

**GREEN'S RELATIONS AND THE NATURAL PARTIAL ORDER FOR
SEMIGROUPS OF TRANSFORMATIONS PRESERVING AN
EQUIVALENCE RELATION ON AN INVARIANT SET**



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ABSTRACT

Let $T(X)$ be the full transformation semigroup on a nonempty set X . For an equivalence relation E on X and a nonempty subset Y of X , let

$$\overline{S}_E(X, Y) = \{\alpha \in T(X) : \forall x, y \in Y, (x, y) \in E \Rightarrow (x\alpha, y\alpha) \in E, x\alpha, y\alpha \in Y\}.$$

Then $\overline{S}_E(X, Y)$ is a subsemigroup of $T(X)$. In this thesis, we study some well-known and very important concepts in semigroup theory, such as regularity, Green's relations, and the natural partial order on $\overline{S}_E(X, Y)$. Moreover, we consider the left regular and right regular elements of the semigroups $T(X, Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\}$ and $S(X, Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}$.

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CHAPTER I

INTRODUCTION

Transformation semigroups stand as foundational mathematical entities, playing a pivotal role in diverse areas of mathematics and theoretical computer science. Primarily, they emerge organically as endomorphism semigroups within various mathematical structures, encapsulating the essence of morphisms within those domains. Their significance extends to theoretical computer science, where the characteristics of languages are intricately tied to the algebraic properties of associated transformation semigroups. The interplay between these semigroups and languages provides a rich framework for understanding computational processes and formal language theory.

In the realm of semigroup theory, transformation semigroups hold paramount importance. Notably, every semigroup can be represented as isomorphic to a transformation semigroup. This fundamental correspondence implies that, to delve into the intricacies of semigroups, one can effectively focus on the study of transformation semigroups. In essence, the study of transformation semigroups becomes a gateway to comprehending the broader landscape of semigroups.

Thus, transformation semigroups serve as a unifying thread in mathematics, connecting various disciplines and offering a powerful lens through which to explore the intricate structures that underlie both algebraic and computational phenomena.

Let X be a nonempty set. We define the full transformation semigroup on X , denoted by $T(X)$, as the semigroup that comprises all mappings from X to X , with the operation of composition of functions. It is well-known that $T(X)$ is a regular semigroup (see [7]), possessing significant algebraic properties and structures. Furthermore, every semigroup can be embedded in $T(X)$ for some appropriate set X . This characterization highlights the importance and versatility of the full transformation semigroup, providing a solid foundation for further exploration and research in the field of semigroup theory.

In 1975, Symons [37] introduced a subsemigroup of $T(X)$, the total transformation semigroup, specifically defined as

$$T(X, Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\},$$

where Y represents a nonempty subset of X . Symons conducted a thorough analysis

of the automorphisms within $T(X, Y)$, determining their precise nature and properties. In 2005, Nenthein [16] focused on the study of regular elements within $T(X, Y)$, particularly when X is a finite set. They provided a comprehensive count of the total number of regular elements in $T(X, Y)$, apply the cardinality of X , the cardinality of Y , and relevant Stirling numbers as essential parameters for describing this count. A few years later, Sanwong and Sommanee [23] explored the concepts of regularity and Green's relations within the semigroup $T(X, Y)$. Their investigation led to the determination of specific conditions under which $T(X, Y)$ becomes a regular semigroup. Furthermore, they introduced a distinctive class of maximal inverse subsemigroups within $T(X, Y)$ in 2008. Notably, they proved that the set $F(X, Y) = \{\alpha \in T(X, Y) : X\alpha = Y\alpha\}$ represents the largest regular subsemigroup of $T(X, Y)$ and provided a detailed analysis of its Green's relations. In [24], Sanwong described Green's relations and found all maximal regular subsemigroups of $F(X, Y)$. In 2009, the focus convert to the examination of maximal and minimal congruences on $T(X, Y)$ by Sanwong et al. [25]. Their research revealed that $T(X, Y)$ possesses only one maximal congruence when X is a finite set. Subsequently, this finding was generalized in Theorem 3.4 for the case where Y is infinite. Additionally, comprehensive characterizations of all minimal congruences on $T(X, Y)$ were presented. Continuing the investigation, in the same year, Sun [34] showed that while the semigroup $T(X, Y)$ is not left abundant, it is right abundant. Lastly, in 2016, Lei Sun and Junling Sun [35] explored the natural partial order inherent within $T(X, Y)$. Their research provided a determination of the maximal and minimal elements present in $T(X, Y)$, contributing to a deeper understanding of its structural properties.

Consider a fixed nonempty subset Y of X , denoted as

$$S(X, Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}.$$

$S(X, Y)$ represents a semigroup of total transformations of X that preserve the subset Y , ensuring its invariance. In 1975, Symons [37] provided a detailed description of the automorphism group of this semigroup, shedding light on its structural properties and fundamental symmetries. In 2005, Nenthein, Youngkhong, and Kemprasit [16] conducted an extensive study on $S(X, Y)$, demonstrating that $S(X, Y)$ is a regular semigroup if and only if Y is either equal to X or contains precisely one element. They further identified $F(X, Y)$ as the set comprising all regular elements within $S(X, Y)$. Moreover, they determined the number of regular elements in $S(X, Y)$ when X is a

finite set, contributing to a comprehensive understanding of its regularity properties. In 2011, Honyam and Sanwong [8] delved into the characterization of $S(X, Y)$, investigating when $S(X, Y)$ is isomorphic to $T(Z)$ for some set Z . They also proved that every semigroup A can be embedded in $S(A^1, A)$, illustrating the flexibility and generality of $S(X, Y)$. Additionally, they provided insightful descriptions of Green's relations, $\mathcal{H}$ -classes, and ideals within the semigroup $S(X, Y)$. In 2013, Choomanee, Honyam, and Sanwong [2] focused on the exploration of left regular, right regular, and intra-regular elements of $S(X, Y)$ and considered the relationships between these elements.. Furthermore, they determined the count of left regular elements in $S(X, Y)$ when X is a finite set, further enhancing our understanding of the semigroup's structural properties. In [32], Sun and Wang investigated the natural partial order within $S(X, Y)$, offering insights into the relationships between elements and identifying those that are compatible. They described the maximal and minimal elements within $S(X, Y)$ and explored the concept of the greatest lower bound for pairs of elements. Additionally, they showed that $S(X, Y)$ is abundant. In [30], Sun and Sun conducted a focused study on the elements within $S(X, Y)$ that are left compatible with respect to the natural partial order, providing further insights into the structural characteristics of $S(X, Y)$. Finally, in [1], Chinram and Baupradist characterized left magnifying elements and right magnifying elements within semigroups of transformations with an invariant set.

Consider E , an equivalence relation on a set X . We examine a subset of $T(X)$, denoted as

$$T_E(X) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in E \Rightarrow (x\alpha, y\alpha) \in E\},$$

which is defined based on this equivalence relation. It is evident that if E is a non-trivial equivalence relation, then $T_E(X)$ constitutes a proper subsemigroup of $T(X)$. However, if E is either the identity or universal relation, then $T_E(X)$ and $T(X)$ coincide. In 1994 and 1996, Pei [18, 19] delved into the study of α -congruences and certain regular subsemigroups that induce a specific lattice structure on $T_E(X)$. These investigations provided valuable insights into the algebraic properties and relationships within $T_E(X)$. In 2005, Pei [17] further explored the regularity of elements and Green's relations within the semigroup $T_E(X)$, contributing to a deeper understanding of its structural characteristics. Additionally, Pei determined the rank of the homeomorphism group and examined the rank of $T_E(X)$ when X is a finite set and each class of the equivalence relation E possesses the same cardinality. Moreover, the rank of $\Gamma(X)$, the semigroup

of all closed function alpha on a topological space X , with E classes forming a basis, was also investigated. In 2008, Sun, Pei, and Cheng [33] characterized the natural partial order on the semigroup $T_E(X)$. Their research involved investigating the compatibility of multiplication and identifying all compatible elements within the semigroup. Furthermore, they determined the maximal, minimal, and covering elements with respect to the order. In 2011, Pei and Zhou [21] focused on the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ within the semigroup $T_E(X)$, as well as the equivalence relations E under which $T_E(X)$ exhibits abundance. These studies provided valuable insights into the interplay of these relations and their impact on the algebraic properties of $T_E(X)$. In 2019, Sun [34] examined the left and right compatibility in relation to the natural partial order on $T_E(X)$. Furthermore, in [9], Kaewnoi, Petapirak, and Chinram conducted a detailed investigation of left and right magnifying elements within the semigroup $T_E(X)$.

In their work, Sangkhanan and Sanwong [22] introduced a subsemigroup $T_E(X, Y)$ of $T(X)$, which generalizes the results obtained by Pei [17], Sanwong and Sommanee [23]. This subsemigroup is defined as a transformation semigroup with restricted range that preserves a given equivalence relation. Specifically, $T_E(X, Y)$ is defined as follows:

$$T_E(X, Y) = \{\alpha \in T(X, Y) : \forall (x, y) \in E, (x\alpha, y\alpha) \in E\} = T_E(X) \cap T(X, Y),$$

where Y is a subset of a set X . It is clear that if $X = Y$, then $T_E(X, Y) = T_E(X)$, which means that $T_E(X)$ is a special case of $T_E(X, Y)$. Furthermore, if E is the universal relation, $E = X \times X$, then $T_E(X, Y)$ becomes $T(X, Y)$. It is not difficult to check that $T_E(X, Y)$ is the semigroup of all continuous self-maps of the topological space X for which all E -classes form a basis carrying X into a subspace Y and is referred to as a semigroup of continuous functions. Furthermore, Sangkhanan and Sanwong investigated the regularity of elements and Green's relations within the semigroup $T_E(X, Y)$. Finally, they determined Green's relations on the regular elements in $T_E(X, Y)$.

The object of this thesis is to introduce a new transformation subsemigroup of $T(X)$ by considering a fixed nonempty subset Y of X . Let

$$\overline{S}_E(X, Y) = \{\alpha \in T(X) : \forall x, y \in Y, (x, y) \in E \Rightarrow (x\alpha, y\alpha) \in E, x\alpha, y\alpha \in Y\}.$$

Then $\overline{S}_E(X, Y)$ is a subsemigroup of $T(X)$ and $S(X, Y)$ and id_X , the identity map on X , belongs to $\overline{S}_E(X, Y)$. It is clear that if $X = Y$, then $\overline{S}_E(X, Y) = T_E(X)$. Moreover,

it is not challenging to verify that both $T_E(X)$ and $S(X, Y)$ are intersecting subsets of this semigroup. Furthermore, it is obvious that $\overline{S}_E(X, Y) \subseteq S(X, Y)$.

The purpose of this thesis is to increase understanding about some algebraic structures of $\overline{S}_E(X, Y)$. Also, we study this semigroup with Green's relations and characterize when two elements of this semigroup are related under equivalence relation $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}$ and $\mathcal{D}$. Moreover, we investigate the regularity of elements of $\overline{S}_E(X, Y)$. Furthermore, we endow $\overline{S}_E(X, Y)$ with the natural partial order and determine when two elements of this semigroup are related under this order. Finally, we discuss minimal and maximal elements of this semigroup.

This thesis is divided into six chapters. Chapter 1 serves as an introduction to research problems. In Chapter 2, we introduce some basic terminologies, notations and recall some useful results that will be applied throughout this thesis. Additionally, we discuss the relationships between the transformation semigroups $S(X, Y)$, $T_E(X)$, and $\overline{S}_E(X, Y)$. In Chapter 3, we find elements of $\overline{S}_E(X, Y)$ which are regular, left regular and right regular, respectively. Furthermore, we discuss left regular, and right regular elements of the semigroups $T(X, Y)$ and $S(X, Y)$. In Chapter 4, we determine the characterization of Green's relations on $\overline{S}_E(X, Y)$. Chapter 5 focuses on establishing the natural partial order on $\overline{S}_E(X, Y)$ and characterizing when two elements of $\overline{S}_E(X, Y)$ are related under this order. Also, we explore the concept of minimal and maximal elements within the semigroup $\overline{S}_E(X, Y)$. Finally, in Chapter 6, we conclude with the summary of the results obtained in this thesis.

CHAPTER II

PRELIMINARIES

In this chapter, we introduce some background concepts that will be used throughout the thesis. We begin by describing the definitions and theorems central to semigroup theory. Following this, we delve into discussions on regularity, Green's relations, the natural partial order, and transformations on a set. Additionally, we explore relationships among subsemigroups of $T(X)$ in the latter part of this chapter.

2.1 Elementary properties of semigroups

In this section, we provide definitions and discuss some results in semigroup theory.

Let S be a nonempty set. By a **binary operation** on S , we mean a function $\cdot$ from $S \times S$ into S , and we called $(S, \cdot)$ is **closed**. The image of an element $(a, b) \in S \times S$ under $\cdot$ is denoted by $a \cdot b$. Frequently, we can write ab for $a \cdot b$.

A binary operation on a set S said to be **associative** if $a(bc) = (ab)c$ is satisfied for all $a, b, c \in S$. A set together with an associative binary operation is called a **semigroup**.

An element x of a semigroup S is called an **idempotent element** if $x^2 = x$. Denote the set of all idempotent elements in S by $E(S)$.

An element e of a semigroup S is called an **identity** of S if $es = s = se$ for all $s \in S$.

For subsets A, B , and C of a semigroup S , AB and ABC are subsets of S defined by

$$AB = \{ab : a \in A \text{ and } b \in B\}$$

and

$$ABC = \{abc : a \in A, b \in B \text{ and } c \in C\}.$$

If $A = \{a\}$, we shall write $\{a\}B$ as aB . Also for the case $B = \{b\}$, we write $A\{b\}$ and $A\{b\}C$ as Ab and AbC , respectively.

A nonempty subset A of a semigroup S is called a **subsemigroup** of S if A is closed under the operation, that is, $ab \in A$ for all $a, b \in A$.

A nonempty subset A of a semigroup S is called a **left ideal** if $SA \subseteq A$, a **right ideal** if $AS \subseteq A$, and an **ideal** if it both a left and a right ideal. Evidently, every ideal (whether left or right) is a subsemigroup of S .

For any semigroup S , the notation S^1 means S itself if S contains the identity element and we let $S^1 = S \cup \{1\}$ if S has no identity, then define the binary operation $\cdot$ on S^1 by

$$1 \cdot 1 = 1, 1 \cdot a = a \cdot 1 = a \text{ and } a \cdot b = ab \text{ for all } a, b \in S.$$

Hence, S^1 becomes a semigroup with the identity element 1.

For a nonempty subset A of a semigroup S , a **left ideal of S generated by A** is the intersection of all left ideals of S containing A , denoted by $(A)_l$. Also, a **right ideal of S generated by A** and **ideal of S generated by A** can be defined in the same way and denoted by $(A)_r$ and $(A)_i$, respectively.

The following theorem gives the the description of $(A)_l$, $(A)_r$, and $(A)_i$.

Theorem 2.1.1. [3] *Let A be a nonempty subset of a semigroup S . Then*

- (1) $(A)_l = A \cup SA = S^1A$;
- (2) $(A)_r = A \cup AS = AS^1$;
- (3) $(A)_i = A \cup SA \cup AS \cup SAS = S^1AS^1$.

If $A = \{a\}$, then $(a)_l = S^1a$ is called the **principal left ideal of S generated by a** . Also, $(a)_r$ and $(a)_i$ are called the **principal right ideal** and the **principal ideal of S generated by a** , respectively.

2.2 Regularity and Green's relations

Now, let's revisit the concept of regularity, covering both left and right regular aspects. Additionally, we will review the definitions of Green's relations on semigroups,

which are pivotal concepts that require primary consideration in understanding the semigroup structure.

An element a of a semigroup S is called a **regular element** if there exists $x \in S$ such that $a = axa$. The set of all regular elements of S is denoted by $Reg(S)$. A semigroup S is called a **regular semigroup** if all elements of S are regular. Let x and y be elements of a semigroup S . We say that y is an **inverse** of x if $x = xyx$ and $y = yxy$. A regular semigroup S is an **inverse semigroup** if and only if $E(S)$ is a commutative subsemigroup.

An element x of S is called **left regular** if $x = yx^2$ for some $y \in S$. An element x of S is called **right regular** if $x = x^2y$ for some $y \in S$. If the element x in a semigroup S is both left and right regular, then x is regular. Denote by $LReg(S)$ and $RReg(S)$ the sets of all left regular elements and right regular elements of S , respectively. A semigroup S is called a **left (right) regular semigroup** if all elements of S are left (right) regular.

Let S be a semigroup and $a, b \in S$. An equivalence relation $\mathcal{L}$ on S is defined by the rule that $a\mathcal{L}b$ if and only if a and b generate the same principle left ideal, that is,

$$a\mathcal{L}b \text{ if and only if } S^1a = S^1b.$$

Dually, an equivalence relation $\mathcal{R}$ on S is defined by the rule that $a\mathcal{R}b$ if and only if a and b generate the same principle right ideal, that is,

$$a\mathcal{R}b \text{ if and only if } aS^1 = bS^1.$$

Analogous to $\mathcal{L}$ and $\mathcal{R}$, an equivalence relation $\mathcal{J}$ on S is defined by the rule that $a\mathcal{J}b$ if and only if a and b generate the same principle ideal, that is,

$$a\mathcal{J}b \text{ if and only if } S^1aS^1 = S^1bS^1.$$

Moreover, the equivalence relation $\mathcal{H}$ and $\mathcal{D}$ are defined to be

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R} \text{ and } \mathcal{D} = \mathcal{L} \circ \mathcal{R}.$$

The relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$, and $\mathcal{J}$ are called **Green's relations** on S . For any element $a \in S$, each of L_a, R_a, H_a, D_a , and J_a denotes the equivalence class of $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$, and $\mathcal{J}$ containing a , respectively.

An alternative characterizations, making the divisibility aspect of theses equivalences more apparent, is given in the following lemma:

Lemma 2.2.1. [6] *Let S be a semigroup and $a, b \in S$. Then the following statements hold:*

- (1) $a\mathcal{L}b$ if and only if $a = xb$ and $b = ya$ for some $x, y \in S^1$.
- (2) $a\mathcal{R}b$ if and only if $a = bx$ and $b = ay$ for some $x, y \in S^1$.
- (3) $a\mathcal{J}b$ if and only if $a = xby$ and $b = uav$ for some $x, y, u, v \in S^1$.

Lemma 2.2.2. [7] *Let S be a semigroup. The $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.*

Lemma 2.2.3. [7] *Let S be any semigroup. Then $\mathcal{L} \subseteq \mathcal{D}, \mathcal{R} \subseteq \mathcal{D}$, and $\mathcal{D} \subseteq \mathcal{J}$.*

From Lemma 2.2.3 and $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, we have

$$\mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J} \text{ and } \mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}.$$

Lemma 2.2.4. [7] *$\mathcal{D} = \mathcal{J}$ in every finite semigroup.*

The next theorem shows the relationship between Green's relations and regularity in any semigroup.

