} + \mathcal{L}_v + \dot{\gamma}_1\Omega_1,
\end{aligned}
\tag{5.60}
$$

where

$$
\mathcal{L}_v = -\frac{(2p)!}{2}(\pi^{i_1\cdots i_{2p}})^2 - (2p)\pi^{i_1\cdots i_{2p}}\partial_{i_1}A_{0i_2\cdots i_{2p}} - \frac{1}{2(2p+1)!}F_{i_1\cdots i_{2p+1}}F^{i_1\cdots i_{2p+1}}. \tag{5.61}
$$

Symplectic variables are

$$
\xi^I = (A_{\mu_1\cdots\mu_{2p}}, \pi^{\mu_1\cdots\mu_{2p}}, \gamma_1). \tag{5.62}
$$

Canonical one-form is given by

$$
\mathcal{A} = \int d^{d-1}x(\pi^{\mu_1\cdots\mu_{2p}}\delta A_{\mu_1\cdots\mu_{2p}} + \Omega_1\delta\gamma_1). \tag{5.63}
$$

Symplectic two-form is

$$
\mathcal{F} = \int d^{d-1}x(\delta\pi^{\mu_1\cdots\mu_{2p}} \wedge \delta A_{\mu_1\cdots\mu_{2p}} + \delta\Omega_1 \wedge \delta\gamma_1). \tag{5.64}
$$

Let us consider

$$
\begin{aligned}
i_z\mathcal{F} = \int d^{d-1}x\Big(& z^{\pi^{\mu_1\cdots\mu_{2p}}}\delta A_{\mu_1\cdots\mu_{2p}} - z^{A_{\mu_1\cdots\mu_{2p}}}\delta\pi^{\mu_1\cdots\mu_{2p}} + z^{\pi^{0i_2\cdots i_{2p}}}\delta(\gamma_1)_{i_2\cdots i_{2p}} \\
& - (z^{\gamma_1})_{i_2\cdots i_{2p}}\delta\pi^{0i_2\cdots i_{2p}}\Big).
\end{aligned}
\tag{5.65}
$$

Demanding $i_z\mathcal{F}=0$, gives

$$z^{\pi^{\mu_1\cdots\mu_{2p}}}=0, \tag{5.66}$$

$$z^{A_{i_1\cdots i_{2p}}}=0, \tag{5.67}$$

and

$$(z^{\gamma_1})^{i_2\cdots i_{2p}}=-(2p)z^{A_{0i_2\cdots i_{2p}}}. \tag{5.68}$$

So zero mode is

$$\begin{aligned} z &= \int d^{d-1}x\Big(z^{A_{\mu_1\cdots\mu_{2p}}}\frac{\delta}{\delta A_{\mu_1\cdots\mu_{2p}}}+z^{\pi^{\mu_1\cdots\mu_{2p}}}\frac{\delta}{\delta\pi^{\mu_1\cdots\mu_{2p}}}+(z^{\gamma_1})^{i_2\cdots i_{2p}}\frac{\delta}{\delta(\gamma_1)^{i_2\cdots i_{2p}}}\Big)\\ &= (2p)\int d^{d-1}x z^{A_{0i_2\cdots i_{2p}}}\Big(\frac{\delta}{\delta A_{0i_2\cdots i_{2p}}}-\frac{\delta}{\delta(\gamma_1)_{i_2\cdots i_{2p}}}\Big). \end{aligned} \tag{5.69}$$

It gives new constraint as

$$i_z\delta\int d^{d-1}x\mathcal{L}_v=(2p)\int d^{d-1}x\partial_{i_1}\pi^{i_1\cdots i_{2p}}z^{A_{0i_2\cdots i_{2p}}}. \tag{5.70}$$

So one obtains secondary constraint which is given by

$$\Omega_2^{i_2\cdots i_{2p}}\equiv\partial_{i_1}\pi^{i_1\cdots i_{2p}}. \tag{5.71}$$

Then let us consider second iteration. First-order form of Lagrangian is given by

$$\mathcal{L}_{FOF}=\pi^{\mu_1\cdots\mu_{2p}}\dot{A}_{\mu_1\cdots\mu_{2p}}+\mathcal{L}_v+\dot{\gamma}_1\Omega_1+\dot{\gamma}_2\Omega_2. \tag{5.72}$$

Symplectic variables are

$$\xi^I=(A_{\mu_1\cdots\mu_{2p}},\pi^{\mu_1\cdots\mu_{2p}},\gamma_1,\gamma_2). \tag{5.73}$$

Canonical one-form is given by

$$\mathcal{A}=\int d^{d-1}x(\pi^{\mu_1\cdots\mu_{2p}}\delta A_{\mu_1\cdots\mu_{2p}}+\Omega_1\delta\gamma_1+\Omega_2\delta\gamma_2). \tag{5.74}$$

So symplectic two-form is

$$\mathcal{F}=\int d^{d-1}x(\delta\pi^{\mu_1\cdots\mu_{2p}}\wedge\delta A_{\mu_1\cdots\mu_{2p}}+\delta\Omega_1\wedge\delta\gamma_1+\delta\Omega_2\wedge\delta\gamma_2). \tag{5.75}$$

Let us consider

$$i_z\mathcal{F} = \int d^{d-1}x\Big(z^{\pi^{\mu_1\cdots\mu_{2p}}}\delta A_{\mu_1\cdots\mu_{2p}} - z^{A_{\mu_1\cdots\mu_{2p}}}\delta\pi^{\mu_1\cdots\mu_{2p}} + z^{\pi^{0i_2\cdots i_{2p}}}\delta(\gamma_1)^{i_2\cdots i_{2p}}$$
$$- (z^{\gamma_1})^{i_2\cdots i_{2p}}\delta\pi^{0i_2\cdots i_{2p}} + \partial_{i_1}z^{\pi^{i_1\cdots i_{2p}}}\delta(\gamma_2)^{i_2\cdots i_{2p}} - (z^{\gamma_2})^{i_2\cdots i_{2p}}\partial_{i_1}\delta\pi^{i_1\cdots i_{2p}}\Big). \tag{5.76}$$

Demanding $i_z\mathcal{F} = 0$, gives

$$z^{\pi^{\mu_1\cdots\mu_{2p}}} = 0, \tag{5.77}$$

$$\partial_{[i_1}(z^{\gamma_2})^{i_1\cdots i_{2p}]} = z^{A_{i_1\cdots i_{2p}}}, \tag{5.78}$$

and

$$(z^{\gamma_1})^{i_2\cdots i_{2p}} = -(2p)z^{A_{0i_2\cdots i_{2p}}}. \tag{5.79}$$

It gives zero mode

$$z = \int d^{d-1}x\Big(\partial_{i_1}(z^{\gamma_2})^{i_2\cdots i_{2p}}\frac{\delta}{\delta A_{i_1\cdots i_{2p}}} - \frac{1}{2p}(z^{\gamma_1})^{i_2\cdots i_{2p}}\frac{\delta}{\delta A_{0i_2\cdots i_{2p}}}$$
$$+ (z^{\gamma_1})^{i_2\cdots i_{2p}}\frac{\delta}{\delta(\gamma_1)^{i_2\cdots i_{2p}}} + (z^{\gamma_2})^{i_2\cdots i_{2p}}\frac{\delta}{\delta(\gamma_2)^{i_2\cdots i_{2p}}}\Big). \tag{5.80}$$

New constraint is given by

$$i_z\delta\int d^{d-1}x\mathcal{L}_v = \frac{1}{(2p)!}\int d^{d-1}x\partial_{i_1}\partial_{i_2}(z^{\gamma_2})^{i_3\cdots i_{2p+1}}F^{i_1\cdots i_{2p+1}} = 0, \tag{5.81}$$

so there is no new constraint generated.

5.4.1 Number of degree of freedom of 2*p*-form Maxwell theory

In this subsection, the number of degree of freedom of $2p$-form Maxwell theory in d dimensions is considered. One obtains the number of phase space variable $A_{\mu_1\cdots\mu_{2p}}$ equal to $\binom{d}{2p}$ and $\pi^{\mu_1\cdots\mu_{2p}}$ also equal to $\binom{d}{2p}$. Therefore, the number of phase space variables equal to $2\binom{d}{2p}$. The number of primary constraint $\pi^{0i_2\cdots i_{2p}}$ is $\binom{d-1}{2p-1}$. The number of secondary constraint is given by

$$\binom{d-1}{2p-1} - \binom{d-1}{2p-2} + \binom{d-1}{2p-3} - \cdots - \binom{d-1}{0} = \binom{d-2}{2p-1}. \tag{5.82}$$

The number of last iteration zero modes $(z^{\gamma_1})^{i_2\cdots i_{2p}}$ are $\binom{d-1}{2p-1}$ and $(z^{\gamma_2})^{i_2\cdots i_{2p}}$ are $\binom{d-2}{2p-1}$. Therefore, the numder of degree of freedom is given by

$$\begin{aligned}
\text{Number of d.o.f.} &= \frac{1}{2}\Big(n_{\text{ps}} - n_{\Omega} - n_z\Big),\\
&= \frac{1}{2}\left(2\binom{d}{2p} - 2\binom{d-1}{2p-1} - 2\binom{d-2}{2p-1}\right)\\
&= \binom{d}{2p} - \binom{d-1}{2p-1} - \binom{d-2}{2p-1}\\
&= \binom{d-1}{2p} - \binom{d-2}{2p-1}\\
&= \binom{d-2}{2p}.
\end{aligned} \tag{5.83}$$

5.5 Sen formulation for chiral fields

In this section, we review the Sen formulation for chiral $(2p)$-form in $4p+2$ dimensional spacetime which was inspired by string field theory [26], [27], [28], [29].

One important key feature of this formulation is that each Sen theory describes a system which is separated into two sectors uncoupled from each other. One sector is called the physical sector. It contains a chiral $(2p)$-form field, the standard metric g, as well as other physical external fields Ψ. The other sector is called the unphysical sector. It contains an unphysical chiral $2p$-form field and the unphysical metric $\bar{g}$.

Each Sen theory describes dynamics of a $(2p)$-form P and a $(2p+1)$-form Q which satisfies $Q = \bar{*}Q$, where $\bar{*}$ is the Hodge star with respect to an external unphysical metric $\bar{g}$. The Lagrangian of a general Sen theory is of the form

$$\begin{aligned}
\mathcal{L} = \frac{(2p)!}{2}\bigg(&\frac{1}{4}dP\wedge\bar{*}dP - Q\wedge dP\\
&+ \frac{2}{(2p)!}\mathcal{L}_I(Q,g,\bar{g},\Psi)d^{4p+2}x\bigg),
\end{aligned} \tag{5.84}$$

where

$$\delta_Q\mathcal{L}_I(Q,g,\bar{g},\Psi) = \frac{(2p)!}{2}\delta Q\wedge R(Q,g,\bar{g},\Psi), \tag{5.85}$$

and we have suppressed $d^{4p+2}x$ on LHS of eq.(5.85). This suppression will be adopted throughout this thesis. The field R is a $(2p+1)$-form which is anti-self-dual with respect to $\bar{*}$, i.e. $R = -\bar{*}R$.

$$\begin{aligned} Q \wedge R_+ &= \bar{*}Q \wedge R_+ \\ &= \bar{*}R_+ \wedge Q \\ &= R_+ \wedge Q \\ &= -Q \wedge R_+ \\ &= 0, \end{aligned} \tag{5.86}$$

and

$$\begin{aligned} Q \wedge R_- &= \bar{*}Q \wedge R_- \\ &= \bar{*}R_- \wedge Q \\ &= -R_+ \wedge Q \\ &= Q \wedge R_-. \end{aligned} \tag{5.87}$$

By using

$$A_p \wedge \bar{*}B_p = B_p \wedge \bar{*}A_p, \tag{5.88}$$

where any p-form in $4p+2$ dimensions.

By considering the equation of motion level [26], [27], [28], [29], it can be seen that the combination

$$H = Q - R \tag{5.89}$$

describes degrees of freedom of physical chiral $(2p)$-form field. The composite field H should satisfy a nonlinear self-duality condition which should only involve H and physical external fields g, Ψ. By using the self-duality condition, one can completely determine R.

It can be seen from the Lagrangian that P partly provides unphysical degrees of freedom due to the wrong sign of the kinetic term. At the equation of motion level, part of P can be combined with Q to form a field

$$H_{(s)} = Q + \frac{1}{2}dP + \frac{1}{2}\bar{*}dP, \tag{5.90}$$

which is closed: $dH_{(s)} = 0$. The field $H_{(s)}$ is the field strength of the unphysical chiral $(2p)$-form field. It is linear self-dual with respect to $\bar{g}$. The generalisation to nonlinear self-duality in the unphysical sector is possible [29]. However, since we will mostly focus on the physical sector in this work, we leave the unphysical sector with linear self-duality.

Note that[4] we are mainly interested in the system of the fields P, Q and treat $g, \bar{g}$, as well as Ψ as external fields. However, we keep in mind the framework in which g and $\bar{g}$ can both be promoted to be dynamical. This is one of the frameworks suggested by [29]. The other framework is to allow g to be promoted to be dynamical while always treating $\bar{g}$ as a fixed background.

It is important to discuss the validity of the framework we are following. In particular, there is an argument against $\bar{g}$ being promoted to be dynamical. The field equation of P and Q for the Lagrangian (5.84) are given by

$$\begin{aligned} \delta_P\mathcal{L} &= \frac{(2p)!}{2}\left(\frac{1}{4}\delta_p dP\wedge\bar{*}dP + \frac{1}{4}dP\wedge\bar{*}\delta_p dP - Q\wedge\delta_p dP\right) \\ &= \frac{(2p)!}{2}\left(\frac{1}{4}\delta_p dP\wedge\bar{*}dP + \frac{1}{4}\delta_p dP\wedge\bar{*}dP + \delta_p dP\wedge Q\right). \end{aligned} \tag{5.91}$$

Let us consider

$$d(\delta_p P\wedge Q) = \delta_p P\wedge dQ - \delta_p dP\wedge Q, \tag{5.92}$$

and

$$d(\delta_p P\wedge\bar{*}dP) = \delta_p P\wedge d(\bar{*}dP) - \delta_p dP\wedge\bar{*}dP. \tag{5.93}$$

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[4]We thank anonymous reviewers for the important question leading to the discussion in this paragraph and the next paragraph.

So it gives

$$\begin{aligned}\delta_p\mathcal{L} &= \frac{(2p)!}{2}\left(\frac{1}{2}(\delta_p P\wedge d(\bar{*}dP) - d(\delta_p P\wedge\bar{*}dP)) + \delta_p P\wedge dQ - d(\delta p P\wedge Q)\right)\\ &= \frac{(2p)!}{2}\left(\delta_p P\wedge(\frac{1}{2}d(\bar{*}dP) + dQ) + \text{tot.}\right).\end{aligned} \tag{5.94}$$

Therefore, equation of motion is given by

$$d\left(\frac{1}{2}\bar{*}dP + Q\right) = 0. \tag{5.95}$$

Then let us consider

$$\begin{aligned}\delta_Q\mathcal{L} &= \frac{(2p)!}{2}\left(-\delta Q\wedge dP\right) + \delta_Q\mathcal{L}_I\\ &= \frac{(2p)!}{2}\left(-\delta Q\wedge dP\right) + \frac{(2p)!}{2}\delta Q\wedge R\\ &= \frac{(2p)!}{2}\left(\delta Q\wedge(-dP + R)\right),\end{aligned} \tag{5.96}$$

and consider

$$\begin{aligned}\delta_Q\mathcal{L} &= \frac{(2p)!}{2}\left(\left(\frac{1+\bar{*}}{2}\right)\delta Q\wedge(-dP + R)\right)\\ &= \frac{(2p)!}{2}\left(\delta Q\wedge\left(\frac{1-\bar{*}}{2}\right)(-dP + R)\right).\end{aligned} \tag{5.97}$$

So equation of motion is

$$\begin{aligned}\left(\frac{1-\bar{*}}{2}\right)(-dP + R) &= 0\\ R &= \frac{1}{2}(dP - \bar{*}dP).\end{aligned} \tag{5.98}$$

By substituting eq.(5.98) into eq.(5.95), we obtain

$$d(Q - R) = 0. \tag{5.99}$$

Since $H = Q - R$ is also non-linear self-dual with respect to $*$, one then interprets H as the field strength of physical chiral $(2p)$-form field. The field H itself depends on $Q, g, \bar{g}, \Psi$. In particular, the dependence on $\bar{g}$ suggests that $\bar{g}$ which couples to unphysical chiral field also couples to physical chiral field. So if $\bar{g}$ is dynamical, then the physical chiral field would couple to unphysical chiral field, which is problematic.

We do not agree with this argument. The process of the substitution of eq.(5.98) into eq.(5.95) suggests that P is eliminated and is given in terms of Q. However, this could potentially give incorrect information on the degrees of freedom. In the equations (5.95)-(5.98), only P is differentiated twice with respect to time. On the other hand Q is differentiated at most once with respect to time. The initial value problem then suggests that while P and $\dot{P}$ can be freely determined at the initial time, the initial data of Q and $\dot{Q}$ is constrained. It is therefore better to eliminate Q as opposed to the elimination of P as what essentially carried out in the previous paragraph. An important upshot is that the physical chiral field is only coupled to g and external physical field Ψ while the unphysical chiral field is only coupled to $\bar{g}$. This suggests the complete separation of the two sectors, and hence even if $\bar{g}$ become dynamical it would not couple to the physical sector.

In order to make it completely clear that $\bar{g}$ can become dynamical without issue, one should consider the full system whose Lagrangian also contains the kinetic terms of g and $\bar{g}$ to see if $\bar{g}$ also decouples from the physical chiral field. Since the dependence of the Sen Lagrangian on g and $\bar{g}$ is complicated, the analysis is anticipated to be quite involved. We expect to attempt on this in a future work.

For definiteness, we will work in this thesis within the framework that $\bar{g}$ is dynamical. We will assume that this framework is valid. Even if it ultimately turns out that this is not valid, the results of this thesis will not be affected. This is because the other framework, in which $\bar{g}$ is fixed, is always valid and is less restrictive. For example, there is no problem with the coupling between $\bar{g}$ and the physical chiral field since $\bar{g}$ has no dynamics.

The role of physical and unphysical sectors can be interchanged [29] simply by introducing an overall minus sign to the Lagrangian. We will not consider this interchange in this thesis.

In [27], it has been shown that by using Hamiltonian analysis of a general

Sen theory with $\bar{g} = \eta$ that the Hamiltonian of the theory can be separated into the sum of the Hamiltonian of the two sectors. However, as we will show in subsection 6.2.1, when $\bar{g}$ is general a more specific form of $\mathcal{L}_I$ is required so that the separation at the Hamiltonian level is realised.

In the original Sen theory as well as various generalisations [28], [61], [62], the field $H_{(s)}$ is singlet $\delta H_{(s)} = 0$ under several kinds of transformation such as diffeomorphism, supersymmetry, kappa symmetry.

The reference [29] provides an insight on this feature at least when concerning diffeomorphism. Consider the case of general $\bar{g}$. There are in fact two types of diffeomorphism transformations[5]. The first type is called ζ-transformation. Physical fields transform as $\delta_\zeta^{\text{zeta}} H = \pounds_\zeta H, \delta_\zeta^{\text{zeta}} g = \pounds_\zeta g, \delta_\zeta^{\text{zeta}} \Psi = \pounds_\zeta \Psi$, where $\pounds_\zeta$ is the Lie derivative along vector field ζ. On the other hand, unphysical fields — $H_{(s)}$ and $\bar{g}$ — are singlet under this transformation. This then makes P and Q transform in a non-standard way. The second type is called χ-transformation. Physical fields are singlet under this transformation while unphysical fields transform as $\delta_\chi^{\text{chi}} H_{(s)} = \pounds_\chi H_{(s)}, \delta_\chi^{\text{chi}} \bar{g} = \pounds_\chi \bar{g}$. The standard diffeomorphism is the diagonal subgroup with $\zeta = \chi$. That is $\delta_\xi^{\text{standard}} = \delta_\xi^{\text{zeta}} + \delta_\xi^{\text{chi}} = \pounds_\xi$ when applied to any field, including P and Q.

In the case $\bar{g} = \eta$, the symmetry under χ-transformation and hence under standard diffeomorphism transformation is broken. Only the symmetry under ζ-transformation remains, giving rise to non-standard transformation rule of P, Q such that $H_{(s)}$ is singlet.

Consider a particular example theory in which H is linear self-dual with respect to the standard curved metric g and with the presence of the external

[5]At the Lagrangian level, these transformation rules are in fact more complicated than presented here. They also involve additional terms [28], [29] which vanish upon imposing equation of motion.

$(2p+1)$-form source J. The linear self-duality condition reads $H^J = *H^J$, where $H^J = Q - R + J$, with $*$ being the Hodge star with respect to the standard metric g. The Lagrangian for this theory is

$$\mathcal{L} = \frac{(2p)!}{2}\left(\frac{1}{4}dP\wedge\bar{*}dP - Q\wedge dP + \frac{1}{2}(Q+J)\wedge R + \frac{1}{2}Q\wedge J\right), \tag{5.100}$$

where

$$\delta_Q\left((Q+J)\wedge R + Q\wedge J\right) = 2\delta Q\wedge R. \tag{5.101}$$

This Lagrangian is invariant under both ζ- and χ-transformation [28], [29]. The invariance under gauge transformation on P is straightforward, whereas the invariance under gauge transformation on J is not realised unless the gauge transformation parameter is restricted [28] or appropriate $(4p+2)$-forms are added into the Lagrangian [29], [61].

Properties of more general theories, in which H is nonlinear self-dual is explored in section 6.2.

CHAPTER VI

NONLINEAR CHIRAL FORMS IN THE SEN FORMULATION

This chapter is largely based on [31]. We consider nonlinear chiral forms in the Sen formulation.

6.1 Conventions

In this section, we introduce useful conventions to be used in the analysis in this thesis. Although our main interest is at the Lagrangian level, it turns out that much insight is gained from the study at the Hamiltonian level. Technical complications arise from the complicated way that the fields P and Q are coupled to the standard metric g. The presence of another metric $\bar{g}$ also adds to the complication. The conventions are introduced to make the calculation at the Hamiltonian level more manageable.

6.1.1 Index notation

Let Greek indices for example $\mu, \nu, \rho, \cdots$ represent $(4p+2)$-dimensional spacetime indices. So they run in $0, 1, 2, \cdots, 4p+1$. Let Roman indices for example $a, b, c, i, j, k, \cdots$ represent spatial indices $0, 1, 2, \cdots, 4p+1$. So the index notation for physical and unphysical metrics are $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$, respectively. We denote $g^{\mu\nu}$ to be the matrix inverse of $g_{\mu\nu}$ as usual. Furthermore, we denote $\bar{g}^{\mu\nu}$ to be matrix inverse of $\bar{g}_{\mu\nu}$. Note that generically, $\bar{g}^{\mu\nu} \neq g^{\mu\rho}\bar{g}_{\rho\sigma}g^{\sigma\nu}$. Since the theory involves two metrics, raising and lowering of the indices will be carefully specified when needed.

Note that we preserve the notations g and $\bar{g}$ to refer to the metric tensors. The determinants of the matrix forms of these tensors will be denoted $\det(g)$ and $\det(\bar{g})$.

It is convenient to define

$$(\partial P)_{\mu_1\cdots\mu_{2p+1}} = (2p+1)\partial_{[\mu_1}P_{\mu_2\cdots\mu_{2p+1}]}, \tag{6.1}$$

where the antisymmetriser is defined with standard weight, for example, $A_{[ab]} = (A_{ab} - A_{ba})/2$. We also denote

$$\widetilde{\partial P}^{a_1\cdots a_{2p}} = -\frac{1}{(2p)!}\partial_{[i}P_{b_1\cdots b_{2p}]}\epsilon^{a_1\cdots a_{2p}ib_1\cdots b_{2p}}, \tag{6.2}$$

where $\epsilon^{i_1\cdots i_{4p+1}} = \epsilon^{0i_1\cdots i_{4p+1}}$ is Levi-Civita symbol for spatial part.

In order to work on Hamiltonian analysis, it is convenient to make use of Arnowitt-Deser-Misner (ADM) decomposition of both metrics. For the decomposition of standard metric, we have

$$\begin{aligned} &g_{00} = -N^2 + \gamma_{ij}N^iN^j, \qquad g_{0i} = g_{i0} = \gamma_{ij}N^j, \\ &g_{ij} = \gamma_{ij}, \end{aligned} \tag{6.3}$$

$$\begin{aligned} &g^{00} = -\frac{1}{N^2}, \qquad g^{0i} = g^{i0} = \frac{N^i}{N^2}, \\ &g^{ij} = \gamma^{ij} - \frac{N^iN^j}{N^2}, \end{aligned} \tag{6.4}$$

and

$$\sqrt{-\det(g)} = N\sqrt{\det(\gamma)}, \tag{6.5}$$

where γ^{ij} is matrix inverse of γ_{ij}. As for the decomposition of unphysical metric, it is given in a way similar to eqs.(6.3)-(6.5) where all the quantities becomes barred, for example $\bar{g}_{\mu\nu}, \bar{N}, \bar{N}^i, \bar{\gamma}_{ij}$, etc.

6.1.2 Multi-index notation and bra-ket notation

In the analysis, it will be convenient to make use of multi-index notation and Dirac bra-ket notation.