Theorem 2.2.5. [39] *Let S be a semigroup and $a \in S$. Then*

- (1) a is left regular if and only if $a\mathcal{L}a^2$.
- (2) a is right regular if and only if $a\mathcal{R}a^2$.

2.3 Natural partial order

The natural partial order within the semigroup serve as the relation employed to elucidate the relationships between every pair of elements in the semigroup. This serve as a valuable means to visualize the alignment within the semigroup. In this section, we will remind the definition of the natural partial order.

Let X be a nonempty set and σ a relation on X . Then

- (1) σ is reflexive if $(x, x) \in \sigma$ for all $x \in X$.
- (2) σ is symmetric if $(x, y) \in \sigma$ implies $(y, x) \in \sigma$ for all $x, y \in X$.
- (3) σ is anti-symmetric if $(x, y), (y, x) \in \sigma$ implies $x = y$ for all $x, y \in X$.
- (4) σ is transitive if $(x, y), (y, z) \in \sigma$ implies $(x, z) \in \sigma$ for all $x, y, z \in X$.

If σ satisfies (1), (3), and (4), then σ is a **partially order** on X . We call σ an **equivalence relation** on X if σ satisfies (1), (2), and (4).

For a nonempty set X and a partially order $\leq$ on X , we say that $(X, \leq)$ is a **partially ordered set**. For $x, y \in X$, we denote $(x, y) \in \leq$ by $x \leq y$ and write $x < y$ if $x \leq y$ and $x \neq y$.

Let X be a nonempty set and E an equivalence relation on X , we denote

$$X/E = \{\{x \in X : (x, a) \in E\} : a \in X\}$$

and call $A \in X/E$ an **equivalence class** or **E -class**.

Let $(X, \leq)$ be a partially ordered set and $x \in X$. Then x is a **maximal element** of $(X, \leq)$ if for all $y \in X$, $x \leq y$ implies $x = y$. Similarly, x is a **minimal element** of $(X, \leq)$ if for all $y \in X$, $y \leq x$ implies $y = x$.

A **partition** of X is a collection D of nonempty subsets of X such that

- (1) $\bigcup_{P \in D} P = X$,
- (2) for every $P, Q \in D$, $P \cap Q \neq \emptyset$ implies $P = Q$.

It is well-known that for any equivalence relation E on a nonempty set X , the quotient set X/E from a partition of a nonempty set X . Conversely, for any a partition D of a nonempty set X , the relation $E = \bigcup_{A \in D} (A \times A)$ defines an equivalence relation on X , where the quotient set X/E is precisely equal to D .

Let ρ be a partial order on a semigroup S . An element $c \in S$ is said to be **left compatible** with ρ if for all $a, b, c \in S$,

$$(a, b) \in \rho \text{ implies } (ca, cb) \in \rho,$$

and **right compatible** with ρ if for all $a, b, c \in S$.

$$(a, b) \in \rho \text{ implies } (ac, bc) \in \rho,$$

It is called **compatible** if it is both left and a right compatible.

In 1952, Vagner [38] introduced a partial order $\leq$ on an inverse semigroup S as follows: for $a, b \in S$,

$$a \leq b \text{ if and only if } a = eb \text{ for some } e \in E(S). \quad (1)$$

Later, Mitsch [14] defined a partial order $\leq$ on an inverse semigroup S as follows: for $a, b \in S$,

$$a \leq b \text{ if and only if } ab' = aa' \quad (2)$$

where a', b' denote the unique inverses of a and b respectively and showed that the partial orders (1) and (2) are equivalent.

In 1980, Nambooripad [15] extended a partial order $\leq$ on a regular semigroup S by

$$a \leq b \text{ if and only if } a = eb = bf \text{ for some } e, f \in E(S). \quad (3)$$

Later in 1986, Kowol and Mitsch [14] extended the above a partial order to any semigroup S by defining $\leq$ on S as follows:

$$a \leq b \text{ if and only if } a = xb = by \text{ and } a = ay \text{ for some } x, y \in S^1.$$

This order coincides with the relation (3) if the semigroup S is regular and this is called the **natural partial order** on S .

2.4 Transformation on a set

Throughout this thesis, we write the functions on the right; in particular, this means that for a composition $\alpha \circ \beta$, the function α is applied first.

Let X be a nonempty set. A **full transformation semigroup** is the set of all functions from X into X under the composition of functions, denoted by

$$T(X) = \{\alpha : \alpha : X \rightarrow X\}.$$

Let $\alpha \in T(X)$ and let E be an equivalence relation on X . The notation $x\alpha$ means the image of x under α and $X\alpha = \{x\alpha : x \in X\}$ is the range of α . Moreover, $\pi(\alpha)$ is the partition of X induced by α , i.e.,

$$\pi(\alpha) = \{x\alpha^{-1} : x \in X\alpha\},$$

where $x\alpha^{-1}$ is the inverse image of x under α . Note that by a slight abuse of notation, we write $P\alpha = x$, where $P = x\alpha^{-1} \in \pi(\alpha)$. Then $\pi(\alpha)$ is a partition of X . Particularly,

$$E(\alpha) = \{A\alpha^{-1} : A \in X/E \text{ and } A \cap X\alpha \neq \emptyset\},$$

where $A\alpha^{-1} = \bigcup \{x\alpha^{-1} : x \in A \cap X\alpha\}$. Obviously, $\pi(\alpha)$ refines $E(\alpha)$, i.e., for each $P \in \pi(\alpha)$, there exists $Q \in E(\alpha)$ such that $P \subseteq Q$. For a nonempty subset Y of X , we define restrictions to Y of $\pi(\alpha)$ and $E(\alpha)$ by

$$\pi_Y(\alpha) = \{P \in \pi(\alpha) : P \cap Y \neq \emptyset\}$$

and

$$E_Y(\alpha) = \{(A \cap Y)\alpha^{-1} : A \in X/E \text{ and } A \cap X\alpha \cap Y \neq \emptyset\}.$$

In our work, we introduce a new transformation subsemigroup of $T(X)$ by letting Y be a fixed nonempty subset of X , E an equivalence relation on X and define

$$\overline{S}_E(X, Y) = \{\alpha \in T(X) : \forall x, y \in Y, (x, y) \in E \Rightarrow (x\alpha, y\alpha) \in E, x\alpha, y\alpha \in Y\}.$$

Then $\overline{S}_E(X, Y)$ is a subsemigroup of $T(X)$. It is clear that $\overline{S}_E(X, Y) \subseteq S(X, Y)$ where $S(X, Y)$ is the full transformation semigroup preserving an invariant set, i.e., $S(X, Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}$.

Example 2.4.1. Let $X = \{1, 2, 3, 4, 5\}$, $Y = \{1, 2, 3\}$ and $X/E = \{\{1, 2\}, \{3, 4\}, \{5\}\}$. Define $\alpha : X \rightarrow X$ by

$$x\alpha = \begin{cases} 1 & \text{if } x = 2, \\ 2 & \text{if } x = 5, \\ x & \text{otherwise.} \end{cases}$$

Note that $Y\alpha = \{1, 3\} \subseteq Y$. Then $\alpha \in \overline{S}_E(X, Y)$. Moreover, $\pi(\alpha) = \{\{1, 2\}, \{3\}, \{4\}, \{5\}\}$, $E(\alpha) = \{\{1, 2, 5\}, \{3, 4\}\}$, $\pi_Y(\alpha) = \{\{1, 2\}, \{3\}\}$ and $E_Y(\alpha) = \{\{1, 2, 5\}, \{3\}\}$.

In the following theorem, we obtain some properties in our semigroup $\overline{S}_E(X, Y)$.

Theorem 2.4.2. *Let $\alpha \in \overline{S}_E(X, Y)$. Then $E_Y(\alpha)$ is a partition of $Y\alpha^{-1}$.*

Proof. Note that $(A \cap Y)\alpha^{-1} \subseteq Y\alpha^{-1}$ for all $A \in X/E$. Then $\bigcup E_Y(\alpha) \subseteq Y\alpha^{-1}$. Clearly, $Y\alpha^{-1} \subseteq \bigcup E_Y(\alpha)$. Hence, $\bigcup E_Y(\alpha) = Y\alpha^{-1}$. Let $U, V \in E_Y(\alpha)$ be such that $U \cap V \neq \emptyset$. Then $U = (A \cap Y)\alpha^{-1}$, $V = (B \cap Y)\alpha^{-1}$ for some $A, B \in X/E$, $A \cap Y \cap X\alpha \neq \emptyset$ and $B \cap Y \cap X\alpha \neq \emptyset$. Choose $x \in U \cap V$. So $x\alpha \in A \cap Y$ and $x\alpha \in B \cap Y$. Therefore, $A \cap B \neq \emptyset$. From X/E is a partition of X , we have $U = (A \cap Y)\alpha^{-1} = (B \cap Y)\alpha^{-1} = V$. Thus $E_Y(\alpha)$ is a partition of $Y\alpha^{-1}$. $\square$

In this thesis, for a nonempty subset Z of X and $a \in X$, we denote a constant mapping whose domain is Z and range is $\{a\}$ by $\overline{Z}_a$. Denote identity on Z by id_Z .

Lemma 2.4.3. *Let $\alpha \in T(X)$ and $a \in Y$. If $\alpha|_Y = \overline{Y}_a$, then $\alpha \in \overline{S}_E(X, Y)$.*

Proof. Assume that $\alpha|_Y = \overline{Y}_a$ and let $x, y \in Y$ be such that $(x, y) \in E$. Then $x\alpha = y\alpha = a \in Y$ and $(x\alpha, y\alpha) = (a, a) \in E$. Hence, $\alpha \in \overline{S}_E(X, Y)$. $\square$

2.5 Relationship between transformation semigroups

We first present the relationships between the semigroup $\overline{S}_E(X, Y)$ and $S(X, Y)$, including $T_E(X)$, as follows:

Theorem 2.5.1. *$\overline{S}_E(X, Y) = S(X, Y)$ if and only if $Y \times Y \subseteq E$ or $(Y \times Y) \cap E = id_Y$.*

Proof. Assume that $Y \times Y \not\subseteq E$ and $(Y \times Y) \cap E \neq id_Y$. Then there exist distinct $a, b \in Y$ such that $(a, b) \notin E$ and there exist distinct $c, d \in Y$ such that $(c, d) \in E$. Define $\alpha : X \rightarrow X$ by

$$x\alpha = \begin{cases} a & \text{if } x = c, \\ b & \text{if } x = d, \\ x & \text{otherwise.} \end{cases}$$

Let $x \in Y$. If $x = c$ or $x = d$, then $x\alpha \in \{a, b\} \subseteq Y$. If $x \in X \setminus \{c, d\}$, then $x\alpha = x \in Y$. Thus $\alpha \in S(X, Y)$. Since $(c\alpha, d\alpha) = (a, b) \notin E$, we obtain $\alpha \notin \overline{S}_E(X, Y)$.

Assume that $Y \times Y \subseteq E$ or $(Y \times Y) \cap E = id_Y$. Clearly, $\overline{S}_E(X, Y) \subseteq S(X, Y)$. Let $\alpha \in S(X, Y)$ and $x, y \in Y$ be such that $(x, y) \in E$. Then $x\alpha, y\alpha \in Y$. If $Y \times Y \subseteq E$, then $(x\alpha, y\alpha) \in E$. If $(Y \times Y) \cap E = id_Y$, then $x = y$, and so $(x\alpha, y\alpha) \in E$. Hence, $\alpha \in \overline{S}_E(X, Y)$. $\square$

Theorem 2.5.2. $\overline{S}_E(X, Y) \subseteq T_E(X)$ if and only if either one of the following hold:

- (1) $E = X \times X$.
- (2) for each $z \in X \setminus Y$, $\{z\} \in X/E$.

Consequently, $\overline{S}_E(X, Y) = T_E(X)$ if and only if $Y = X$.

Proof. Assume that $E \neq X \times X$ and there exists $z \in X \setminus Y$ such that $\{z\} \notin X/E$. From X/E is a partition of X , there exists $A \in X/E$ such that $z \in A$. Since $\{z\} \notin X/E$, we have $A \neq \{z\}$. Choose $a \in A \setminus \{z\}$. Since $E \neq X \times X$, there exists $b \in X$ such that $(z, b) \notin E$. There exists $B \in X/E$ such that $b \in B$.

Case 1: $B \cap Y \neq \emptyset$. Assume that $b \in B \cap Y$ and $\alpha : X \rightarrow X$ be defined by

$$x\alpha = \begin{cases} z & \text{if } x = z, \\ b & \text{otherwise.} \end{cases}$$

From $\alpha|_Y = \overline{Y}_b$ and Lemma 2.4.3, we obtain $\alpha \in \overline{S}_E(X, Y)$. However, $\alpha \notin T_E(X)$ since $(z\alpha, a\alpha) = (z, b) \notin E$.

Case 2: $B \cap Y = \emptyset$. Choose $c \in Y$. Define $\alpha : X \rightarrow X$ by

$$x\alpha = \begin{cases} b & \text{if } x = z, \\ c & \text{otherwise.} \end{cases}$$

From $\alpha|_Y = \overline{Y}_c$ and Lemma 2.4.3, we obtain $\alpha \in \overline{S}_E(X, Y)$. However, $\alpha \notin T_E(X)$ since $(z\alpha, a\alpha) = (b, c) \notin E$.

Conversely, if $E = X \times X$, then we let $\alpha \in \overline{S}_E(X, Y)$ and $x, y \in X$ such that $(x, y) \in E$. Thus $(x\alpha, y\alpha) \in E$. Hence, $\alpha \in T_E(X)$ and so $\overline{S}_E(X, Y) \subseteq T_E(X)$. Assume (2) holds. Let $\alpha \in \overline{S}_E(X, Y)$ and let $x, y \in X$ be such that $(x, y) \in E$. If $x \notin Y$, then $\{x\} \in X/E$ by (2). So $x = y$. Thus $(x\alpha, y\alpha) \in E$. If $x, y \in Y$, then $(x\alpha, y\alpha) \in E$ since $\alpha \in \overline{S}_E(X, Y)$. Hence, $\alpha \in T_E(X)$. So $\overline{S}_E(X, Y) \subseteq T_E(X)$.

Consequently, if $Y \subsetneq X$, there exists $z \in X \setminus Y$, then $\overline{X}_z \in T_E(X)$. Therefore $\overline{X}_z \notin \overline{S}_E(X, Y)$ and so $T_E(X) \neq \overline{S}_E(X, Y)$. If $X = Y$, then (2) is true and so $\overline{S}_E(X, Y) \subseteq T_E(X)$. Let $\alpha \in T_E(X)$ and let $x, y \in Y$ be such that $(x, y) \in E$. Then $(x\alpha, y\alpha) \in E$ and $x\alpha, y\alpha \in X = Y$ since $\alpha \in T_E(X)$. Therefore $\alpha \in \overline{S}_E(X, Y)$ and $\overline{S}_E(X, Y) = T_E(X)$. $\square$

The following example shows that, in general, $\overline{S}_E(X, Y) \subsetneq T_E(X)$, $T_E(X) \subsetneq \overline{S}_E(X, Y)$, and $S(X, Y) \subsetneq \overline{S}_E(X, Y)$.

Example 2.5.3. Let $X = \{1, 2, 3, 4, 5\}$, $Y = \{1, 2, 3\}$ and $X/E = \{\{1, 2\}, \{3, 4\}, \{5\}\}$. Define $\alpha, \beta, \gamma: X \rightarrow X$ by

$$x\alpha = \begin{cases} 2 & \text{if } x = 1, \\ 3 & \text{if } x \in \{2, 4\}, \\ x & \text{otherwise} \end{cases}, \quad x\beta = \begin{cases} 4 & \text{if } x \in \{1, 2\}, \\ 1 & \text{otherwise} \end{cases}, \quad \text{and } x\gamma = \begin{cases} 2 & \text{if } x = 1, \\ 5 & \text{if } x = 4, \\ x & \text{otherwise.} \end{cases}$$

Clearly, $Y\alpha = \{2, 3\} \subseteq Y$ and $Y\gamma = \{1, 2, 3\} = Y$. Note that $(1\gamma, 2\gamma) = (2, 2) \in E$ and $(3\gamma, 3\gamma) = (3, 3) \in E$. Then $\gamma \in \overline{S}_E(X, Y)$ and $\alpha \in S(X, Y)$. Since $(3\gamma, 4\gamma) = (3, 5) \notin E$ and $(1\alpha, 2\alpha) = (2, 3) \notin E$, we have $\gamma \notin T_E(X)$ and $\alpha \notin \overline{S}_E(X, Y)$. Next, we consider $(1\beta, 2\beta) = (4, 4) \in E$ and $(3\beta, 4\beta) = (1, 1) \in E$. Therefore $\beta \in T_E(X)$ but $\beta \notin \overline{S}_E(X, Y)$ since $1\beta = 4 \notin Y$.

CHAPTER III

Regularity

In this chapter, we start by examining the regularity of $\overline{S}_E(X, Y)$, including regular, left regular, and right regular elements of the semigroup $\overline{S}_E(X, Y)$. Additionally, we explore left regular and right regular elements of the transformation semigroups $T(X, Y)$ and $S(X, Y)$.

3.1 Regularity of $\overline{S}_E(X, Y)$

In this section, we examine the regular elements in $\overline{S}_E(X, Y)$ and subsequently provide necessary and sufficient conditions for $\overline{S}_E(X, Y)$ to be left regular and right regular. To accomplish this, the following lemma serves as a crucial tool.

Lemma 3.1.1. *Let $\alpha \in \overline{S}_E(X, Y)$. Then, for each $A \in X/E$, there exists $B \in X/E$ such that $(A \cap Y)\alpha \subseteq B \cap Y \neq \emptyset$.*

Proof. Let $A \in X/E$. If $A \cap Y = \emptyset$, then $(A \cap Y)\alpha = \emptyset \subseteq B \cap Y$ for some $B \in X/E$. Let $x \in A \cap Y$. Since $x \in Y$ and $\alpha \in \overline{S}_E(X, Y)$, we have $x\alpha \in Y$. From X/E is a partition of X , there exists $B \in X/E$ such that $x\alpha \in B \cap Y$. To show $(A \cap Y)\alpha \subseteq B \cap Y$, we let $y \in A \cap Y$. Since x and y belong to the same partition, we have $(x, y) \in E$. Since $x, y \in Y$ and $\alpha \in \overline{S}_E(X, Y)$, we get $(x\alpha, y\alpha) \in E$ and $y\alpha \in Y$. Hence, $y\alpha \in B \cap Y$, as required. $\square$

Theorem 3.1.2. *Let $\alpha \in \overline{S}_E(X, Y)$. Then α is regular if and only if, for each $A \in X/E$, there exists $B \in X/E$ such that $B \cap Y \neq \emptyset$ and $(A \cap Y) \cap X\alpha \subseteq (B \cap Y)\alpha$.*

Proof. Assume α is regular and let $A \in X/E$. Then $\alpha = \alpha\beta\alpha$ for some $\beta \in \overline{S}_E(X, Y)$. From Lemma 3.1.1, there exists $B \in X/E$ such that $(A \cap Y)\beta \subseteq B \cap Y \neq \emptyset$. For each $x \in (A \cap Y) \cap X\alpha$, we have $x = a\alpha$ for some $a \in X$, and so $x = a\alpha = a\alpha\beta\alpha = x\beta\alpha \in (B \cap Y)\alpha$. Hence, $(A \cap Y) \cap X\alpha \subseteq (B \cap Y)\alpha$.

Conversely, assume that the condition hold. For each $A \in X/E$ in which $A \cap Y \neq \emptyset$, by the assumption, we can fix $B_A \in X/E$ such that $B_A \cap Y \neq \emptyset$ and $(A \cap Y) \cap X\alpha \subseteq (B_A \cap Y)\alpha$. Let $y_A \in B_A \cap Y$ be fixed. Consider $x \in A \cap Y$. If $x \in X\alpha$, then we choose

$x' \in B_A \cap Y$ such that $x'\alpha = x$. If $x \notin X\alpha$, then we set $x' = y_A$. For each $x \in X\alpha \setminus Y$, we choose $x' \in X$ such that $x'\alpha = x$. Now, define $\beta : X \rightarrow X$ by

$$x\beta = \begin{cases} x' & \text{if } x \in X\alpha \cup Y, \\ x & \text{otherwise.} \end{cases}$$

To show $\beta \in \overline{S}_E(X, Y)$, let $x, y \in Y$ in which $(x, y) \in E$. Thus x and y belong to the same equivalence class, say A . Then $x\beta = x'$ and $y\beta = y'$ are both in $B_A \cap Y$, that is, $\beta \in \overline{S}_E(X, Y)$. To show $\alpha = \alpha\beta\alpha$, let $x \in X$. Then $x\alpha\beta\alpha = (x\alpha)\beta\alpha = (x\alpha)'\alpha = x\alpha$. This completes the proof. $\square$

Theorem 3.1.3. $\overline{S}_E(X, Y)$ is a regular semigroup if and only if one of the following conditions hold:

- (1) $|Y| = 1$.
- (2) $Y = X$ and $E = X \times X$.
- (3) $Y = X$ and $E = id_X$.

Proof. Assume that all conditions are false.

Case 1: $|Y| \neq 1$ and $Y \neq X$. Then there exist distinct $a, b \in Y$ and $X \setminus Y \neq \emptyset$. Define $\alpha : X \rightarrow X$ by

$$x\alpha = \begin{cases} b & \text{if } x \in Y, \\ a & \text{otherwise.} \end{cases}$$

Note that $\alpha|_Y = \overline{Y}_b$ and $X\alpha = \{a, b\}$ since $X \setminus Y \neq \emptyset$. By Lemma 2.4.3, we get $\alpha \in \overline{S}_E(X, Y)$. Let $A \in X/E$ be such that $a \in A$. For each $B \in X/E$, we have $(B \cap Y)\alpha \subseteq Y\alpha = \{b\}$. Since $a \in (A \cap Y) \cap X\alpha$, we obtain $(A \cap Y) \cap X\alpha \not\subseteq (B \cap Y)\alpha$ for all $B \in X/E$. By Theorem 3.1.2, α is not regular.

Case 2: $|Y| \neq 1$, $Y = X$, $E \neq X \times X$, and $E \neq id_X$. Then there exists $(a, b) \in E$ and $c \in X$ such that $a \neq b$ and $(a, c) \notin E$. Thus, there exists a unique $A \in X/E$ such that $a, b \in A$. Define $\alpha : X \rightarrow X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in A, \\ b & \text{otherwise.} \end{cases}$$

We note from $X\alpha = \{a, b\} \subseteq X = Y$ and $(a, b) \in E$ that $\alpha \in \overline{S}_E(X, Y)$. It is clear that $(B \cap Y)\alpha = B\alpha$ is a singleton set or empty for all $B \in X/E$. Since $a, b \in (A \cap Y) \cap X\alpha$, we get $(A \cap Y) \cap X\alpha \not\subseteq (B \cap Y)\alpha$. By Theorem 3.1.2, α is not regular.

Conversely, assume one of three aforementioned conditions holds. Let $\alpha \in \overline{S}_E(X, Y)$. If $|Y| = 1$, then we let $Y = \{y\}$. Hence, there exists $B \in X/E$ such that $y \in B$. In this case, $(A \cap Y) \cap X\alpha \subseteq \{y\} = (B \cap Y)\alpha$ for all $A \in X/E$, and so α is regular. For the case of $Y = X$ and $E = X \times X$, we have X is exactly one equivalence class in X/E and $(X \cap Y) \cap X\alpha = X\alpha = (X \cap Y)\alpha$. This implies α is regular. Finally, if $Y = X$ and $E = id_X$, then each equivalence class in X/E is a singleton set of elements in $Y = X$. Consider $\{x\} \in X/E$. If $x \notin X\alpha$, then $(\{x\} \cap Y) \cap X\alpha = \emptyset \subseteq (\{x\} \cap Y)\alpha$. If $x \in X\alpha$, then $x = x'\alpha$ for some $x' \in X$. Hence, $(\{x\} \cap Y) \cap X\alpha = (\{x'\} \cap X)\alpha$, and so α is regular. Therefore, $\overline{S}_E(X, Y)$ is regular, as required. $\square$

If $X = Y$, then $\overline{S}_E(X, Y) = T_E(X)$, and we have the following corollaries, which first appeared in [17].

Corollary 3.1.4. [17] $\alpha \in T_E(X)$ is regular if and only if for each $A \in X/E$, there exists $B \in X/E$ such that $A \cap X\alpha \subseteq B\alpha$.

Corollary 3.1.5. [17] $T_E(X)$ is regular if and only if $E = id_X$ or $X \times X$.

Next, we consider left and right regular elements of the semigroup $\overline{S}_E(X, Y)$.

Theorem 3.1.6. Let $\alpha \in \overline{S}_E(X, Y)$. Then α is left regular if and only if the following two conditions hold:

- (1) for each $A \in X/E$, there exists $B \in X/E$ such that for all $P \in \pi(\alpha)$, if $P \cap A \cap Y \neq \emptyset$, then $x\alpha \in P \cap A \cap Y$ for some $x \in B \cap Y$.
- (2) for each $P \in \pi(\alpha)$, if $P \cap Y = \emptyset$, then $x\alpha \in P$ for some $x \in X \setminus Y$.

Proof. Assume that α is left regular. Then $\alpha = \beta\alpha^2$ for some $\beta \in \overline{S}_E(X, Y)$.

(1) Let $A \in X/E$. By Lemma 3.1.1, we have $(A \cap Y)\beta \subseteq B \cap Y$ for some $B \in X/E$. Let $P \in \pi(\alpha)$ and let $x \in P \cap A \cap Y$. So $x\beta \in B \cap Y$. Note that $x\alpha = x\beta\alpha^2 = (x\beta\alpha)\alpha$. Thus $x\beta\alpha \in x\alpha\alpha^{-1} = P$.

(2) Let $P \in \pi(\alpha)$ and suppose $P \cap Y = \emptyset$. Let $x \in P$. Note that $x\beta\alpha \in x\alpha\alpha^{-1} = P$ since $x\beta\alpha^2 = x\alpha$. If $x\beta \in Y$, then $x\beta\alpha \in Y$ from $\overline{S}_E(X, Y) \subseteq S(X, Y)$. Therefore $x\beta\alpha \in P \cap Y$ which is a contradiction. Hence, $x\beta \in X \setminus Y$.

Conversely, assume (1) and (2) hold. For each $A \in X/E$, we fix $A' \in X/E$ such that for all $P \in \pi(\alpha)$, if $P \cap A \cap Y \neq \emptyset$, then $x\alpha \in P \cap A \cap Y$ for some $x \in A' \cap Y$. Let $x \in Y$. Then there exist a unique $A \in X/E$ and $P \in \pi(\alpha)$ such that $x \in P \cap A$. Thus, we choose $x' \in A' \cap Y$ with $x'\alpha \in P \cap A \cap Y$. Thus $x'\alpha\alpha = x\alpha$ since $x, x'\alpha \in P$. Let $x \in X \setminus Y$. Then there exists $P \in \pi(\alpha)$ such that $x \in P$. If $P \cap Y \neq \emptyset$, then we choose $B \in X/E$ such that $P \cap B \cap Y \neq \emptyset$. We put $x' \in B' \cap Y$ with $x'\alpha \in P \cap B \cap Y$. Therefore, $x'\alpha\alpha = x\alpha$. If $P \cap Y = \emptyset$, then we choose $x' \in X \setminus Y$ such that $x'\alpha \in P$. Thus $x'\alpha\alpha = x\alpha$. Define $\beta : X \rightarrow X$ by

$$x\beta = x' \text{ for all } x \in X.$$

Clearly, for $x \in X$, we get $x\beta\alpha^2 = x'\alpha^2 = x'\alpha\alpha = x\alpha$. Let $x, y \in Y$ be such that $(x, y) \in E$. Then $x, y \in C$ for some $C \in X/E$. This implies that $x', y' \in C' \cap Y$ where $C' \in X/E$. Therefore, $\beta \in \overline{S}_E(X, Y)$. Hence, α is left regular. $\square$

To investigate the right regularity of $\overline{S}_E(X, Y)$, we require the following theorem.

Theorem 3.1.7. [10] *Let $\alpha \in T(X)$. Then α is right regular if and only if $\alpha|_{X\alpha}$ is injective.*

Theorem 3.1.8. *Let $\alpha \in \overline{S}_E(X, Y)$. Then α is right regular if and only if*

(1) $\alpha|_{X\alpha}$ is injective.

(2) for each $x, y \in X\alpha$, if $(x\alpha, y\alpha) \in E$ and $x\alpha, y\alpha \in Y$, then $(x, y) \in E$ and $x, y \in Y$.

Proof. Assume that α is right regular. Then $\alpha = \alpha^2\beta$ for some $\beta \in \overline{S}_E(X, Y)$. From α is right regular in $T(X)$, it follows from Theorem 3.1.7 that $\alpha|_{X\alpha}$ is injective. Let $x, y \in X\alpha$. Suppose that $(x\alpha, y\alpha) \in E$ and $x\alpha, y\alpha \in Y$. Then $x = x'\alpha$ and $y = y'\alpha$ for some $x', y' \in X$. Since $x\alpha, y\alpha \in Y$, $(x\alpha, y\alpha) \in E$ and $\beta \in \overline{S}_E(X, Y)$, we have $(x\alpha\beta, y\alpha\beta) \in E$ and $x\alpha\beta, y\alpha\beta \in Y$. Therefore $(x, y) = (x'\alpha, y'\alpha) = (x'\alpha^2\beta, y'\alpha^2\beta) = (x\alpha\beta, y\alpha\beta) \in E$ and $x, y \in Y$.

Conversely, assume (1) and (2) hold. Let $A \in X/E$ be such that $X\alpha^2 \cap (A \cap Y) \neq \emptyset$. We choose and fix an element $x_A \in X\alpha^2 \cap (A \cap Y)$. For each $x \in X\alpha^2 \cap (A \cap Y)$,

there exists a unique $x' \in X\alpha$ such that $x = x'\alpha$ since $\alpha|_{X\alpha}$ is injective. Note that $(x'\alpha, x'_A\alpha) = (x, x_A) \in E$. It follows from assumption that $(x', x'_A) \in E$ and $x', x'_A \in Y$. If $x \in X\alpha^2 \cap A \cap (X \setminus Y)$, then there is a unique $x' \in X\alpha$ such that $x = x'\alpha$. Clearly, $x' \in X \setminus Y$. Define $\beta_A : A \rightarrow X$ by

$$x\beta_A = \begin{cases} x' & \text{if } x \in X\alpha^2, \\ x'_A & \text{if } x \in Y \setminus X\alpha^2, \\ x & \text{otherwise.} \end{cases}$$

Let $A \in X/E$ be such that $X\alpha^2 \cap (A \cap Y) = \emptyset$ and let $x \in A$. If $x \in X\alpha^2$, then there exists a unique $x' \in X\alpha$ such that $x = x'\alpha$. Clearly, $x' \notin Y$. Define $\beta_A : A \rightarrow X$ by

$$x\beta_A = \begin{cases} x' & \text{if } x \in X\alpha^2, \\ x & \text{otherwise.} \end{cases}$$

Thus, we define the map $\beta : X \rightarrow X$ by $x\beta = x\beta_A$ where $x \in A$ for some $A \in X/E$. Let $x, y \in Y$ and $(x, y) \in E$. Then $x, y \in A$ for some $A \in X/E$. By the definition of β , we have $(x\beta, y\beta) = (x\beta_A, y\beta_A)$. If $X\alpha^2 \cap (A \cap Y) = \emptyset$, then $(x\beta, y\beta) = (x\beta_A, y\beta_A) = (x, y) \in E$. Suppose that $X\alpha^2 \cap (A \cap Y) \neq \emptyset$. If $x, y \notin X\alpha^2$, then $(x\beta, y\beta) = (x'_A, x'_A) \in E$ and $x\beta, y\beta \in Y$. If $x \in X\alpha^2$ and $y \notin X\alpha^2$, then $(x\beta, y\beta) = (x', x'_A) \in E$ and $x\beta, y\beta \in Y$. If $x, y \in X\alpha^2$, then $(x\beta, y\beta) = (x\beta_A, y\beta_A) = (x', y')$. Note that $(x\beta, x'_A) = (x', x'_A) \in E$ and $(y\beta, x'_A) = (y', x'_A) \in E$. By transitivity and symmetry of E , we have $(x', y') \in E$ and $x', y' \in Y$. Therefore, $\beta \in \overline{S}_E(X, Y)$. For each $x \in X$, we have $x\alpha^2 \in X\alpha^2$. This implies that $x\alpha^2\beta_A = (x\alpha^2)'$ where $(x\alpha^2)'\alpha = x\alpha^2 = (x\alpha)\alpha$ where $x\alpha^2 \in A$ and $A \in X/E$. By the uniqueness of $(x\alpha^2)'$, we obtain $(x\alpha^2)' = x\alpha$. Thus $x\alpha^2\beta = x\alpha^2\beta_A = (x\alpha^2)' = x\alpha$. Hence, $\alpha = \alpha^2\beta$ and so α is right regular. $\square$

3.2 Regularity of $T(X, Y)$

In this section, we initially review a theorem on regular elements from [16]. Subsequently, we examine the conditions under which an element of $T(X, Y)$ is both left and right regular. Furthermore, we demonstrate that every left regular element of $T(X, Y)$ is regular, and when Y is finite, the set of all left regular elements of $T(X, Y)$ is identical to the set of all right regular elements of $T(X, Y)$.

Theorem 3.2.1. [16] For $\alpha \in T(X, Y)$, the following statements are equivalent:

- (1) α is regular.
- (2) $X\alpha = Y\alpha$.
- (3) for each $P \in \pi(\alpha)$, $P \cap Y \neq \emptyset$.
- (4) $x\alpha^{-1} \cap Y \neq \emptyset$ for every $x \in X\alpha$.

Corollary 3.2.2. [16] *The semigroup $T(X, Y)$ is regular if and only if either $X = Y$ or $|Y| = 1$.*

Now, we will explain left regularity and right regularity in $T(X, Y)$.

Theorem 3.2.3. *Let $\alpha \in T(X, Y)$. Then the following statements are equivalent.:*

- (1) α is left regular.
- (2) $Y\alpha^2 = X\alpha$.
- (3) for any $P \in \pi(\alpha)$, $Y\alpha \cap P \neq \emptyset$.

Proof. (1) $\Rightarrow$ (2) Suppose that α is left regular. Thus there exists $\beta \in T(X, Y)$ satisfying $\alpha = \beta\alpha^2$. This implies that $Y\alpha^2 \subseteq X\alpha = X\beta\alpha^2 \subseteq Y\alpha^2$. Thus $Y\alpha^2 = X\alpha$.

(2) $\Rightarrow$ (3) Suppose that $Y\alpha^2 = X\alpha$ and let $P \in \pi(\alpha)$. Thus there is $y \in X\alpha$ satisfying $y\alpha^{-1} = P$. By assumption, we have $y \in Y\alpha^2$. Thus there exists $z \in Y$ such that $y = z\alpha^2$, that is, $z\alpha \in y\alpha^{-1} = P$. It follows that $z\alpha \in Y\alpha \cap P$. Therefore, $Y\alpha \cap P \neq \emptyset$.

(3) $\Rightarrow$ (1) Assume that (3) holds. For $x \in X$, there is a unique $P \in \pi(\alpha)$ satisfying $x \in P$. From assumption, we have $Y\alpha \cap P \neq \emptyset$. So there exists $x' \in Y$ such that $x'\alpha \in P = (x\alpha)\alpha^{-1}$. Define $\beta : X \rightarrow X$ by $x\beta = x'$ for all $x \in X$. It is obvious that $\beta \in T(X, Y)$. Let $x \in X$. This implies that $x\beta\alpha^2 = x'\alpha^2 = (x'\alpha)\alpha = x\alpha$. Hence, $\alpha = \beta\alpha^2$ and so α is left regular. $\square$

If we replace Y with X in Theorem 3.2.3, we have the following corollary.

Corollary 3.2.4. *Let $\alpha \in T(X)$. Then α is left regular if and only if for each $P \in \pi(\alpha)$, $X\alpha \cap P \neq \emptyset$.*

Proof. By taking $X = Y$, we obtain $T(X, Y) = T(X, X) = T(X)$ and $X\alpha = Y\alpha$. By Theorem 3.2.3(3), we get α is left regular if and only if for each $P \in \pi(\alpha)$, $X\alpha \cap P \neq \emptyset$. $\square$

Theorem 3.2.5. *Let $\alpha \in T(X, Y)$. Then α is right regular if and only if $\alpha|_{X\alpha}$ is injective.*

Proof. Assume that α is right regular. Then there exists $\beta \in T(X, Y)$ such that $\alpha = \alpha^2\beta$. We will show that $\alpha|_{X\alpha}$ is injective. Let $x, y \in X\alpha$ be such that $x\alpha = y\alpha$. Thus there exist $x', y' \in X$ such that $x = x'\alpha$ and $y = y'\alpha$. We obtain that

$$x = x'\alpha = x'\alpha^2\beta = (x'\alpha)\alpha\beta = x\alpha\beta = y\alpha\beta = y'\alpha^2\beta = y'\alpha = y.$$

Therefore, $\alpha|_{X\alpha}$ is injective.