Let an index with angled bracket $\langle\cdot\rangle$ stands for the collection of $2p$ indices. For example $\langle a\rangle = (a_1\cdots a_{2p})$. Let an index with square bracket $[\cdot]$ stands for the collection of $2p+1$ indices. For example $[i] = (i_1 \cdots i_{2p+1})$, $[\mu] = (\mu_1 \cdots \mu_{2p+1})$. Furthermore,

$$dx^{\langle a \rangle} \equiv dx^{a_1} \wedge \cdots \wedge dx^{a_{2p}}, \tag{6.6}$$

$$dx^{[i]} \equiv dx^{i_1} \wedge \cdots \wedge dx^{i_{2p+1}}. \tag{6.7}$$

Using multi-index notation, we have

$$\begin{aligned} \bar{*}(dt \wedge dx^{\langle a \rangle}) = & -\frac{1}{(2p)!}\epsilon_{0\langle b \rangle [i]} w^{0\langle a \rangle, 0\langle b \rangle} dx^{[i]} \\ & + \frac{1}{(2p)!}\epsilon_{0\langle b \rangle [i]} w^{0\langle a \rangle, [i]} dt \wedge dx^{\langle b \rangle}, \end{aligned} \tag{6.8}$$

$$\begin{aligned} \bar{*}dx^{[i]} = & -\frac{1}{(2p)!}\epsilon_{0\langle b \rangle [j]} w^{0\langle b \rangle, [i]} dx^{[j]} \\ & + \frac{1}{(2p)!}\epsilon_{0\langle b \rangle [j]} w^{[i],[j]} dt \wedge dx^{\langle b \rangle}, \end{aligned} \tag{6.9}$$

where

$$w^{\mu_1 \cdots \mu_{2p+1}, \nu_1 \cdots \nu_{2p+1}} \equiv \sqrt{-\det(\bar{g})}\,\bar{g}^{[\mu_1|\nu_1|} \cdots \bar{g}^{\mu_{2p+1}]\nu_{2p+1}}, \tag{6.10}$$

where indices $\mu_1 \cdots \mu_{2p+1}$ are totally antisymmetrised and indices $\nu_1 \cdots \nu_{2p+1}$ are also totally antisymmetrised. So

$$w^{\mu_1 \cdots \mu_{2p+1}, \nu_1 \cdots \nu_{2p+1}} = w^{\nu_1 \cdots \nu_{2p+1}, \mu_1 \cdots \mu_{2p+1}}. \tag{6.11}$$

We also have

$$\widetilde{\partial P}^{\langle a \rangle} = -\frac{1}{(2p+1)!}\epsilon^{0\langle a \rangle [i]} (\partial P)_{[i]}. \tag{6.12}$$

Let us define

$$\hat{\bar{\epsilon}}_{\mu_1 \cdots \mu_{4p+2}} \equiv \sqrt{-\det(\bar{g})}\,\epsilon_{\mu_1 \cdots \mu_{4p+2}}, \tag{6.13}$$

which is the Levi-Civita tensor with respect to $\bar{g}$. Indices of $\hat{\bar{\epsilon}}$ are raised and lowered by the unphysical metric $\bar{g}$. In the calculations, we will often need the identity

$$\hat{\bar{\epsilon}}^{[\mu]}{}_{[\rho]}\hat{\bar{\epsilon}}_{[\mu]}{}^{[\nu]} = (2p+1)\hat{\bar{\epsilon}}^{0\langle a \rangle}{}_{[\rho]}\hat{\bar{\epsilon}}_{0\langle a \rangle}{}^{[\nu]} + \hat{\bar{\epsilon}}^{[i]}{}_{[\rho]}\hat{\bar{\epsilon}}_{[i]}{}^{[\nu]}. \tag{6.14}$$

Let us now make use of Dirac bra-ket notation. We will suppress the indices of the form $\langle a \rangle$ (which might also include index 0 when applicable). Quantities with one set of index $0\langle a\rangle$ are represented by ket or bra. In particular, we denote

$$\begin{aligned} |\epsilon^{[i]}\rangle &= \left(\frac{\epsilon^{0\langle a\rangle[i]}}{(2p+1)!}\right) = \langle\epsilon^{[i]}|, \\ |\epsilon_{[i]}\rangle &= \left(-\frac{\epsilon_{0\langle a\rangle[i]}}{(2p)!}\right) = \langle\epsilon_{[i]}|, \end{aligned} \tag{6.15}$$

$$\begin{aligned} |dT\rangle &= \left(dt\wedge dx^{\langle a\rangle}\right) = \langle dT|, \\ |w^{[i]}\rangle &= \left(w^{0\langle a\rangle,[i]}\right) = \langle w^{[i]}|. \end{aligned} \tag{6.16}$$

The normalisation of $|\epsilon^{[i]}\rangle$ and $|\epsilon_{[i]}\rangle$ are introduced for the convenience of inner and outer products. Quantities with two sets of index $0\langle a\rangle$ are suppressed and are considered as linear operators acting on bra or ket. In particular,

$$w = (w^{0\langle a\rangle,0\langle b\rangle}). \tag{6.17}$$

Note that the indices of the form $[i]$ are not suppressed. For convenience, we will denote

$$w^{[i][j]} \equiv w^{[i],[j]}. \tag{6.18}$$

The contraction of unsuppressed indices is defined as usual. For example, eq.(6.12) can be written as

$$|\widetilde{\partial P}\rangle = -|\epsilon^{[i]}\rangle(\partial P)_{[i]}. \tag{6.19}$$

The contraction of suppressed indices are defined such that one of the indices is upper while the other is lower. For example,

$$\langle\epsilon^{[i]}|\epsilon_{[j]}\rangle = -\frac{1}{(2p)!(2p+1)!}\epsilon^{0\langle a\rangle[i]}\epsilon_{0\langle a\rangle[j]} = \delta^{[i]}_{[j]}, \tag{6.20}$$

$$|\epsilon^{[i]}\rangle\langle\epsilon_{[i]}| = \left(-\frac{1}{(2p)!(2p+1)!}\epsilon^{0\langle a\rangle[i]}\epsilon_{0\langle b\rangle[i]}\right) = \mathbb{1}, \tag{6.21}$$

$$w|\epsilon_{[i]}\rangle = \left(-\frac{w^{0\langle a\rangle,0\langle b\rangle}\epsilon_{0\langle b\rangle[i]}}{(2p)!}\right), \tag{6.22}$$

whereas the quantity $w|\epsilon^{[i]}\rangle$ is undefined because the suppressed indices are all upper indices, so they cannot be contracted.

Let us separate the identity (6.14) into time and space components. It is convenient to separate $\hat{\bar{\epsilon}}^{[\mu]}{}_{[\nu]}$ as follows.

$$\begin{aligned}
\hat{\bar{\epsilon}}^{0\langle a\rangle}{}_{[k]} &= \hat{\bar{\epsilon}}^{0a_1\cdots a_{2p}}{}_{[k]} \\
&= \bar{g}^{0\mu_1}\bar{g}^{a_1\mu_2}\cdots\bar{g}^{a_{2p}\mu_{2p+1}}\hat{\bar{\epsilon}}_{\mu_1\cdots\mu_{2p+1}[k]} \\
&= \bar{g}^{0\mu_1}\bar{g}^{a_1\mu_2}\cdots\bar{g}^{a_{2p}\mu_{2p+1}}\sqrt{-\det(\bar{g})}\epsilon_{\mu_1\cdots\mu_{2p+1}[k]} \\
&= w^{0a_1\cdots a_{2p}\mu_1\cdots\mu_{2p+1}}\epsilon_{\mu_1\cdots\mu_{2p+1}[k]} \\
&= (2p+1)w^{0\langle a\rangle 0\langle b\rangle}\epsilon_{0\langle b\rangle[k]},
\end{aligned} \tag{6.23}$$

$$\begin{aligned}
\hat{\bar{\epsilon}}_{0\langle a\rangle}{}^{[j]} &= \hat{\bar{\epsilon}}_{0\langle a\rangle}{}^{j_1\cdots j_{2p+1}} \\
&= \bar{g}^{j_1\mu_1}\cdots\bar{g}^{j_{2p+1}\mu_{2p+1}}\hat{\bar{\epsilon}}_{0\langle a\rangle\mu_1\cdots\mu_{2p+1}} \\
&= \bar{g}^{j_1\mu_1}\cdots\bar{g}^{j_{2p+1}\mu_{2p+1}}\sqrt{-\det(\bar{g})}\epsilon_{0\langle a\rangle\mu_1\cdots\mu_{2p+1}} \\
&= w^{j_1\cdots j_{2p+1}\mu_1\cdots\mu_{2p+1}}\epsilon_{0\langle a\rangle\mu_1\cdots\mu_{2p+1}} \\
&= w^{[i][j]}\epsilon_{0\langle a\rangle[i]},
\end{aligned} \tag{6.24}$$

$$\begin{aligned}
\hat{\bar{\epsilon}}^{[i]}{}_{[k]} &= \hat{\bar{\epsilon}}^{i_1\cdots i_{2p+1}}{}_{[k]} \\
&= \bar{g}^{i_1\mu_1}\cdots\bar{g}^{i_{2p+1}\mu_{2p+1}}\hat{\bar{\epsilon}}_{\mu_1\cdots\mu_{2p+1}[k]} \\
&= \bar{g}^{i_1\mu_1}\cdots\bar{g}^{i_{2p+1}\mu_{2p+1}}\sqrt{-\det(\bar{g})}\epsilon_{\mu_1\cdots\mu_{2p+1}[k]} \\
&= (2p+1)w^{[i]0\langle a\rangle}\epsilon_{0\langle a\rangle[k]},
\end{aligned} \tag{6.25}$$

$$\begin{aligned}
\hat{\bar{\epsilon}}_{0\langle a\rangle}{}^{0\langle b\rangle} &= \hat{\bar{\epsilon}}_{0\langle a\rangle}{}^{0b_1\cdots b_{2p}} \\
&= \bar{g}^{0\mu_1}\cdots\bar{g}^{b_{2p}\mu_{2p+1}}\hat{\bar{\epsilon}}_{0\langle a\rangle\mu_1\cdots\mu_{2p+1}} \\
&= \bar{g}^{0\mu_1}\cdots\bar{g}^{b_{2p}\mu_{2p+1}}\sqrt{-\det(\bar{g})}\epsilon_{0\langle a\rangle\mu_1\cdots\mu_{2p+1}} \\
&= w^{0b_1\cdots b_{2p}\mu_1\cdots\mu_{2p+1}}\epsilon_{0\langle a\rangle\mu_1\cdots\mu_{2p+1}} \\
&= w^{0\langle b\rangle[i]}\epsilon_{0\langle a\rangle[i]}.
\end{aligned} \tag{6.26}$$

The identity (6.14) can then be separated into four cases as follows.

Case I: $\nu = j, \rho = k$

$$\hat{\hat{\epsilon}}^{[\mu]}{}_{[k]}\hat{\hat{\epsilon}}_{[\mu]}{}^{[j]} = (2p+1)^2 w^{0\langle a\rangle 0\langle b\rangle}\epsilon_{0\langle b\rangle[k]}w^{[i][j]}\epsilon_{0\langle a\rangle[i]} - (2p+1)^2 w^{[i]0\langle a\rangle}\epsilon_{0\langle a\rangle[k]}w^{[j]0\langle b\rangle}\epsilon_{0\langle b\rangle[i]}$$
$$= (2p)!(2p)!(2p+1)^2\left(\langle\epsilon_{[i]}|w|\epsilon_{[k]}\rangle w^{[i][j]} - \langle w^{[i]}|\epsilon_{[k]}\rangle\langle w^{[j]}|\epsilon_{[i]}\rangle\right), \tag{6.27}$$

and

$$\hat{\hat{\epsilon}}^{[\mu]}{}_{[k]}\hat{\hat{\epsilon}}_{[\mu]}{}^{[j]} = -(2p+1)!(2p+1)!\delta^{[j]}_{[k]}. \tag{6.28}$$

So we obtain

$$\langle\epsilon_{[i]}|w|\epsilon_{[k]}\rangle w^{[i][j]} - \langle w^{[i]}|\epsilon_{[k]}\rangle\langle w^{[j]}|\epsilon_{[i]}\rangle = -\delta^{[j]}_{[k]}. \tag{6.29}$$

Case II: $\nu = 0\langle b\rangle, \rho = k$

$$\hat{\hat{\epsilon}}^{[\mu]}{}_{[k]}\hat{\hat{\epsilon}}_{[\mu]}{}^{0\langle b\rangle} = (2p+1)\hat{\hat{\epsilon}}^{0\langle a\rangle}{}_{[k]}\hat{\hat{\epsilon}}_{0\langle a\rangle}{}^{0\langle b\rangle} + \hat{\hat{\epsilon}}^{[i]}{}_{[k]}\hat{\hat{\epsilon}}_{[i]}{}^{0\langle b\rangle}$$
$$= (2p)!(2p)!(2p+1)^2\left(|w^{[i]}\rangle\langle\epsilon_{[i]}|w|\epsilon_{[k]}\rangle - \langle w^{[i]}|\epsilon_{[k]}\rangle w|\epsilon_{[i]}\rangle\right), \tag{6.30}$$

and

$$\hat{\hat{\epsilon}}^{[\mu]}{}_{[k]}\hat{\hat{\epsilon}}_{[\mu]}{}^{0\langle b\rangle} = -(2p+1)!(2p+1)!\delta^{0\langle b\rangle}_{[k]} = 0. \tag{6.31}$$

So we obtain

$$|w^{[i]}\rangle\langle\epsilon_{[i]}|w|\epsilon_{[k]}\rangle - w|\epsilon_{[i]}\rangle\langle w^{[i]}|\epsilon_{[k]}\rangle = 0. \tag{6.32}$$

Case III: $\nu = j, \rho = 0\langle b\rangle$

$$\hat{\hat{\epsilon}}^{[\mu]}{}_{0\langle b\rangle}\hat{\hat{\epsilon}}_{[\mu]}{}^{[j]} = (2p+1)\hat{\hat{\epsilon}}^{0\langle a\rangle}{}_{0\langle b\rangle}\hat{\hat{\epsilon}}_{0\langle a\rangle}{}^{[j]} + \hat{\hat{\epsilon}}^{[i]}{}_{0\langle b\rangle}\hat{\hat{\epsilon}}_{[i]}{}^{[j]}$$
$$= (2p)!(2p)!(2p+1)\left(w^{[i][k]}\langle\epsilon_{[i]}|w^{[j]}\rangle|\epsilon_{[k]}\rangle - w^{[i][j]}\langle\epsilon_{[i]}|w^{[k]}\rangle|\epsilon_{[k]}\rangle\right), \tag{6.33}$$

and

$$\hat{\hat{\epsilon}}^{[\mu]}{}_{0\langle b\rangle}\hat{\hat{\epsilon}}_{[\mu]}{}^{[j]} = -(2p+1)!(2p+1)!\delta^{[j]}_{0\langle b\rangle} = 0. \tag{6.34}$$

So we obtain

$$w^{[i][k]}\langle\epsilon_{[i]}|w^{[j]}\rangle|\epsilon_{[k]}\rangle - w^{[i][j]}\langle\epsilon_{[i]}|w^{[k]}\rangle|\epsilon_{[k]}\rangle = 0. \tag{6.35}$$

Case IV: $\nu = 0\langle a\rangle, \rho = 0\langle b\rangle$

$$\hat{\hat{\epsilon}}^{[\mu]}{}_{0\langle b\rangle}\hat{\hat{\epsilon}}_{[\mu]}{}^{0\langle a\rangle} = (2p+1)\hat{\hat{\epsilon}}^{0\langle c\rangle}{}_{0\langle b\rangle}\hat{\hat{\epsilon}}_{0\langle c\rangle}{}^{0\langle a\rangle} + \hat{\hat{\epsilon}}^{[i]}{}_{0\langle b\rangle}\hat{\hat{\epsilon}}_{[i]}{}^{0\langle a\rangle}$$
$$= (2p+1)\Big(w^{0\langle c\rangle[i]}\epsilon_{[i]0\langle b\rangle}w^{0\langle a\rangle[j]}\epsilon_{0\langle c\rangle[j]} + w^{[i][j]}\epsilon_{[j]0\langle b\rangle}w^{0\langle a\rangle 0\langle c\rangle}\epsilon_{[i]0\langle c\rangle}\Big)$$
$$= (2p)!(2p+1)!\Big(w^{[j][i]}w|\epsilon_{[j]}\rangle - |w^{[j]}\rangle\langle w^{[i]}|\epsilon_{[j]}\rangle\Big)\langle\epsilon_{[i]}|, \tag{6.36}$$

and

$$\hat{\hat{\epsilon}}^{[\mu]}{}_{0\langle b\rangle}\hat{\hat{\epsilon}}_{[\mu]}{}^{0\langle a\rangle} = -(2p)!(2p+1)!\delta^{\langle a\rangle}_{\langle b\rangle}. \tag{6.37}$$

So we obtain

$$w|\epsilon_{[j]}\rangle w^{[i][j]} = -|\epsilon^{[i]}\rangle + |w^{[j]}\rangle\langle w^{[i]}|\epsilon_{[j]}\rangle. \tag{6.38}$$

The identity (6.14) implies

$$w|\epsilon_{[j]}\rangle w^{[i][j]} = -|\epsilon^{[i]}\rangle + |w^{[j]}\rangle\langle w^{[i]}|\epsilon_{[j]}\rangle, \tag{6.39}$$

and

$$|\epsilon_{[i]}\rangle\langle w^{[i]}|\epsilon_{[k]}\rangle w^{[j][k]} = |\epsilon_{[k]}\rangle\langle w^{[j]}|\epsilon_{[i]}\rangle w^{[i][k]}. \tag{6.40}$$

Let $m_{[i][j]}$ be the inverse of $w^{[i][j]}$, that is

$$m_{[i][j]}w^{[j][k]} = \delta^{[k]}_{[i]}. \tag{6.41}$$

With this, eqs.(6.39)-(6.40) are respectively equivalent to

$$w = -|\epsilon^{[i]}\rangle m_{[i][j]}\langle\epsilon^{[j]}| + |w^{[j]}\rangle m_{[i][k]}\langle\epsilon^{[k]}|\langle w^{[i]}|\epsilon_{[j]}\rangle, \tag{6.42}$$

$$\langle w^{[j]}|\epsilon_{[k]}\rangle m_{[j][i]} = \langle w^{[j]}|\epsilon_{[i]}\rangle m_{[j][k]}. \tag{6.43}$$

In order to write Q and R, it is convenient to work out the eigenforms of $\bar{*}$ of the form. Let us define $\xi^{\pm}$ to be (2p+1)-forms such that $\xi^{\pm} = \pm\bar{*}\xi^{\pm}$. We use $|\xi^{\pm}_{\downarrow}\rangle$ to represent $\xi^{\pm}_{0\langle a\rangle}$ and $|\xi^{\pm}_{\uparrow}\rangle$ to represent $\xi^{\pm 0\langle a\rangle}$. From self-duality condition, we obtain

$$\xi^{(\pm)[\mu]} = \frac{\pm 1}{(2p+1)!}\frac{\epsilon^{[\mu][\nu]}}{\sqrt{-\det(\bar{g})}}\xi^{(\pm)}{}_{[\nu]}. \tag{6.44}$$

Raising index of $\xi^{\pm}_{[\mu]}$ can be expressed using $w^{[\mu][\nu]}$ in eq.(6.10) as

$$\xi^{(\pm)[\mu]} = \frac{w^{[\mu][\nu]}}{\sqrt{-\det(\bar{g})}}\xi^{(\pm)}{}_{[\nu]}. \tag{6.45}$$

Let us consider the case $[\mu] = [i]$

$$\begin{aligned} \xi^{(\pm)[i]} &= \frac{\pm 1}{(2p+1)!}(2p+1)\frac{\epsilon^{[i]0\langle a\rangle}}{\sqrt{-\det(\bar{g})}}\xi^{(\pm)}{}_{0\langle a\rangle} \\ &= \frac{\mp}{(2p)!}\frac{\epsilon^{0\langle a\rangle[i]}}{\sqrt{-\det(\bar{g})}}\xi^{(\pm)}{}_{0\langle a\rangle} \\ &= \mp\frac{(2p+1)}{\sqrt{-\det(\bar{g})}}\langle\epsilon^{[i]}|\xi^{\pm}_{\downarrow}\rangle, \end{aligned} \tag{6.46}$$

and

$$\begin{aligned} \xi^{(\pm)[i]} &= (2p+1)\frac{w^{[i]0\langle a\rangle}}{\sqrt{-\det(\bar{g})}}\xi^{(\pm)}{}_{0\langle a\rangle} + \frac{w^{[i][j]}}{\sqrt{-\det(\bar{g})}}\xi^{(\pm)}{}_{[j]} \\ &= (2p+1)\frac{\langle w^{[i]}|\xi^{(\pm)}_{\downarrow}\rangle}{\sqrt{-\det(\bar{g})}} + \frac{w^{[i][j]}}{\sqrt{-\det(\bar{g})}}\xi^{(\pm)}{}_{[j]}. \end{aligned} \tag{6.47}$$

Equating eq.(6.46) with eq.(6.47) gives

$$\frac{w^{[i][j]}}{\sqrt{-\det(\bar{g})}}\xi^{(\pm)}{}_{[j]} = \mp\frac{(2p+1)}{\sqrt{-\det(\bar{g})}}\langle\epsilon^{[i]}|\xi^{(\pm)}_{\downarrow}\rangle - \frac{(2p+1)}{\sqrt{-\det(\bar{g})}}\langle w^{[i]}|\xi^{(\pm)}_{\downarrow}\rangle. \tag{6.48}$$

So it gives

$$\xi^{(\pm)}{}_{[i]} = (2p+1)m_{[i][j]}\Big(- \langle w^{[j]}| \mp \langle\epsilon^{[j]}|\Big)|\xi^{(\pm)}_{\downarrow}\rangle, \tag{6.49}$$

which can be written as

$$\xi^{(\pm)}{}_{[i]} = (2p+1)\langle\epsilon_{[i]}|W_{\pm}|\xi^{(\pm)}_{\downarrow}\rangle, \tag{6.50}$$

where the symmetric $W_{\pm}$ is given by

$$\begin{aligned} W^T_{\pm} &= W_{\pm} \\ &= \mp|\epsilon^{[i]}\rangle m_{[i][j]}\langle\epsilon^{[j]}| - |w^{[i]}\rangle m_{[i][j]}\langle\epsilon^{[j]}|. \end{aligned} \tag{6.51}$$

Then consider the case $[\mu] = 0\,\langle a\rangle$

$$\xi^{(\pm)0\langle a\rangle} = \pm\frac{1}{(2p+1)!}\frac{\epsilon^{0\langle a\rangle[i]}}{\sqrt{-\det(\bar{g})}}\xi^{(\pm)}_{[i]}. \tag{6.52}$$

It can be written in braket notation as

$$|\xi_{\uparrow}^{(\pm)}\rangle = \pm\frac{|\epsilon^{[i]}\rangle}{\sqrt{-\det(\bar{g})}}\xi_{[i]}^{(\pm)}. \tag{6.53}$$

Substituting eq.(6.50) into eq.(6.53), we obtain

$$|\xi_{\uparrow}^{(\pm)}\rangle = \pm\frac{(2p+1)}{\sqrt{-\det(\bar{g})}}W_{\pm}|\xi_{\downarrow}^{(\pm)}\rangle. \tag{6.54}$$

We can write the components of $w^{[\mu]}$ in term of $W_{\pm}$ and write $W_{\pm}^{-1}$ in term of w^{-1} as

$$\begin{aligned} W_{\pm}^{T} &= W_{\pm} \\ &= \mp|\epsilon^{[i]}\rangle m_{[i][j]}\langle\epsilon^{[j]}| - |w^{[i]}\rangle m_{[i][j]}\langle\epsilon^{[j]}|, \end{aligned} \tag{6.55}$$

$$w^{[i][j]} = 2\langle\epsilon^{[i]}|(W_{-} - W_{+})^{-1}|\epsilon^{[j]}\rangle, \tag{6.56}$$

$$|w^{[i]}\rangle = -(W_{+} + W_{-})(W_{-} - W_{+})^{-1}|\epsilon^{[i]}\rangle, \tag{6.57}$$

$$w = 2(W_{+}^{-1} - W_{-}^{-1})^{-1}, \tag{6.58}$$

$$W_{\pm}^{-1} = \pm w^{-1} - w^{-1}|w^{[i]}\rangle\langle\epsilon_{[i]}|, \tag{6.59}$$

where w^{-1} is the inverse operator of w. These in turn imply identities which we will also often make use of:

$$\langle w^{[i]}|w^{-1}|w^{[j]}\rangle - w^{[i][j]} = \langle\epsilon^{[i]}|w^{-1}|\epsilon^{[j]}\rangle, \tag{6.60}$$

$$\langle\epsilon^{[i]}| + (\langle w^{[i]}| - \langle\epsilon^{[i]}|)w^{-1}W_{+} = 0. \tag{6.61}$$

It will also be useful to consider differential $(2p+1)$-forms which are self-dual or anti-self-dual with respect to standard metric g. For this, we define

$$v^{\mu_1\cdots\mu_{2p+1},\nu_1\cdots\nu_{2p+1}} \equiv \sqrt{-\det(g)}g^{[\mu_1|\nu_1|}\cdots g^{\mu_{2p+1}]\nu_{2p+1}}, \tag{6.62}$$

and $V_{\pm}$ such that

$$|dT\rangle + V_{\pm}|\epsilon_{[i]}\rangle dx^{[i]} = \pm * (|dT\rangle + V_{\pm}|\epsilon_{[i]}\rangle dx^{[i]}). \tag{6.63}$$

6.2 Analysis of a general nonlinear Sen theory

6.2.1 Hamiltonian analysis of a general nonlinear Sen theory

In this subsection, we present Hamiltonian analysis of a general Sen theory, in which the self-duality condition in the physical sector can be nonlinear, while the unphysical sector has linear self-duality condition. Furthermore, the metrics $g_{\mu\nu}, \bar{g}_{\mu\nu}$ are general. Apart from being technically more involved, the analysis will closely follow the idea of the case $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ given in [27], in which the decoupling between physical and unphysical fields at the Hamiltonian level is shown. However, for the case of general $\bar{g}_{\mu\nu}$, the Lagrangian should take a certain form in order for the decoupling to be realised. We derive this form in this subsection.