Conversely, suppose $\alpha|_{X\alpha}$ is injective. Let $z \in X\alpha^2$. Then there exists a unique $z' \in X\alpha$ such that $z'\alpha = z$. We choose $y \in Y$. Define $\beta : X \rightarrow X$ by

$$z\beta = \begin{cases} z' & \text{if } z \in X\alpha^2, \\ y & \text{otherwise.} \end{cases}$$

From $X\alpha \subseteq Y$, we obtain that $X\beta \subseteq X\alpha \cup \{y\} \subseteq Y$. Then $\beta \in T(X, Y)$. Let $x \in X$. Note that $x\alpha^2 \in X\alpha^2$ and $(x\alpha^2)\beta = (x\alpha^2)'$ where $(x\alpha^2)'\alpha = x\alpha^2 = x\alpha\alpha$. Since $\alpha|_{X\alpha}$ is injective, we have $(x\alpha^2)' = x\alpha$. So $x\alpha^2\beta = (x\alpha^2)' = x\alpha$. Hence, $\alpha^2\beta = \alpha$ and so α is right regular. $\square$

Notice that for $|X| \leq 2$, we get that $T(X, Y)$ is left and right regular. Now, we consider the other case.

Theorem 3.2.6. *Let $X \neq Y$ be such that $|X| \geq 3$. Then $T(X, Y)$ is a right regular semigroup if and only if $|Y| = 1$.*

Proof. If $|Y| = 1$, then $|T(X, Y)| = 1$ and hence $T(X, Y)$ is a right regular semigroup. Assume that $T(X, Y)$ is a right regular semigroup and suppose $|Y| \neq 1$. Let $a, b, c \in X$ be distinct elements and $a, b \in Y$. Define $\alpha : X \rightarrow X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in \{a, b\}, \\ b & \text{otherwise.} \end{cases}$$

Note that $X\alpha = \{a, b\} \subseteq Y$. Then $\alpha \in T(X, Y)$ and $\alpha|_{X\alpha}$ is not injective. Thus α is not right regular which is a contradiction. Hence, $|Y| = 1$. $\square$

Theorem 3.2.7. *Let $X \neq Y$ be such that $|X| \geq 3$. Then $T(X, Y)$ is a left regular semigroup if and only if $|Y| = 1$.*

Proof. If $|Y| = 1$, then $|T(X, Y)| = 1$ and so $T(X, Y)$ is a left regular semigroup. Assume that $T(X, Y)$ is a left regular semigroup and $|Y| \neq 1$. Let $a, b, c \in X$ be distinct elements and $a, b \in Y$. Define $\alpha : X \rightarrow X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in \{a, b\}, \\ b & \text{otherwise.} \end{cases}$$

Then $\alpha \in T(X, Y)$. Note that $a, b \notin b\alpha^{-1}$ and $Y\alpha \subseteq \{a, b\}$. So $Y\alpha \cap b\alpha^{-1} = \emptyset$. Therefore α is not left regular which is a contradiction. Hence, $|Y| = 1$. $\square$

From Theorem 2.4 of [23], we note that $F(X, Y) = \{\alpha \in T(X, Y) : X\alpha = Y\alpha\}$ is the largest regular subsemigroup of $T(X, Y)$.

Corollary 3.2.8. *Every left regular element of a semigroup $T(X, Y)$ is regular.*

Proof. We first note that $F(X, Y)$ is the largest regular subsemigroup of $T(X, Y)$. Let α be a left regular element of $T(X, Y)$. It follows from Theorem 3.2.3(2) that $X\alpha = Y\alpha^2 = (Y\alpha)\alpha \subseteq Y\alpha \subseteq X\alpha$. Thus, $X\alpha = Y\alpha$ and so $\alpha \in F(X, Y)$. Therefore, α is a regular element of $T(X, Y)$. $\square$

Example 3.2.9. *Let $X = \mathbb{N}$ and $Y = 2\mathbb{N}$. Define $\alpha : X \rightarrow Y$ by*

$$x\alpha = \begin{cases} x + 2 & \text{if } x \in 2\mathbb{N}, \\ x + 3 & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in T(X, Y)$. Consider $X\alpha = Y\alpha = 2\mathbb{N} \setminus \{2\}$ and $Y\alpha^2 = 2\mathbb{N} \setminus \{2, 4\}$. Then α is regular, but it is not left regular since Theorem 3.2.1 and Theorem 3.2.3.

Example 3.2.10. *Let $X = \mathbb{N}$ and $Y = 2\mathbb{N} + 1$. Define $\alpha : X \rightarrow Y$ by*

$$x\alpha = \begin{cases} x + 1 & \text{if } x \in 2\mathbb{N}, \\ x + 2 & \text{otherwise.} \end{cases}$$

It is verify that $\alpha \in T(X, Y)$. We obtain that $\alpha|_{X\alpha}$ is injective and $X\alpha \neq Y\alpha$. Thus, α is right regular, but it is not regular since Theorem 3.2.1 and Theorem 3.2.5.

Finally, we consider the sets of all left regular elements, denoted as $LReg(T(X, Y))$, and all right regular elements, denoted as $RReg(T(X, Y))$, of $T(X, Y)$. We begin with the following example.

Example 3.2.11. Let $X = \{1, 2, 3, 4, 5\}$ and $Y = \{1, 2, 3, 4\}$. We define mappings $\alpha, \beta : X \rightarrow Y$ by

$$x\alpha = \begin{cases} 4 & \text{if } x = 4, \\ 3 & \text{otherwise} \end{cases} \quad \text{and} \quad x\beta = \begin{cases} 2 & \text{if } x \in \{1, 4\}, \\ 1 & \text{otherwise.} \end{cases}$$

We note that $\alpha, \beta \in T(X, Y)$. Moreover, $\alpha|_{X\alpha}$ and $\beta|_{X\beta}$ are injective, that is, $\alpha, \beta \in RReg(T(X, Y))$, as shown in Theorem 3.2.5. We note that $\alpha\beta|_{X\alpha\beta}$ is not injective and so $\alpha\beta \notin RReg(T(X, Y))$. In this case, we obtain that $RReg(T(X, Y))$ is not a semigroup.

Theorem 3.2.12. Let $|X| \geq 3$. Then $RReg(T(X, Y))$ is a semigroup if and only if $|Y| \leq 2$.

Proof. Suppose that $|Y| \geq 3$. Let $a, b, c \in Y$ be distinct elements. Define $\alpha : X \rightarrow Y$ by

$$x\alpha = \begin{cases} a & \text{if } x = b, \\ b & \text{if } x = a, \\ c & \text{otherwise.} \end{cases}$$

We see that $X\alpha = \{a, b, c\}$ and $\alpha|_{X\alpha}$ is injective. Clearly, $\alpha \in T(X, Y)$. Define $\beta : X \rightarrow Y$ by

$$x\beta = \begin{cases} a & \text{if } x = a, \\ c & \text{otherwise.} \end{cases}$$

Then $X\beta = \{a, c\}$ and $\beta|_{X\beta}$ is injective. Clearly, $\beta \in T(X, Y)$. Since $b\alpha\beta = a$ and $c\alpha\beta = c$, we get $a, c \in X\alpha\beta$. From $a\alpha\beta = c = c\alpha\beta$, we conclude that $\alpha\beta|_{X\alpha\beta}$ is not injective. Therefore, $\alpha\beta \notin RReg(T(X, Y))$ and that $RReg(T(X, Y))$ is not closed.

Assume that $|Y| \leq 2$. If $|Y| = 1$, then $T(X, Y)$ is a right regular semigroup. Therefore $RReg(T(X, Y))$ is a semigroup. Suppose that $|Y| = 2$. Let $Y = \{a, b\}$ and $\alpha, \beta \in RReg(T(X, Y))$. We will show that $\alpha\beta|_{X\alpha\beta}$ is injective. Let $x, y \in X\alpha\beta$ be such that $x\alpha\beta = y\alpha\beta$. If α or β is a constant mapping, then $\alpha\beta$ is a constant mapping. Thus $\alpha\beta|_{X\alpha\beta}$ is injective. Suppose that α and β are not constant mappings. Then $X\alpha = Y = X\beta$. We observe that $x, y \in X\alpha\beta \subseteq X\beta = X\alpha$. If $x \neq y$, then $x\alpha \neq y\alpha$ since $\alpha|_{X\alpha}$ is injective. Note that $x\alpha, y\alpha \in Y = X\beta$. Then $(x\alpha)\beta \neq (y\alpha)\beta$ since $\beta|_{X\beta}$ is injective which is a contradiction. Hence, $x = y$. So $\alpha\beta|_{X\alpha\beta}$ is injective. Therefore $RReg(T(X, Y))$ is closed. $\square$

Theorem 3.2.13. *Let $|X| \geq 3$. Then $LReg(T(X, Y))$ is a semigroup if and only if $|Y| \leq 2$.*

Proof. Suppose that $|Y| \geq 3$. Let $a, b, c \in Y$ be distinct elements. Define $\alpha : X \rightarrow Y$ by

$$x\alpha = \begin{cases} b & \text{if } x = c, \\ c & \text{otherwise.} \end{cases}$$

We define $\beta : X \rightarrow Y$ by

$$x\beta = \begin{cases} a & \text{if } x = b, \\ b & \text{otherwise.} \end{cases}$$

Then $Y\alpha^2 = \{b, c\}\alpha = \{b, c\} = X\alpha$ and $Y\beta^2 = \{a, b\}\beta = \{a, b\} = X\beta$. Thus $\alpha, \beta \in LReg(T(X, Y))$. It is easy to verify that

$$x\alpha\beta = \begin{cases} a & \text{if } x = c, \\ b & \text{otherwise} \end{cases}$$

such that $\pi(\alpha\beta) = \{\{c\}, X \setminus \{c\}\}$ and $Y\alpha\beta = \{a, b\}$. Clearly, $Y\alpha\beta \cap \{c\} = \emptyset$. Hence, $\alpha\beta \notin LReg(T(X, Y))$. So $\alpha\beta$ is not left regular.

Conversely, suppose $|Y| \leq 2$. If $|Y| = 1$, then $T(X, Y)$ is a left regular semigroup. We have $LReg(T(X, Y))$ is a semigroup. Assume that $Y = \{a, b\}$. Let $\alpha, \beta \in LReg(T(X, Y))$. If $|X\alpha| = 1$ or $|X\beta| = 1$, then $\alpha\beta$ is a constant mapping. Suppose that $|X\alpha| = |X\beta| = 2$. Then $\pi(\alpha) = \{a\alpha^{-1}, b\alpha^{-1}\}$ and so it is a left regular

element. From α is a left regular element, we get $Y\alpha \cap a\alpha^{-1} \neq \emptyset$ and $Y\alpha \cap b\alpha^{-1} \neq \emptyset$. This implies that $|a\alpha^{-1} \cap Y| = 1 = |b\alpha^{-1} \cap Y|$ and $X\alpha = Y\alpha = \{a, b\}$. Similarly, $|a\beta^{-1} \cap Y| = 1 = |b\beta^{-1} \cap Y|$ and $X\beta = Y\beta = \{a, b\}$. It is easy to verify that $\pi(\alpha\beta) = \pi(\alpha)$ and $Y\alpha\beta = \{a, b\} = Y\alpha$. Hence, $Y\alpha\beta \cap a\alpha^{-1} \neq \emptyset$ and $Y\alpha\beta \cap b\alpha^{-1} \neq \emptyset$. Therefore, $\alpha\beta$ is a left regular element. $\square$

As shown in Example 3.2.10, $RReg(T(X, Y))$ and $LReg(T(X, Y))$ are not generally identical. However, if Y is finite, we obtain the following result even if X is infinite.

Theorem 3.2.14. *If Y is finite, then $RReg(T(X, Y)) = LReg(T(X, Y))$.*

Proof. Assume that Y is finite. Let α be a left regular element of $T(X, Y)$. Then $Y\alpha^2 = X\alpha$. It follows that $X\alpha = Y\alpha^2 \subseteq X\alpha^2 \subseteq X\alpha$, that is, $X\alpha^2 = X\alpha$. From $(X\alpha)\alpha = X\alpha^2 = X\alpha$, we obtain $\alpha|_{X\alpha} : X\alpha \rightarrow X\alpha$ is surjective. Since $X\alpha \subseteq Y$ and Y is a finite set, we get $\alpha|_{X\alpha}$ is an injection. So α is right regular.

Assume that α is right regular. Thus $\alpha|_{X\alpha} : X\alpha \rightarrow X\alpha$ is injective and also $\alpha|_{X\alpha} : X\alpha \rightarrow X\alpha$ is surjective since $X\alpha$ is finite. This means that $(X\alpha)\alpha = X\alpha$. We see that $X\alpha = (X\alpha)\alpha \subseteq Y\alpha \subseteq X\alpha$. Hence $X\alpha = Y\alpha$ and so $Y\alpha^2 = X\alpha^2 = X\alpha$. Therefore α is a left regular element of $T(X, Y)$. $\square$

3.3 Regularity of $S(X, Y)$

In this section, we begin by recalling a theorem on regular elements, encompassing both left and right elements, from [16] and [2]. We then explore the conditions that determine the left and right regularity of elements in $S(X, Y)$. Finally, we describe some properties associated with left and right regular elements.

Theorem 3.3.1. [16] *For $\alpha \in S(X, Y)$, the following statements are equivalent:*

- (1) α is regular.
- (2) $X\alpha \cap Y = Y\alpha$.
- (3) $y\alpha^{-1} \cap Y \neq \emptyset$ for every $y \in X\alpha \cap Y$.
- (4) $x\alpha^{-1} \cap Y \neq \emptyset$ for every $x \in X\alpha \cap Y$.

Corollary 3.3.2. [16] *The semigroup $S(X, Y)$ is regular if and only if either $X = Y$ or $|Y| = 1$.*

Theorem 3.3.3. [2] *Let $\alpha \in S(X, Y)$. Then α is left regular if and only if $X\alpha = X\alpha^2$ and $Y\alpha = Y\alpha^2$.*

Theorem 3.3.4. [2] *Let $\alpha \in S(X, Y)$. Then α is right regular if and only if $\pi(\alpha) = \pi(\alpha^2)$ and $\sigma(\alpha) = \sigma(\alpha^2)$ where $\sigma(\alpha) = \{y\alpha^{-1} : y \in X\alpha \cap Y\}$.*

Although the left and right regular elements of $S(X, Y)$ was characterized in [2], but in this section we obtain the different results, see the following theorems.

Theorem 3.3.5. *Let $\alpha \in S(X, Y)$. Then α is a left regular element if and only if for every $P \in \pi(\alpha)$, $P \cap X\alpha \neq \emptyset$ and for every $P \in \pi_Y(\alpha)$, $P \cap Y\alpha \neq \emptyset$.*

Proof. Assume that α is a left regular element. Thus $\alpha = \beta\alpha^2$ for some $\beta \in S(X, Y)$. Let $P \in \pi(\alpha)$ and let $x \in P$. Then $P = (x\alpha)\alpha^{-1}$ and $x\alpha = x\beta\alpha^2 = [(x\beta)\alpha]\alpha$. We see that $(x\beta)\alpha \in (x\alpha)\alpha^{-1} = P$ and $(x\beta)\alpha \in X\alpha$. Therefore, $P \cap X\alpha \neq \emptyset$. Let $P \in \pi_Y(\alpha)$ and let $y \in P \cap Y$. Then $P = (y\alpha)\alpha^{-1}$, $y\beta \in Y$ and $y\alpha = y\beta\alpha^2 = [(y\beta)\alpha]\alpha$. We note that $(y\beta)\alpha \in Y\alpha$ and $(y\beta)\alpha \in (y\alpha)\alpha^{-1} = P$. Hence, $P \cap Y\alpha \neq \emptyset$.

Conversely, suppose the conditions hold. Let $x \in X$. Since $\pi(\alpha)$ is a partition of X , there is a unique $P_x \in \pi(\alpha)$ satisfying $x \in P_x$. If $P_x \cap Y \neq \emptyset$, then $P_x \in \pi_Y(\alpha)$ and $P_x \cap Y\alpha \neq \emptyset$ by our assumption. So, we choose $x' \in Y$ satisfying $x'\alpha \in P_x$. If $P_x \cap Y = \emptyset$, then since $P_x \cap X\alpha \neq \emptyset$, we choose $x' \in X$ satisfying $x'\alpha \in P_x$. Define $\beta : X \rightarrow X$ by $x\beta = x'$ for all $x \in X$. Then β is well-defined and $x\beta\alpha^2 = x'\alpha^2 = (x'\alpha)\alpha = x\alpha$. Let $y \in Y$. Then there is a unique $P_y \in \pi(\alpha)$ such that $y \in P_y$. Thus $P_y \cap Y \neq \emptyset$. Therefore, $y\beta = y' \in Y$, that is, $Y\beta \subseteq Y$. So α is left regular. $\square$

Theorem 3.3.6. *Let $\alpha \in S(X, Y)$. Then α is a right regular element if and only if $\alpha|_{X\alpha}$ is injective and $(X\alpha \setminus Y)\alpha \subseteq X \setminus Y$.*

Proof. Assume that α is right regular. Thus α is also a right regular element in $T(X)$. From Corollary 3.1.7, we have $\alpha|_{X\alpha}$ is injective. Next, we will show that $(X\alpha \setminus Y)\alpha \subseteq X \setminus Y$. Let $z \in X\alpha \setminus Y$. Then $z = z'\alpha$ for some $z' \in X$. Thus $z = z'\alpha^2\beta$ for some $\beta \in S(X, Y)$ since α is a right regular element in $S(X, Y)$. If $z'\alpha^2 \in Y$, then $z = (z'\alpha^2)\beta \in Y\beta \subseteq Y$ which is a contradiction. Hence $z\alpha = z'\alpha^2 \notin Y$ and so $(X\alpha \setminus Y)\alpha \subseteq X \setminus Y$.