As explained in section 5.5, the action of a general Sen theory encodes the dynamics of the fields P and Q along with external unphysical metric $\bar{g}_{\mu\nu}$ and can also include physical external fields g, Ψ. Due to $\bar{*}$-self-duality of Q, one may choose half of the components of Q to appear in the Lagrangian without loss of generality. In particular, let us choose the components $Q_{0a_1\cdots a_{2p}}$, which will be denoted in bra-ket notation as $|Q\rangle$ or $\langle Q|$. Similarly, since R is $\bar{*}$-anti-self-dual, we choose to present only the components $R_{0a_1\cdots a_{2p}}$, which will be denoted in bra-ket notation as $|R\rangle$ or $\langle R|$. We then express

$$Q = \frac{(-1)^p}{(2p)!}(\langle Q|dT\rangle + \langle Q|W_+|\epsilon_{[i]}\rangle dx^{[i]}), \tag{6.64}$$

$$R = \frac{(-1)^p}{(2p)!}(\langle R|dT\rangle + \langle R|W_-|\epsilon_{[i]}\rangle dx^{[i]}). \tag{6.65}$$

Note that

$$\begin{aligned} Q_{[i]} &= (2p+1)\langle Q|W_+|\epsilon_{[i]}\rangle, \\ R_{[i]} &= (2p+1)\langle R|W_-|\epsilon_{[i]}\rangle. \end{aligned} \tag{6.66}$$

In this subsection, we study properties of a general form of Sen theory. We investigate the condition to be imposed on the Lagrangian so that the Hamiltonian is separated into the two sectors. The Lagrangian of a general Sen theory can be expressed using bra-ket notation as

$$
\begin{aligned}
\mathcal{L} = & \frac{1}{8}(2p+1)\langle \partial P|w|\partial P\rangle + \frac{1}{4}\langle \partial P|w^{[i]}\rangle(\partial P)_{[i]} \\
& + \frac{1}{8}\frac{1}{2p+1}(\partial P)_{[i]}(\partial P)_{[j]}w^{[i][j]} \\
& - \frac{1}{2}\langle Q|\epsilon^{[i]}\rangle(\partial P)_{[i]} + \frac{1}{2}(2p+1)\langle Q|W_+|\partial P\rangle \\
& + \mathcal{L}_I(Q,g,\bar{g},\Psi),
\end{aligned}
\tag{6.67}
$$

with

$$
\begin{aligned}
& \delta_Q\mathcal{L}_I(Q,g,\bar{g},\Psi) \\
& \quad = -\frac{1}{2}(2p+1)\langle \delta Q|(W_+ - W_-)|R(Q,g,\bar{g},\Psi)\rangle.
\end{aligned}
\tag{6.68}
$$

The Lagrangian of a general Sen theory can be expressed using bra-ket notation as

$$
\begin{aligned}
\mathcal{L} = & \frac{1}{8}(2p+1)\langle \partial P|w|\partial P\rangle + \frac{1}{4}\langle \partial P|w^{[i]}\rangle(\partial P)_{[i]} \\
& + \frac{1}{8}\frac{1}{2p+1}(\partial P)_{[i]}(\partial P)_{[j]}w^{[i][j]} \\
& - \frac{1}{2}\langle Q|\epsilon^{[i]}\rangle(\partial P)_{[i]} + \frac{1}{2}(2p+1)\langle Q|W_+|\partial P\rangle \\
& + \mathcal{L}_I(Q,g,\bar{g},\Psi),
\end{aligned}
\tag{6.69}
$$

with

$$
\begin{aligned}
& \delta_Q\mathcal{L}_I(Q,g,\bar{g},\Psi) \\
& \quad = -\frac{1}{2}(2p+1)\langle \delta Q|(W_+ - W_-)|R(Q,g,\bar{g},\Psi)\rangle.
\end{aligned}
\tag{6.70}
$$

The combination $H = Q - R$ describes degrees of freedom of physical chiral $(2p)$-form field. It should satisfy a nonlinear self-duality condition which only involves H, g, and Ψ.

Let us now work out the Hamiltonian analysis. The analysis in this subsection is a direct generalisation of the analysis given in [27] in which $\bar{g} = \eta$ is imposed. Here, we work with the case where $\bar{g}$ is a general unphysical metric.

Conjugate momenta for $P_{\mu_1\cdots\mu_{2p}}$ are denoted

$$\pi^{\mu_1\cdots\mu_{2p}} = \frac{\partial\mathcal{L}}{\partial\partial_0 P_{\mu_1\cdots\mu_{2p}}}. \tag{6.71}$$

After a direct calculation, we obtain

$$\begin{aligned}|\pi\rangle = {}&\frac{1}{4}(2p+1)w|\partial P\rangle + \frac{1}{4}(\partial P)_{[i]}|w^{[i]}\rangle \\ &+ \frac{1}{2}(2p+1)W_+|Q\rangle,\end{aligned} \tag{6.72}$$

and

$$\pi^{0a_1\cdots a_{2p-1}} = 0, \tag{6.73}$$

where $|\pi\rangle$ represents $\pi^{a_1\cdots a_{2p}}$. Conjugate momenta for $Q_{0a_1\cdots a_{2p}}$ are

$$\pi_Q^{a_1\cdots a_{2p}} = \frac{\partial\mathcal{L}}{\partial\partial_0 Q_{0a_1\cdots a_{2p}}} = 0. \tag{6.74}$$

After some calculations, and by defining

$$\pi_\pm^{a_1\cdots a_{2p}} = \pi^{a_1\cdots a_{2p}} \pm \frac{1}{4}\widetilde{\partial P}^{a_1\cdots a_{2p}}, \tag{6.75}$$

or

$$|\pi_\pm\rangle = |\pi\rangle \pm \frac{1}{4}|\widetilde{\partial P}\rangle, \tag{6.76}$$

we obtain the total Hamiltonian

$$\begin{aligned}\mathcal{H} = {}&\frac{1}{2p+1}\langle\pi_-|W_+^{-1}|\pi_-\rangle - \frac{1}{2p+1}\langle\pi_+|W_-^{-1}|\pi_+\rangle \\ &- \frac{1}{4}(2p+1)\langle Q|(W_+ - W_-)W_-^{-1}W_+|Q\rangle \\ &+ \langle\pi_+|W_-^{-1}(W_+ - W_-)|Q\rangle - \mathcal{L}_I \\ &+ 2p\partial_{a_1}\pi^{a_1a_2\cdots a_{2p}}P_{a_2\cdots a_{2p}0} \\ &+ \zeta_{a_1\cdots a_{2p-1}}\pi^{0a_1\cdots a_{2p-1}} + u_{a_1\cdots a_{2p}}\pi_Q^{a_1\cdots a_{2p}},\end{aligned} \tag{6.77}$$

where $\zeta_{a_1\cdots a_{2p-1}}$ and $u_{a_1\cdots a_{2p}}$ are Lagrange multipliers which enforce the primary constraints $\pi^{0a_1\cdots a_{2p-1}} \approx 0$ and $\pi_Q^{a_1\cdots a_{2p}} \approx 0$. Next, requiring time derivative of the primary constraints to vanish gives rise to secondary constraints

$$\partial_{a_1}\pi^{a_1\cdots a_{2p}} \approx 0, \tag{6.78}$$

$$-(2p+1)\langle Q|W_+ + (2p+1)\langle R|W_- + 2\langle\pi_+| \approx 0. \tag{6.79}$$

Using $H = Q - R$, we may express eq.(6.79) alternatively as

$$H_{[i]} \approx 2\langle\pi_+|\epsilon_{[i]}\rangle. \tag{6.80}$$

Next, by computing time derivative of secondary constraints, it turns out that there is no further constraint.

Classification of constraints suggests that first-class constraints are $\pi^{0a_1\cdots a_{2p-1}} \approx 0$ and $\partial_{a_1}\pi^{a_1\cdots a_{2p}} \approx 0$, whereas second-class constraints are $\pi_Q^{a_1\cdots a_{2p}} \approx 0$ and $H_{[i]} \approx 2\langle\pi_+|\epsilon_{[i]}\rangle$. After constraints are classified, the number of degrees of freedom can be determined. It turns out that the number of degrees of freedom for Sen action is equal to twice of the number of degrees of chiral $(2p)$-form fields. This is as expected since Sen theory contains two chiral $(2p)$-form fields, one of which is physical while the other is unphysical with the wrong sign of kinetic energy.

Next, one may then solve second-class constraints to reduce the number of phase space variables. In particular, it is natural to eliminate $\pi_Q^{a_1\cdots a_{2p}}$ and $Q_{0a_1\cdots a_{2p}}$. This is by solving the second-class constraints for $\pi_Q^{a_1\cdots a_{2p}}$ and $Q_{a_1\cdots a_{2p}}$ and substituting into the Hamiltonian. Eliminating $\pi_Q^{a_1\cdots a_{2p}}$ is straightforward. As for eliminating $Q_{a_1\cdots a_{2p}}$, let us first use the definition $H = Q - R$, which gives $\langle H| = \langle Q| - \langle R|$. So eq.(6.79) gives

$$\begin{aligned}\langle Q| \approx &- \langle H|W_-(W_+ - W_-)^{-1} \\ &+ \frac{2}{2p+1}\langle\pi_+|(W_+ - W_-)^{-1}\end{aligned} \tag{6.81}$$

$$\begin{aligned}\langle R| \approx &- \langle H|W_+(W_+ - W_-)^{-1} \\ &+ \frac{2}{2p+1}\langle\pi_+|(W_+ - W_-)^{-1}.\end{aligned} \tag{6.82}$$

One may then eliminate $\langle H|$ by using nonlinear self-duality condition to write it in terms of $H_{[i]}, g$, and Ψ then apply to eq.(6.80).

The resulting Hamiltonian after the substitution of $|Q\rangle$ as a function of

$|\pi_+\rangle, g, \bar{g}, \Psi$ and after all the constraints are eliminated shows the separation between $|\pi_-\rangle$ and $|\pi_+\rangle$. This means that the Hamiltonian is of the form

$$\mathcal{H} = \mathcal{H}_- + \mathcal{H}_+, \tag{6.83}$$

where

$$\mathcal{H}_- = \frac{1}{2p+1}\langle\pi_-|W_+^{-1}|\pi_-\rangle, \tag{6.84}$$

$$\begin{aligned}\mathcal{H}_+ &= \mathcal{H}_+(|\pi_+\rangle, g, \bar{g}, \Psi)\\ &= -\frac{1}{2p+1}\langle\pi_+|W_-^{-1}|\pi_+\rangle\\ &\quad -\frac{1}{4}(2p+1)\langle Q|(W_+ - W_-)W_-^{-1}W_+|Q\rangle\\ &\quad + \langle\pi_+|W_-^{-1}(W_+ - W_-)|Q\rangle - \mathcal{L}_I,\end{aligned} \tag{6.85}$$

subject to $\partial_{a_1}\pi^{a_1\cdots a_{2p}} \approx 0$, and where $\langle Q|$ is determined from eq.(6.81) along with the nonlinear self-duality condition. The Hamiltonian $\mathcal{H}_-$ has the wrong sign. So $|\pi_-\rangle$ is unphysical. Furthermore, $|\pi_-\rangle$ only couples to $\bar{g}$ (through W_+). As for the Hamiltonian $\mathcal{H}_+$, it is a function of $|\pi_+\rangle, g, \bar{g}, \Psi$. In the case $\bar{g} = \eta$ originally considered in [27], $\mathcal{H}_+$ only contains physical fields. So it looks as if any form of $\mathcal{L}_I$ can be chosen so that the Hamiltonian separates into two disconnected sectors. However, in the case of general $\bar{g}$, this metric is unphysical since it couples to unphysical mode $|\pi_-\rangle$. So $\mathcal{H}_+$ should in fact be independent from $\bar{g}$, otherwise the unphysical and physical sector will be mixed. In this case, the form of $\mathcal{L}_I$ cannot be given arbitrarily.

In order to determine $\mathcal{L}_I$, let us note that by using eq.(6.79) $\mathcal{H}_+$ can be expressed as

$$\mathcal{H}_+ = -\frac{1}{4}(2p+1)\langle Q|(W_+ - W_-)|R\rangle - \frac{1}{2}\langle\pi_+|H\rangle - \mathcal{L}_I. \tag{6.86}$$

It is clear that $\mathcal{L}_I$ should take the form

$$\mathcal{L}_I = \mathcal{L}_I^{(1)}(Q, g, \bar{g}, \Psi) + \mathcal{L}_I^{(2)}(H, g, \Psi). \tag{6.87}$$

The term $\mathcal{L}_I^{(1)}(Q, g, \bar{g}, \Psi)$ should be such that $\mathcal{H}_+$ is independent from $\bar{g}$ while the term $\mathcal{L}_I^{(2)}(H, g, \Psi)$ only contains physical degrees of freedom. In particular, one can set without the loss of generality,

$$\mathcal{L}_I = -\frac{1}{4}(2p+1)\langle Q|(W_+ - W_-)|R\rangle + \mathcal{L}_I^{(2)}(H, g, \Psi), \tag{6.88}$$

which can be expressed using differential form as

$$\mathcal{L}_I = \frac{(2p)!}{4} Q \wedge R + \mathcal{L}_I^{(2)}(H, g, \Psi). \tag{6.89}$$

In the case of linear self-duality and in the absence of source, we have $\mathcal{L}_I^{(2)}(H, g, \Psi) = 0$.

Having determined the form of general Lagrangian, we can now see that the Hamiltonian (6.77) takes the form, after all secondary constraints are eliminated

$$\begin{aligned}\mathcal{H} \approx \frac{1}{2p+1}\langle \pi_- | W_+^{-1} | \pi_- \rangle - \frac{1}{2}\langle \pi_+ | H \rangle \\ - \mathcal{L}_I^{(2)}(H, g, \Psi) \\ + 2p\partial_{a_1}\pi^{a_1 a_2 \cdots a_{2p}} P_{a_2 \cdots a_{2p} 0} \\ + \zeta_{a_1 \cdots a_{2p-1}} \pi^{0 a_1 \cdots a_{2p-1}},\end{aligned} \tag{6.90}$$

where H is related to $|\pi_+\rangle$ by $H_{[i]} \approx 2\langle \pi_+ | \epsilon_{[i]} \rangle$ and $H_{0\langle a \rangle} = H_{0\langle a \rangle}(\pi_+^{\langle b \rangle}, g, \Psi)$, in which the latter is obtained from the nonlinear self-duality condition for H.

Dirac bracket relations are identified with Poisson bracket relations[6]

$$\begin{aligned}&[\pi_\pm^{a_1 \cdots a_{2p}}(t, \mathbf{x}), \pi_\pm^{b_1 \cdots b_{2p}}(t, \mathbf{x}')] \\ &\quad = \mp \frac{1}{2(2p)!} \frac{\partial}{\partial x^i} \delta^{(4p+1)}(\mathbf{x} - \mathbf{x}') \epsilon^{a_1 \cdots a_{2p} b_1 \cdots b_{2p} i}, \\ &[\pi_\pm^{a_1 \cdots a_{2p}}(t, \mathbf{x}), \pi_\mp^{b_1 \cdots b_{2p}}(t, \mathbf{x}')] = 0.\end{aligned} \tag{6.91}$$

---

[6] A quick check is to consider first-order Lagrangian corresponding to the Hamiltonian (6.90). After eliminating all the constraints, the terms containing time derivative are $\pi^{a_1 \cdots a_{2p}} \dot{P}_{a_1 \cdots a_{2p}}$. By the procedure of Faddeev-Jackiw [4], [5], one immediately sees that Dirac brackets are simply given by Poisson brackets.

It is useful to discuss the relationship between the Hamiltonian variables with Lagrangian variables. From the constraint (6.80), we have

$$(\pi_+)^{a_1\cdots a_{2p}} = \frac{1}{2}\epsilon^{0a_1\cdots a_{2p}i_1\cdots i_{2p+1}} H_{i_1\cdots i_{2p+1}}. \tag{6.92}$$

Or

$$(\pi_+)^{a_1\cdots a_{2p}} = \frac{1}{2}\sqrt{-\det(g)}(*H)^{0a_1\cdots a_{2p}}, \tag{6.93}$$

where indices of $*H$ are raised by g. Next, we note that eq.(5.90) implies

$$\begin{aligned}\langle H_{(s)}| = \langle Q| + \frac{1}{2}\langle \partial P|wW_+^{-1} \\ + \frac{1}{2p+1}\langle\widetilde{\partial P}|(W_- - W_+)^{-1}.\end{aligned} \tag{6.94}$$

Then by using eq.(6.72) and identities (6.55)-(6.59), we obtain

$$|\pi_-\rangle = \frac{1}{2}(2p+1)W_+|H_{(s)}\rangle, \tag{6.95}$$

or in index notation,

$$\pi_-^{a_1\cdots a_{2p}} = \frac{1}{2}\sqrt{-\det(\bar{g})}(H_{(s)})^{0a_1\cdots a_{2p}}, \tag{6.96}$$

where indices of $H_{(s)}$ are raised by $\bar{g}$. The relationships (6.92) and (6.96) are direct generalisation to the analysis of [28], as one would expect.

Given the Lagrangian of a Sen theory, i.e. with the form of $\mathcal{L}_I^{(2)}$ given, the Hamiltonian can immediately be written down by using eq.(6.90). However, we still need to determine how $H_{0\langle a\rangle}$ is related to $\pi_+^{\langle b\rangle}, g, \Psi$, which should be obtained from self-duality condition of the given theory. This is the only task left to do for each given Sen theory.

6.2.2 Duality and decoupling at Lagrangian level

In Sen theory, the decoupling [26], [27], [28] between physical and unphysical chiral ($2p$)-form fields can be shown at the level of Hamiltonian as well as equations of motion. The decoupling at the Lagrangian level, however, is left to be explicitly shown. A hint that this could be possible comes from the result of dimensional reduction of Sen theory and its generalisations [27], [28], [63], [64] that unphysical and physical fields are decoupled in the Lagrangian of the reduced theories.

In this subsection, we will present the decoupling of physical and unphysical fields at the Lagrangian level. We apply the standard approach [65], [66] (see also [67]), which makes use of field redefinition on phase space variables to decouple chiral $(2p)$-form fields. In passing, this approach also makes a duality present in the action.

More explicitly, the approach of [65], [66] is introduced to separate a two-dimensional Klein-Gordon scalar to two chiral bosons. This approach can easily be extended to help recognising a $(2p)$-form electrodynamics in $(4p+2)$-dimensional spacetime as a theory of two chiral $(2p)$-forms.

The $p = 0$ case can be explained as follows. One starts by writing the first-order form of Klein-Gordon Lagrangian: $\mathcal{L} = \pi\partial_0\phi - \pi^2/2 - (\partial_1\phi)^2/2$, where ϕ is the scalar field and π is the conjugate momentum of ϕ. Then one makes a field redefinition $\phi = \phi_+ + \phi_-, \pi = \partial_1\phi_+ - \partial_1\phi_-$ giving $\mathcal{L} = \partial_0\phi_+\partial_1\phi_+ - (\partial_1\phi_+)^2 - \partial_0\phi_-\partial_1\phi_- - (\partial_1\phi_-)^2$. With this form, one easily sees the decoupling between the chiral bosons ϕ_+ and ϕ_-. The Dirac bracket relations for ϕ_+ and ϕ_- can directly be obtained from the Poisson bracket relations for ϕ and π. One has $[\partial_x\phi_\pm(t,x), \partial_{x'}\phi_\pm(t,x')] = \pm\partial_x\delta(x-x')/2$ and $[\partial_x\phi_\pm(t,x), \partial_{x'}\phi_\mp(t,x')] = 0$, which take the expected form.

In fact, before making the field redefinition as explained above, one may replace π by $\partial_1\chi$. The Lagrangian then becomes $\mathcal{L} = \partial_1\chi\partial_0\phi - (\partial_1\chi)^2/2 - (\partial_1\phi)^2/2$. The dualisation $\phi \leftrightarrow \chi$ leaves this Lagrangian invariant up to total derivative terms. Note that the dualisation corresponds to $\phi_+ \to \phi_+, \phi_- \to -\phi_-$.

Let us now turn to the Lagrangian (5.84) of general Sen theory. The Hamiltonian analysis as discussed in subsection 6.2.1 suggests that each of the physical and unphysical sector contain a chiral $(2p)$-form. However, the Hamiltonian of the unphysical chiral $(2p)$-form field has the wrong sign. Nevertheless, the same idea of field redefinition still applies. Let us start from the first-order Lagrangian corresponding to the total Hamiltonian (6.90):

$$\begin{aligned}\mathcal{L} = \pi^{a_1\cdots a_{2p}}\partial_0 P_{a_1\cdots a_{2p}} \\ + 2p\pi^{0a_1\cdots a_{2p-1}}\partial_0 P_{0a_1\cdots a_{2p-1}} - \mathcal{H}.\end{aligned} \tag{6.97}$$

By eliminating $\pi^{0a_2\cdots a_{2p}}$ and $\zeta_{a_2\cdots a_{2p}}$ using their equations of motion, the first-order Lagrangian reduces to

$$\begin{aligned}\mathcal{L} = \pi^{a_1\cdots a_{2p}}\partial_0 P_{a_1\cdots a_{2p}} \\ - \frac{\bar{N}^i}{(2p)!}\epsilon_{ia_1\cdots a_{2p}b_1\cdots b_{2p}}\pi_-^{a_1\cdots a_{2p}}\pi_-^{b_1\cdots b_{2p}} \\ + \frac{\bar{N}}{\sqrt{\det(\bar{\gamma})}}\bar{\gamma}_{a_1b_1}\cdots\bar{\gamma}_{a_{2p}b_{2p}}\pi_-^{a_1\cdots a_{2p}}\pi_-^{b_1\cdots b_{2p}} \\ + \frac{1}{2}\langle\pi_+|H\rangle + \mathcal{L}_I^{(2)}(H, g, \Psi),\end{aligned} \tag{6.98}$$

such that $H_{[i]} \approx 2\langle\pi_+|\epsilon_{[i]}\rangle$ and, due to the nonlinear self-duality condition, $H_{0\langle a\rangle} = H_{0\langle a\rangle}(\pi_+^{\langle b\rangle}, g, \Psi)$. Furthermore, $\pi^{a_1a_2\cdots a_{2p}}$ should satisfy

$$\partial_{a_1}\pi^{a_1a_2\cdots a_{2p}} = 0, \tag{6.99}$$

which can be solved by setting

$$\begin{aligned}\pi^{a_1a_2\cdots a_{2p}} &= \frac{1}{4}\widetilde{\partial\phi}^{a_1a_2\cdots a_{2p}} \\ &\equiv -\frac{1}{4(2p)!}\partial_{[i}\phi_{b_1\cdots b_{2p}]}\epsilon^{a_1\cdots a_{2p}ib_1\cdots b_{2p}}.\end{aligned} \tag{6.100}$$

This leads to

$$\pi_\pm^{\langle a\rangle} = \frac{1}{4}(\widetilde{\partial\phi}^{\langle a\rangle} \pm \widetilde{\partial P}^{\langle a\rangle}), \tag{6.101}$$

which can be substituted into the first-order Lagrangian (6.98) giving

$$
\begin{aligned}
\mathcal{L} = & \frac{1}{4}\widetilde{\partial\phi}\partial_0 P_{a_1\cdots a_{2p}} \\
& - \frac{\bar{N}^i \epsilon_{ia_1\cdots a_{2p}b_1\cdots b_{2p}}}{16(2p)!}(\widetilde{\partial\phi} - \widetilde{\partial P})^{a_1\cdots a_{2p}}(\widetilde{\partial\phi} - \widetilde{\partial P})^{b_1\cdots b_{2p}} \\
& + \frac{\bar{N}\bar{\gamma}_{a_1b_1}\cdots\bar{\gamma}_{a_{2p}b_{2p}}}{16\sqrt{\det(\bar{\gamma})}}(\widetilde{\partial\phi} - \widetilde{\partial P})^{a_1\cdots a_{2p}}(\widetilde{\partial\phi} - \widetilde{\partial P})^{b_1\cdots b_{2p}} \\
& + \tilde{\mathcal{L}}\left(\frac{\widetilde{\partial\phi} + \widetilde{\partial P}}{2}, g, \Psi\right),
\end{aligned}
\tag{6.102}
$$

where $\tilde{\mathcal{L}}((\widetilde{\partial\phi} + \widetilde{\partial P})/2, g, \Psi)$ is obtained from $\langle\pi_+|H\rangle/2 + \mathcal{L}_I^{(2)}(H, g, \Psi)$ after the appropriate substitutions. The Lagrangian (6.102) is unchanged under the duality transformation $P \leftrightarrow \phi$ which corresponds to $\pi_+ \to \pi_+, \pi_- \to -\pi_-$, which can in turn be identified with duality transformation $H \to H, H_{(s)} \to -H_{(s)}$ that is discovered in [68] by directly dualising P on the original Sen action.

Let us now consider the field redefinitions

$$P_{a_1\cdots a_{2p}} = \rho_{a_1\cdots a_{2p}} + \sigma_{a_1\cdots a_{2p}}, \tag{6.103}$$

$$\phi_{a_1\cdots a_{2p}} = \rho_{a_1\cdots a_{2p}} - \sigma_{a_1\cdots a_{2p}}, \tag{6.104}$$

with

$$\widetilde{\partial\rho}^{a_1\cdots a_{2p}} \equiv -\frac{1}{(2p)!}\partial_{[i}\rho_{b_1\cdots b_{2p}]}\epsilon^{a_1\cdots a_{2p}ib_1\cdots b_{2p}}, \tag{6.105}$$

$$\widetilde{\partial\sigma}^{a_1\cdots a_{2p}} \equiv -\frac{1}{(2p)!}\partial_{[i}\sigma_{b_1\cdots b_{2p}]}\epsilon^{a_1\cdots a_{2p}ib_1\cdots b_{2p}}. \tag{6.106}$$

With this, the Lagrangian (6.98) becomes

$$
\begin{aligned}
\mathcal{L} = & -\frac{1}{4}\widetilde{\partial\sigma}^{a_1\cdots a_{2p}}\dot{\sigma}_{a_1\cdots a_{2p}} \\
& - \frac{1}{4}\frac{\bar{N}^i}{(2p)!}\epsilon_{ia_1\cdots a_{2p}b_1\cdots b_{2p}}\widetilde{\partial\sigma}^{a_1\cdots a_{2p}}\widetilde{\partial\sigma}^{b_1\cdots b_{2p}} \\
& + \frac{1}{4}\frac{\bar{N}}{\sqrt{\det(\bar{\gamma})}}\bar{\gamma}_{a_1b_1}\cdots\bar{\gamma}_{a_{2p}b_{2p}}\widetilde{\partial\sigma}^{a_1\cdots a_{2p}}\widetilde{\partial\sigma}^{b_1\cdots b_{2p}} \\
& + \frac{1}{4}\widetilde{\partial\rho}^{a_1\cdots a_{2p}}\dot{\rho}_{a_1\cdots a_{2p}} + \tilde{\mathcal{L}}(\widetilde{\partial\rho}, g, \Psi),
\end{aligned}
\tag{6.107}
$$

which clearly shows the decoupling between the two sectors at the Lagrangian level. Furthermore, the unphysical chiral $(2p)$-form field is described by Henneaux-Teitelboim Lagrangian [22] but with the wrong sign of kinetic term and only couples to the unphysical metric $\bar{g}_{\mu\nu}$. We will show explicitly in section 6.3 that for the case of quadratic Sen theory, the physical chiral $(2p)$-form field is also described by Henneaux-Teitelboim Lagrangian with the correct sign of kinetic term.

6.3 Separation of the two sectors of quadratic Sen theory

Although the analysis in the previous section provides some useful insights on the properties of a general form of Sen theory, in practice one would also work directly with explicit theories. As discussed above, what remains to be done in order to obtain the explicit final form of the Hamiltonian and Lagrangian for each specific theory is to solve the self-duality condition and determine the corresponding $\mathcal{L}_I^{(2)}$.

In this section, we will consider the quadratic Lagrangian (5.100) and derive the explicit form at the Hamiltonian and Lagrangian levels showing the separations of the two sectors. As a consistency check, we will also derive diffeomorphism rules from the Hamiltonian.

It is convenient to write $\mathcal{L}_I^{(2)}$ for the quadratic Lagrangian (5.100) as a function of $H^J = H + J$. We have

$$\mathcal{L}_I^{(2)} = \frac{1}{4}\langle H^J|\epsilon^{[i]}\rangle J_{[i]} - \frac{1}{4}\langle J|\epsilon^{[i]}\rangle H^J_{[i]}. \tag{6.108}$$

6.3.1 Decoupling at Hamiltonian level of quadratic Sen theory

Self-duality condition $H^J = *H^J$ gives

$$\langle H^J| = \frac{1}{2p+1} H^J_{[i]}\langle\epsilon^{[i]}|V_+^{-1}. \tag{6.109}$$

We can relate $H^J_{[i]}$ with $\langle\pi_+|$ by using eq.(6.80), which after introducing source is equivalent to

$$H^J_{[i]} \approx 2\langle\pi_+^J|\epsilon_{[i]}\rangle, \tag{6.110}$$

where

$$
\begin{aligned}
|\pi_+^J\rangle &\equiv |\pi_+\rangle + \frac{2p+1}{2}(W_+|J_+\rangle + W_-|J_-\rangle) \\
&= |\pi_+\rangle + \frac{1}{2}J_{[i]}|\epsilon^{[i]}\rangle.
\end{aligned}
\tag{6.111}
$$

Substituting eqs.(6.108)-(6.110) into eq.(6.90), we obtain

$$
\begin{aligned}
\mathcal{H} \approx & \frac{1}{2p+1}\langle\pi_-|W_+^{-1}|\pi_-\rangle - \frac{1}{2p+1}\langle\pi_+^J|V_+^{-1}|\pi_+^J\rangle \\
& + \langle\pi_+^J|J\rangle - \frac{1}{4}J_{[i]}\langle\epsilon^{[i]}|J\rangle \\
& + 2p\partial_{a_1}\pi^{a_1a_2\cdots a_{2p}}P_{a_2\cdots a_{2p}0} \\
& + \zeta_{a_1\cdots a_{2p-1}}\pi^{0a_1\cdots a_{2p-1}}.
\end{aligned}
\tag{6.112}
$$

It is now useful to express the Hamiltonian in terms of index notation. For this, let us express W_+^{-1} and V_+^{-1} in index notation using ADM variables. Direct calculation gives

$$
\begin{aligned}
&(W_\pm^{-1})_{a_1\cdots a_{2p}b_1\cdots b_{2p}} \\
&\quad = \mp\frac{\bar{N}}{\sqrt{\det(\bar{\gamma})}}(2p+1)\bar{\gamma}_{[a_1|b_1|}\cdots\bar{\gamma}_{a_{2p}]b_{2p}} \\
&\quad\quad + \frac{2p+1}{(2p)!}\bar{N}^i\epsilon_{a_1\cdots a_{2p}ib_1\cdots b_{2p}},
\end{aligned}
\tag{6.113}
$$

and

$$
\begin{aligned}
&(V_\pm^{-1})_{a_1\cdots a_{2p}b_1\cdots b_{2p}} \\
&\quad = \mp\frac{N}{\sqrt{\det(\gamma)}}(2p+1)\gamma_{[a_1|b_1|}\cdots\gamma_{a_{2p}]b_{2p}} \\
&\quad\quad + \frac{2p+1}{(2p)!}N^i\epsilon_{a_1\cdots a_{2p}ib_1\cdots b_{2p}}.
\end{aligned}
\tag{6.114}
$$

The Hamiltonian then becomes

$$\begin{aligned}\mathcal{H} \approx & \frac{\bar{N}^i}{(2p)!}\epsilon_{ia_1\cdots a_{2p}b_1\cdots b_{2p}}\pi_-^{a_1\cdots a_{2p}}\pi_-^{b_1\cdots b_{2p}} - \frac{\bar{N}}{\sqrt{\det(\bar{\gamma})}}\bar{\gamma}_{a_1b_1}\cdots\bar{\gamma}_{a_{2p}b_{2p}}\pi_-^{a_1\cdots a_{2p}}\pi_-^{b_1\cdots b_{2p}} \\ & - \frac{N^i}{(2p)!}\epsilon_{ia_1\cdots a_{2p}b_1\cdots b_{2p}}(\pi_+^J)^{a_1\cdots a_{2p}}(\pi_+^J)^{b_1\cdots b_{2p}} \\ & + \frac{N}{\sqrt{\det(\gamma)}}\gamma_{a_1b_1}\cdots\gamma_{a_{2p}b_{2p}}(\pi_+^J)^{a_1\cdots a_{2p}}(\pi_+^J)^{b_1\cdots b_{2p}} \\ & + (\pi_+^J)^{a_1\cdots a_{2p}}J_{0a_1\cdots a_{2p}} - \frac{1}{4}\frac{1}{(2p+1)!}J_{i_1\cdots i_{2p+1}}\epsilon^{a_1\cdots a_{2p}i_1\cdots i_{2p+1}}J_{0a_1\cdots a_{2p}} \\ & + 2p\partial_{a_1}\pi^{a_1a_2\cdots a_{2p}}P_{a_2\cdots a_{2p}0} + \zeta_{a_1\cdots a_{2p-1}}\pi^{0a_1\cdots a_{2p-1}}.\end{aligned} \tag{6.115}$$

The explicit form of the Hamiltonian in the case of static background $g_{0i} = 0, \bar{g}_{\mu\nu} = \eta_{\mu\nu}$ is presented in [28]. In our eq.(6.115) we provide the generalisation to the case of general curved metrics $g_{\mu\nu}, \bar{g}_{\mu\nu}$. Alternative to directly working out the Hamiltonian, one could also follow [28] by expressing the Hamiltonian in terms of Lagrangian variables and then generalise the static spacetime case to the case of general metric with the help of the identity

$$\begin{aligned}H^J_{0a_1\cdots a_{2p}} = & \frac{N\sqrt{\det(\gamma)}N^i}{(2p)!}\epsilon_{ia_1\cdots a_{2p}b_1\cdots b_{2p}}(H^J)^{0b_1\cdots b_{2p}} \\ & - N^2\gamma_{a_1b_1}\cdots\gamma_{a_{2p}b_{2p}}(H^J)^{0b_1\cdots b_{2p}},\end{aligned} \tag{6.116}$$

for $H^J = *H^J$. In either of these methods, one would see that the π_- and π_+^J part of the Hamiltonian separately coincide with Henneaux-Teitelboim Hamiltonian [22] for chiral $(2p)$-form. The explicit form of eq.(6.115) suggests that the theory indeed describes two chiral $(2p)$-form with π_+ corresponding to physical chiral field, while π_- corresponds to unphysical chiral field since its Hamiltonian has the wrong sign.

6.3.2 Decoupling at Lagrangian level of quadratic Sen theory

By applying the procedure of subsection 6.2.2 to the Hamiltonian (6.115), we obtain

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}\widetilde{\partial\sigma}^{a_1\cdots a_{2p}}\dot{\sigma}_{a_1\cdots a_{2p}} \\ &- \frac{\bar{N}^i\epsilon_{ia_1\cdots a_{2p}b_1\cdots b_{2p}}}{4(2p)!}\widetilde{\partial\sigma}^{a_1\cdots a_{2p}}\widetilde{\partial\sigma}^{b_1\cdots b_{2p}} \\ &+ \frac{1}{4}\frac{\bar{N}}{\sqrt{\det(\bar{\gamma})}}\bar{\gamma}_{a_1b_1}\cdots\bar{\gamma}_{a_{2p}b_{2p}}\widetilde{\partial\sigma}^{a_1\cdots a_{2p}}\widetilde{\partial\sigma}^{b_1\cdots b_{2p}} \\ &+ \frac{1}{4}(\widetilde{\partial\rho} - \tilde{J})^{a_1\cdots a_{2p}}(\dot{\rho}_{a_1\cdots a_{2p}} - J_{0a_1\cdots a_{2p}}) \\ &+ \frac{N^i\epsilon_{ia_1\cdots a_{2p}b_1\cdots b_{2p}}}{4(2p)!}(\widetilde{\partial\rho} - \tilde{J})^{a_1\cdots a_{2p}}(\widetilde{\partial\rho} - \tilde{J})^{b_1\cdots b_{2p}} \\ &- \frac{N\gamma_{a_1b_1}\cdots\gamma_{a_{2p}b_{2p}}}{4\sqrt{\det(\gamma)}}(\widetilde{\partial\rho} - \tilde{J})^{a_1\cdots a_{2p}}(\widetilde{\partial\rho} - \tilde{J})^{b_1\cdots b_{2p}} \\ &+ \frac{1}{4}\tilde{J}^{a_1\cdots a_{2p}}(\dot{\rho}_{a_1\cdots a_{2p}} - J_{0a_1\cdots a_{2p}}) \\ &- \frac{1}{4}(\widetilde{\partial\rho} - \tilde{J})^{a_1\cdots a_{2p}}J_{0a_1\cdots a_{2p}},\end{aligned} \tag{6.117}$$

which describes a theory with an unphysical chiral $(2p)$-form σ uncoupled with a physical chiral $(2p)$-form ρ. The equation (6.117) shows explicitly that these chiral $(2p)$-form fields are described by Henneaux-Teitelboim Lagrangian [22] such that σ has the wrong sign of kinetic term and that it is coupled only to the unphysical metric $\bar{g}$, whereas ρ has the correct sign of kinetic term and that it is coupled to the curved metric g and source J.

CHAPTER VII

CONCLUSION AND DISCUSSION

In this thesis, we have studied constraint analysis in generalised Proca theories and chiral boson theories. In the first half, we obtain the sufficient conditions for multi-field generalised Proca theory, coupled to non-dynamical external fields. This theory satisfies the special Hessian condition (4.48), free of Ostrogradski instability and satisfies diffeomorphism invariance requirement. Moreover, we require that this theory satisfy the secondary-constraint enforcing relations (4.70) (or equivalently, eq.(4.75)) and the completion requirements (4.104) which can be computed using eqs.(4.145)-(4.149).

We used Faddeev-Jackiw constraint analysis and cross checked by Lagrangian constraint analysis. Then we obtained all of constraints and count the number of degree of freedom. In the analysis, diffeomorphism invariance requirements, eqs.(C.9), (C.15)-(C.16) are needed. The diffeomorphism invariance requirements are not extra conditions. They are in fact conditions for which every diffeomorphism invariance theory is satisfied. If one analyses each specific theory one by one, it can be explicitly seen that these requirements are automatically satisfied. However, if one analyses a class of theories at a time, diffeomorphism invariance is less manifest as, by the nature of constraint analysis, time and space are not treated on equal footing. In this case, diffeomorphism invariance requirements help to realise the diffeomorphism invariance that every theory in the class possesses. These requirements are especially useful in simplifying key expressions in intermediate steps. Let us provide two example instances where the usefulness of diffeomorphism invariance requirements when analysing a class of theories are shown.

The first example is that, if the secondary-constraint enforcing relations

(4.70) is imposed, and if one does not know that theories which are diffeomorphism invariant should satisfy eq.(C.9), one would not be able to see, when analysing a class of theories, that eq.(4.67) is trivial, and hence would impose eq.(C.9) as another, but it is in fact obsolete, secondary-constraint enforcing relations. Another notable example is that diffeomorphism invariance requirements allow us to realise the connection between results from Faddeev-Jackiw constraint analysis and Lagrangian constraint analysis. The diffeomorphism invariance requirements have been helping in simplifying $\mathcal{C}_{0\alpha\beta}, \mathcal{C}^{i}_{1\alpha\beta}, \mathcal{C}^{ij}_{2\alpha\beta}$ and allowing us to realise that these expressions also appear, after transforming to tangent bundle, in Lagrangian constraint analysis.

Secondary-constraint enforcing relations we have obtained in this thesis is a correction to [8], [9]. This means that behaviour of some theories are previously misjudged. We have shown in subsection 4.4.1 an example of a legitimate theory previously misinterpreted as containing extra degrees of freedom as well as an example of undesired theory with extra degrees of freedom previously misinterpreted as being legitimate. We leave the work of identifying or constructing all of the theories which pass the secondary-constraint enforcing relations and the completion relations for future. Nevertheless, a consequence can readily be discussed and is provided in subsection 4.4.2 which points out that legitimate terms previously misjudged could be reintroduced into models to investigate cosmological implications.

An important future work is to analyse a larger class of theories, not necessarily restricted to those describing only vector fields. In fact, an important step has already been laid out by [10], which gives criteria for counting the number of degrees of freedom for theories with Lagrangians as functions of up to first order derivative in fields. These criteria, however, should be revised because as points out by [52], the analysis of [10] is not correct even in the case of the standard Proca theory. Additionally, as reported in this thesis, the analysis of [10] when specialised to multi-field generalised Proca theories misses terms in intermediate steps, for example $\partial_i \ddot{Q}^\beta$ and $\partial_i \partial_j \ddot{Q}^\beta$ within $\dot{\phi}_\alpha$. The corrections are required to address these issues. Once they are taken care of, we expect that the analysis would benefit from the help of diffeomorphism conditions. This is because in constraint analysis, even for Lagrangian constraint analysis, time and space are not treated on an equal footing. So the manifestation of diffeomorphism invariance (or, in case of flat spacetime, Lorentz isometry) is lost in the steps. The manifestation could be recovered with the use of diffeomorphism invariance requirements.

In particular, since the external fields, including gravity, considered in this thesis are all non-dynamical, one might attempt to extend the analysis of this thesis by considering n vector fields coupled to dynamical gravity and see if it is possible to obtain the criteria for the theory to describe n-field Proca theory, or even Maxwell-Proca theory, coupled to dynamical gravity.

In the second half of this thesis, we have studied aspects of non-linear chiral $(2p)$-form theory in the Sen formulation [26], [27], [28], [29] by using Dirac constraint analysis. Firstly, we review chiral field theories. There are Floreanini-Jackiw theory, Henneaux-Teitelboim formulation, $2p$-form Maxwell theory, and Sen formulation.

The scope of the theories we investigate is as follows. At the Lagrangian level, the theory encodes the dynamics of a $(2p)$-form P and a $(2p+1)$-form Q. Any external physical field can be introduced. In particular, the standard metric g is introduced as part of external physical fields. As for external unphysical field, only the unphysical metric $\bar{g}$ is introduced. The field strength of the unphysical chiral $(2p)$-form field is always linear self-dual, whereas the field strength of the physical chiral $(2p)$-form field can be nonlinear self-dual.

In particular, we work out the separation of the physical and unphysical sectors at the Hamiltonian and Lagrangian level. We started by considering the general Sen theories within the scope explained above. This gives properties in which these theories have in common.

We have shown that in order for the physical and unphysical sectors of a general Sen theory to be separated at the Hamiltonian level, the Lagrangian should take the form of eq.(5.84) with $\mathcal{L}_I$ given by eq.(6.89). Furthermore, the Hamiltonian for the chiral ($2p$)-form in the unphysical sector is described by the Henneaux-Teitelboim Hamiltonian [22] but with the incorrect overall sign.

The decoupling between the physical and unphysical sectors at the Lagrangian can also be shown. The idea is similar to the separation at the Lagrangian level of a 2D Klein-Gordon into two chiral scalar fields. That is, one considers the first-order Lagrangian of a general Sen theory. By eliminating all of the conjugate momenta and constraints followed by making field redefinition (6.103)-(6.104), we obtain the Lagrangian (6.107). It turns out that the unphysical chiral ($2p$)-form field is described by Henneaux-Teitelboim Lagrangian [22] but with the wrong sign of the kinetic term.

An example usage of our results is as follows. If one is given a nonlinear Sen theory whose Lagrangian is described by eq.(5.84), eq.(6.89) with $\mathcal{L}_I^{(2)}$ specified, the separation of the two sectors at the Hamiltonian level and at the Lagrangian level are given by eq.(6.90) and eq.(6.107) respectively. For each theory, the only remaining task to do to obtain the explicit form of these Hamiltonian and Lagrangian is to solve the nonlinear self-duality condition for $H_{0\langle a\rangle}$ in terms of $H_{[i]}$.

Having worked out common features of the general Sen theories, we then considered explicit example theories to obtain properties specific to each theory. We analysed the quadratic Sen theory whose Lagrangian is (5.100) and found that the Hamiltonian and Lagrangian of the physical chiral ($2p$)-form are described by the Henneaux-Teitelboim Hamiltonian and Lagrangian.

As a consistency check, we have also worked out the diffeomorphism rules from the Hamiltonian of the quadratic Sen theory. The derivation can easily be carried out in the reduced phase space since the Hamiltonian is up to linear powers in lapse functions and shift vectors.

As a future work, it would be interesting to see whether it is possible to obtain the diffeomorphism rules in the full phase space on which the Hamiltonian is a complicated function of lapse functions and shift vectors. Looking further ahead, it would also be interesting to see whether the Sen formulation is directly related at the Lagrangian level to the standard PST formulation [23], [24], [25], [69], [70] and its alternatives [71], [72], [73], [74], [75]. It can then be hoped that if this is successful, one would obtain more insights into solving some problems for example the difficulties in constructing the $2 + 4$ formulation of M5-brane action [76]. It is also interesting to see whether the Sen formulation is related at the Lagrangian level to the clone field formulation [77], [78], [79], [80].

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APPENDIX

APPENDIX A DIFFEOMORPHISM INVARIANCE IN SINGLE-FIELD GENERALISED PROCA THEORIES

Diffeomorphism invariance mean that physical law should be independent of coordinate system. So the action should be invariant under any coordinate transformation. The infinitesimal form of diffeomorphism transformation is

$$x^\mu \mapsto x^\mu - \epsilon^\mu(x) \tag{A.1}$$

where $\epsilon^\mu(x)$ are arbitrary functions of space time. Under this transformation, the fields transform as Lie derivative which shown below. We will show diffeomorphism transformation of tensor field $T^\mu{}_\nu$ as

$$\begin{aligned}\tilde{T}^\mu{}_\nu(\tilde{x}) &= \frac{\partial \tilde{x}^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu}T^\rho{}_\sigma(x) \\ &= \frac{\partial(x^\mu - \epsilon^\mu(x))}{\partial x^\rho}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu}T^\rho{}_\sigma(x).\end{aligned} \tag{A.2}$$

Let us consider

$$\begin{aligned}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu} &= \frac{\partial(\tilde{x}^\sigma + \epsilon^\sigma(x))}{\partial \tilde{x}^\nu} \\ &= \frac{\partial \tilde{x}^\sigma}{\partial \tilde{x}^\nu} + \frac{\partial \epsilon^\sigma(x)}{\partial \tilde{x}^\nu} \\ &= \delta^\sigma_\nu + \frac{\partial \epsilon^\sigma(x)}{\partial x^\mu}\frac{\partial x^\mu}{\partial \tilde{x}^\nu},\end{aligned} \tag{A.3}$$

so one obtains

$$\frac{\partial x^\sigma}{\partial \tilde{x}^\nu} - \frac{\partial \epsilon^\sigma(x)}{\partial x^\mu}\frac{\partial x^\mu}{\partial \tilde{x}^\nu} = \delta^\sigma_\nu. \tag{A.4}$$

Taking $\delta^\sigma{}_\mu$ into the first term, it gives

$$\begin{aligned}\delta^\sigma{}_\mu\frac{\partial x^\mu}{\partial \tilde{x}^\nu} - \frac{\partial \epsilon^\sigma(x)}{\partial x^\mu}\frac{\partial x^\mu}{\partial \tilde{x}^\nu} &= \delta^\sigma_\nu \\ \left(\delta^\sigma{}_\mu - \frac{\partial \epsilon^\sigma(x)}{\partial x^\mu}\right)\frac{\partial x^\mu}{\partial \tilde{x}^\nu} &= \delta^\sigma_\nu.\end{aligned} \tag{A.5}$$

In order to

$$\delta^\sigma{}_\mu \delta^\mu{}_\nu = \delta^\sigma{}_\nu, \tag{A.6}$$

the term $\partial x^\mu/\partial\tilde{x}^\nu$ in eq.(A.5) is the inverse of the term $\left(\delta^\sigma{}_\mu - \frac{\partial\epsilon^\sigma(x)}{\partial x^\mu}\right)$. So we will guess their solution which is written as

$$\frac{\partial x^\mu}{\partial\tilde{x}^\nu} = \delta^\mu{}_\nu - V^\mu{}_\nu. \tag{A.7}$$

One left only first order derivative. Let us substitute eq.(A.7) into eq.(A.5), one obtains

$$\begin{aligned}\left(\delta^\sigma{}_\mu - \frac{\partial\epsilon^\sigma(x)}{\partial x^\mu}\right)\left(\delta^\mu{}_\nu - V^\mu{}_\nu\right) &= \delta^\sigma{}_\nu \\ \delta^\sigma{}_\nu - \delta^\sigma{}_\mu V^\mu{}_\nu(x) - \delta^\mu{}_\nu\frac{\partial\epsilon^\sigma(x)}{\partial x^\mu} &= \delta^\sigma{}_\nu \\ V^\sigma{}_\nu(x) &= -\delta^\mu{}_\nu\frac{\partial\epsilon^\sigma(x)}{\partial x^\mu}.\end{aligned} \tag{A.8}$$

Therefore, eq.(A.7) can be written as

$$\frac{\partial x^\mu}{\partial\tilde{x}^\nu} = \delta^\mu{}_\nu + \delta^\sigma{}_\nu\frac{\partial\epsilon^\mu(x)}{\partial x^\sigma}. \tag{A.9}$$

One substitutes eq.(A.9) back to eq.(A.2), it gives

$$\begin{aligned}\tilde{T}^\mu{}_\nu(\tilde{x}) &= \frac{\partial(x^\mu - \epsilon^\mu(x))}{\partial x^\rho}\left(\delta^\sigma{}_\nu + \delta^\kappa{}_\nu\frac{\partial\epsilon^\sigma(x)}{\partial x^\kappa}\right)T^\rho{}_\sigma(x) \\ &= \left(\delta^\mu{}_\rho - \frac{\partial\epsilon^\mu(x)}{\partial x^\rho}\right)\left(T^\rho{}_\nu(x) + \frac{\partial\epsilon^\sigma(x)}{\partial x^\nu(x)}T^\rho{}_\sigma(x)\right) \\ &= T^\mu{}_\nu(x) + \frac{\partial\epsilon^\sigma(x)}{\partial x^\nu}T^\mu{}_\sigma(x) - \frac{\partial\epsilon^\mu(x)}{\partial x^\rho}T^\rho{}_\nu(x).\end{aligned} \tag{A.10}$$