Assume that $\alpha|_{X\alpha}$ is injective and $(X\alpha \setminus Y)\alpha \subseteq X \setminus Y$. Let β be defined in the converse part of Theorem 3.2.5, we note that $\alpha = \alpha^2\beta$. It is enough to verify that $\beta \in S(X, Y)$. Let $x \in Y$. If $x \notin X\alpha^2$, then by the definition of β , we have $x\beta \in Y$. Assume that $x \in X\alpha^2$. There is a unique $x' \in X\alpha$ satisfying $x'\alpha = x$. If $x' \notin Y$, then $x' \in X\alpha \setminus Y$. By assumption, we have $x'\alpha \in X \setminus Y$ which is a contradiction. This means that $x\beta = x' \in Y$. Hence, $Y\beta \subseteq Y$ and so $\beta \in S(X, Y)$. $\square$

Example 3.3.7. Let $X = \mathbb{N}$ and $Y = 2\mathbb{N}$. Define $\alpha : X \rightarrow Y$ by

$$x\alpha = \begin{cases} 2 & \text{if } x \in 2\mathbb{N}, \\ 4 & \text{if } x = 1, \\ x - 2 & \text{otherwise.} \end{cases}$$

Then $X\alpha = \{2, 4\} \cup \{2n - 1 : n \in \mathbb{N}\}$ and $Y\alpha = \{2\} \subseteq Y$. So $\alpha \in S(X, Y)$. Moreover, we get $\pi(\alpha) = \{Y\} \cup \{\{2n - 1\} : n \in \mathbb{N}\}$ and $\pi_Y(\alpha) = \{Y\}$. It is clear that $P \cap X\alpha \neq \emptyset$ for every $P \in \pi(\alpha)$ and $P \cap Y\alpha \neq \emptyset$ for all $P \in \pi_Y(\alpha)$. From Theorem 3.3.5, α is left regular. Note that $X\alpha \cap Y = \{2, 4\} \neq Y\alpha$. By Theorem 3.3.1, α is also not regular.

Example 3.3.8. Let $X = \mathbb{N}$ and $Y = 2\mathbb{N}$. Define $\alpha : X \rightarrow Y$ by

$$x\alpha = \begin{cases} x + 2 & \text{if } x \in 2\mathbb{N}, \\ x + 3 & \text{otherwise.} \end{cases}$$

Then $Y\alpha \subseteq Y$ and also $X\alpha \cap Y = Y\alpha$. Thus $\alpha \in S(X, Y)$ and α is regular since Theorem 3.3.1. Note that $4\alpha^{-1} = \{1, 2\}$ and $X\alpha \cap 4\alpha^{-1} = \emptyset$. Hence, α is not a left regular element of $S(X, Y)$ since Theorem 3.3.5.

Example 3.3.9. Let $X = \mathbb{N}$ and $Y = 2\mathbb{N}$. Define $\alpha : X \rightarrow Y$ by

$$x\alpha = \begin{cases} 4 & \text{if } x \in \{1, 2, 3, 4\}, \\ x - 2 & \text{if } x \in 2\mathbb{N} \setminus \{2, 4\}, \\ x + 1 & \text{otherwise.} \end{cases}$$

Then $Y\alpha \subseteq Y$ and so $\alpha \in S(X, Y)$. Consider $Y \cap X\alpha = 2\mathbb{N} \setminus \{2\} = Y\alpha$ and $\alpha|_{X\alpha}$ is not injective. Hence, α is regular, but not right regular of $S(X, Y)$ since Theorem 3.3.1 and Theorem 3.3.6.

Example 3.3.10. Let $X = \mathbb{N}$ and $Y = 2\mathbb{N}$. Define $\alpha : X \rightarrow Y$ by

$$x\alpha = \begin{cases} x+2 & \text{if } x \in 2\mathbb{N}, \\ x+1 & \text{otherwise.} \end{cases}$$

Then $Y\alpha \subseteq Y$ and so $\alpha \in S(X, Y)$. We see that $\alpha|_{X\alpha}$ is injective and $(X\alpha \setminus Y)\alpha = \emptyset \subseteq X \setminus Y$. From Theorem 3.3.6, we obtain α is right regular. Consider $X\alpha \cap Y = 2\mathbb{N}$ and $Y\alpha = 2\mathbb{N} \setminus \{2\}$. Hence $X\alpha \cap Y \neq Y\alpha$. From Theorem 3.3.1, we obtain α is not regular.

Theorem 3.3.11. *The following statements are equivalent:*

- (1) $S(X, Y) = LReg(S(X, Y))$.
- (2) $S(X, Y) = RReg(S(X, Y))$.
- (3) $|X| \leq 2$.

Proof. (1) $\Leftrightarrow$ (3) Assume that $S(X, Y) = LReg(S(X, Y))$. We will show that $|X| \leq 2$. Suppose that $|X| \geq 3$. Let $a, b, c \in X$ be distinct elements and $a \in Y$. Define $\alpha : X \rightarrow X$ by

$$x\alpha = \begin{cases} b & \text{if } x = c, \\ a & \text{if } x \neq c. \end{cases}$$

Claim that $\alpha \in S(X, Y)$. If $c \notin Y$, then $Y\alpha = \{a\} \subseteq Y$. If $b, c \in Y$, then $Y\alpha = \{a, b\} \subseteq Y$. Thus, $\alpha \in S(X, Y)$ when $c \notin Y$ or $b, c \in Y$. Note that $\pi(\alpha) = \{\{c\}, X \setminus \{c\}\}$ and $X\alpha = \{a, b\}$. Clearly, $X\alpha \cap \{c\} = \emptyset$. Hence, α is not left regular. For the case $b \notin Y$ and $c \in Y$, we define $\beta : X \rightarrow X$ by

$$x\beta = \begin{cases} c & \text{if } x = b, \\ a & \text{if } x \neq b. \end{cases}$$

Then $Y\beta = \{a\} \subseteq Y$, $\pi(\beta) = \{\{b\}, X \setminus \{b\}\}$ and $X\beta = \{a, c\}$. It follows that $\beta \in S(X, Y)$ but $\beta \notin LReg(S(X, Y))$. We conclude that $S(X, Y) \neq LReg(S(X, Y))$, which is a contradiction. So $|X| \leq 2$.

Conversely, assume that $|X| \leq 2$. Then, it is easy to verify that $S(X, Y)$ is a left regular semigroup. Hence, $S(X, Y) = LReg(S(X, Y))$.

(2) $\Leftrightarrow$ (3) Assume that $S(X, Y) = RReg(S(X, Y))$. We will show that $|X| \leq 2$. Suppose that $|X| \geq 3$. We consider α, β from above condition (1) $\Leftrightarrow$ (3). Then $\alpha, \beta \in S(X, Y)$. Since $X\alpha = \{a, b\}$ and $\alpha|_{X\alpha}$ is not injective, we have α is not right regular. Similarly, $X\beta = \{a, c\}$ and $\beta|_{X\beta}$ is not injective. So β is not right regular, which is a contradiction. Hence, $|X| \leq 2$.

Conversely, suppose that $|X| \leq 2$. Then, it is clear that $S(X, Y)$ is a right regular semigroup. Hence, $S(X, Y) = RReg(S(X, Y))$. $\square$

Theorem 3.3.12. *The following statements are equivalent:*

- (1) $LReg(S(X, Y))$ is a semigroup.
- (2) $RReg(S(X, Y))$ is a semigroup.
- (3) $|X| \leq 2$.

Proof. (1) $\Leftrightarrow$ (3) Suppose that $|X| \geq 3$. Let $a, b, c \in X$ be distinct elements and $a \in Y$. It is enough to consider only two cases.

Case 1: $\{b, c\} \subseteq Y$. Recall α, β from Theorem 3.2.13, we have $\alpha, \beta \in S(X, Y)$. Note that $\pi(\alpha) = \pi_Y(\alpha) = \{\{c\}, X \setminus \{c\}\}$ and $X\alpha = \{b, c\}$. Clearly, $Y\alpha \cap \{c\} \neq \emptyset$ and $Y\alpha \cap X \setminus \{c\} \neq \emptyset$. Thus α is left regular, that is, $\alpha \in LReg(S(X, Y))$. Similarly, $\beta \in LReg(S(X, Y))$. Consider $\{c\} \in \pi(\alpha\beta)$ and $X\alpha\beta = \{a, b\}$. Therefore, $X\alpha\beta \cap \{c\} = \emptyset$ and so $\alpha\beta$ is not left regular, that is, $\alpha\beta \notin LReg(S(X, Y))$. Hence, $LReg(S(X, Y))$ is not a semigroup.

Case 2: $\{b, c\} \not\subseteq Y$. Assume that $c \notin Y$. Define $\alpha : X \rightarrow X$ by

$$x\alpha = \begin{cases} c & \text{if } x = c, \\ a & \text{otherwise.} \end{cases}$$

Then $Y\alpha = \{a\} \subseteq Y$. So $\alpha \in S(X, Y)$. Note that $\pi(\alpha) = \{\{c\}, X \setminus \{c\}\}$, $\pi_Y(\alpha) = \{X \setminus \{c\}\}$ and $X\alpha = \{a, c\}$. Clearly, $X\alpha \cap \{c\} \neq \emptyset$, $X\alpha \cap (X \setminus \{c\}) \neq \emptyset$ and $Y\alpha \cap (X \setminus \{c\}) \neq \emptyset$. Therefore, α is a left regular element of $S(X, Y)$. Define $\beta : X \rightarrow X$ by

$$x\beta = \begin{cases} b & \text{if } x \in \{b, c\}, \\ a & \text{otherwise.} \end{cases}$$

If $b \in Y$, then $Y\beta = \{a, b\} \subseteq Y$. Thus $\beta \in S(X, Y)$. If $b \notin Y$, then $Y\beta = \{a\} \subseteq Y$. So $\beta \in S(X, Y)$. We can show that $\beta \in LReg(S(X, Y))$. Then, we note that $\pi(\alpha\beta) = \{\{c\}, X \setminus \{c\}\}$ and $X\alpha\beta = \{a, b\}$. Clearly, $X\alpha\beta \cap \{c\} = \emptyset$. Hence, $\alpha\beta \notin LReg(S(X, Y))$ and so $LReg(S(X, Y))$ is not a semigroup.

Conversely, suppose $|X| \leq 2$. Then, we have $LReg(S(X, Y)) = S(X, Y)$ is a semigroup from Theorem 3.3.11(1).

(2) $\Leftrightarrow$ (3) Suppose that $|X| \geq 3$. Let $a, b, c \in X$ be distinct elements and $a \in Y$. Recall α and β from proof of (1) $\Leftrightarrow$ (3) Case 2. It is enough to show that α and β are right regular elements of $S(X, Y)$. Clearly, $\alpha|_{X\alpha}$ and $\beta|_{X\beta}$ are injective. Consider

$$(X\alpha \setminus Y)\alpha = \{c\} \subseteq X \setminus Y \text{ and } (X\beta \setminus Y)\beta = \begin{cases} \emptyset & \text{if } b \in Y, \\ \{b\} & \text{if } b \notin Y \end{cases} \subseteq X \setminus Y.$$

Then α and β are right regular elements of $S(X, Y)$, that is, $\alpha, \beta \in RReg(S(X, Y))$. Note that $\alpha\beta|_{X\alpha\beta}$ is not injective. We conclude that $\alpha\beta$ is not right regular. Thus, $\alpha\beta \notin RReg(S(X, Y))$ and so $RReg(S(X, Y))$ is not a semigroup.

Conversely, suppose $|X| \leq 2$. Then, we have $RReg(S(X, Y)) = S(X, Y)$ is a semigroup from Theorem 3.3.11(2). $\square$

CHAPTER IV

GREEN'S RELATIONS

In this chapter, we characterize Green's relations on $\overline{S}_E(X, Y)$, starting with the relation $\mathcal{L}$.

4.1 Green's relations on $\overline{S}_E(X, Y)$

To determine the relation $\mathcal{L}$ on $\overline{S}_E(X, Y)$, we establish that for any $\alpha, \beta \in \overline{S}_E(X, Y)$, there exists $\gamma \in \overline{S}_E(X, Y)$ such that $\alpha = \gamma\beta$.

Lemma 4.1.1. *Let $\alpha, \beta \in \overline{S}_E(X, Y)$. Then $\alpha = \gamma\beta$ for some $\gamma \in \overline{S}_E(X, Y)$ if and only if $X\alpha \subseteq X\beta$ and, for each $A \in X/E$, there exists $B \in X/E$ such that $(A \cap Y)\alpha \subseteq (B \cap Y)\beta$.*

Proof. Assume $\alpha = \gamma\beta$ for some $\gamma \in \overline{S}_E(X, Y)$. Clearly, $X\alpha = (X\gamma)\beta \subseteq X\beta$. Let $A \in X/E$. By Lemma 3.1.1, there exists $B \in X/E$ such that $(A \cap Y)\gamma \subseteq B \cap Y$. Hence, $(A \cap Y)\alpha = (A \cap Y)\gamma\beta \subseteq (B \cap Y)\beta$.

Conversely, assume the conditions hold. For each $A \in X/E$, we fix $A' \in X/E$ such that $(A \cap Y)\alpha \subseteq (A' \cap Y)\beta$. Let $x \in X$. Then there exists $A \in X/E$ such that $x \in A$. If $x \in Y$, we choose $x' \in A' \cap Y$ such that $x\alpha = x'\beta$. If $x \notin Y$, then there exists $x' \in X$ such that $x\alpha = x'\beta$. Define $\gamma : X \rightarrow X$ by $x\gamma = x'$ for all $x \in X$. To show $\gamma \in \overline{S}_E(X, Y)$, let $x, y \in Y$ in which $(x, y) \in E$. Then $x, y \in B$ for some $B \in X/E$. Thus $x', y' \in B'$ where $B' \in X/E$. Therefore, $(x\gamma, y\gamma) = (x', y') \in E$ and $x\gamma, y\gamma \in Y$. For each $x \in X$, we have $x\gamma\beta = x'\beta = x\alpha$. Hence, $\alpha = \gamma\beta$. This completes the proof. $\square$

Theorem 4.1.2. *Let $\alpha \in \overline{S}_E(X, Y)$. Then $\alpha \mathcal{L} \beta$ if and only if $X\alpha = X\beta$ and, for each $A \in X/E$, there exist $B, C \in X/E$ such that $(A \cap Y)\alpha \subseteq (B \cap Y)\beta$ and $(A \cap Y)\beta \subseteq (C \cap Y)\alpha$.*

Recall that for any two collections $\mathcal{A}, \mathcal{B}$ of subsets of a nonempty set X . If for each $B \in \mathcal{B}$, there exists some $A \in \mathcal{A}$ such that $B \subseteq A$, then $\mathcal{B}$ refines $\mathcal{A}$. Similar to the $\mathcal{L}$ -relation, we first need the following.

Lemma 4.1.3. *Let $\alpha, \beta \in \overline{S}_E(X, Y)$. Then $\alpha = \beta\gamma$ for some $\gamma \in \overline{S}_E(X, Y)$ if and only if $\pi(\beta)$ refines $\pi(\alpha)$ and $E_Y(\beta)$ refines $E_Y(\alpha)$.*

Proof. Assume that $\alpha = \beta\gamma$ for some $\gamma \in \overline{S}_E(X, Y)$. It is clear that $\pi(\beta)$ refines $\pi(\alpha)$. To show that $E_Y(\beta)$ refines $E_Y(\alpha)$, let $U \in E_Y(\beta)$. Then $U = A \cap Y\beta^{-1}$ and $(A \cap Y) \cap X\beta \neq \emptyset$ for some $A \in X/E$. Thus $U\beta \subseteq A \cap Y$. Since $\gamma \in \overline{S}_E(X, Y)$, we get $(A \cap Y)\gamma \subseteq B \cap Y \neq \emptyset$ for some $B \in X/E$. This implies $U\alpha = U\beta\gamma \subseteq (A \cap Y)\gamma \subseteq B \cap Y$. Hence, $U \subseteq (B \cap Y)\alpha^{-1} \in E_Y(\alpha)$, as required.

On the other hand, assume the conditions hold. For each $x \in X\beta$, choose $x' \in X$ such that $x = x'\beta$. For each $A \in X/E$ in which $(A \cap Y) \cap X\beta \neq \emptyset$, choose $x_A \in (A \cap Y) \cap X\beta$. Define $\gamma : X \rightarrow X$ by

$$x\gamma = \begin{cases} x'\alpha, & \text{if } x \in X\beta, \\ x'_A\alpha, & \text{if } x \in A \cap Y \setminus X\beta \text{ where } (A \cap Y) \cap X\beta \neq \emptyset, \\ x, & \text{otherwise.} \end{cases}$$

To show $\gamma \in \overline{S}_E(X, Y)$, let $x, y \in Y$ and $(x, y) \in E$. Then there exists $A \in X/E$ such that $x, y \in A$. If $(A \cap Y) \cap X\beta = \emptyset$, then $(x\gamma, y\gamma) = (x, y) \in E$. If $(A \cap Y) \cap X\beta \neq \emptyset$, then $(x, x_A), (y, x_A) \in E$. Assume $x \in X\beta$ and $y \notin X\beta$ (the other cases can be proved similar). Then $x, x_A \in (A \cap Y) \cap X\beta$, and so $x\beta^{-1}, x_A\beta^{-1} \subseteq (A \cap Y)\beta^{-1} \in E_Y(\beta)$. Since $E_Y(\beta)$ refines $E_Y(\alpha)$, $(A \cap Y)\beta^{-1} \subseteq (B \cap Y)\alpha^{-1}$ for some $B \in X/E$. Note from $x' \in x\beta^{-1}$ and $x'_A \in x_A\beta^{-1}$ that $x', x'_A \in (A \cap Y)\beta^{-1} \subseteq (B \cap Y)\alpha^{-1}$. Hence, $x'\alpha, x'_A\alpha \in B \cap Y$ which yields $(x\gamma, y\gamma) = (x'\alpha, x'_A\alpha) \in E$ and $x'\alpha, x'_A\alpha \in Y$. Let $x \in X$ be such that $x\beta = y$. Since $y'\beta = y$, $x, y' \in y\beta^{-1} \in \pi(\beta)$ and $\pi(\beta)$ refines $\pi(\alpha)$, we obtain $x\alpha = y'\alpha$. This implies that $x\beta\gamma = y\gamma = y'\alpha = x\alpha$ and so $\alpha = \beta\gamma$. $\square$

Theorem 4.1.4. *Let $\alpha \in \overline{S}_E(X, Y)$. Then $\alpha \mathcal{R} \beta$ if and only if $\pi(\beta) = \pi(\alpha)$ and $E_Y(\beta) = E_Y(\alpha)$.*

As an immediate consequence of the previous theorems, we obtain the following theorem.

Theorem 4.1.5. *Let $\alpha, \beta \in \overline{S}_E(X, Y)$. Then the following statements are equivalent:*

- (1) $(\alpha, \beta) \in \mathcal{H}$.