Let us consider $\tilde{T}^\mu{}_\nu(\tilde{x})$ by using Talor series expansion,

$$\begin{aligned}\tilde{T}^\mu{}_\nu(\tilde{x}) &= \tilde{T}^\mu{}_\nu(x^\mu - \epsilon^\mu(x)) \\ &= \tilde{T}^\mu{}_\nu(x) - \epsilon^\rho(x)\partial_\rho\tilde{T}^\mu{}_\nu(x) \\ &= \tilde{T}^\mu{}_\nu(x) - \epsilon^\rho(x)\partial_\rho(T^\mu{}_\nu(x) + \mathcal{O}(x)) \\ &\approx \tilde{T}^\mu{}_\nu(x) - \epsilon^\rho(x)\partial_\rho T^\mu{}_\nu(x)\end{aligned} \tag{A.11}$$

where $\mathcal{O}(\epsilon)$ are terms of at least the first order in ϵ. Therefore, eq.(A.10) can be rewritten as

$$\begin{aligned}\tilde{T}^{\mu}{}_{\nu}(x)-\epsilon^{\rho}(x)\partial_{\rho}T^{\mu}_{\nu}(x)&=T^{\mu}{}_{\nu}(x)+\frac{\partial\epsilon^{\sigma}(x)}{\partial x^{\nu}}T^{\mu}{}_{\sigma}(x)-\frac{\partial\epsilon^{\mu}(x)}{\partial x^{\rho}}T^{\rho}{}_{\nu}(x)\\ \tilde{T}^{\mu}{}_{\nu}(x)-T^{\mu}{}_{\nu}(x)&=\epsilon^{\rho}(x)\partial_{\rho}T^{\mu}{}_{\nu}(x)+\frac{\partial\epsilon^{\sigma}(x)}{\partial x^{\nu}}T^{\mu}{}_{\sigma}(x)-\frac{\partial\epsilon^{\mu}(x)}{\partial x^{\rho}}T^{\rho}{}_{\nu}(x)\\ \delta_{\epsilon}T^{\mu}{}_{\nu}(x)&=\epsilon^{\rho}(x)\partial_{\rho}T^{\mu}{}_{\nu}(x)+\partial_{\nu}\epsilon^{\sigma}(x)T^{\mu}{}_{\sigma}(x)-\partial_{\rho}\epsilon^{\mu}(x)T^{\rho}{}_{\nu}(x)\\ \delta_{\epsilon}T^{\mu}{}_{\nu}(x)&=\mathcal{L}_{\epsilon}T^{\mu}{}_{\nu}(x).\end{aligned} \tag{A.12}$$

Similarly, one obtains

$$\delta_{\epsilon}A_{\mu}=\mathcal{L}_{\epsilon}A_{\mu}=\epsilon^{\rho}\partial_{\rho}A_{\mu}+A_{\rho}\partial_{\mu}\epsilon^{\rho}, \tag{A.13}$$

$$\delta_{\epsilon}g_{\mu\nu}=\mathcal{L}_{\epsilon}g_{\mu\nu}=\epsilon^{\rho}\partial_{\rho}g_{\mu\nu}+2g_{\rho(\nu}\partial_{\mu)}\epsilon^{\rho}. \tag{A.14}$$

The field variation δ_{ϵ} commutes with partial derivatives. Then the Lagrangian should transform as

$$\delta_{\epsilon}\mathcal{L}=\epsilon^{\mu}\partial_{\mu}\mathcal{L}+\mathcal{L}\partial_{\mu}\epsilon^{\mu}. \tag{A.15}$$

Let us consider Lagrangian eq.(3.14). It can be seen that $\delta_{\epsilon}\mathcal{L}-\partial_{\mu}(\epsilon^{\mu}\mathcal{L})$ can be given as a quadratic polynomial in $\partial_0 A_0$. It can be written as

$$\delta_{\epsilon}\mathcal{L}-\partial_{\mu}(\epsilon^{\mu}\mathcal{L})=a_2\partial_0A_0\partial_0A_0+a_1\partial_0A_0+a_0 \tag{A.16}$$

where a_2, a_1, and a_0 are quantities which are independent from $\partial_0 A_0$. However, $\partial_{\mu}(\epsilon^{\mu}\mathcal{L})=\partial_{\mu}\epsilon^{\mu}\mathcal{L}+\epsilon^{\mu}\partial_{\mu}\mathcal{L}$ which is caused eq.(A.16) equal to zero. So the coefficient of each order in ∂_0A_0 in (A.16) should vanish. One obtains $a_2=0, a_1=0$, and $a_0=0$ which can be thought of as conditions arising from the diffeomorphism requirement. Using eq.(3.14) and eq.(A.13), one first considers $\delta_{\epsilon}\mathcal{L}$

$$
\begin{aligned}
\delta_\epsilon \mathcal{L} &= \delta_\epsilon (T + U\partial_0 A_0) \\
&= \delta_\epsilon T + (\delta_\epsilon U)\partial_0 A_0 + U\delta_\epsilon(\partial_0 A_0) \\
&= \frac{\partial T}{\partial A_\nu}\delta_\epsilon A_\nu + \frac{\partial T}{\partial\partial_j A_\nu}\delta\partial_j A_\nu + \frac{\partial T}{\partial\partial_0 A_j}\delta_\epsilon\partial_0 A_j + \cdots + (\delta_\epsilon U)\partial_0 A_0 + U\partial_0(\delta_\epsilon A_0) \\
&= \frac{\partial T}{\partial A_\nu}(\epsilon^\rho\partial_\rho A_\nu + A_\rho\partial_\nu\epsilon^\rho) + \frac{\partial T}{\partial\partial_j A_\nu}\partial_j(\epsilon^\rho\partial_\rho A_\nu + A_\rho\partial_\nu\epsilon^\rho) \\
&\quad + \frac{\partial T}{\partial\partial_0 A_j}\partial_0(\epsilon^\rho\partial_\rho A_j + A_\rho\partial_j\epsilon^\rho) + \cdots + \delta_\epsilon U\partial_0 A_0 + U\partial_0(\epsilon^\rho\partial_\rho A_0 + A_\rho\partial_0\epsilon^\rho).
\end{aligned}
\tag{A.17}
$$

Then one considers $\partial_\mu(\epsilon^\mu \mathcal{L})$

$$
\begin{aligned}
\partial_\mu(\epsilon^\mu\mathcal{L}) &= \partial_\mu\epsilon^\mu(T + U\partial_0 A_0) + \epsilon^\mu\partial_\mu(T + U\partial_0 A_0) \\
&= T\partial_\mu\epsilon^\mu + U\partial_\mu\epsilon^\mu\partial_0 A_0 + \epsilon^\mu\partial_\mu T + \epsilon^\mu\partial_\mu U\partial_0 A_0 + \epsilon^\mu U\partial_\mu\partial_0 A_0 \\
&= T\partial_\mu\epsilon^\mu + U\partial_\mu\epsilon^\mu\partial_0 A_0 + \epsilon^\mu\Big(\frac{\partial T}{\partial A_\nu}\partial_\mu A_\nu + \frac{\partial T}{\partial\partial_j A_\nu}\partial_\mu\partial_j A_\nu \\
&\quad + \frac{\partial T}{\partial\partial_0 A_j}\partial_\mu\partial_0 A_j + \cdots\Big) \\
&= \epsilon^\mu\Big(\frac{\partial U}{\partial A_\nu}\partial_\mu A_\nu + \frac{\partial U}{\partial\partial_i A_\nu}\partial_\mu\partial_i A_\nu + \cdots\Big)\partial_0 A_0 + \epsilon^\mu U\partial_\mu\partial_0 A_0.
\end{aligned}
\tag{A.18}
$$

See the difference of eq.(A.17) and eq.(A.18), one obtains

$$
\begin{aligned}
\delta_\epsilon\mathcal{L} - \partial_\mu(\epsilon^\mu\mathcal{L}) &= \left(\frac{\partial T}{\partial A_0}\epsilon^0 + \frac{\partial T}{\partial\partial_j A_0}\partial_j\epsilon_0 + \frac{\partial T}{\partial\partial_0 A_j}\partial_j\epsilon^0 + \delta_\epsilon U + 2U\partial_0\epsilon^0\right)\partial_0 A_0 + \cdots \\
&\quad - \left(\frac{\partial T}{\partial A_0}\epsilon^0 + U\partial_\mu\epsilon^\mu\right)\partial_0 A_0 - \epsilon^\mu\left(\frac{\partial U}{\partial\partial_i A_\nu}\partial_\mu\partial_i A_\nu + \cdots\right)\partial_0 A_0 + \cdots \\
&\quad - \left(\frac{\partial U}{\partial A_0}\epsilon^0\right)\partial_0 A_0\partial_0 A_0 + \cdots .
\end{aligned}
\tag{A.19}
$$

In addition, one considers $\delta_\epsilon U$ by using eq.(A.13)

$$\begin{aligned}
\delta_\epsilon U &= \frac{\partial U}{\partial A_\nu}\delta_\epsilon A_\nu + \frac{\partial U}{\partial\partial_i A_\nu}\delta_\epsilon \partial_i A_\nu + \cdots \\
&= \frac{\partial U}{\partial A_\nu}(\epsilon^\rho\partial_\rho A_\nu + A_\rho\partial_\nu\epsilon^\rho) + \frac{\partial U}{\partial\partial_i A_\nu}\partial_i(\epsilon^\rho\partial_\rho A_\nu + A_\rho\partial_\nu\epsilon^\rho) + \cdots \\
&= \frac{\partial U}{\partial A_\nu}(\epsilon^\rho\partial_\rho A_\nu + A_\rho\partial_\nu\epsilon^\rho) + \frac{\partial U}{\partial\partial_i A_\nu}(\partial_i\epsilon^\rho\partial_\rho A_\nu + \epsilon^\rho\partial_\rho\partial_i A_\nu \\
&\quad + \partial_i A_\rho\partial_\nu\epsilon^\rho + A_\rho\partial_\nu\partial_i\epsilon^\rho) + \cdots \\
&= \left(\frac{\partial U}{\partial A_\nu}\epsilon^\rho + \frac{\partial U}{\partial\partial_i A_\nu}\partial_i\epsilon^\rho\right)\partial_\rho A_\nu + \cdots .
\end{aligned} \tag{A.20}$$

Then let us consider the coefficient of $\dot{A}_0\dot{A}_0$. Using eq.(A.16), eq.(A.19) and eq.(A.20), one obtains

$$\frac{\partial U}{\partial\partial_i A_0} = 0. \tag{A.21}$$

Then one considers the coefficient of $\dot{A}_0$.

$$\frac{\partial T}{\partial\partial_j A_0}\partial_j\epsilon^0 + \frac{\partial T}{\partial\partial_0 A_j}\partial_j\epsilon^0 + 2U\partial_0 e^0 - U\partial_\mu\epsilon^\mu + \delta_\epsilon U\mid_{\partial_0 A_0=0} = 0. \tag{A.22}$$

Then let us find two useful conditions from diffeomorphism invariance. One of them is obtained by taking derivative of eq. (A.22) with respect to $\partial_0 A_j$. It gives

$$\begin{aligned}
0 &= \frac{\partial^2 T}{\partial\partial_0 A_j\partial\partial_k A_0}\partial_k\epsilon^0 + \frac{\partial^2 T}{\partial\partial_0 A_j\partial\partial_0 A_k}\partial_k\epsilon^0 \\
&\quad + \frac{\partial}{\partial\partial_0 A_j}\left[\left(\frac{\partial U}{\partial A_\nu}\epsilon^\rho + \frac{\partial U}{\partial\partial_i A_\nu}\partial_i\epsilon^\rho\right)\partial_\rho A_\nu\right]_{\partial_0 A_0=0} \\
&= \frac{\partial^2 T}{\partial\dot{A}_j\partial\partial_k A_0}\partial_k\epsilon^0 + \frac{\partial^2 T}{\partial\dot{A}_j\partial\dot{A}_k}\partial_k\epsilon^0 + \left(\frac{\partial U}{\partial A_\nu}\epsilon^\rho + \frac{\partial U}{\partial\partial_i A_\nu}\partial_i\epsilon^\rho\right)\frac{\partial\partial_\rho A_\nu}{\partial\partial_0 A_j} \\
&= \frac{\partial^2 T}{\partial\dot{A}_j\partial\partial_k A_0}\partial_k\epsilon^0 + \frac{\partial^2 T}{\partial\dot{A}_j\partial\dot{A}_k}\partial_k\epsilon^0 + \frac{\partial U}{\partial A_j}\epsilon^0 + \frac{\partial U}{\partial\partial_i A_j}\partial_i\epsilon^0.
\end{aligned} \tag{A.23}$$

So there are conditions

$$\frac{\partial U}{\partial A_j} = 0, \tag{A.24}$$

and

$$\frac{\partial^2 T}{\partial\dot{A}_i\partial\dot{A}_j} + \frac{\partial^2 T}{\partial\dot{A}_i\partial\partial_j A_0} + \frac{\partial U}{\partial\partial_j A_i} = 0. \tag{A.25}$$

However, the useful condition is eq.(A.25) which will be used in next subsection. Another condition which will be made use can be obtained by taking derivative of eq.(A.22) with respect to $\partial_j A_0$ and use eq.(A.21). It can be written as

$$\begin{aligned}
0 &= \frac{\partial^2 T}{\partial\partial_j A_0 \partial\partial_k A_0}\partial_k\epsilon^0 + \frac{\partial^2 T}{\partial\partial_j A_0 \partial\dot{A}_k}\partial_k\epsilon^0 + \frac{\partial}{\partial\partial_j A_0}\left[\left(\frac{\partial U}{\partial A_\nu}\epsilon^\rho + \frac{\partial U}{\partial\partial_i A_\nu}\partial_i\epsilon^\rho\right)\partial_\rho A_\nu|_{\partial_0 A_0=0} \right. \\
&\quad \left. + \frac{\partial U}{\partial A_\nu}A_\rho\partial_\nu\epsilon^\rho\right] + \frac{\partial}{\partial\partial_j A_0}\left[\frac{\partial U}{\partial\partial_i A_\nu}\left(\epsilon^\rho\partial_\rho\partial_i A_\nu + \partial_i A_\rho\partial_\nu\epsilon^\rho + A_\rho\partial_\nu\partial_i\epsilon^\rho\right)\right] \\
&= \frac{\partial^2 T}{\partial\partial_j A_0 \partial\partial_k A_0}\partial_k\epsilon^0 + \frac{\partial^2 T}{\partial\partial_j A_0 \partial\dot{A}_k}\partial_k\epsilon^0 + \left(\frac{\partial U}{\partial A_\nu}\epsilon^\rho + \frac{\partial U}{\partial\partial_i A_\nu}\partial_i\epsilon^\rho\right)\frac{\partial\partial_\rho A_\nu}{\partial\partial_j A_0}|_{\partial_0 A_0=0} \\
&\quad + \frac{\partial U}{\partial\partial_i A_\nu}\frac{\partial}{\partial\partial_j A_0}\left(\partial_i A_\rho\partial_\nu\epsilon^\rho\right) \\
&= \frac{\partial^2 T}{\partial\partial_j A_0 \partial\partial_k A_0}\partial_k\epsilon^0 + \frac{\partial^2 T}{\partial\partial_j A_0 \partial\dot{A}_k}\partial_k\epsilon^0 + \frac{\partial U}{\partial\partial_i A_\nu}\delta_i^j\delta_\rho^0\partial_\nu\epsilon^\rho + \frac{\partial U}{\partial A_0}\epsilon^j + \frac{\partial U}{\partial\partial_i A_0}\partial_i\epsilon^j \\
&= \frac{\partial^2 T}{\partial\partial_j A_0 \partial\partial_k A_0}\partial_k\epsilon^0 + \frac{\partial^2 T}{\partial\partial_j A_0 \partial\dot{A}_k}\partial_k\epsilon^0 + \frac{\partial U}{\partial\partial_j A_\nu}\partial_\nu\epsilon^0 + \frac{\partial U}{\partial A_0}\epsilon^j.
\end{aligned} \tag{A.26}$$

Then one consider the third term

$$\begin{aligned}
\frac{\partial U}{\partial\partial_j A_\nu}\partial_\nu\epsilon^0 &= \frac{\partial U}{\partial\partial_j A_i}\partial_i\epsilon^0 + \frac{\partial U}{\partial\partial_j A_0}\partial_0\epsilon^0 \\
&= \frac{\partial U}{\partial\partial_j A_i}\partial_i\epsilon^0
\end{aligned} \tag{A.27}$$

where $\frac{\partial U}{\partial\partial_j A_0}\partial_\nu\epsilon^0 = 0$. Using eq.(A.21), So one obtains

$$\frac{\partial^2 T}{\partial\partial_j A_0 \partial\partial_k A_0}\partial_k\epsilon^0 + \frac{\partial^2 T}{\partial\partial_j A_0 \partial\dot{A}_k}\partial_k\epsilon^0 + \frac{\partial U}{\partial\partial_j A_k}\partial_k\epsilon^0 + \frac{\partial U}{\partial A_0}\epsilon^j = 0. \tag{A.28}$$

There are two conditoins here namely

$$\frac{\partial U}{\partial A_0} = 0, \tag{A.29}$$

and

$$\frac{\partial^2 T}{\partial\partial_i A_0 \partial\partial_j A_0} + \frac{\partial^2 T}{\partial\dot{A}_i \partial\partial_j A_0} + \frac{\partial U}{\partial\partial_j A_i} = 0. \tag{A.30}$$

Actually, the diffeomorphism invariance requirement gives several conditions. However, the useful ones are eq.(A.25) and eq.(A.30).

APPENDIX B CONDITION FROM DIFFEOMORPHISM INVARIANCE IN SINGLE-FIELD GENERALISED PROCA THEORIES

One shows that diffeomorphism invariance conditions imply that $\mathcal{C}_2^{ij}$ and $\mathcal{C}_1^i$ vanish which given in eq.(4.44) and eq.(4.45). In addition to eq.(4.97) one also uses diffeomorphism invariant conditions to rewrite $\mathcal{C}_0$ in simple case namely, in term of configuration space variables.

B.1 The condition $\mathcal{C}_2^{ij} = 0$

One imposes diffeomorphism invariance to show the condition in eq.(4.44) is trivially satisfied. It automatically vanish when one use below conditions. One need eq.(A.21) for the proof. However, one also need eq.(A.25) and eq.(A.30) in the phase space versions which given as

$$\frac{\partial^2 \mathcal{T}}{\partial \Lambda_i \partial \Lambda_j} + \frac{\partial^2 \mathcal{T}}{\partial \Lambda_i \partial \partial_j A_0} + \frac{\partial U}{\partial \partial_j A_i} = 0, \tag{B.1}$$

and

$$\frac{\partial^2 \mathcal{T}}{\partial \partial_i A_0 \partial \partial_j A_0} + \frac{\partial^2 \mathcal{T}}{\partial \Lambda_i \partial \partial_j A_0} + \frac{\partial U}{\partial \partial_j A_i} = 0, \tag{B.2}$$

where one substitutes T by $\mathcal{T}$ and $\dot{A}_i$ by Λ_i. However, one also defines the conditions by taking partial derivative of $\pi^i = \partial \mathcal{T} / \partial \Lambda_i$ with respect to phase space variables, namely π^k, A_ν and $\partial_j A_\nu$. One obtains

$$\begin{aligned} \frac{\partial \pi^i}{\partial \pi^k} &= \frac{\partial}{\partial \pi^k}\left(\frac{\partial \mathcal{T}}{\partial \Lambda_i}\right) = \frac{\partial^2 \mathcal{T}}{\partial \Lambda_j \partial \Lambda_i} \frac{\partial \Lambda_j}{\partial \pi^k} \\ \delta^i_k &= \frac{\partial^2 \mathcal{T}}{\partial \Lambda_i \partial \Lambda_j} \frac{\partial \Lambda_j}{\partial \pi^k}, \end{aligned} \tag{B.3}$$

$$\begin{aligned} \frac{\partial \pi^i}{\partial A_\nu} &= \frac{\partial}{\partial A_\nu}\left(\frac{\partial \mathcal{T}}{\partial \Lambda_i}\right) = \frac{\partial^2 \mathcal{T}}{\partial \Lambda_i \partial A_\mu} \frac{\partial A_\mu}{\partial A_\nu} + \frac{\partial^2 \mathcal{T}}{\partial \Lambda_i \partial \Lambda_k} \frac{\partial \Lambda_k}{\partial A_\nu} \\ 0 &= \frac{\partial^2 \mathcal{T}}{\partial \Lambda_i \partial A_\nu} + \frac{\partial^2 \mathcal{T}}{\partial \Lambda_i \partial \Lambda_k} \frac{\partial \Lambda_k}{\partial A_\nu}, \end{aligned} \tag{B.4}$$

and

$$\begin{aligned}\frac{\partial\pi^i}{\partial\partial_j A_\nu} &= \frac{\partial}{\partial\partial_j A_\nu}\left(\frac{\partial\mathcal{T}}{\partial\Lambda_i}\right) = \frac{\partial^2\mathcal{T}}{\partial\Lambda_i\partial\partial_k A_\mu}\frac{\partial\partial_k A_\mu}{\partial\partial_j A_\nu} + \frac{\partial^2\mathcal{T}}{\partial\Lambda_i\partial\Lambda_k}\frac{\partial\Lambda_k}{\partial\partial_j A_\nu} \\ 0 &= \frac{\partial^2\mathcal{T}}{\partial\Lambda_i\partial\partial_k A_\nu} + \frac{\partial^2\mathcal{T}}{\partial\Lambda_i\partial\Lambda_k}\frac{\partial\Lambda_k}{\partial\partial_j A_\nu}.\end{aligned} \quad \text{(B.5)}$$

To show eq.(4.44), one express derivatives of Ω_1 and Ω_2 with respect to phase space variables in term of derivatives of U and $\mathcal{T}$. Then one substitutes them into eq.(4.40). For $\Omega_1 = \pi^0 - U$, it is easily proof. Then one focuses on the derivatives of Ω_2. In eq.(4.40), one only considers derivatives of Ω_2 with respect to $\partial_i\partial_j A_0$ and $\partial_i\pi_j$. One sees that, in eq.(4.28), the expressions $\partial_i\partial_j A_0$ and $\partial_i\pi^j$ appear in Ω_2 only through

$$-\partial_i\left(\frac{\partial_i\mathcal{T}}{\partial\partial_i A_0}\right) - \partial_i\Lambda_j\frac{\partial U}{\partial\partial_i A_j} \subset \Omega_2. \quad \text{(B.6)}$$

Then let us consider eq.(B.6) for simple calculation. It gives

$$\begin{aligned}\Omega_2 &\supset -\partial_i\left(\frac{\partial\mathcal{T}}{\partial\partial_i A_0}\right) - \partial_i\Lambda_j\frac{\partial U}{\partial\partial_i A_j} \\ &= \left(-\frac{\partial^2\mathcal{T}}{\partial\partial_j A_0\partial\partial_i A_0}\partial_i\partial_j A_0 - \frac{\partial^2\mathcal{T}}{\partial\Lambda_j\partial\partial_i A_0}\partial_i\Lambda_j\right) - \left(\frac{\partial\Lambda_i}{\partial\partial_k A_0}\partial_j\partial_k A_0 + \frac{\partial\Lambda_i}{\partial\pi^k}\partial_j\pi^k\right)\frac{\partial U}{\partial\partial_j A_i} \\ &= \left(-\frac{\partial^2\mathcal{T}}{\partial\partial_j A_0\partial\partial_i A_0}\partial_i\partial_j A_0 - \frac{\partial^2\mathcal{T}}{\partial\Lambda_j\partial\partial_i A_0}\left(\frac{\partial\Lambda_j}{\partial\partial_k A_0}\partial_i\partial_k A_0 + \frac{\partial\Lambda_j}{\partial\pi^k}\partial_i\pi^k\right)\right) \\ &\quad - \left(\frac{\partial U}{\partial\partial_i A_k}\frac{\partial\Lambda_k}{\partial\partial_j A_0}\partial_i\partial_j A_0 + \frac{\partial\Lambda_k}{\partial\pi^j}\partial_i\pi^j\right)\frac{\partial U}{\partial\partial_i A_k} \\ &= -\left(\frac{\partial^2\mathcal{T}}{\partial\partial_j A_0\partial\partial_i A_0} + \frac{\partial^2\mathcal{T}}{\partial\Lambda_k\partial\partial_i A_0}\frac{\partial\Lambda_k}{\partial\partial_j A_0} + \frac{\partial U}{\partial\partial_i A_k}\frac{\partial\Lambda_k}{\partial\partial_j A_0}\right)\partial_i\partial_j A_0 \\ &\quad - \left(\frac{\partial^2\mathcal{T}}{\partial\Lambda_k\partial\partial_i A_0}\frac{\partial\Lambda_k}{\partial\pi^j} + \frac{\partial U}{\partial\partial_i A_k}\frac{\partial\Lambda_k}{\partial\pi^j}\right)\partial_i\pi^j.\end{aligned} \quad \text{(B.7)}$$

Then using eq.(B.1)-(B.5), it can be seen that the expressions containing $\partial_i\partial_j A_0$ and $\partial_i\pi^j$ only appear in Ω_2 as

$$\begin{aligned}\Omega_2 &\supset -\left(\frac{\partial^2\mathcal{T}}{\partial\partial_j A_0\partial\partial_i A_0} - \frac{\partial^2\mathcal{T}}{\partial\Lambda_i\partial\Lambda_k}\frac{\partial\Lambda_k}{\partial\partial_j A_0}\right)\partial_i\partial_j A_0 + \frac{\partial^2\mathcal{T}}{\partial\Lambda_i\partial\Lambda_k}\frac{\partial\Lambda_k}{\partial\pi^j}\partial_i\partial^j \\ &= -\left(\frac{\partial^2\mathcal{T}}{\partial\partial_j A_0\partial\partial_i A_0} + \frac{\partial^2\mathcal{T}}{\partial\Lambda_i\partial\partial_j A_0}\right)\partial_i\partial_j A_0 + \delta^i_j\partial_i\pi^j \\ &= \frac{\partial U}{\partial\partial_j A_i}\partial_i\partial_j A_0 + \partial_i\pi^i.\end{aligned} \quad \text{(B.8)}$$

Substituting all results into eq.(4.40) gives

$$\begin{aligned}
\mathcal{C}_2^{jk} &\equiv -\frac{\partial \Omega_2}{\partial \partial_j \partial_k A_0} - \frac{\partial \Omega_1}{\partial \partial_j A_i}\frac{\partial \Omega_2}{\partial \partial_k \pi^i} \\
&= -\frac{\partial}{\partial \partial_j \partial_k A_0}\left(\frac{\partial U}{\partial \partial_l A_i}\partial_i \partial_l A_0 + \partial_i \pi^i\right) - \frac{\partial(\pi^0 - U)}{\partial \partial_{(j} A_{|i|}}\frac{\partial}{\partial \partial_{k)} \pi^i}\left(\frac{\partial U}{\partial \partial_l A_m}\partial_m \partial_l A_0 + \partial_m \pi^m\right) \\
&= -\frac{\partial U}{\partial \partial_l A_i}\frac{\partial \partial_i \partial_l A_0}{\partial \partial_j \partial_k A_0} + \frac{\partial U}{\partial \partial_{(j} A_{|i|}}\frac{\partial \partial_m \pi^m}{\partial \partial_{k)} \pi^i} \\
&= -\frac{1}{2}\left(\frac{\partial U}{\partial \partial_k A_j} + \frac{\partial U}{\partial \partial_j A_k}\right) + \frac{1}{2}\left(\frac{\partial U}{\partial \partial_j A_k} + \frac{\partial U}{\partial \partial_k A_j}\right) \\
&= 0.
\end{aligned} \tag{B.9}$$

This means that eq.(4.44) is automatically satisfied.