- (2) $\pi(\alpha) = \pi(\beta)$, $E_Y(\alpha) = E_Y(\beta)$, $X\alpha = X\beta$ and for each $A \in X/E$, there exist $B, C \in X/E$ such that $(A \cap Y)\alpha \subseteq (B \cap Y)\beta$ and $(A \cap Y)\beta \subseteq (C \cap Y)\alpha$.

Next, we consider Green's relation $\mathcal{D}$.

Theorem 4.1.6. *Let $\alpha, \beta \in \overline{S}_E(X, Y)$. Then $(\alpha, \beta) \in \mathcal{D}$ if and only if there exists a bijection $\varphi : \pi(\alpha) \rightarrow \pi(\beta)$ with satisfy the following properties:*

- (1) *for each $A \in X/E$, there exist $B, C \in X/E$ such that $(\pi_{A \cap Y}(\alpha))\varphi \subseteq \pi_{B \cap Y}(\beta)$ and $(\pi_{A \cap Y}(\beta)) \subseteq \pi_{C \cap Y}(\alpha)\varphi$,*
- (2) *for each $A \in X/E$, there exist $B, C \in X/E$ such that $((A \cap Y)\alpha^{-1})\varphi = (B \cap Y)\beta^{-1}$ and $(A \cap Y)\beta^{-1} = (C \cap Y)\alpha^{-1}\varphi$ where $W\alpha^{-1}\varphi = \bigcup_{x \in W} x\alpha^{-1}\varphi$.*

Proof. Assume that $(\alpha, \beta) \in \mathcal{D}$. Then $(\alpha, \gamma) \in \mathcal{L}$ and $(\gamma, \beta) \in \mathcal{R}$ for some $\gamma \in \overline{S}_E(X, Y)$. By Theorem 4.1.1, we have $X\alpha = X\gamma$ and for each $A \in X/E$, there exist $B, C \in X/E$ such that $(A \cap Y)\alpha \subseteq (B \cap Y)\gamma$ and $(A \cap Y)\gamma \subseteq (C \cap Y)\alpha$. By Theorem 4.1.3, we have $\pi(\gamma) = \pi(\beta)$ and $E_Y(\gamma) = E_Y(\beta)$. For each $P \in \pi(\alpha)$, there exists a unique $y \in X\alpha$ such that $P = y\alpha^{-1}$. Since $y \in X\alpha = X\gamma$, we have $y\gamma^{-1} \in \pi(\gamma) = \pi(\beta)$. Define $\varphi : \pi(\alpha) \rightarrow \pi(\beta)$ by $(y\alpha^{-1})\varphi = y\gamma^{-1}$ for all $y \in X\alpha$. Let $y_1, y_2 \in X\alpha$ be such that $(y_1\alpha^{-1})\varphi = (y_2\alpha^{-1})\varphi$. Then $y_1\gamma^{-1} = y_2\gamma^{-1}$. This implies that $y_1 = (y_1\gamma^{-1})\gamma = (y_2\gamma^{-1})\gamma = y_2$. Therefore, $y_1\alpha^{-1} = y_2\alpha^{-1}$. Hence, φ is injective. Let $P \in \pi(\beta) = \pi(\gamma)$. Then $P = y\gamma^{-1}$ for some $y \in X\gamma = X\alpha$. Thus $(y\alpha^{-1})\varphi = y\gamma^{-1} = P$ and so φ is surjective. Hence, φ is bijective. Let $A \in X/E$ be such that $A \cap Y \neq \emptyset$. Then there exists $B, C \in X/E$ such that $(A \cap Y)\alpha \subseteq (B \cap Y)\gamma$ and $(A \cap Y)\gamma \subseteq (C \cap Y)\alpha$. Let $P \in \pi_{A \cap Y}(\alpha)$. Then $P \in \pi(\alpha)$ such that $P \cap (A \cap Y) \neq \emptyset$. Choose $y \in P \cap (A \cap Y)$. Thus $y\alpha \in (A \cap Y)\alpha \subseteq (B \cap Y)\gamma$. Therefore, $y\alpha = b\gamma$ for some $b \in B \cap Y$. We obtain that $b \in y\alpha\gamma^{-1}$. Then $y\alpha\gamma^{-1} \cap B \cap Y \neq \emptyset$. Hence, $P\varphi = (y\alpha\gamma^{-1})\varphi = y\alpha\gamma^{-1} \in \pi_{B \cap Y}(\beta)$ and so $(\pi_{A \cap Y}(\alpha))\varphi \subseteq \pi_{B \cap Y}(\beta)$. Let $P \in \pi_{A \cap Y}(\beta) = \pi_{A \cap Y}(\gamma)$. Then $P \in \pi(\gamma)$ and $P \cap (A \cap Y) \neq \emptyset$. Let $y \in P \cap A \cap Y$. Then $y\gamma \in (A \cap Y)\gamma \subseteq (C \cap Y)\alpha$. Thus $y\gamma = c\alpha$ for some $c \in C \cap Y$. So $c \in y\gamma\alpha^{-1}$ and then $y\gamma\alpha^{-1} \cap C \cap Y \neq \emptyset$. This means that $P = y\gamma\gamma^{-1} = (y\gamma\alpha^{-1})\varphi \in (\pi_{C \cap Y}(\alpha))\varphi$. Hence, $\pi_{A \cap Y}(\beta) \subseteq \pi_{C \cap Y}(\alpha)\varphi$. Consider

$$((A \cap Y)\alpha^{-1})\varphi = \bigcup_{x \in A \cap Y} (x\alpha^{-1})\varphi = \bigcup_{x \in A \cap Y} x\gamma^{-1} = (A \cap Y)\gamma^{-1} = (B \cap Y)\beta^{-1},$$

where $B \in X/E$ since $E_Y(\gamma) = E_Y(\beta)$. Similarly, there exists $C \in X/E$ such that $(A \cap Y)\beta^{-1} = (C \cap Y)\gamma^{-1}$ since $E_Y(\beta) = E_Y(\gamma)$. Consider

$$(A \cap Y)\beta^{-1} = (C \cap Y)\gamma^{-1} = \bigcup_{x \in C \cap Y} x\gamma^{-1} = \bigcup_{x \in C \cap Y} x\alpha^{-1}\varphi = (C \cap Y)\alpha^{-1}\varphi.$$

Hence, $(A \cap Y)\beta^{-1}\varphi^{-1} = (C \cap Y)\alpha^{-1}$.

Conversely, suppose that there exists a bijection $\varphi : \pi(\alpha) \rightarrow \pi(\beta)$ which satisfying (1) and (2). For each $x \in X$, there exists a unique $P_x \in \pi(\beta)$ such that $x \in P_x$. Define $\gamma : X \rightarrow X$ by $x\gamma = P_x\varphi^{-1}\alpha$ for all $x \in X$. Since φ is a bijection, we have γ is well-defined. Note that $\pi(\beta) = \pi(\gamma)$. Let $x, y \in Y$ be such that $(x, y) \in E$. Then there exists $A \in X/E$ such that $x, y \in A \cap Y$. From (1), there exists $C \in X/E$ such that $\pi_{A \cap Y}(\beta) \subseteq \pi_{C \cap Y}(\alpha)\varphi$. Clearly, $P_x, P_y \in \pi_{A \cap Y}(\beta)$. So $P_x = P'\varphi$ for some $P' \in \pi_{C \cap Y}(\alpha)$. Since φ is a bijection, we have $P_x\varphi^{-1} \in \pi_{C \cap Y}(\alpha)$. Similarly, $P_y\varphi^{-1} \in \pi_{C \cap Y}(\alpha)$. We choose $a \in P_x\varphi^{-1} \cap (C \cap Y)$ and $b \in P_y\varphi^{-1} \cap (C \cap Y)$. Thus $(a\alpha, b\alpha) \in E$ and $a\alpha, b\alpha \in Y$. Therefore, $(x\gamma, y\gamma) = (P_x\varphi^{-1}\alpha, P_y\varphi^{-1}\alpha) = (a\alpha, b\alpha) \in E$ and $x\gamma, y\gamma \in Y$. Hence, $\gamma \in \overline{S}_E(X, Y)$. Next, we will show that $(\alpha, \gamma) \in \mathcal{L}$. It is obvious that $X\gamma \subseteq X\alpha$. Let $x \in X\alpha$. Note that $x\alpha^{-1} \in \pi(\alpha)$ and so $x\alpha^{-1}\varphi \in \pi(\beta)$. For each $a \in x\alpha^{-1}\varphi$, we get $P_a = x\alpha^{-1}\varphi$. Thus $x = x\alpha^{-1}\alpha = P_a\varphi^{-1}\alpha = a\gamma \in X\gamma$. Then $X\gamma = X\alpha$. Let $A \in X/E$. By (1), there exists $B \in X/E$ such that $(\pi_{A \cap Y}(\alpha))\varphi \subseteq \pi_{B \cap Y}(\beta)$. Then $\pi_{A \cap Y}(\alpha) \subseteq (\pi_{B \cap Y}(\beta))\varphi^{-1}$. This implies that $(A \cap Y)\alpha = (\bigcup \pi_{A \cap Y}(\alpha))\alpha \subseteq (\bigcup \pi_{B \cap Y}(\beta))\varphi^{-1}\alpha = (B \cap Y)\gamma$. Therefore, $(\alpha, \gamma) \in \mathcal{L}$. Let $A \in X/E$ be such that $(A \cap Y) \cap X\alpha \neq \emptyset$. Consider

$$\begin{aligned} (A \cap Y)\gamma^{-1} &= \bigcup \{P \in \pi(\gamma) : P\gamma \in A \cap Y\} \\ &= \bigcup \{P \in \pi(\beta) : P\varphi^{-1}\alpha \in A \cap Y\} \\ &= \bigcup \{P \in \pi(\beta) : P \in (A \cap Y)\alpha^{-1}\varphi\} \\ &= \bigcup \{P \in \pi(\beta) : P \in (B \cap Y)\beta^{-1}\} \end{aligned}$$

for some $B \in X/E$ since (2) holds. Therefore $(A \cap Y)\gamma^{-1} = (B \cap Y)\beta^{-1}$ and hence $E_Y(\gamma) \subseteq E_Y(\beta)$. Now, we consider $(A \cap Y)\beta^{-1} = (C \cap Y)\alpha^{-1}\varphi$ for some $C \in X/E$, by (2). Then

$$(A \cap Y)\beta^{-1} = \bigcup \{x\alpha^{-1}\varphi : x \in C \cap Y\} = \bigcup \{x\gamma^{-1} : x \in C \cap Y\}.$$

Therefore, $(A \cap Y)\beta^{-1} = (C \cap Y)\gamma^{-1}$ and hence $E_Y(\gamma) = E_Y(\beta)$. It follows that $(\gamma, \beta) \in \mathcal{R}$ and so $(\alpha, \beta) \in \mathcal{D}$. $\square$

The disscussion of the last relation $\mathcal{J}$ is more complicated than the others. We first observe a sufficient condition for two elements of $\overline{S}_E(X, Y)$ to be $\mathcal{J}$ -related.

Theorem 4.1.7. Let $\alpha, \beta \in \overline{S}_E(X, Y)$ and suppose that there exists a mapping $\varphi : X\alpha \rightarrow X\beta$ with satisfy the following properties:

- (1) for each $A \in X/E$, there exists $B \in X/E$ such that $((A \cap Y) \cap X\alpha)\varphi \subseteq B \cap Y$,
- (2) for each $A \in X/E$, there exists $B \in X/E$ such that $(A \cap Y)\beta \subseteq (B \cap Y)\alpha\varphi$ and $(X \setminus Y)\beta \subseteq (X \setminus Y)\alpha\varphi$.

Then $\beta = \gamma\alpha\delta$ for some $\gamma, \delta \in \overline{S}_E(X, Y)$.

Proof. Let $A \in X/E$ be such that $(A \cap Y) \cap X\alpha \neq \emptyset$. By (1), we let $A' \in X/E$ such that $((A \cap Y) \cap X\alpha)\varphi \subseteq A' \cap Y$ and fix $x_A \in A' \cap Y$. Define $\delta : X \rightarrow X$ by

$$x\delta = \begin{cases} x\varphi & \text{if } x \in X\alpha, \\ x_A & \text{if } x \in A \cap Y \setminus X\alpha \text{ and } (A \cap Y) \cap X\alpha \neq \emptyset \text{ for some } A \in X/E, \\ x & \text{otherwise.} \end{cases}$$

Let $x, y \in Y$ be such that $(x, y) \in E$. Then there exists $A \in X/E$ such that $x, y \in A$. If $(A \cap Y) \cap X\alpha = \emptyset$, then $(x\delta, y\delta) = (x, y) \in E$ and $x\delta, y\delta \in Y$. Suppose that $(A \cap Y) \cap X\alpha \neq \emptyset$. If $x \in X\alpha$, then $x\delta = x\varphi \in ((A \cap Y) \cap X\alpha)\varphi \subseteq A' \cap Y$. Similarly, $y\delta \in A' \cap Y$. If $x \notin X\alpha$, then $x\delta, y\delta = x_A \in A' \cap Y$. Hence, $\delta \in \overline{S}_E(X, Y)$. For each $A \in X/E$, we fix $A' \in X/E$ such that $(A \cap Y)\beta \subseteq (A' \cap Y)\alpha\varphi$. Let $A \in X/E$ and $x \in A \cap Y$. Choose $x' \in A' \cap Y$ such that $x\beta = x'\alpha\varphi$. Let $x \in X \setminus Y$. By (2), $(X \setminus Y)\beta \subseteq (X \setminus Y)\alpha\varphi$, so there exists $x' \in X \setminus Y$ such that $x\beta = x'\alpha\varphi$. Define $\gamma : X \rightarrow X$ by $x\gamma = x'$ for all $x \in X$. Let $x, y \in Y$ and $(x, y) \in E$. Then $x, y \in A$ for some $A \in X/E$. Thus $x', y' \in A' \cap Y$. This implies that $(x\gamma, y\gamma) = (x', y') \in E$ and $x\gamma, y\gamma \in Y$. Finally, let $x \in X$. Then $x\gamma\alpha\delta = x'\alpha\delta = x'\alpha\varphi = x\beta$. Hence, $\beta = \gamma\alpha\delta$. $\square$

Corollary 4.1.8. If $\alpha, \beta \in \overline{S}_E(X, Y)$ and there exist mappings $\varphi : X\alpha \rightarrow X\beta$ and $\psi : X\beta \rightarrow X\alpha$ such that

- (1) for each $A \in X/E$, there exist $B, C \in X/E$ such that $((A \cap Y) \cap X\alpha)\varphi \subseteq B \cap Y$ and $((A \cap Y) \cap X\beta)\psi \subseteq C \cap Y$ and
- (2) for each $A \in X/E$, there exist $B, C \in X/E$ such that $(A \cap Y)\beta \subseteq (B \cap Y)\alpha\varphi$, $(A \cap Y)\alpha \subseteq (C \cap Y)\beta\psi$, $(X \setminus Y)\beta \subseteq (X \setminus Y)\alpha\varphi$ and $(X \setminus Y)\alpha \subseteq (X \setminus Y)\beta\psi$,

then $(\alpha, \beta) \in \mathcal{J}$.

CHAPTER V

THE NATURAL PARTIAL ORDER

In this chapter, we investigate the condition under which $\alpha \leq \beta$ for two elements $\alpha, \beta \in \overline{S}_E(X, Y)$ and give a necessary and sufficient condition for elements in $\overline{S}_E(X, Y)$ to be minimal and maximal elements with respect to order.

5.1 Natural partial order on $\overline{S}_E(X, Y)$

In this section, let's recall the natural partial order on the semigroup $\overline{S}_E(X, Y)$ as follows: for $\alpha, \beta \in \overline{S}_E(X, Y)$,

$$\alpha \leq \beta \text{ if and only if } \alpha = \delta\beta = \beta\gamma \text{ and } \alpha = \alpha\gamma \text{ for some } \delta, \gamma \in \overline{S}_E(X, Y).$$

The following theorem gives a characterization of this partial order.

Theorem 5.1.1. *Let $\alpha, \beta \in \overline{S}_E(X, Y)$. Then $\alpha \leq \beta$ if and only if the following conditions are satisfied:*

- (1) $\pi(\beta)$ refines $\pi(\alpha)$ and $E_Y(\beta)$ refines $E_Y(\alpha)$.
- (2) $X\alpha \subseteq X\beta$.
- (3) for any $x \in X$, if $x\beta \in X\alpha$, then $x\alpha = x\beta$.
- (4) for every $A \in X/E$, there exists $B \in X/E$ such that $(A \cap Y)\alpha \subseteq (B \cap Y)\beta$.

Proof. Assume that $\alpha \leq \beta$. Then there exist $\gamma, \delta \in \overline{S}_E(X, Y)$ such that $\alpha = \beta\gamma = \delta\beta$ and $\alpha = \alpha\gamma$. The relationship $\alpha = \beta\gamma$ implies that $\pi(\beta)$ refines $\pi(\alpha)$. Let $U \in E_Y(\beta)$ be expressed as $U = (A \cap Y)\beta^{-1}$, where $A \in X/E$ and $A \cap Y \cap X\beta \neq \emptyset$. This implies $U\alpha = U\beta\gamma \subseteq (A \cap Y)\gamma$. Since $\gamma \in \overline{S}_E(X, Y)$, we have $(A \cap Y)\gamma \subseteq B \cap Y$ for some $B \in X/E$. Consequently, $U \subseteq (B \cap Y)\alpha^{-1}$, indicating that $E_Y(\beta)$ refines $E_Y(\alpha)$, and condition (1) holds. To demonstrate the satisfaction of condition (2), let $x \in X\alpha$. Then $x = x'\alpha$ for some $x' \in X$. Thus $x = x'\alpha = x'\delta\beta = (x'\delta)\beta \in X\beta$. Therefore, $X\alpha \subseteq X\beta$. Let $z \in X$ be such that $z\beta \in X\alpha$. Then $z\beta = z'\alpha$ for some $z' \in X$. Hence, $z\alpha = z\beta\gamma = z'\alpha\gamma = z'\alpha = z\beta$, confirming the validity of condition (3). To establish

condition (4), let $A \in X/E$. Since $\delta \in \overline{S}_E(X, Y)$, we have $(A \cap Y)\delta \subseteq B \cap Y$ for some $B \in X/E$. Consequently, $(A \cap Y)\alpha = (A \cap Y)\delta\beta \subseteq (B \cap Y)\beta$.

Conversely, suppose conditions (1)–(4) are satisfied. For each $x \in X\beta$, there exists $x' \in X$ such that $x = x'\beta$. Additionally, for each $A \in X/E$ with $A \cap Y \cap X\beta \neq \emptyset$, we choose $x_A \in A \cap Y \cap X\beta$. Now, define γ on X by

$$x\gamma = \begin{cases} x'\alpha & \text{if } x \in X\beta, \\ x'_A\alpha & \text{if } x \in (A \cap Y) \setminus X\beta, \text{ where } A \cap Y \cap X\beta \neq \emptyset, \\ x & \text{otherwise.} \end{cases}$$

By condition (1), we have $\pi(\beta)$ refines $\pi(\alpha)$ and this implies $\gamma \in T(X)$. To demonstrate that $\gamma \in \overline{S}_E(X, Y)$, consider $x, y \in Y$ with $(x, y) \in E$. Then $x, y \in A$ for some $A \in X/E$. If $A \cap Y \cap X\beta = \emptyset$, then $(x\gamma, y\gamma) = (x, y) \in E$. Assume that $A \cap Y \cap X\beta \neq \emptyset$. Thus $(x, x_A), (y, x_A) \in E$. Two cases arise.

Case 1: Both x and y are in $X\beta$. Then $x\beta^{-1}, y\beta^{-1} \subseteq (A \cap Y)\beta^{-1} \in E_Y(\beta)$. Since $x' \in x\beta^{-1}$ and $y' \in y\beta^{-1}$, it follows that $x', y' \in (A \cap Y)\beta^{-1} \subseteq (B \cap Y)\alpha^{-1}$ for some $B \in X/E$ due to the refinement property of $E_Y(\beta)$ over $E_Y(\alpha)$. Consequently, $x'\alpha, y'\alpha \in B \cap Y$. This implies that $(x\gamma, y\gamma) = (x'\alpha, y'\alpha) \in E$, and $x\gamma, y\gamma \in Y$.

Case 2: Either x or y is not in $X\beta$. Without loss of generality, assume that $x \in X\beta$ and $y \notin X\beta$. Then $y\gamma = x'_A\alpha = x_A\gamma$. Since $x, x_A \in X\beta$, by Case 1, we get $(x\gamma, y\gamma) = (x\gamma, x_A\gamma) \in E$ and $x\gamma, y\gamma \in Y$.

From both cases, we conclude that $\gamma \in \overline{S}_E(X, Y)$. Let $x \in X$ and setting $y = x\beta$. Note that $y = y'\beta$. Thus $x, y' \in y\beta^{-1}$. Since $y\beta^{-1} \in \pi(\beta)$ and $\pi(\beta)$ refines $\pi(\alpha)$, we have $x\alpha = y'\alpha$. This implies that $x\beta\gamma = y\gamma = y'\alpha = x\alpha$, resulting in $\alpha = \beta\gamma$. Furthermore, by condition (2), we have $x\alpha \in X\alpha \subseteq X\beta$ and then, by our setting, $(x\alpha)'\beta = x\alpha \in X\alpha$. By condition (3), we obtain $x\alpha\gamma = (x\alpha)'\alpha = (x\alpha)'\beta = x\alpha$. This implies $\alpha = \alpha\gamma$.

To define δ , for each $A \in X/E$, by condition (4), we choose $A' \in X/E$ such that $(A \cap Y)\alpha \subseteq (A' \cap Y)\beta$. For any $x \in X$, if $x \in Y$, then there exists $A \in X/E$ such that $x \in A \cap Y$. In this case, we choose $x' \in A' \cap Y$ such that $x\alpha = x'\beta$. If $x \in X \setminus Y$, given that $X\alpha \subseteq X\beta$, we choose $x' \in X$ such that $x\alpha = x'\beta$. Define $\delta \in T(X)$ by $x\delta = x'$. It is clear that $x\delta \in Y$ for all $x \in Y$. To establish that $\delta \in \overline{S}_E(X, Y)$, consider $x, y \in Y$ with $(x, y) \in E$. Consequently, $x, y \in A \cap Y$ for some $A \in X/E$. This implies

$x', y' \in A' \cap Y$, indicating that $(x\delta, y\delta) = (x', y') \in E$. Finally, consider any $x \in X$. Then $x\delta\beta = x'\beta = x\alpha$, and thus $\alpha = \delta\beta$. Therefore, $\alpha \leq \beta$, as required. $\square$

Remark 5.1.2. If $X = Y$, then $\overline{S}_E(X, Y) = T_E(X)$, and we obtain the characterization of $\leq$ on $\overline{S}_E(X, Y)$, as initially presented in [33, Theorem 2.1].

Directly following from Theorem 5.1.1, we derive the following corollary.

Corollary 5.1.3. Let $\alpha, \beta \in \overline{S}_E(X, Y)$ with $\alpha \leq \beta$. If $X\alpha = X\beta$, then $\alpha = \beta$.

5.2 Minimal and maximal elements of $\overline{S}_E(X, Y)$

In this section, we discuss the minimal and maximal elements in $\overline{S}_E(X, Y)$. The following theorem describes the minimal elements with respect to the natural partial order.

Theorem 5.2.1. Let $\alpha \in \overline{S}_E(X, Y)$. Then α is minimal if and only if α is a constant map.

Proof. Assume that α is minimal and suppose $|X\alpha| \geq 2$. Let $a \in Y\alpha$. Define $\beta : X \rightarrow X$ by $x\beta = y$ for all $x \in X$. Then $\beta \in \overline{S}_E(X, Y)$. Note that $\pi(\beta) = \{X\} = E_Y(\beta)$. Hence $E_Y(\alpha)$ refines $E_Y(\beta)$ and $\pi(\alpha)$ refines $\pi(\beta)$. Clearly, $X\beta \subseteq X\alpha$. Let $x \in X$ be such that $x\alpha \in X\beta$. Then $x\alpha = y = x\beta$. Let $A \in X/E$. Note that there exists $a \in Y$ such that $a\alpha = y$. There exists $A' \in X/E$ such that $a \in A'$. Therefore, $(A \cap Y)\beta = \{y\} \subseteq (A' \cap Y)\alpha$. This means that $\beta \leq \alpha$ which is a contradiction.

Conversely, assume $\alpha = \overline{Y}_a$, for some $a \in Y$. Let $\beta \in \overline{S}_E(X, Y)$ be such that $\beta \leq \alpha$. Consequently, $X\beta \subseteq X\alpha = \{a\}$ and this implies that $X\beta = X\alpha$. Hence, by Corollary 5.1.3, $\beta = \alpha$, and thus, α is minimal. $\square$

Definition 5.2.2. Let $\alpha \in \overline{S}_E(X, Y)$ and $U \in E_Y(\alpha)$. Then U is termed *divisible* if there exist subsets C and K of X such that $C \in X/E$ with $C \cap Y \cap X\alpha = \emptyset$, and $\emptyset \neq K \subseteq U$ with $|K\alpha| \leq |C \cap Y|$, satisfying one of the following conditions:

- (1) $K \subseteq U \setminus Y$ and $(U - K)\alpha = U\alpha$.
- (2) $K = A \cap Y$ for some $A \in X/E$ and there exists $B \in X/E$ such that $B \neq A$ with $(A \cap Y)\alpha \subseteq (B \cap Y)\alpha$.

To begin the examination, we identify the necessary conditions for elements in $\overline{S}_E(X, Y)$ to be maximal with respect to the natural partial order.

Lemma 5.2.3. *Let $\alpha \in \overline{S}_E(X, Y)$ be maximal. Then U is not divisible for all $U \in E_Y(\alpha)$.*

Proof. Assume, to the contrary, that there exists $U \in E_Y(\alpha)$ such that U is divisible. Then there exist subsets C and K of X satisfying the properties in Definition 5.2.2. Let $\varphi : K\alpha \rightarrow C \cap Y$ be injective. Define $\beta \in \overline{S}_E(X, Y)$ as follows:

$$x\beta = \begin{cases} x\alpha\varphi & \text{if } x \in K, \\ x\alpha & \text{otherwise.} \end{cases}$$

Then $\pi(\beta) = \{P \setminus K : P \in \pi(\alpha) \text{ and } P \not\subseteq K\} \cup \{P \cap K : P \in \pi(\alpha) \text{ and } P \cap K \neq \emptyset\}$ refines $\pi(\alpha)$, and $E_Y(\beta) = (E_Y(\alpha) \setminus \{U\}) \cup \{U \setminus K, K\}$ refines $E_Y(\alpha)$. So $\beta \in \overline{S}_E(X, Y)$. Furthermore, by properties of K , we have $X\beta = X\alpha \cup K\alpha\varphi$, resulting in $X\alpha \subsetneq X\beta$ and $x\alpha = x\beta$ for all $x\beta \in X\alpha$. To verify that α and β satisfy condition (4) of Theorem 5.1.1, we consider two cases.

Case 1: $K \subseteq U \setminus Y$ and $(U - K)\alpha = U\alpha$. Then $(D \cap Y)\alpha = (D \cap Y)\beta$ for all $D \in X/E$.

Case 2: $K = A \cap Y$ for some $A \in X/E$ and there exists $B \in X/E$ such that $B \neq A$ and $(A \cap Y)\alpha \subseteq (B \cap Y)\alpha$. In this case, we have $(A \cap Y)\alpha \subseteq (B \cap Y)\alpha = (B \cap Y)\beta$ and $(D \cap Y)\alpha = (D \cap Y)\beta$ for all $D \in X/E$ with $D \neq A$.

Hence, α and β satisfy all conditions in Theorem 5.1.1 and $\alpha \neq \beta$. Therefore, $\alpha < \beta$ which contradicts to the maximality of α . $\square$

Lemma 5.2.4. *Let $\alpha \in \overline{S}_E(X, Y)$ be maximal. If $X \setminus Y \not\subseteq X\alpha$, then $\{z\} \in \pi(\alpha)$ for all $z \in X \setminus Y$.*

Proof. Suppose that $X \setminus Y \not\subseteq X\alpha$. Then there exists $c \in (X \setminus Y) \setminus X\alpha$. Assume, to the contrary, that there exists $z \in X \setminus Y$ such that $\{z\} \notin \pi(\alpha)$. Then $z \in P$ for some $P \in \pi(\alpha)$ and $|P| > 1$. Define $\beta : X \rightarrow X$ as follows:

$$x\beta = \begin{cases} c & \text{if } x = z, \\ x\alpha & \text{otherwise.} \end{cases}$$

Since $z, c \notin Y$ and $|P| > 1$, we get $\beta \in \overline{S}_E(X, Y)$ and $X\beta = X\alpha \cup \{c\}$, implying that $\alpha \neq \beta$. Note that $\pi(\beta) = (\pi(\alpha) \setminus \{P\}) \cup \{P \setminus \{z\}, \{z\}\}$ refines $\pi(\alpha)$ and $E_Y(\alpha) = E_Y(\beta)$. Clearly, $x\beta \in X\alpha$ implying $x\alpha = x\beta$, and $(A \cap Y)\alpha = (A \cap Y)\beta$ for all $A \in X/E$. Thus $\alpha < \beta$, contradicting the maximality of α . $\square$

Lemma 5.2.5. *Let $\alpha \in \overline{S}_E(X, Y)$ be maximal and let $U \in E_Y(\alpha)$ be such that $U\alpha \notin X/E$.*

- (1) *If $A, B \in X/E$ with $A \cap Y \neq \emptyset$ and $(A \cap Y)\alpha \subseteq (B \cap Y)\alpha \subseteq U\alpha$, then $A = B$.*
- (2) *If $P \in \pi(\alpha)$ with $|P| > 1$, then $P \cap (U \setminus Y) = \emptyset$ and $|P \cap U \cap (A \cap Y)| \leq 1$ for all $A \in X/E$.*

Proof. Since $U \in E_Y(\alpha)$, we can express $U = (C \cap Y)\alpha^{-1}$ for some $C \in X/E$. Due to $U\alpha \notin X/E$, we have $U\alpha \subsetneq C \cap Y$. Choose $c \in (C \cap Y) \setminus U\alpha$. Note that $c \notin X\alpha$.

- (1) Let $A, B \in X/E$ be such that $A \cap Y \neq \emptyset$ and $(A \cap Y)\alpha \subseteq (B \cap Y)\alpha \subseteq U\alpha$. Choose $a \in A \cap Y$ and define $\beta \in \overline{S}_E(X, Y)$ as follows:

$$x\beta = \begin{cases} c & \text{if } x = a, \\ x\alpha & \text{otherwise.} \end{cases}$$

If $A \neq B$, then $X\beta = X\alpha \cup \{c\}$, implying $X\alpha \subsetneq X\beta$. As before, we can see that α and β satisfy all conditions in Theorem 5.1.1. Hence, $\alpha < \beta$, contradicting the maximality of α . Therefore, $A = B$, as required.

- (2) Let $P \in \pi(\alpha)$ be such that $|P| > 1$. Assume, to the contrary, that $P \cap (U \setminus Y) \neq \emptyset$ or $|P \cap U \cap (A \cap Y)| > 1$ for some $A \in X/E$.

Case 1: $P \cap (U \setminus Y) \neq \emptyset$. In this situation, there exists $z \in P \cap (U \setminus Y)$. Define $\beta \in \overline{S}_E(X, Y)$ as follows:

$$x\beta = \begin{cases} c & \text{if } x = z, \\ x\alpha & \text{otherwise.} \end{cases}$$

Since $|P| > 1$, we have $X\beta = X\alpha \cup \{c\}$; furthermore, α and β satisfy all conditions in Theorem 5.1.1. This implies that $\alpha < \beta$, contradicting the maximality of α .

Case 2: $|P \cap U \cap (A \cap Y)| > 1$ for some $A \in X/E$. Let $a \in P \cap U \cap (A \cap Y)$. Define $\beta : X \rightarrow X$ as follows:

$$x\beta = \begin{cases} c & \text{if } x = a, \\ x\alpha & \text{otherwise.} \end{cases}$$

Similar to Case 1, $\beta \in \overline{S}_E(X, Y)$ and $\alpha < \beta$, a contradiction. $\square$

Theorem 5.2.6. *Let $\alpha \in \overline{S}_E(X, Y)$. Then α is maximal if and only if α satisfies all of the following conditions:*

- (1) U is not divisible for all $U \in E_Y(\alpha)$.
- (2) if $X \setminus Y \not\subseteq X\alpha$, then $\{z\} \in \pi(\alpha)$ for all $z \in X \setminus Y$.
- (3) if $U \in E_Y(\alpha)$ with $U\alpha \not\subseteq X/E$, then $(A \cap Y)\alpha \subseteq (B \cap Y)\alpha \subseteq U\alpha$ implies $A = B$ for all $A, B \in X/E$ with $A \cap Y \neq \emptyset$.
- (4) if $U \in E_Y(\alpha)$ with $U\alpha \not\subseteq X/E$, then $P \cap (U \setminus Y) = \emptyset$ and $|P \cap U \cap (A \cap Y)| < 1$ for all $P \in \pi(\alpha)$ with $|P| > 1$, and for all $A \in X/E$.

Proof. By Lemmas 5.2.3 - 5.2.5 collectively imply the truth of conditions (1) through (4).

Conversely, assume that conditions (1) through (4) hold. Let $\beta \in \overline{S}_E(X, Y)$ be such that $\alpha \leq \beta$. To demonstrate that $\alpha = \beta$, consider an arbitrary element $x \in X$. Then $x\alpha \in X\alpha \subseteq X\beta$. Consequently, $x\alpha = z\beta$ for some $z \in X$. Note that $x \in P$ for some $P \in \pi(\alpha)$. Since $z\beta \in X\alpha$, we deduce $x\alpha = z\beta = z\alpha$, and thus, $z \in P$. If $|P| = 1$, then $x = z$, leading to $x\alpha = x\beta$. For the case where $|P| > 1$, we consider two cases.

Case 1: $x\alpha \notin Y$. Then $x \in X \setminus Y$. According to condition (2) and the fact that $|P| > 1$, it follows that $X \setminus Y \subseteq X\alpha$. If $x\beta \in Y$, then $x \in U$ for some $U \in E_Y(\beta)$. Since $E_Y(\beta)$ refines $E_Y(\alpha)$, we have $x \in U \subseteq V$ for some $V \in E_Y(\alpha)$. This implies that $x\alpha \in V\alpha \subseteq Y$, which is impossible. Thus, $x\beta \in X \setminus Y \subseteq X\alpha$, and so $x\beta = x\alpha$.

Case 2: $x\alpha \in Y$. Then $x \in U = (A \cap Y)\alpha^{-1}$ for some $A \in X/E$. Let $V = (A \cap Y)\beta^{-1} \in E_Y(\beta)$. If $x\beta \notin Y$, then $x \notin Y$. Since $|P| > 1$, by (2), we obtain that $X \setminus Y \subseteq X\alpha$. This implies that $x\beta \in X\alpha$, and so $x\beta = x\alpha \in Y$, which is a contradiction. Hence, $x\beta \in Y$.

Subcase 2.1: $U\alpha \in X/E$. Then $U\alpha = A \cap Y$. From $x\alpha = z\beta = z\alpha$, we have $z \in U$ and $z \in V$. As $E_Y(\beta)$ refines $E_Y(\alpha)$, we conclude that $V \subseteq U$. Since $A \cap Y = U\alpha \subseteq X\alpha \subseteq X\beta$, we deduce that $V\beta = A \cap Y \subseteq X\alpha$, implying $V\beta = V\alpha$. To show $x \in V$, we assume, to the contrary, that $x \notin V$. Since $x\beta \in Y$, it follows that $x \in W = (C \cap Y)\beta^{-1}$ for some $C \in X/E$, where $C \neq A$. Consequently, $\emptyset \neq W \subseteq U$. We claim that U is divisible, and so we will examine properties of C while choosing a suitable K . If $(C \cap Y) \cap X\alpha \neq \emptyset$, then $(C \cap Y)\alpha^{-1} \in E_Y(\alpha)$. Let $c \in (C \cap Y)\alpha^{-1}$. Thus, $c\alpha \in (C \cap Y) \cap X\beta$, so $c\alpha = c'\beta = c'\alpha$ for some $c' \in X$. Hence, $c, c' \in (C \cap Y)\alpha^{-1}$ and $c' \in (C \cap Y)\beta^{-1} = W$. Thus, $W \subseteq (C \cap Y)\alpha^{-1}$. Since $W \subseteq U = (A \cap Y)\alpha^{-1}$, it follows that $A = C$, which is a contradiction. Therefore, $C \cap Y \cap X\alpha = \emptyset$.

If $W \cap Y = \emptyset$, then we choose $K = W$. Since $\pi(\beta)$ refines $\pi(\alpha)$, we obtain $|K\alpha| = |W\alpha| \leq |W\beta| \leq |C \cap Y|$. Furthermore, $A \cap Y = V\alpha \subseteq (U - K)\alpha \subseteq A \cap Y$ since $V, K \subseteq U$ and $V \cap K = \emptyset$. Thus $K \subseteq U \setminus Y$ such that $(U - K)\alpha = U\alpha$. Hence, C and K satisfy the properties in Definition 5.2.2.