B.2 The condition $\mathcal{C}_1^i = 0$

The condition (4.45) can easily be shown to be automatically satisfied after the diffeomorphism invariance conditions are used. One considers eq.(4.41) to show the derivative of Ω_2 with respect to $\partial_i \partial_j A_0, \partial_i \pi^j, \pi^i$ and to $\partial_i A_0$. One already have the derivatives of Ω_2 with respect to $\partial_i \partial_j A_0$ and $\partial_i \pi^j$ in eq.(B.8). Then one will present the left two terms as

$$\frac{\partial \Omega_2}{\partial \pi^i}, \frac{\partial \Omega_2}{\partial \partial_i A_0}. \tag{B.10}$$

One takes derivative of eq.(4.28) with respect to π^i. First of all, one will expand Ω_2 for properly taking derivative of π^i and $\partial \partial_i A_0$. It gives

$$\begin{aligned}
\Omega_2 &= \frac{\partial \mathcal{T}}{\partial A_0} - \partial_i\left(\frac{\partial \mathcal{T}}{\partial \partial_i A_0}\right) - \Lambda_i \frac{\partial U}{\partial A_i} - \partial_j \Lambda_i \frac{\partial U}{\partial \partial_j A_i} \\
&= \frac{\partial \mathcal{T}}{\partial A_0} - \left(\frac{\partial^2 \mathcal{T}}{\partial A_0 \partial \partial_i A_0}\partial_i A_0 + \frac{\partial^2 \mathcal{T}}{\partial \partial_j A_0 \partial \partial_i A_0}\partial_i \partial_j A_0 + \frac{\partial^2 \mathcal{T}}{\partial \Lambda_j \partial \partial_i A_0}\partial_i \Lambda_j + \dots\right) \\
&\quad - \Lambda_i \frac{\partial U}{\partial A_i} - \left(\frac{\partial \Lambda_i}{\partial A_0}\partial_j A_0 + \frac{\partial \Lambda_i}{\partial \partial_k A_0}\partial_j \partial_k A_0 + \frac{\partial \Lambda_i}{\partial \pi^k}\partial_j \pi^k + \dots\right)\frac{\partial U}{\partial \partial_j A_i}.
\end{aligned} \tag{B.11}$$

Let us consider $\partial_i \Lambda_j$ which gives

$$\partial_i \Lambda_j = \frac{\partial \Lambda_j}{\partial A_0}\partial_i A_0 + \frac{\partial \Lambda_j}{\partial \partial_k A_0}\partial_i \partial_k A_0 + \frac{\partial \Lambda_j}{\partial \pi^k}\partial_i \pi^k + \dots. \tag{B.12}$$

Then one can rewrite Ω_2 as

$$\begin{aligned}\Omega_2 = \frac{\partial\mathcal{T}}{\partial A_0} - \Bigg(&\frac{\partial^2\mathcal{T}}{\partial A_0\partial\partial_i A_0}\partial_i A_0 + \frac{\partial^2\mathcal{T}}{\partial\partial_j A_0\partial\partial_i A_0}\partial_i\partial_j A_0 + \frac{\partial^2\mathcal{T}}{\partial\Lambda_j\partial\partial_i A_0}\\ &\times\left(\frac{\partial\Lambda_j}{\partial A_0}\partial_i A_0 + \frac{\partial\Lambda_j}{\partial\partial_k A_0}\partial_i\partial_k A_0 + \frac{\partial\Lambda_j}{\partial\pi^k}\partial_i\pi^k + R_{ij}\right) + R_{\mathcal{T}}\Bigg) - \Lambda_i\frac{\partial U}{\partial A_i}\\ &-\left(\frac{\partial\Lambda_i}{\partial A_0}\partial_j A_0 + \frac{\partial\Lambda_i}{\partial\partial_k A_0}\partial_j\partial_k A_0 + \frac{\partial\Lambda_i}{\partial\pi^k}\partial_j\pi^k + R_{ji}\right)\frac{\partial U}{\partial\partial_j A_i},\end{aligned} \tag{B.13}$$

where

$$R_{ij} \equiv \frac{\partial\Lambda_j}{\partial\mathcal{X}}\partial_i\mathcal{X}; \tag{B.14}$$

$\mathcal{X} \equiv A_i, \partial_j A_i, \pi^i, g_{\mu\nu}, \ldots, K$, and

$$R_{\mathcal{T}} \equiv \frac{\partial^2\mathcal{T}}{\partial\partial_i A_0\partial\mathcal{X}'}\partial_i\mathcal{X}'; \tag{B.15}$$

$\mathcal{X}' \equiv A_i, \partial_j A_i, g_{\mu\nu}, \ldots, K$. Therefore, one obtains

$$\begin{aligned}\frac{\partial\Omega_2}{\partial\pi^i} = \frac{\partial^2\mathcal{T}}{\partial\Lambda_j\partial A_0}\frac{\partial\Lambda_j}{\partial\pi^i} - \Bigg(&\frac{\partial^3\mathcal{T}}{\partial\Lambda_k\partial A_0\partial\partial_j A_0}\frac{\partial\Lambda_k}{\partial\pi^i}\partial_j A_0 + \frac{\partial^3\mathcal{T}}{\partial\Lambda_k\partial\partial_l A_0\partial\partial_j A_0}\frac{\partial\Lambda_k}{\partial\pi^i}\partial_l\partial_j A_0\\ &+\frac{\partial^3\mathcal{T}}{\partial\Lambda_k\partial\Lambda_l\partial\partial_j A_0}\frac{\partial\Lambda_k}{\partial\pi^i}\partial_j\Lambda_l + \frac{\partial^2\mathcal{T}}{\partial\Lambda_k\partial\partial_j A_0}\Bigg(\frac{\partial^2\Lambda_k}{\partial\pi^l\partial A_0}\frac{\partial\pi^l}{\partial\pi^i}\partial_j A_0\\ &+\frac{\partial^2\Lambda_k}{\partial\pi^m\partial\partial_l A_0}\frac{\partial\pi^m}{\partial\pi^i}\partial_j\partial_l A_0 + \frac{\partial^2\Lambda_k}{\partial\pi^m\partial\pi^l}\frac{\partial\pi^m}{\partial\pi^i}\partial_j\pi^l + \frac{\partial R_{jk}}{\partial\pi^i}\Bigg) + \frac{\partial R_{\mathcal{T}}}{\partial\pi^i}\Bigg)\\ &-\frac{\partial\Lambda_j}{\partial\pi^i}\frac{\partial U}{\partial A_j} - \Bigg(\frac{\partial^2\Lambda_j}{\partial\pi^k\partial A_0}\frac{\partial\pi^k}{\partial\pi^i}\partial_l A_0 + \frac{\partial^2\Lambda_j}{\partial\pi^m\partial\partial_k A_0}\frac{\partial\pi^m}{\partial\pi^i}\partial_l\partial_k A_0\\ &+\frac{\partial^2\Lambda_j}{\partial\pi^m\partial\pi^j}\frac{\partial\pi^m}{\partial\pi^i}\partial_l\pi^j + \frac{\partial R_{lj}}{\partial\pi^i}\Bigg)\frac{\partial U}{\partial\partial_l A_j}.\end{aligned} \tag{B.16}$$

Using total derivatives properties

$$\partial_k\left(\frac{\partial^2\mathcal{T}}{\partial\partial_k A_0\partial\Lambda_j}\frac{\partial\Lambda_j}{\partial\pi_i}\right) = \partial_k\left(\frac{\partial^2\mathcal{T}}{\partial\partial_k A_0\partial\Lambda_j}\right)\frac{\partial\Lambda_j}{\partial\pi_i} + \frac{\partial^2\mathcal{T}}{\partial\partial_k A_0\partial\Lambda_j}\partial_k\left(\frac{\partial\Lambda_j}{\partial\pi_i}\right), \tag{B.17}$$

$$\begin{aligned}\partial_j\left(\frac{\partial^2\mathcal{T}}{\partial\partial_j A_0\partial\Lambda_k}\right) &= \frac{\partial^3\mathcal{T}}{\partial\Lambda_k\partial A_0\partial\partial_j A_0}\partial_j A_0 + \frac{\partial^3\mathcal{T}}{\partial\Lambda_k\partial\partial_l A_0\partial\partial_j A_0}\partial_l\partial_j A_0\\ &\quad+\frac{\partial^3\mathcal{T}}{\partial\Lambda_k\partial\Lambda_l\partial\partial_j A_0}\partial_j\Lambda_l + \ldots,\end{aligned} \tag{B.18}$$

$$\partial_l\left(\frac{\partial\Lambda_j}{\partial\pi_i}\right) = \frac{\partial^2\Lambda_j}{\partial\pi^i\partial A_0}\partial_l A_0 + \frac{\partial^2\Lambda_j}{\partial\pi^i\partial\partial_k A_0}\partial_l\partial_k A_0 + \frac{\partial^2\Lambda_j}{\partial\pi^i\partial\pi^j}\partial_l\pi^j + \ldots, \tag{B.19}$$

then eqs.(B.1)-(B.5) finally gives

$$
\begin{aligned}
\frac{\partial \Omega_2}{\partial \pi^i} &= \left(\frac{\partial^2 \mathcal{T}}{\partial A_0 \partial \Lambda_j} - \frac{\partial U}{\partial A_j}\right)\frac{\partial \Lambda_j}{\partial \pi^i} - \frac{\partial U}{\partial \partial_k A_j}\partial_k\left(\frac{\partial \Lambda_j}{\partial \pi^i}\right) - \partial_k\left(\frac{\partial^2 \mathcal{T}}{\partial \partial_k A_0 \partial \Lambda_j}\right)\frac{\partial \Lambda_j}{\partial \pi^i} \\
&\quad - \frac{\partial^2 \mathcal{T}}{\partial \partial_k A_0 \partial \Lambda_j}\partial_k\left(\frac{\partial \Lambda_j}{\partial \pi^i}\right) \\
&= \left(\frac{\partial^2 \mathcal{T}}{\partial A_0 \partial \Lambda_j} - \frac{\partial U}{\partial A_j}\right)\frac{\partial \Lambda_j}{\partial \pi^i} - \frac{\partial U}{\partial \partial_k A_j}\partial_k\left(\frac{\partial \Lambda_j}{\partial \pi^i}\right) + \partial_k\left(\frac{\partial^2 \mathcal{T}}{\partial \Lambda_k \partial \Lambda_j} + \frac{\partial U}{\partial \partial_k A_j}\right)\frac{\partial \Lambda_j}{\partial \pi^i} \\
&\quad + \left(\frac{\partial^2 \mathcal{T}}{\partial \Lambda_k \partial \Lambda_j} + \frac{\partial U}{\partial \partial_k A_j}\right)\partial_k\left(\frac{\partial \Lambda_j}{\partial \pi^i}\right) \\
&= \left(\frac{\partial^2 \mathcal{T}}{\partial A_0 \partial \Lambda_j} - \frac{\partial U}{\partial A_j} + \partial_k\left(\frac{\partial U}{\partial \partial_k A_j}\right)\right)\frac{\partial \Lambda_j}{\partial \pi^i},
\end{aligned} \tag{B.20}
$$

where

$$
\partial_k\left(\frac{\partial^2 \mathcal{T}}{\partial \Lambda_k \partial \Lambda_j}\right)\frac{\partial \Lambda_j}{\partial \pi^i} = -\frac{\partial^2 \mathcal{T}}{\partial \Lambda_j \partial \Lambda_k}\partial_k\left(\frac{\partial \Lambda_j}{\partial \pi^i}\right). \tag{B.21}
$$

Eq.(B.21) can be solved from the eq.(B.3):

$$
\begin{aligned}
\frac{\partial^2 \mathcal{T}}{\partial \Lambda_i \partial \Lambda_j}\frac{\partial \Lambda_j}{\partial \pi^k} &= \delta^i_k \\
\partial_i\left(\frac{\partial^2 \mathcal{T}}{\partial \Lambda_i \partial \Lambda_j}\frac{\partial \Lambda_j}{\partial \pi^k}\right) &= \partial_i\left(\delta^i_k\right) \\
\partial_i\left(\frac{\partial^2 \mathcal{T}}{\partial \Lambda_i \partial \Lambda_j}\right)\frac{\partial \Lambda_j}{\partial \pi^k} + \frac{\partial^2 \mathcal{T}}{\partial \Lambda_i \partial \Lambda_j}\partial_i\left(\frac{\partial \Lambda_j}{\partial \pi^k}\right) &= 0.
\end{aligned} \tag{B.22}
$$

Similarly, by directly taking derivative of eq.(B.13) with respect to $\partial_i A_0$ and imposing eq.(A.21), one obtains

$$
\begin{aligned}
\frac{\partial \Omega_2}{\partial \partial_i A_0} &= \frac{\partial^2 \mathcal{T}}{\partial \partial_i A_0 \partial A_0} + \frac{\partial^2 \mathcal{T}}{\partial \Lambda_k \partial A_0}\frac{\partial \Lambda_k}{\partial \partial_i A_0} - \bigg(\bigg(\frac{\partial^3 \mathcal{T}}{\partial \partial_i A_0 \partial A_0 \partial \partial_j A_0} + \frac{\partial^3 \mathcal{T}}{\partial \Lambda_k \partial A_0 \partial \partial_j A_0}\frac{\partial \Lambda_k}{\partial \partial_i A_0}\bigg) \\
&\quad \times \partial_j A_0 + \frac{\partial^2 \mathcal{T}}{\partial A_0 \partial \partial_j A_0}\frac{\partial \partial_j A_0}{\partial \partial_i A_0} + \bigg(\frac{\partial^3 \mathcal{T}}{\partial \partial_i A_0 \partial \partial_j A_0 \partial \partial_l A_0} + \frac{\partial^3 \mathcal{T}}{\partial \Lambda_k \partial \partial_j A_0 \partial \partial_l A_0}\frac{\partial \Lambda_k}{\partial \partial_i A_0}\bigg) \\
&\quad \times \partial_j \partial_l A_0 + \bigg(\frac{\partial^3 \mathcal{T}}{\partial \partial_i A_0 \partial \Lambda_k \partial \partial_j A_0} + \frac{\partial^3 \mathcal{T}}{\partial \Lambda_l \partial \Lambda_k \partial \partial_j A_0}\frac{\partial \Lambda_l}{\partial \partial_i A_0}\bigg)\partial_j \Lambda_k \\
&\quad + \frac{\partial^2 \mathcal{T}}{\partial \Lambda_k \partial \partial_j A_0}\bigg(\frac{\partial^2 \Lambda_k}{\partial \partial_i A_0 \partial A_0}\partial_j A_0 + \frac{\partial \Lambda_k}{\partial A_0}\frac{\partial \partial_j A_0}{\partial \partial_i A_0} + \frac{\partial^2 \Lambda_k}{\partial \partial_i A_0 \partial \partial_l A_0}\partial_j \partial_l A_0 + \frac{\partial R_{jk}}{\partial \partial_i A_0}\bigg) \\
&\quad + \frac{\partial R_{\mathcal{T}}}{\partial \partial_i A_0} + \frac{\partial R_{\mathcal{T}}}{\partial \Lambda_k}\frac{\partial \Lambda_k}{\partial \partial_i A_0}\bigg) - \frac{\partial \Lambda_j}{\partial \partial_i A_0}\frac{\partial U}{\partial A_j} - \Lambda_j \frac{\partial^2 U}{\partial \partial_i A_0 \partial A_j} - \bigg(\frac{\partial^2 \Lambda_j}{\partial \partial_i A_0 \partial A_0}\partial_k A_0 \\
&\quad + \frac{\partial \Lambda_j}{\partial A_0}\frac{\partial \partial_k A_0}{\partial \partial_i A_0} + \frac{\partial^2 \Lambda_j}{\partial \partial_i A_0 \partial \partial_l A_0}\partial_k \partial_l A_0 + \frac{\partial R_{kj}}{\partial \partial_i A_0}\bigg)\frac{\partial U}{\partial \partial_k A_j} + \partial_j A_k \frac{\partial^2 U}{\partial \partial_i A_0 \partial \partial_j A_k}.
\end{aligned} \tag{B.23}
$$

Then one considers

$$\partial_j\left(\frac{\partial^2\mathcal{T}}{\partial\partial_iA_0\partial\partial_jA_0}\right) = \frac{\partial^3\mathcal{T}}{\partial A_0\partial\partial_iA_0\partial\partial_jA_0}\partial_jA_0 + \frac{\partial^3\mathcal{T}}{\partial\partial_kA_0\partial\partial_iA_0\partial\partial_jA_0}\partial_j\partial_kA_0 + \frac{\partial^3\mathcal{T}}{\partial\Lambda_k\partial\partial_iA_0\partial\partial_jA_0}\partial_j\Lambda_k + \frac{\partial R_{\mathcal{T}}}{\partial\partial_iA_0}, \tag{B.24}$$

$$\partial_j\left(\frac{\partial^2\mathcal{T}}{\partial\partial_jA_0\partial\Lambda_k}\right) = \frac{\partial^3\mathcal{T}}{\partial A_0\partial\partial_jA_0\partial\Lambda_k}\partial_jA_0 + \frac{\partial^3\mathcal{T}}{\partial\partial_lA_0\partial\partial_jA_0\partial\Lambda_k}\partial_j\partial_lA_0 + \frac{\partial^3\mathcal{T}}{\partial\Lambda_l\partial\partial_jA_0\partial\Lambda_k}\partial_j\Lambda_l + \frac{\partial R_{\mathcal{T}}}{\partial\Lambda_k}, \tag{B.25}$$

and

$$\partial_j\left(\frac{\partial\Lambda_k}{\partial\partial_iA_0}\right) = \frac{\partial^2\Lambda_k}{\partial A_0\partial\partial_iA_0}\partial_jA_0 + \frac{\partial^2\Lambda_k}{\partial\partial_lA_0\partial\partial_iA_0}\partial_j\partial_lA_0 + \frac{\partial R_{jk}}{\partial\partial_iA_0}. \tag{B.26}$$

One can rewrite $\partial\Omega_2/\partial\partial_iA_0$ after using eq.(B.1)-eq.(B.5), it finally gives

$$\begin{aligned}\frac{\partial\Omega_2}{\partial\partial_iA_0} &= \left(\frac{\partial^2\mathcal{T}}{\partial A_0\partial\Lambda_j} - \frac{\partial U}{\partial A_j}\right)\frac{\partial\Lambda_j}{\partial\partial_iA_0} - \frac{\partial U}{\partial\partial_kA_j}\partial_k\left(\frac{\partial\Lambda_j}{\partial\partial_iA_0}\right) + \partial_j\left(\frac{\partial^2\mathcal{T}}{\partial\Lambda_j\partial\partial_iA_0} + \frac{\partial U}{\partial\partial_iA_j}\right)\\ &\quad + \partial_j\left(\frac{\partial^2\mathcal{T}}{\partial\Lambda_j\partial\Lambda_k}\frac{\partial\Lambda_k}{\partial\partial_iA_0}\right) + \partial_j\left(\frac{\partial U}{\partial\partial_jA_k}\frac{\partial\Lambda_k}{\partial\partial_iA_0}\right) + \frac{\partial^2\mathcal{T}}{\partial\Lambda_i\partial\Lambda_j}\frac{\partial\Lambda_j}{\partial A_0}\\ &= \left(\frac{\partial^2\mathcal{T}}{\partial A_0\partial\Lambda_j} - \frac{\partial U}{\partial A_j}\right)\frac{\partial\Lambda_j}{\partial\partial_iA_0} - \frac{\partial U}{\partial\partial_kA_j}\partial_k\left(\frac{\partial\Lambda_j}{\partial\partial_iA_0}\right) + \partial_j\left(\frac{\partial^2\mathcal{T}}{\partial\Lambda_j\partial\partial_iA_0} + \frac{\partial U}{\partial\partial_iA_j}\right)\\ &\quad - \partial_j\left(\frac{\partial^2\mathcal{T}}{\partial\Lambda_j\partial\partial_iA_0}\right) + \partial_j\left(\frac{\partial U}{\partial\partial_jA_k}\right)\frac{\partial\Lambda_k}{\partial\partial_iA_0} + \frac{\partial U}{\partial\partial_jA_k}\partial_j\left(\frac{\partial\Lambda_k}{\partial\partial_iA_0}\right) + \frac{\partial^2\mathcal{T}}{\partial\Lambda_i\partial\Lambda_j}\frac{\partial\Lambda_j}{\partial A_0}\\ &= \left(\frac{\partial^2\mathcal{T}}{\partial A_0\partial\Lambda_j} - \frac{\partial U}{\partial A_j} + \partial_k\left(\frac{\partial U}{\partial\partial_kA_j}\right)\right)\frac{\partial\Lambda_j}{\partial\partial_iA_0} + \partial_j\left(\frac{\partial U}{\partial\partial_iA_j}\right) - \frac{\partial^2\mathcal{T}}{\partial\Lambda_i\partial A_0}.\end{aligned} \tag{B.27}$$

Let us substitutes all results into eq.(4.41), one obtains

$$\begin{aligned}\mathcal{C}_1^i &= \frac{\partial U}{\partial A_j}\frac{\partial\partial_k\pi^k}{\partial\partial_i\pi^j} - \frac{\partial U}{\partial\partial_iA_j}\left(\frac{\partial^2\mathcal{T}}{\partial A_0\partial\Lambda_k} - \frac{\partial U}{\partial A_k} + \partial_l\left(\frac{\partial U}{\partial\partial_lA_k}\right)\right)\frac{\partial\Lambda_k}{\partial\pi^j}\\ &\quad + \frac{\partial U}{\partial\partial_jA_k}\partial_j\left(\frac{\partial\partial_l\pi^l}{\partial\partial_i\pi^k}\right) + \frac{\partial U}{\partial\partial_iA_k}\partial_j\left(\frac{\partial\partial_l\pi^l}{\partial\partial_j\pi^k}\right) + \left(\frac{\partial^2\mathcal{T}}{\partial A_0\partial\Lambda_j} - \frac{\partial U}{\partial A_j}\right.\\ &\quad \left.+ \partial_k\left(\frac{\partial U}{\partial\partial_kA_j}\right)\right)\frac{\partial\Lambda_j}{\partial\partial_iA_0} + \partial_j\left(\frac{\partial U}{\partial\partial_iA_j}\right) - \frac{\partial^2\mathcal{T}}{\partial\Lambda_i\partial A_0} - 2\partial_k\left(\frac{\partial U}{\partial\partial_lA_j}\frac{\partial\partial_j\partial_lA_0}{\partial\partial_i\partial_kA_0}\right)\\ &= \left(-\delta^i_j + \frac{\partial\Lambda_j}{\partial\partial_iA_0} - \frac{\partial U}{\partial\partial_iA_k}\frac{\partial\Lambda_j}{\partial\pi^k}\right)\left(\frac{\partial^2\mathcal{T}}{\partial A_0\partial\Lambda_j} - \frac{\partial U}{\partial A_j} + \partial_l\left(\frac{\partial U}{\partial\partial_lA_j}\right)\right).\end{aligned} \tag{B.28}$$

To see that $\mathcal{C}_1^i$ identically vanish, one will show that first term of $\mathcal{C}_1^i$ equals to zero by using eq.(B.3) and

$$\frac{\partial^2\mathcal{T}}{\partial\Lambda_j\partial\Lambda_k}\frac{\partial\Lambda_k}{\partial\partial_i A_0} = -\frac{\partial^2\mathcal{T}}{\partial\Lambda_j\partial\partial_i A_0}. \tag{B.29}$$

It gives

$$\begin{aligned}\frac{\partial^2\mathcal{T}}{\partial\Lambda_j\partial\Lambda_m}\left(-\delta_j^i+\frac{\partial\Lambda_j}{\partial\partial_i A_0}-\frac{\partial U}{\partial\partial_i A_k}\frac{\partial\Lambda_j}{\partial\pi^k}\right) &= -\frac{\partial^2\mathcal{T}}{\partial\Lambda_i\partial\Lambda_m}+\frac{\partial^2\mathcal{T}}{\partial\Lambda_j\partial\Lambda_m}\frac{\partial\Lambda_j}{\partial\partial_i A_0}\\ &\quad -\frac{\partial U}{\partial\partial_i A_k}\frac{\partial^2\mathcal{T}}{\partial\Lambda_j\partial\Lambda_m}\frac{\partial\Lambda_j}{\partial\pi^k}\\ &= -\frac{\partial^2\mathcal{T}}{\partial\Lambda_i\partial\Lambda_m}-\frac{\partial^2\mathcal{T}}{\partial\Lambda_m\partial\partial_i A_0}-\frac{\partial U}{\partial\partial_i A_k}\delta_k^m\\ &= 0.\end{aligned} \tag{B.30}$$

In eq.(B.3), it show that the matrix $\partial^2\mathcal{T}/\partial\Lambda_i\partial\Lambda_j$ is invertible. One can rewrite eq.(B.30) as

$$-\delta_j^i+\frac{\partial\Lambda_j}{\partial\partial_i A_0}-\frac{\partial U}{\partial\partial_i A_k}\frac{\partial\Lambda_j}{\partial\pi^k} = 0. \tag{B.31}$$

Then one substitutes it back into $\mathcal{C}_1^i$. It gives

$$\mathcal{C}_1^i = 0. \tag{B.32}$$

Therefore, the condition in eq.(4.45) is automatically satisfied by diffeomorphism invariance.

B.3 Simplication of $\mathcal{C}_0$

In this section, one will rewrite $\mathcal{C}_0$ in simply way by using diffeomorphism invariance conditions. Let us consider eq.(4.42), one needs derivatives of Ω_2 with respect to $\partial_i\partial_j A_0, \partial_i\pi^j, \pi^i, \partial_i A_0$ and A_0. One already has the results from the last section except A_0. So let us consider $\partial\Omega_2/\partial A_0$ by taking derivative of eq.(B.13) with respect to A_0,

$$\frac{\partial\Omega_2}{\partial A_0} = \frac{\partial^2\mathcal{T}}{\partial A_0\partial A_0} + \frac{\partial^2\mathcal{T}}{\partial\Lambda_i\partial A_0}\frac{\partial\Lambda_i}{\partial A_0} - \Bigg(\bigg(\frac{\partial^3\mathcal{T}}{\partial A_0\partial A_0\partial\partial_i A_0} + \frac{\partial^3\mathcal{T}}{\partial\Lambda_j\partial A_0\partial\partial_i A_0}\frac{\partial\Lambda_j}{\partial A_0}\bigg)\partial_i A_0 + \bigg(\frac{\partial^3\mathcal{T}}{\partial A_0\partial\partial_j A_0\partial\partial_i A_0} + \frac{\partial^3\mathcal{T}}{\partial\Lambda_k\partial\partial_j A_0\partial\partial_i A_0}\frac{\partial\Lambda_k}{\partial A_0}\bigg)\partial_i\partial_j A_0 + \bigg(\frac{\partial^3\mathcal{T}}{\partial A_0\partial\Lambda_l\partial\partial_i A_0} + \frac{\partial^3\mathcal{T}}{\partial\Lambda_k\partial\Lambda_l\partial\partial_i A_0}\frac{\partial\Lambda_k}{\partial A_0}\bigg)\partial_i\Lambda_l + \frac{\partial^2\mathcal{T}}{\partial\Lambda_k\partial\partial_i A_0}\bigg(\frac{\partial^2\Lambda_k}{\partial A_0\partial A_0}\partial_i A_0 + \frac{\partial^2\Lambda_k}{\partial A_0\partial\partial_j A_0}\partial_i\partial_j A_0 + \frac{\partial^2\Lambda_k}{\partial A_0\partial\pi^j}\partial_i\pi^j + \frac{\partial R_{ik}}{\partial A_0}\bigg) + \frac{\partial R_{\mathcal{T}}}{\partial A_0}\Bigg) - \frac{\partial\Lambda_i}{\partial A_0}\frac{\partial U}{\partial A_i} - \Lambda_i\frac{\partial^2 U}{\partial A_0\partial A_i} - \bigg(\frac{\partial^2\Lambda_i}{\partial A_0\partial A_0}\partial_j A_0 + \frac{\partial^2\Lambda_i}{\partial A_0\partial\partial_k A_0}\partial_j\partial_k A_0 + \frac{\partial^2\Lambda_i}{\partial A_0\partial\pi^l}\partial_j\pi^l + \frac{\partial R_{ji}}{\partial A_0}\bigg)\frac{\partial U}{\partial\partial_j A_i} - \partial_j\Lambda_i\frac{\partial^2 U}{\partial A_0\partial\partial_j A_i}. \tag{B.33}$$

One considers

$$\partial_i\bigg(\frac{\partial^2\mathcal{T}}{\partial\partial_i A_0\partial A_0}\bigg) = \frac{\partial^3\mathcal{T}}{\partial A_0\partial\partial_i A_0\partial A_0}\partial_i A_0 + \frac{\partial^3\mathcal{T}}{\partial\partial_j A_0\partial\partial_i A_0\partial A_0}\partial_i\partial_j A_0 + \frac{\partial^3\mathcal{T}}{\partial\Lambda_l\partial\partial_i A_0\partial A_0}\partial_i\Lambda_l + \frac{\partial R_{\mathcal{T}}}{\partial A_0}, \tag{B.34}$$

$$\partial_i\bigg(\frac{\partial^2\mathcal{T}}{\partial\partial_i A_0\partial\Lambda_k}\bigg) = \frac{\partial^3\mathcal{T}}{\partial A_0\partial\partial_i A_0\partial\Lambda_k}\partial_i A_0 + \frac{\partial^3\mathcal{T}}{\partial\partial_j A_0\partial\partial_i A_0\partial\Lambda_k}\partial_i\partial_j A_0 + \frac{\partial^3\mathcal{T}}{\partial\Lambda_l\partial\partial_i A_0\partial\Lambda_k}\partial_i\Lambda_l + \frac{\partial R_{\mathcal{T}}}{\partial\Lambda_k}, \tag{B.35}$$

and

$$\partial_i\bigg(\frac{\partial\Lambda_k}{\partial A_0}\bigg) = \frac{\partial^2\Lambda_k}{\partial A_0\partial A_0}\partial_i A_0 + \frac{\partial^2\Lambda_k}{\partial\partial_j A_0\partial A_0}\partial_i\partial_j A_0 + \frac{\partial R_{ik}}{\partial A_0}. \tag{B.36}$$

So one can rewrite $\partial\Omega_2/\partial A_0$ by using eqs.(B.1)-(B.5) as

$$\begin{aligned}\frac{\partial\Omega_2}{\partial A_0} &= \frac{\partial^2\mathcal{T}}{\partial A_0\partial A_0} + \bigg(\frac{\partial^2\mathcal{T}}{\partial A_0\partial\Lambda_i} - \frac{\partial U}{\partial A_i}\bigg)\frac{\partial\Lambda_i}{\partial A_0} - \partial_i\bigg(\frac{\partial^2\mathcal{T}}{\partial\partial_i A_0\partial A_0}\bigg) - \partial_i\bigg(\frac{\partial^2\mathcal{T}}{\partial\Lambda_k\partial\partial_i A_0}\frac{\partial\Lambda_k}{\partial A_0}\bigg) \\ &\quad - \partial_j\bigg(\frac{\partial\Lambda_i}{\partial A_0}\bigg)\frac{\partial U}{\partial\partial_j A_i} - \frac{\partial^2 U}{\partial A_0\partial\partial_j A_i}\partial_j\Lambda_i - \frac{\partial^2 U}{\partial A_0\partial A_i}\Lambda_i \\ &= \frac{\partial^2\mathcal{T}}{\partial A_0\partial A_0} + \bigg(\frac{\partial^2\mathcal{T}}{\partial A_0\partial\Lambda_i} - \frac{\partial U}{\partial A_i} + \partial_j\bigg(\frac{\partial U}{\partial\partial_j A_i}\bigg)\bigg)\frac{\partial\Lambda_i}{\partial A_0} - \partial_i\bigg(\frac{\partial^2\mathcal{T}}{\partial\partial_i A_0\partial A_0} + \frac{\partial^2\mathcal{T}}{\partial\Lambda_i\partial A_0}\bigg) \\ &\quad - \frac{\partial^2 U}{\partial A_0\partial\partial_i A_j}\partial_i\Lambda_j - \frac{\partial^2 U}{\partial A_0\partial A_i}\Lambda_i.\end{aligned} \tag{B.37}$$

After substituting these results into eq.(4.42), one sees that $\mathcal{C}_0$ is now given by

$$\begin{aligned}\mathcal{C}_0 &= \frac{\partial\Omega_1}{\partial A_i}\frac{\partial\Omega_2}{\partial\pi^i} - \frac{\partial\Omega_1}{\partial A_i}\partial_j\left(\frac{\partial\Omega_2}{\partial\partial_j\pi^i}\right) + \frac{\partial\Omega_1}{\partial\partial_i A_j}\partial_i\left(\frac{\partial\Omega_2}{\partial\pi^j}\right) - \frac{\partial\Omega_1}{\partial\partial_i A_j}\partial_i\partial_k\left(\frac{\partial\Omega_2}{\partial\partial_k\pi^j}\right)\\ &\quad - \frac{\partial\Omega_2}{\partial A_0} + \partial_j\left(\frac{\partial\Omega_2}{\partial\partial_j A_0}\right) - \partial_j\partial_k\left(\frac{\partial\Omega_2}{\partial\partial_j\partial_k A_0}\right)\\ &= -\frac{\partial U}{\partial A_i}\left(\frac{\partial^2\mathcal{T}}{\partial A_0\partial\Lambda_k} - \frac{\partial U}{\partial A_k} + \partial_l\left(\frac{\partial U}{\partial\partial_l A_k}\right)\right)\frac{\partial\Lambda_k}{\partial\pi^i} - \frac{\partial U}{\partial\partial_i A_j}\partial_i\left(\left(\frac{\partial^2\mathcal{T}}{\partial A_0\partial\Lambda_k}\right.\right.\\ &\quad\left.\left. - \frac{\partial U}{\partial A_k} + \partial_l\left(\frac{\partial U}{\partial\partial_l A_k}\right)\right)\frac{\partial\Lambda_k}{\partial\pi^j}\right) - \frac{\partial\Omega_2}{\partial A_0} + \partial_j\left(\left(\frac{\partial^2\mathcal{T}}{\partial A_0\partial\Lambda_k} - \frac{\partial U}{\partial A_k}\right.\right.\\ &\quad\left.\left. + \partial_l\left(\frac{\partial U}{\partial\partial_l A_k}\right)\right)\frac{\partial\Lambda_k}{\partial\partial_j A_0} + \partial_k\left(\frac{\partial U}{\partial\partial_j A_k}\right) - \frac{\partial^2\mathcal{T}}{\partial\Lambda_j\partial A_0}\right) - \partial_j\partial_k\left(\frac{\partial U}{\partial\partial_j A_k}\right)\\ &= -\frac{\partial U}{\partial A_i}\Pi^k\frac{\partial\Lambda_k}{\partial\pi^i} - \frac{\partial U}{\partial\partial_i A_j}\partial_i\left(\Pi^k\frac{\partial\Lambda_k}{\partial\pi^j}\right) - \frac{\partial\Omega_2}{\partial A_0} + \partial_j\left(\Pi^k\frac{\partial\Lambda_k}{\partial\partial_j A_0} - \frac{\partial^2\mathcal{T}}{\partial\Lambda_j\partial A_0}\right),\end{aligned} \tag{B.38}$$

where

$$\Pi^k \equiv \frac{\partial^2\mathcal{T}}{\partial A_0\partial\Lambda_k} - \frac{\partial U}{\partial A_k} + \partial_l\left(\frac{\partial U}{\partial\partial_l A_k}\right). \tag{B.39}$$

It gives

$$\begin{aligned}\mathcal{C}_0 = -\frac{\partial U}{\partial A_i}\Pi^k\frac{\partial\Lambda_k}{\partial\pi^i} - \frac{\partial U}{\partial\partial_i A_j}\partial_i\Pi^k\frac{\partial\Lambda_k}{\partial\pi^j} - \frac{\partial U}{\partial\partial_i A_j}\Pi^k\partial_i\frac{\partial\Lambda_k}{\partial\pi^j} - \frac{\partial\Omega_2}{\partial A_0}\\ + \partial_j\Pi^k\frac{\partial\Lambda_k}{\partial\partial_j A_0} + \Pi^k\partial_j\frac{\partial\Lambda_k}{\partial\partial_j A_0} - \partial_j\left(\frac{\partial^2\mathcal{T}}{\partial\Lambda_j\partial A_0}\right).\end{aligned} \tag{B.40}$$

Then one consider these two terms

$$\begin{aligned}\partial_i\Pi^k\left(-\frac{\partial U}{\partial\partial_i A_j}\frac{\partial\Lambda_k}{\partial\pi^j} + \frac{\partial\Lambda_k}{\partial\partial_i A_0}\right) &= \partial_i\Pi^k\left(\frac{\partial^2\mathcal{T}}{\partial\Lambda_j\partial\Lambda_i}\frac{\partial\Lambda_k}{\partial\pi^j} + \frac{\partial^2\mathcal{T}}{\partial\Lambda_j\partial\partial_i A_0}\frac{\partial\Lambda_k}{\partial\pi_j} + \frac{\partial\Lambda_k}{\partial\partial_i A_0}\right)\\ &= \partial_i\Pi^k\left(\delta^i_k - \frac{\partial^2\mathcal{T}}{\partial\Lambda_j\partial\Lambda_l}\frac{\partial\Lambda_l}{\partial\partial_i A_0}\frac{\partial\Lambda_k}{\partial\pi^j} + \frac{\partial\Lambda_k}{\partial\partial_i A_0}\right)\\ &= \partial_i\Pi^i.\end{aligned} \tag{B.41}$$

Let us substitute them back into $\mathcal{C}_0$, it can be written as

$$\begin{aligned}\mathcal{C}_0 = -\frac{\partial U}{\partial A_i}\Pi^k\frac{\partial\Lambda_k}{\partial\pi^i} + \partial_i\Pi^i - \frac{\partial U}{\partial\partial_i A_j}\Pi^k\partial_i\left(\frac{\partial\Lambda_k}{\partial\pi^j}\right) - \frac{\partial\Omega_2}{\partial A_0} + \Pi^k\partial_j\left(\frac{\partial\Lambda_k}{\partial\partial_j A_0}\right)\\ - \partial_j\left(\frac{\partial^2\mathcal{T}}{\partial\Lambda_j\partial A_0}\right).\end{aligned} \tag{B.42}$$

Then one considers

$$\partial_j\left(\frac{\partial\Lambda_k}{\partial\partial_j A_0}\right)-\frac{\partial U}{\partial\partial_i A_j}\partial_i\left(\frac{\partial\Lambda_k}{\partial\pi^j}\right)=\partial_j\left(\frac{\partial\Lambda_k}{\partial\partial_j A_0}-\frac{\partial U}{\partial\partial_j A_m}\frac{\partial\Lambda_k}{\partial\pi^m}\right)+\partial_j\left(\frac{\partial U}{\partial\partial_j A_m}\right)\frac{\partial\Lambda_k}{\partial\pi^m}$$
$$=\partial_j\left(\frac{\partial U}{\partial\partial_j A_m}\right)\frac{\partial\Lambda_k}{\partial\pi^m}. \tag{B.43}$$

It gives

$$\begin{aligned}\mathcal{C}_0&=-\frac{\partial U}{\partial A_i}\Pi^k\frac{\partial\Lambda_k}{\partial\pi^i}+\Pi^k\partial_j\left(\frac{\partial U}{\partial\partial_j A_i}\right)\frac{\partial\Lambda_k}{\partial\pi^i}-\frac{\partial\Omega_2}{\partial A_0}-\partial_j\left(\frac{\partial^2\mathcal{T}}{\partial\Lambda_j\partial A_0}\right)\\&\quad+\partial_i\left(\Pi^i-\frac{\partial^2\mathcal{T}}{\partial\Lambda_i\partial A_0}\right)\\&=\partial_i\left(-\frac{\partial U}{\partial A_i}+\partial_j\left(\frac{\partial U}{\partial\partial_j A_i}\right)\right)+\Pi^k\frac{\partial\Lambda_k}{\partial\pi^i}\left(-\frac{\partial U}{\partial A_i}+\partial_j\left(\frac{\partial U}{\partial\partial_j A_i}\right)\right)-\frac{\partial\Omega_2}{\partial A_0}.\end{aligned} \tag{B.44}$$

Let us substitutes $\partial\Omega_2/\partial A_0$ into $\mathcal{C}_0$, one obtains

$$\begin{aligned}\mathcal{C}_0=\partial_i&\left(\frac{\partial^2\mathcal{T}}{\partial\partial_i A_0\partial A_0}+\frac{\partial^2\mathcal{T}}{\partial\Lambda_i\partial A_0}-\frac{\partial U}{\partial A_i}+\partial_k\left(\frac{\partial U}{\partial\partial_k A_i}\right)\right)\\&+\Pi^k\frac{\partial\Lambda_k}{\partial\pi^i}\left(-\frac{\partial U}{\partial A_i}+\partial_j\left(\frac{\partial U}{\partial\partial_j A_i}\right)\right)-\frac{\partial^2\mathcal{T}}{\partial A_0\partial A_0}\\&+\frac{\partial^2 U}{\partial A_0\partial\partial_i A_j}\partial_i\Lambda_j-\frac{\partial^2 U}{\partial A_0\partial A_i}\Lambda_i+\Pi^i\frac{\partial\Lambda_i}{\partial A_0}.\end{aligned} \tag{B.45}$$

Then one considers

$$\begin{aligned}\frac{\partial^2\mathcal{T}}{\partial A_0\partial\Lambda_j}\frac{\partial\Lambda_j}{\partial\pi^i}&=-\frac{\partial^2\mathcal{T}}{\partial\Lambda_j\partial\Lambda_k}\frac{\partial\Lambda_k}{\partial A_0}\frac{\partial\Lambda_j}{\partial\pi^i}\\&=\frac{\partial\Lambda_i}{\partial A_0}.\end{aligned} \tag{B.46}$$

Substituting back into $\mathcal{C}_0$, one finally obtains

$$\begin{aligned}\mathcal{C}_0=\partial_i&\left(\frac{\partial^2\mathcal{T}}{\partial\partial_i A_0\partial A_0}+\frac{\partial^2\mathcal{T}}{\partial\Lambda_i\partial A_0}-\frac{\partial U}{\partial A_i}+\partial_k\left(\frac{\partial U}{\partial\partial_k A_i}\right)\right)+\Pi^k\frac{\partial\Lambda_k}{\partial\pi^i}\left(\frac{\partial^2\mathcal{T}}{\partial A_0\partial\Lambda_i}\right.\\&\left.-\frac{\partial U}{\partial A_i}+\partial_j\left(\frac{\partial U}{\partial\partial_j A_i}\right)\right)-\frac{\partial^2\mathcal{T}}{\partial A_0\partial A_0}+\frac{\partial^2 U}{\partial A_0\partial\partial_i A_j}\partial_i\Lambda_j-\frac{\partial^2 U}{\partial A_0\partial A_i}\Lambda_i\\=\partial_i&\left(\frac{\partial^2\mathcal{T}}{\partial\partial_i A_0\partial A_0}+\frac{\partial^2\mathcal{T}}{\partial\Lambda_i\partial A_0}-\frac{\partial U}{\partial A_i}+\partial_k\left(\frac{\partial U}{\partial\partial_k A_i}\right)\right)\\&+\frac{\partial\Lambda_k}{\partial\pi^i}\left(\frac{\partial^2\mathcal{T}}{\partial A_0\partial\Lambda_k}-\frac{\partial U}{\partial A_k}+\partial_j\left(\frac{\partial U}{\partial\partial_j A_k}\right)\right)\left(\frac{\partial^2\mathcal{T}}{\partial A_0\partial\Lambda_i}-\frac{\partial U}{\partial A_i}+\partial_l\left(\frac{\partial U}{\partial\partial_l A_i}\right)\right)\\&-\frac{\partial^2\mathcal{T}}{\partial A_0\partial A_0}+\frac{\partial^2 U}{\partial A_0\partial\partial_i A_j}\partial_i\Lambda_j-\frac{\partial^2 U}{\partial A_0\partial A_i}\Lambda_i.\end{aligned} \tag{B.47}$$

It is possible to rewrite $\mathcal{C}_0$ in configuration space. This is simply by replacing $\Lambda_i \to \dot{A}_i$. Therefore

$$\mathcal{C}_0 = \partial_i\left(\frac{\partial^2 T}{\partial\partial_i A_0 \partial A_0} + \frac{\partial^2 T}{\partial \dot{A}_i \partial A_0} - \frac{\partial U}{\partial A_i} + \partial_j\left(\frac{\partial U}{\partial\partial_j A_i}\right)\right)$$
$$+ (T^{-1})_{ki}\left(\frac{\partial^2 T}{\partial A_0 \partial \dot{A}_k} - \frac{\partial U}{\partial A_k} + \partial_j\left(\frac{\partial U}{\partial\partial_j A_k}\right)\right)\left(\frac{\partial^2 T}{\partial A_0 \partial \dot{A}_i} - \frac{\partial U}{\partial A_i} + \partial_j\left(\frac{\partial U}{\partial\partial_j A_i}\right)\right)$$
$$- \frac{\partial^2 T}{\partial A_0 \partial A_0} + \frac{\partial^2 U}{\partial A_0 \partial\partial_i A_j}\partial_i \dot{A}_j - \frac{\partial^2 U}{\partial A_0 \partial A_i}\dot{A}_i, \tag{B.48}$$

where $(T^{-1})_{ki}$ is the matrix inverse of $T^{ki} \equiv \partial^2 T/\partial \dot{A}_k \partial \dot{A}_i$. The form of $\mathcal{C}_0$ in eq.(B.48) makes it easier to check whether any given Lagrange in eq.(3.14) satisfies eq.(4.97).

APPENDIX C CONDITIONS FROM DIFFEOMORPHISM INVARIANCE IN MULTI-FIELD GENERALISED PROCA THEORIES

In this appendix, we consider a class of theories described in section 4.2. Since these theories are diffeomorphism invariant, their Lagrangians would satisfy the conditions to be presented in this appendix.

Under diffeomorphism $x^\mu \mapsto x^\mu - \epsilon^\mu(x)$, the vector fields transform as

$$\delta_\epsilon A^\alpha_\mu = \epsilon^\nu \partial_\nu A^\alpha_\mu + A^\alpha_\nu \partial_\mu \epsilon^\nu, \tag{C.1}$$

and the external fields K transform under standard diffeomorphism. The Lagrangian density transforms as eq.(A.15). Demanding the expression $\delta_\epsilon \mathcal{L} - \partial_\mu(\epsilon^\mu \mathcal{L})$ to vanish will give rise to useful conditions. In order to evaluate this expression, we begin by recall that $\mathcal{L} = T + U_\alpha \dot{A}^\alpha_0$. Then we consider

$$\delta_\epsilon T = \frac{\partial T}{\partial A^\alpha_\nu}\delta_\epsilon A^\alpha_\nu + \frac{\partial T}{\partial \partial_k A^\alpha_\nu}\partial_k \delta_\epsilon A^\alpha_\nu + \frac{\partial T}{\partial \dot{A}^\alpha_k}\partial_0 \delta_\epsilon A^\alpha_k + \frac{\partial T}{\partial K}\delta_\epsilon K. \tag{C.2}$$

Next, let us consider

$$\begin{aligned} -\partial_\mu(\epsilon^\mu T) = -\partial_\mu \epsilon^\mu T - \epsilon^\mu \left(\frac{\partial T}{\partial A^\alpha_\nu}\partial_\mu A^\alpha_\nu + \frac{\partial T}{\partial \partial_k A^\alpha_\nu}\partial_k \partial_\mu A^\alpha_\nu + \frac{\partial T}{\partial \dot{A}^\alpha_k}\partial_0 \partial_\mu A^\alpha_k \right) \\ - \epsilon^\mu \frac{\partial T}{\partial K}\partial_\mu K. \end{aligned} \tag{C.3}$$

Combining the two expressions, we obtain

$$\begin{aligned} \delta_\epsilon T - \partial_\mu(\epsilon^\mu T) = {} & \frac{\partial T}{\partial A^\alpha_\nu} A^\alpha_\mu \partial_\nu \epsilon^\mu + \frac{\partial T}{\partial \partial_k A^\alpha_\nu}\left(\partial_k \epsilon^\mu \partial_\mu A^\alpha_\nu + \partial_k A^\alpha_\mu \partial_\nu \epsilon^\mu + A^\alpha_\mu \partial_k \partial_\nu \epsilon^\mu\right) \\ & + \frac{\partial T}{\partial \dot{A}^\alpha_k}\left(\dot{\epsilon}^\mu \partial_\mu A^\alpha_k + \dot{A}^\alpha_\mu \partial_k \epsilon^\mu + A^\alpha_\mu \partial_k \dot{\epsilon}^\mu\right) - \partial_\mu \epsilon^\mu T \\ & + \frac{\partial T}{\partial K}\delta_\epsilon K - \epsilon^\mu \frac{\partial T}{\partial K}\partial_\mu K. \end{aligned} \tag{C.4}$$

Let us also compute

$$\begin{aligned} \delta_\epsilon(U_\beta \dot{A}^\beta_0) = {} & \frac{\partial U_\beta}{\partial A^\alpha_\nu}(\epsilon^\mu \partial_\mu A^\alpha_\nu + A^\alpha_\mu \partial_\nu \epsilon^\mu)\dot{A}^\beta_0 + \frac{\partial U_\beta}{\partial \partial_i A^\alpha_\nu}\partial_i(\epsilon^\mu \partial_\mu A^\alpha_\nu + A^\alpha_\mu \partial_\nu \epsilon^\mu)\dot{A}^\beta_0 \\ & + \dot{A}^\beta_0 \frac{\partial U_\beta}{\partial K}\delta_\epsilon K + U_\beta \partial_0(\epsilon^\mu \partial_\mu A^\beta_0 + A^\beta_\nu \partial_0 \epsilon^\nu), \end{aligned} \tag{C.5}$$

and

$$-\partial_\mu(\epsilon^\mu U_\beta \dot{A}_0^\beta) = -\partial_\mu \epsilon^\mu U_\beta \dot{A}_0^\beta - \epsilon^\mu \left(\frac{\partial U_\beta}{\partial A_\nu^\alpha} \partial_\mu A_\nu^\alpha + \frac{\partial U_\beta}{\partial \partial_i A_\nu^\alpha} \partial_i \partial_\mu A_\nu^\alpha \right) \dot{A}_0^\beta \\ - \epsilon^\mu \dot{A}_0^\beta \frac{\partial U_\beta}{\partial K} \partial_\mu K - \epsilon^\mu U_\beta \partial_\mu \dot{A}_0^\beta. \tag{C.6}$$

So we have

$$\delta_\epsilon(U_\beta \dot{A}_0^\beta) - \partial_\mu(\epsilon^\mu U_\beta \dot{A}_0^\beta) = \frac{\partial U_\beta}{\partial A_\nu^\alpha} A_\mu^\alpha \partial_\nu \epsilon^\mu \dot{A}_0^\beta + U_\beta(\dot{\epsilon}^\mu \partial_\mu A_0^\beta + \epsilon^\mu \partial_\mu \dot{A}_0^\beta + \dot{A}_\nu^\beta \partial_0 \epsilon^\nu + A_\nu^\beta \ddot{\epsilon}^\nu) \\ - \partial_\mu \epsilon^\mu U_\beta \dot{A}_0^\beta - \epsilon^\mu U_\beta \partial_\mu \dot{A}_0^\beta + \dot{A}_0^\beta \frac{\partial U_\beta}{\partial K} \delta_\epsilon K - \epsilon^\mu \dot{A}_0^\beta \frac{\partial U_\beta}{\partial K} \partial_\mu K \\ + \frac{\partial U_\beta}{\partial \partial_i A_\nu^\alpha} (\partial_i \epsilon^\mu \partial_\mu A_\nu^\alpha + \partial_i A_\mu^\alpha \partial_\nu \epsilon^\mu + A_\mu^\alpha \partial_i \partial_\nu \epsilon^\mu) \dot{A}_0^\beta. \tag{C.7}$$

Combining eq.(C.4) with eq.(C.7), we obtain the expression for $\delta_\epsilon \mathcal{L} - \partial_\mu(\epsilon^\mu \mathcal{L})$.

We are now ready to obtain useful conditions. Let us note that the expression $\delta_\epsilon \mathcal{L} - \partial_\mu(\epsilon^\mu \mathcal{L})$ is a polynomial in $\dot{A}_0^\alpha$ up to degree two. Consider the term containing $\dot{A}_0^\alpha \dot{A}_0^\beta$ in $\delta_\epsilon \mathcal{L} - \partial_\mu(\epsilon^\mu \mathcal{L})$. It can easily be seen that there is only one term which is

$$\frac{\partial U_\beta}{\partial \partial_i A_0^\alpha} \partial_i \epsilon^0 \dot{A}_0^\alpha \dot{A}_0^\beta. \tag{C.8}$$

Demanding this expression to vanish gives

$$\frac{\partial U_\alpha}{\partial \partial_i A_0^\beta} + \frac{\partial U_\beta}{\partial \partial_i A_0^\alpha} = 0. \tag{C.9}$$

Let us next turn to the coefficients of $\dot{A}_0^\beta$. For this, it would be convenient to consider eq.(C.4) and eq.(C.7) and collect the terms proportional to $\dot{A}_0^\beta$. We have

$$\delta_\epsilon T - \partial_\mu(\epsilon^\mu T) \ni \frac{\partial T}{\partial \partial_k A_0^\alpha} \partial_k \epsilon^0 \dot{A}_0^\alpha + \frac{\partial T}{\partial \dot{A}_k^\alpha} \dot{A}_0^\alpha \partial_k \epsilon^0, \tag{C.10}$$

and

$$\delta_\epsilon(U_\beta \dot{A}_0^\beta) - \partial_\mu(\epsilon^\mu U_\beta \dot{A}_0^\beta) \ni \frac{\partial U_\beta}{\partial A_\nu^\alpha} A_\mu^\alpha \partial_\nu \epsilon^\mu \dot{A}_0^\beta + U_\beta(\dot{\epsilon}^0 \dot{A}_0^\beta + \dot{A}_0^\beta \dot{\epsilon}^0) - \partial_\mu \epsilon^\mu U_\beta \dot{A}_0^\beta \\ + \frac{\partial U_\beta}{\partial \partial_i A_\nu^\alpha} (\partial_i \epsilon^\mu \partial_\mu A_\nu^\alpha + \partial_i A_\mu^\alpha \partial_\nu \epsilon^\mu + A_\mu^\alpha \partial_i \partial_\nu \epsilon^\mu) \Big|_{\dot{A}_0^\alpha = 0} \dot{A}_0^\beta \\ + \dot{A}_0^\beta \frac{\partial U_\beta}{\partial K} \delta_\epsilon K - \epsilon^\mu \dot{A}_0^\beta \frac{\partial U_\beta}{\partial K} \partial_\mu K. \tag{C.11}$$

Therefore, we have

$$\frac{\partial T}{\partial \partial_k A_0^\beta}\partial_k \epsilon^0 + \frac{\partial T}{\partial \dot{A}_k^\beta}\partial_k \epsilon^0 + \frac{\partial U_\beta}{\partial A_\nu^\alpha} A_\mu^\alpha \partial_\nu \epsilon^\mu + 2U_\beta \dot{\epsilon}^0 - \partial_\mu \epsilon^\mu U_\beta + \frac{\partial U_\beta}{\partial K}\delta_\epsilon K$$
$$+ \frac{\partial U_\beta}{\partial \partial_i A_\nu^\alpha}(\partial_i \epsilon^\mu \partial_\mu A_\nu^\alpha + \partial_i A_\mu^\alpha \partial_\nu \epsilon^\mu + A_\mu^\alpha \partial_i \partial_\nu \epsilon^\mu)\bigg|_{\dot{A}_0^\alpha = 0} - \epsilon^\mu \frac{\partial U_\beta}{\partial K}\partial_\mu K = 0. \tag{C.12}$$

Although the above equation looks complicated especially due to the explicit presence of external fields, we will only extract some parts of this equation to obtain the conditions that we will need. These conditions will look much more simple. For example, the dependence on the external fields and their derivatives are only through T and U_β. We may derive these conditions as follows. Taking derivative of eq.(C.12) with respect to $\dot{A}_j^\alpha$ gives

$$\frac{\partial^2 T}{\partial \partial_k A_0^\beta \partial \dot{A}_j^\alpha} + \frac{\partial^2 T}{\partial \dot{A}_k^\beta \partial \dot{A}_j^\alpha} + \frac{\partial U_\beta}{\partial \partial_k A_j^\alpha} = 0. \tag{C.13}$$

Let us take derivative of eq.(C.12) with respect to $\partial_j A_0^\alpha$, then swap the indices α and β, add it to the original equation, and use eq.(C.9), we obtain

$$2\frac{\partial^2 T}{\partial \partial_j A_0^{(\alpha} \partial \partial_k A_0^{\beta)}} + 2\frac{\partial^2 T}{\partial \partial_j A_0^{(\alpha} \partial \dot{A}_k^{\beta)}} + \frac{\partial U_\beta}{\partial \partial_j A_k^\alpha} + \frac{\partial U_\alpha}{\partial \partial_j A_k^\beta} = 0. \tag{C.14}$$

Expressing in phase space, the conditions eqs.(C.13)-(C.14) become

$$\frac{\partial^2 \mathcal{T}}{\partial \partial_k A_0^\beta \partial \Lambda_j^\alpha} + \frac{\partial^2 \mathcal{T}}{\partial \Lambda_k^\beta \partial \Lambda_j^\alpha} + \frac{\partial U_\beta}{\partial \partial_k A_j^\alpha} = 0, \tag{C.15}$$

and

$$2\frac{\partial^2 \mathcal{T}}{\partial \partial_j A_0^{(\alpha} \partial \partial_k A_0^{\beta)}} + 2\frac{\partial^2 \mathcal{T}}{\partial \partial_j A_0^{(\alpha} \partial \Lambda_k^{\beta)}} + \frac{\partial U_\beta}{\partial \partial_j A_k^\alpha} + \frac{\partial U_\alpha}{\partial \partial_j A_k^\beta} = 0. \tag{C.16}$$

By substituting eq.(C.15) into eq.(C.16), we obtain

$$\frac{\partial^2 \mathcal{T}}{\partial \partial_{(j|} A_0^\alpha \partial \partial_{|k)} A_0^\beta} - \frac{\partial^2 \mathcal{T}}{\partial \Lambda_{(j}^\alpha \partial \Lambda_{k)}^\beta} = 0. \tag{C.17}$$

It can easily be seen that the analysis in this appendix is valid as long as $\partial_i \epsilon^0 \neq 0$. In particular, it is valid when the theories are diffeomorphism invariant. It is also valid when the theory is put in flat spacetime, in which the Lorentz isometry, with Killing vector $\epsilon^\mu = \omega^\mu{}_\nu x^\nu$ for $\omega_{\mu\nu} = -\omega_{\nu\mu}$, is required.

APPENDIX D EXPRESSIONS OF $\partial\phi_\alpha/\partial\partial_\mathcal{I}\dot{Q}^\beta$ IN PHASE SPACE

In this appendix, we outline necessary steps to express $\partial\phi_\alpha/\partial\partial_\mathcal{I}\dot{Q}^\beta$ in phase space. We use the same set-up and notations as those given in sections 4.2-4.3. For convenient, let us denote P_A and Λ^A as collective for π^i_α and Λ^α_i, respectively.

The idea is to first express $\partial\phi_\alpha/\partial\partial_\mathcal{I}\dot{Q}^\beta$ in terms of α_M. This can be achieved by recalling from section 4.3 the equation (4.133). Recall also that diffeomorphism invariance requirements and demanding $\dot{\alpha}_\alpha = 0$ to not introduce further dynamics on the vector fields and imply that $\partial\alpha_\alpha/\partial\dot{Q}^\beta = 0 = \partial\alpha_\alpha/\partial\partial_i\dot{Q}^\beta$. Furthermore, due to the form of the Lagrangian of interest, we also have $\partial\alpha_\alpha/\partial\partial_{i_1}\partial_{i_2}\cdots\partial_{i_l}\dot{Q}^\beta = 0$ for $l \geq 2$. So

$$\frac{\partial\alpha_\alpha}{\partial\partial_\mathcal{I}\dot{Q}^\beta} = 0. \tag{D.1}$$

Then since $\phi_\alpha = \dot{\alpha}_\alpha$ we have, from eq.(4.133) and eq.(D.1)

$$\phi_\alpha = \sum_{|\mathcal{I}|=0}^{2} \frac{\partial\alpha_\alpha}{\partial\partial_\mathcal{I}Q^M}\partial_\mathcal{I}\dot{Q}^M - \sum_{|\mathcal{I}|=0}^{1} \frac{\partial\alpha_\alpha}{\partial\partial_\mathcal{I}\dot{Q}^B}\partial_\mathcal{I}(M^{BC}\alpha_C) + \frac{\partial\alpha_\alpha}{\partial K}\dot{K}. \tag{D.2}$$

Due to eq.(D.1), it can be seen that ϕ_α depend on $\partial_\mathcal{I}\dot{Q}^\beta$ only through the expressions $\partial_\mathcal{I}\dot{Q}^M$ and $\partial_\mathcal{I}(M^{BC}\alpha_C)$ which appear in the above equation. This gives

$$\frac{\partial\phi_\alpha}{\partial\partial_\mathcal{I}\dot{Q}^\beta} = \frac{\partial\alpha_\alpha}{\partial\partial_\mathcal{I}Q^\beta} - \frac{\partial\alpha_\alpha}{\partial\partial_\mathcal{J}\dot{Q}^A}\frac{\partial\partial_\mathcal{J}(M^{AB}\alpha_B)}{\partial\partial_\mathcal{I}\dot{Q}^\beta}. \tag{D.3}$$

We then need to compute each expression on RHS of eq.(D.3). For this, let us directly express α_A in terms of Lagrangian then transforming to phase space, but transform α_α to $-\tilde{\Omega}_\alpha$ (cf. eq.(4.139)). Direct calculations can be given as follows. In order to evaluate $\partial\partial_\mathcal{J}(M^{AB}\alpha_B)/\partial\partial_\mathcal{I}\dot{Q}^\beta$ we note that $\partial M^{AB}/\partial\partial_\mathcal{I}\dot{Q}^\beta = 0$ for $|\mathcal{I}| \geq 0$ whereas α_B depends on $\dot{Q}^\beta$ and $\partial_i\dot{Q}^\beta$ but not on $\partial_\mathcal{I}\dot{Q}^\beta$ where $|\mathcal{I}| \geq 2$. By writing $\partial_k\alpha_B$ by using chain rule and taking derivative with respect to $\partial_\mathcal{I}\dot{Q}^\beta$, we obtain

$$\frac{\partial\partial_k\alpha_B}{\partial\partial_i\partial_j\dot{Q}^\beta} = \frac{\partial\alpha_B}{\partial\partial_l\dot{Q}^\beta}\delta^i_{(k}\delta^j_{l)}, \qquad \frac{\partial\partial_j\alpha_B}{\partial\partial_i\dot{Q}^\beta} = \partial_j\frac{\partial\alpha_B}{\partial\partial_i\dot{Q}^\beta} + \delta^i_j\frac{\partial\alpha_B}{\partial\dot{Q}^\beta}, \qquad \frac{\partial\partial_j\alpha_B}{\partial\dot{Q}^\beta} = \partial_j\frac{\partial\alpha_B}{\partial\dot{Q}^\beta}. \tag{D.4}$$

This gives

$$\begin{aligned}\frac{\partial\partial_k(M^{AB}\alpha_B)}{\partial\partial_i\partial_j\dot{Q}^\beta} &= \frac{\partial((\partial_k M^{AB})\alpha_B + M^{AB}\partial_k\alpha_B)}{\partial\partial_i\partial_j\dot{Q}^\beta} \\ &= M^{AB}\frac{\partial\partial_k\alpha_B}{\partial\partial_i\partial_j\dot{Q}^\beta} \\ &= M^{AB}\frac{\partial\alpha_B}{\partial\partial_l\dot{Q}^\beta}\delta^i_{(k}\delta^j_{l)},\end{aligned} \tag{D.5}$$

$$\frac{\partial\partial_j(M^{AB}\alpha_B)}{\partial\partial_i\dot{Q}^\beta} = \partial_j\left(M^{AB}\frac{\partial\alpha_B}{\partial\partial_i\dot{Q}^\beta}\right) + \delta^i_j M^{AB}\frac{\partial\alpha_B}{\partial\dot{Q}^\beta}, \tag{D.6}$$

$$\frac{\partial\partial_j(M^{AB}\alpha_B)}{\partial\dot{Q}^\beta} = \partial_j\left(M^{AB}\frac{\partial\alpha_B}{\partial\dot{Q}^\beta}\right). \tag{D.7}$$

Next, let us express $\partial\alpha_B/\partial\partial_{\mathcal{I}}\dot{Q}^\beta$ in terms of phase space variables. The calculations will involve $\partial(\partial_j(\partial\mathcal{L}/\partial\partial_j Q^B))/\partial\partial_{\mathcal{I}}\dot{Q}^\beta$, which can be computed by first using the chain rule for ∂_j and then taking derivative with respect to $\partial_{\mathcal{I}}\dot{Q}^\beta$. The relevant results are

$$\frac{\partial}{\partial\dot{Q}^\beta}\left(\partial_i\left(\frac{\partial\mathcal{L}}{\partial\partial_i Q^B}\right)\right) = \partial_i\left(\frac{\partial^2\mathcal{L}}{\partial\dot{Q}^\beta\partial\partial_i Q^B}\right), \quad \frac{\partial}{\partial\partial_i\dot{Q}^\beta}\left(\partial_j\left(\frac{\partial\mathcal{L}}{\partial\partial_j Q^B}\right)\right) = \frac{\partial^2\mathcal{L}}{\partial\dot{Q}^\beta\partial\partial_i Q^B}. \tag{D.8}$$

So

$$\begin{aligned}\frac{\partial\alpha_B}{\partial\dot{Q}^\beta} &= \frac{\partial^2\mathcal{L}}{\partial\dot{Q}^B\partial Q^\beta} + \partial_i\left(\frac{\partial^2\mathcal{L}}{\partial\dot{Q}^\beta\partial\partial_i Q^B}\right) - \frac{\partial^2\mathcal{L}}{\partial\dot{Q}^\beta\partial Q^B} \\ &= \frac{\partial^2\mathcal{T}}{\partial Q^\beta\partial\Lambda^B} + \partial_i\left(\frac{\partial U_\beta}{\partial\partial_i Q^B}\right) - \frac{\partial U_\beta}{\partial Q^B},\end{aligned} \tag{D.9}$$

$$\begin{aligned}\frac{\partial\alpha_B}{\partial\partial_i\dot{Q}^\beta} &= \frac{\partial^2\mathcal{L}}{\partial\partial_i Q^\beta\partial\dot{Q}^B} + \frac{\partial^2\mathcal{L}}{\partial\partial_i Q^B\partial\dot{Q}^\beta} \\ &= \frac{\partial^2\mathcal{T}}{\partial\partial_i Q^\beta\partial\Lambda^B} + \frac{\partial U_\beta}{\partial\partial_i Q^B}.\end{aligned} \tag{D.10}$$

Next, let us express $\partial\alpha_\alpha/\partial\partial_{\mathcal{I}}Q^\beta$ in phase space. For this, we first use eq.(4.139) to transform α_α to $-\tilde{\Omega}_\alpha$. More precisely, this is

$$\alpha_\alpha = -\tilde{\Omega}_\alpha(Q^M, \partial_i Q^M, \partial_i\partial_j Q^M, P_B, \partial_i P_B, K), \tag{D.11}$$

such that $P_B = P_B(Q^M, \partial_i Q^M, \dot{Q}^B, K)$, in which both sides of eq.(B11) are both functions on the tangent bundle. So when taking derivative of α_α with respect to $\partial_{\mathcal{I}} Q^\beta$, we need to also take into account that P_B and $\partial_i P_B$ also depend on $\partial_{\mathcal{I}} Q^\beta$. As part of the intermediate calculations, we need to compute $\partial\partial_k P_B/\partial\partial_{\mathcal{I}} Q^\beta$, which can be done by first writing $\partial_k P_B$ using chain rule, then taking derivative with respect to $\partial_{\mathcal{I}} Q^\beta$. We have

$$\frac{\partial\partial_k P_B}{\partial\partial_i\partial_j Q^\beta} = \frac{\partial P_B}{\partial\partial_l Q^\beta}\delta^i_{(k}\delta^j_{l)}, \quad \frac{\partial\partial_k P_B}{\partial\partial_i Q^\beta} = \partial_k\left(\frac{\partial P_B}{\partial\partial_i Q^\beta}\right) + \delta^i_k \frac{\partial P_B}{\partial Q^\beta}, \quad \frac{\partial\partial_k P_B}{\partial Q^\beta} = \partial_k\left(\frac{\partial P_B}{\partial Q^\beta}\right). \tag{D.12}$$

Then we use eq.(4.80), which is equivalent to $P_B = \partial\mathcal{T}/\partial\Lambda^B$. Keeping these in mind, we have

$$\begin{aligned}\frac{\partial\alpha_\alpha}{\partial\partial_i\partial_j Q^\beta} &= -\frac{\partial\tilde{\Omega}_\alpha}{\partial\partial_i\partial_j Q^\beta} - \frac{\partial\tilde{\Omega}_\alpha}{\partial\partial_{(i|}P_B}\frac{\partial P_B}{\partial\partial_{|j)}Q^\beta} \\ &= -\frac{\partial\tilde{\Omega}_\alpha}{\partial\partial_i\partial_j Q^\beta} - \frac{\partial\tilde{\Omega}_\alpha}{\partial\partial_{(i|}P_B}\frac{\partial^2\mathcal{T}}{\partial\partial_{|j)}Q^\beta\partial\Lambda^B},\end{aligned} \tag{D.13}$$

$$\begin{aligned}\frac{\partial\alpha_\alpha}{\partial\partial_i Q^\beta} &= -\frac{\partial\tilde{\Omega}_\alpha}{\partial\partial_i Q^\beta} - \frac{\partial\tilde{\Omega}_\alpha}{\partial\partial_{\mathcal{I}} P_B}\partial_{\mathcal{I}}\left(\frac{\partial P_B}{\partial\partial_i Q^\beta}\right) - \frac{\partial\tilde{\Omega}_\alpha}{\partial\partial_i P_B}\frac{\partial P_B}{\partial Q^\beta} \\ &= -\frac{\partial\tilde{\Omega}_\alpha}{\partial\partial_i Q^\beta} - \frac{\partial\tilde{\Omega}_\alpha}{\partial\partial_{\mathcal{I}} P_B}\partial_{\mathcal{I}}\left(\frac{\partial^2\mathcal{T}}{\partial\partial_i Q^\beta\partial\Lambda^B}\right) - \frac{\partial\tilde{\Omega}_\alpha}{\partial\partial_i P_B}\frac{\partial^2\mathcal{T}}{\partial Q^\beta\partial\Lambda^B},\end{aligned} \tag{D.14}$$

$$\begin{aligned}\frac{\partial\alpha_\alpha}{\partial Q^\beta} &= -\frac{\partial\tilde{\Omega}_\alpha}{\partial Q^\beta} - \frac{\partial\tilde{\Omega}_\alpha}{\partial\partial_{\mathcal{I}} P_B}\partial_{\mathcal{I}}\left(\frac{\partial P_B}{\partial Q^\beta}\right) \\ &= -\frac{\partial\tilde{\Omega}_\alpha}{\partial Q^\beta} - \frac{\partial\tilde{\Omega}_\alpha}{\partial\partial_{\mathcal{I}} P_B}\partial_{\mathcal{I}}\left(\frac{\partial^2\mathcal{T}}{\partial Q^\beta\partial\Lambda^B}\right).\end{aligned} \tag{D.15}$$

Finally, let us compute $\partial\alpha_\alpha/\partial\partial_{\mathcal{I}} Q^\beta$. For this, as intermediate steps we compute

$$\frac{\partial P_B}{\partial\dot{Q}^A} = \frac{\partial^2\mathcal{L}}{\partial\dot{Q}^A\partial\dot{Q}^B} = W_{AB}, \qquad \frac{\partial\partial_k P_B}{\partial\partial_i\dot{Q}^A} = \delta^i_k\frac{\partial P_B}{\partial\dot{Q}^A} = W_{AB}\delta^i_k, \qquad \frac{\partial\partial_k P_B}{\partial\dot{Q}^A} = \partial_k W_{AB}. \tag{D.16}$$

Then we have

$$\frac{\partial\alpha_\alpha}{\partial\partial_i\dot{Q}^A} = -\frac{\partial\tilde{\Omega}_\alpha}{\partial\partial_i P_B}W_{AB}, \tag{D.17}$$

$$\frac{\partial\alpha_\alpha}{\partial\dot{Q}^A} = -\frac{\partial\tilde{\Omega}_\alpha}{\partial\partial_i P_B}\partial_i W_{BA} - \frac{\partial\tilde{\Omega}_\alpha}{\partial P_B}W_{BA}. \tag{D.18}$$

Then by substituting eq.(D.5) - eq.(D.7), eq.(D.9) - eq.(D.10), eq.(D.13) - eq.(D.15), and eq.(D.17) - eq.(D.18) into eq.(D.3), we obtain

$$\begin{aligned} \frac{\partial \phi_\alpha}{\partial \dot{Q}^\beta} &= -\mathcal{C}_{0\beta\alpha} + \partial_i \mathcal{C}^i_{1\beta\alpha} - \partial_i \partial_j \mathcal{C}^{ij}_{2\beta\alpha}, \\ \frac{\partial \phi_\alpha}{\partial \partial_i \dot{Q}^\beta} &= \mathcal{C}^i_{1\beta\alpha} - 2\partial_j \mathcal{C}^{ij}_{2\beta\alpha}, \\ \frac{\partial \phi_\alpha}{\partial \partial_i \partial_j \dot{Q}^\beta} &= -\mathcal{C}^{ij}_{2\beta\alpha}. \end{aligned} \tag{D.19}$$

By using diffeomorphism invariance requirements, eq.(4.100) is realised. This simplifies eq.(D.19). Further simplifications are possible. For this, let us note that using eq.(D.3)-(D.9) and diffeomorphism invariance requirements, one obtains

$$\frac{\partial \phi_\alpha}{\partial \dot{Q}^\beta} - \frac{\partial \phi_\beta}{\partial \dot{Q}^\alpha} + \partial_i \left(\frac{\partial \phi_\beta}{\partial \partial_i \dot{Q}^\alpha} \right) = \frac{\partial \alpha_\alpha}{\partial Q^\beta} - \frac{\partial \alpha_\beta}{\partial Q^\alpha} + \partial_i \left(\frac{\partial \alpha_\beta}{\partial \partial_i Q^\alpha} + \frac{\partial \alpha_\beta}{\partial \dot{Q}^\alpha_i} + \frac{\partial \alpha^i_\alpha}{\partial \dot{Q}^\beta} \right). \tag{D.20}$$

Then by expressing α_M in terms of Lagrangian and using diffeomorphism invariance and secondary-constraint enforcing relations, we obtain

$$\frac{\partial \phi_\alpha}{\partial \dot{Q}^\beta} - \frac{\partial \phi_\beta}{\partial \dot{Q}^\alpha} + \partial_i \left(\frac{\partial \phi_\beta}{\partial \partial_i \dot{Q}^\alpha} \right) = 0, \tag{D.21}$$

which is equivalent to the phase space expression

$$\mathcal{C}_{0\alpha\beta} = \mathcal{C}_{0\beta\alpha} - \partial_i \mathcal{C}^i_{1\beta\alpha}. \tag{D.22}$$

Finally, this gives

$$\frac{\partial \phi_\alpha}{\partial \dot{Q}^\beta} = -\mathcal{C}_{0\alpha\beta}, \quad \frac{\partial \phi_\alpha}{\partial \partial_i \dot{Q}^\beta} = -\mathcal{C}^i_{1\alpha\beta}, \quad \frac{\partial \phi_\alpha}{\partial \partial_i \partial_j \dot{Q}^\beta} = 0. \tag{D.23}$$

BIOGRAPHY