If $W \cap Y \neq \emptyset$, then there exists $D \in X/E$ such that $D \cap Y \subseteq W$. Choose $K = D \cap Y$. Since $\alpha \leq \beta$, $(D \cap Y)\alpha \subseteq (B \cap Y)\beta$ for some $B \in X/E$. If $D = B$, then let $d \in D \cap Y$. Hence, $d\alpha = b\beta$ for some $b \in B \cap Y = D \cap Y \subseteq W$. Thus, $d\alpha \in U\alpha = A \cap Y$ and $b\beta \in W\beta \subseteq C \cap Y$. Therefore, $A = C$, which is a contradiction. It follows that $D \neq B$ and $(D \cap Y)\alpha \subseteq (B \cap Y)\alpha$ since $a\beta = a\alpha$ for all $a\beta \in (D \cap Y)\alpha$. Since $\pi(\beta)$ refines $\pi(\alpha)$, we get $|(D \cap Y)\alpha| \leq |(D \cap Y)\beta| \leq |W\beta| \leq |C \cap Y|$. Hence, C and K satisfy the properties in Definition 5.2.2.

Thus, U being divisible leads to a contradiction. This implies that $x \in V$, and subsequently, $x\beta \in V\beta = A \cap Y \subseteq X\alpha$. This results in $x\beta = x\alpha$.

Subcase 2.2: $U\alpha \notin X/E$. Since $|P| > 1$, according to (4), we can conclude that $x \in Y$. Let $B \in X/E$ be such that $x \in B \cap Y$. Then $(B \cap Y)\alpha \subseteq (D \cap Y)\beta$ for some $D \in X/E$. Furthermore, since $a\beta = a\alpha$ for all $a\beta \in (B \cap Y)\alpha$, we obtain $(B \cap Y)\alpha \subseteq (D \cap Y)\alpha \subseteq U\alpha$. It follows from (3) that $B = D$, and thus $x\alpha = d\beta = d\alpha$ for some $d \in D \cap Y = B \cap Y$. Hence, $x, d \in P \cap U \cap B \cap Y$. It follows from (4) that $|P \cap U \cap B \cap Y| < 1$, that is, $x = d$, and so $x\beta = x\alpha$. $\square$

Corollary 5.2.7. *Let $\alpha \in \overline{S}_E(X, Y)$. Then the following statements hold:*

- (1) *if α is surjective, then α is a maximal element.*
- (2) *if α is injective, then α is a maximal element.*

Proof. (1) Assume that α is surjective and $\alpha \leq \beta$. Then $X = X\alpha \subseteq X\beta$. Thus $X\alpha = X\beta$. Let $x \in X$. It follows that $x\beta = x\alpha$. Hence, $\alpha = \beta$.

(2) Assume that α is injective and $\alpha \leq \beta$. There exist $\delta, \gamma \in \overline{S}_E(X, Y)$ such that $\alpha = \beta\gamma = \delta\beta$ and $\alpha = \alpha\gamma$. From α is injective, we have γ is the identity mapping. Therefore, $\alpha = \beta id_X = \beta$. $\square$

5.3 Compatibility of $\overline{S}_E(X, Y)$

In this section, for any a within the set Y , it is evident that $\overline{X}_a$ possesses the property of right-compatibility. It is clear that $\pi(id_X) = \{\{x\} : x \in X\}$ and $E_Y(id_X) = \{A \cap Y : A \in X/E\}$, consequently, $\pi(id_X)$ refines $\pi(\alpha)$ while $E_Y(id_X)$ refines $E_Y(\alpha)$ for all $\alpha \in \overline{S}_E(X, Y)$. Subsequently, we proceed to establish a necessary condition for elements in $\overline{S}_E(X, Y)$ that are not constant maps, ensuring their right-compatibility.

Lemma 5.3.1. *Let $\gamma \in \overline{S}_E(X, Y)$ be right-compatible such that $|X\gamma| > 1$. Then, for all $z \in X \setminus Y$, $\{z\} \in \pi(\gamma)$.*

Proof. Assume, to the contrary, that there exist $z \in X \setminus Y$ and $z \neq w \in X$ such that $z\gamma = w\gamma$. Since $|X\gamma| > 1$, there exists $k \in X$ such that $k\gamma \neq z\gamma$. Define $\alpha : X \rightarrow X$ by

$$x\alpha = \begin{cases} k & \text{if } x = z, \\ x & \text{otherwise.} \end{cases}$$

It is evident that $\alpha \in \overline{S}_E(X, Y)$, and $\alpha \leq id_X$. Since γ is right-compatible, we have $\alpha\gamma \leq id_X\gamma = \gamma$. However, $w\alpha\gamma = w\gamma = z\gamma \neq k\gamma = z\alpha\gamma$ while $z\gamma = w\gamma$, leading to the conclusion that $\pi(\gamma)$ does not refine $\pi(\alpha\gamma)$, resulting in a contradiction. $\square$

Lemma 5.3.2. *Let $\gamma \in \overline{S}_E(X, Y)$ be right-compatible. Then $(X \setminus Y)\gamma \subseteq Y$ or $(X \setminus Y)\gamma \subseteq X \setminus Y$.*

Proof. Assume, to the contrary, that $(X \setminus Y)\gamma \not\subseteq Y$ and $(X \setminus Y)\gamma \not\subseteq X \setminus Y$. Then there exist $w, z \in X \setminus Y$ such that $z\gamma \in Y$ and $w\gamma \in X \setminus Y$. Thus, $z \in (A \cap Y)\gamma^{-1}$ for some $A \in X/E$. Fix $a \in A \cap Y$ and define $\alpha : X \rightarrow X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in Y, \\ w & \text{otherwise.} \end{cases}$$

It is evident that $\alpha \in \overline{S}_E(X, Y)$ and $\alpha \leq id_X$. Since γ is right-compatible, we have $\alpha\gamma \leq id_X\gamma = \gamma$. However, $z\alpha\gamma = w\gamma \notin Y$. Therefore, $z \notin (B \cap Y)(\alpha\gamma)^{-1}$ for all $B \in X/E$. We conclude that $E_Y(\gamma)$ does not refine $E_Y(\alpha\gamma)$, which leads to a contradiction. $\square$

Lemma 5.3.3. *Let $\gamma \in \overline{S}_E(X, Y)$ be right-compatible, and let $U \in E_Y(\gamma)$. If $U \cap (X \setminus Y) \neq \emptyset$ and $|U| > 1$, then $X \setminus Y \subseteq U$.*

Proof. Assume that there exist $z \in X \setminus Y$ and $z \neq w \in X$ such that $z\gamma, w\gamma \in A \cap Y$ for some $A \in X/E$. Let $a \in X \setminus Y$. According to Lemma 5.3.2, $a\gamma \in B \cap Y$ for some $B \in X/E$. Define $\alpha : X \rightarrow X$ as follows:

$$x\alpha = \begin{cases} a & \text{if } x = z, \\ x & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in \overline{S}_E(X, Y)$ and $\alpha \leq id_X$. As γ is right-compatible, it follows that $\alpha\gamma \leq \gamma$. According to Theorem 5.1.1, we conclude that $E_Y(\gamma)$ refines $E_Y(\alpha\gamma)$. Since $w\alpha\gamma = w\gamma \in A \cap Y$ and $z\alpha\gamma = a\gamma \in B \cap Y$, it follows that $w \in (A \cap Y)(\alpha\gamma)^{-1}$ and $z \in (B \cap Y)(\alpha\gamma)^{-1}$. Furthermore, since w and z belong to $(A \cap Y)\gamma^{-1}$ and $E_Y(\gamma)$ refines $E_Y(\alpha\gamma)$, we deduce that $(A \cap Y)\gamma^{-1} \subseteq (A \cap Y)(\alpha\gamma)^{-1}$ and $(A \cap Y)\gamma^{-1} \subseteq (B \cap Y)(\alpha\gamma)^{-1}$. Consequently, $(A \cap Y)(\alpha\gamma)^{-1} = (B \cap Y)(\alpha\gamma)^{-1}$, establishing $A = B$. This result implies $a\gamma \in A \cap Y$, as required. $\square$

Lemma 5.3.4. *Let $\gamma \in \overline{S}_E(X, Y)$ be right-compatible. If there exist distinct $A, B \in X/E$ such that $(A \cap Y)\gamma, (B \cap Y)\gamma \subseteq C \cap Y$ for some $C \in X/E$, then $Y\gamma \subseteq C \cap Y$.*

Proof. Assume that there exist distinct $A, B \in X/E$ such that $(A \cap Y)\gamma, (B \cap Y)\gamma \subseteq C \cap Y$ for some $C \in X/E$. Let $y \in Y$. Then there exists $D \in X/E$ such that $y\gamma \in D \cap Y$. Define $\alpha : X \rightarrow X$ by

$$x\alpha = \begin{cases} y & \text{if } x \in B \cap Y, \\ x & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in \overline{S}_E(X, Y)$ and $\alpha \leq id_X$. Since γ is right-compatible, it follows that $\alpha\gamma \leq \gamma$. According to Theorem 5.1.1, we deduce that $E_Y(\gamma)$ refines $E_Y(\alpha\gamma)$. For every $a \in A \cap Y$ and $b \in B \cap Y$, it holds that $a, b \in (C \cap Y)\gamma^{-1}$. This implies $a\alpha\gamma = a\gamma \in C \cap Y$ and $b\alpha\gamma = y\gamma \in D \cap Y$. Consequently, $a \in (C \cap Y)(\alpha\gamma)^{-1}$ and $b \in (D \cap Y)(\alpha\gamma)^{-1}$. As

$E_Y(\gamma)$ refines $E_Y(\alpha\gamma)$, we conclude that $(C \cap Y)\gamma^{-1} \subseteq (C \cap Y)(\alpha\gamma)^{-1}$ and $(C \cap Y)\gamma^{-1} \subseteq (D \cap Y)(\alpha\gamma)^{-1}$. Thus, $(C \cap Y)(\alpha\gamma)^{-1} = (D \cap Y)(\alpha\gamma)^{-1}$ leads to $C = D$. Consequently, $y\gamma \in C \cap Y$, and thus, $Y\gamma \subseteq C \cap Y$, as stipulated. $\square$

Lemma 5.3.5. *Let $\gamma \in \overline{S}_E(X, Y)$ be right-compatible, and let $U \in E_Y(\gamma)$. If $|U \cap (X \setminus Y)| > 1$, then $E_Y(\gamma) = \{X\}$.*

Proof. Assume the existence of distinct z and w in $X \setminus Y$ such that $z\gamma$ and $w\gamma$ both belong to $A \cap Y$ for some $A \in X/E$. Utilizing Lemma 5.3.3, we deduce that $a\gamma \in A \cap Y$ for all $a \in X \setminus Y$. To demonstrate $Y\gamma \subseteq A \cap Y$, we assume, to the contrary, that there exists $a \in Y$ such that $a\gamma \in B \cap Y$ for some $A \neq B \in X/E$. Define the function $\alpha : X \rightarrow X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in Y \cup \{z\}, \\ w & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in \overline{S}_E(X, Y)$ and $\alpha \leq id_X$. Since γ is right-compatible, we have $\alpha\gamma \leq \gamma$. By Theorem 5.1.1, we deduce that $E_Y(\gamma)$ refines $E_Y(\alpha\gamma)$. Considering $X\alpha\gamma = \{a\gamma, w\gamma\}$, we find $E_Y(\alpha\gamma) = \{Y \cup \{z\}, X \setminus (Y \cup \{z\})\}$. However, $X \setminus Y \subseteq (A \cap Y)\gamma^{-1} \in E_Y(\gamma)$, which implies that $E_Y(\gamma)$ does not refine $E_Y(\alpha\gamma)$, leading to a contradiction. Hence, $Y\gamma \subseteq A \cap Y$, allowing us to conclude that $X\gamma \subseteq A \cap Y$. $\square$

Lemma 5.3.6. *Let $\gamma \in \overline{S}_E(X, Y)$ be right-compatible, with $|Y\gamma| > 1$, and let $A \in X/E$. Then $\gamma|_{A \cap Y}$ is constant or injection.*

Proof. Assume, to the contrary, that $\gamma|_{A \cap Y}$ is neither constant nor an injection. Then there exist distinct $a, b, c \in A \cap Y$ such that $a\gamma = b\gamma \neq c\gamma$. Define $\alpha : X \rightarrow X$ by

$$x\alpha = \begin{cases} c & \text{if } x = a, \\ x & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in \overline{S}_E(X, Y)$ and $\alpha \leq id_X$. Since γ is right-compatible, we have $\alpha\gamma \leq \gamma$. However, $a\gamma = b\gamma$ while $a\alpha\gamma = c\gamma \neq b\gamma = b\alpha\gamma$. This contradiction arises from the fact that $\pi(\gamma)$ refines $\pi(\alpha\gamma)$. This completes the proof. $\square$

Lemma 5.3.7. *Let $\gamma \in \overline{S}_E(X, Y)$ be right-compatible such that $|X\gamma| > 1$. Then $\pi(\gamma)$ is in one of the following forms.*

- (1) $\pi(\gamma) = \{\{x\} : x \in X\}$.
- (2) $\pi(\gamma) = \{Y\} \cup \{\{z\} : z \in X \setminus Y\}$.
- (3) $\pi(\gamma) = \{A \cap Y : A \in X/E\} \cup \{\{z\} : z \in X \setminus Y\}$.

Proof. Since γ is right-compatible and $|X\gamma| > 1$, according to Lemma 5.3.1, we have $\{\{z\} : z \in X \setminus Y\} \subseteq \pi(\gamma)$. If γ is injective, it is evident that $\pi(\gamma)$ is in the form (1). In the case where γ is not injective, two cases arise.

Case 1: $|Y\gamma| = 1$. Then $Y \in \pi(\gamma)$ and $\pi(\gamma)$ is in the form (2).

Case 2: $|Y\gamma| > 1$. From γ is not injection and $\{z\} \in \pi(\gamma)$ for all $z \in X \setminus Y$. There exist two distinct $a, b \in Y$ such that $a\gamma = b\gamma$. This implies that $a \in A$ for some $A \in X/E$. To demonstrate that $b \in A$, we assume that $b \in B$ for some $B \in X/E$ and $B \neq A$. Since $|Y\gamma| \neq 1$, there exists $y' \in Y$ such that $y'\gamma \neq a\gamma$. Define $\alpha : X \rightarrow X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in A \cap Y, \\ y' & \text{if } x \in B \cap Y, \\ x & \text{otherwise.} \end{cases}$$

Thus, $\alpha \in \overline{S}_E(X, Y)$ and $\alpha \leq id_X$. Since γ is right-compatible, we have $\alpha\gamma \leq \gamma$. By Theorem 5.1.1, we obtain $\pi(\gamma)$ refines $\pi(\alpha\gamma)$. However, $a\gamma = b\gamma$ while $a\alpha\gamma = a\gamma \neq y'\gamma = b\alpha\gamma$, leading to a contradiction. Hence, $b \in A$ and so $\gamma|_{A \cap Y}$ is constant. To show that $\gamma|_{B \cap Y}$ is constant for every $B \in X/E$. Assume that there exists $B \in X/E$ such that $B \neq A$ and $\gamma|_{B \cap Y}$ is a injection. Let b_1 and b_2 be two distinct elements in $B \cap Y$. Define $\alpha : X \rightarrow X$ by

$$x\alpha = \begin{cases} b_1 & \text{if } x = a, \\ b_2 & \text{otherwise.} \end{cases}$$

From $X\alpha \subseteq B \cap Y$, we get $\alpha \in \overline{S}_E(X, Y)$. Similarly, $a\gamma = b\gamma$ and $a\alpha\gamma = b_1\gamma \neq b_2\gamma = b\alpha\gamma$ which is a contradiction. Thus $A \cap Y \in E_Y(\gamma)$ for all $A \in X/E$. Then, we conclude that $\pi(\gamma)$ is in from (3). $\square$

Lemma 5.3.8. *Let $\gamma \in \overline{S}_E(X, Y)$ be right-compatible such that $|X\gamma| > 1$. Then $E_Y(\gamma)$ is one of the following forms.*

- (1) $E_Y(\gamma) = \{Y\}$.

$$(2) E_Y(\gamma) = \{A \cap Y : A \in X/E\}.$$

$$(3) E_Y(\gamma) = \{X\}.$$

$$(4) E_Y(\gamma) = \{Y\} \cup \{\{z\} : z \in X \setminus Y\}.$$

$$(5) E_Y(\gamma) = \{A \cap Y : A \in X/E\} \cup \{\{z\} : z \in X \setminus Y\}.$$

Proof. By Lemma 5.3.2, we have $(X \setminus Y)\gamma \subseteq Y$ or $(X \setminus Y)\gamma \subseteq X \setminus Y$.

Case 1: $(X \setminus Y)\gamma \subseteq X \setminus Y$. We consider two subcases.

Subcase 1.1: $Y\gamma \subseteq A \cap Y$ for some $A \in X/E$. In this case $Y = (A \cap Y)\gamma^{-1}$, and $E_Y(\gamma)$ is in the form (1).

Subcase 1.2: $Y\gamma \not\subseteq A \cap Y$ for all $A \in X/E$. Hence, $|Y\gamma| > 1$. Considering any $A \in X/E$, there exists $A' \in X/E$ such that $(A \cap Y)\gamma \subseteq A' \cap Y$. Then $A \cap Y \subseteq (A' \cap Y)\gamma^{-1} \in E_Y(\gamma)$. To demonstrate that $A \cap Y = (A' \cap Y)\gamma^{-1}$, we assume, to the contrary, that there exists $a' \in (A' \cap Y)\gamma^{-1} \setminus (A \cap Y)$. Then $a' \in B \cap Y$ for some $B \in X/E$, and $a'\gamma \in A' \cap Y$. By Lemma 5.3.4, we conclude that $Y\gamma \subseteq A' \cap Y$, leading to a contradiction. Thus, $A \cap Y = (A' \cap Y)\gamma^{-1} \in E_Y(\gamma)$. Since A is arbitrary, we obtain $\{A \cap Y : A \in X/E\} \subseteq E_Y(\gamma)$ and the equality holds due to $(X \setminus Y)\gamma \subseteq X \setminus Y$. Therefore, $E_Y(\gamma)$ takes the form (2).

Case 2: $(X \setminus Y)\gamma \subseteq Y$. Two cases arise.

Subcase 2.1: $(X \setminus Y)\gamma \subseteq A \cap Y$ for some $A \in X/E$. If $|X \setminus Y| > 1$, then Lemma 5.3.5, $E_Y(\gamma) = \{X\}$. Thus, we can conclude that $E_Y(\gamma)$ is in the form (3). If $|X \setminus Y| = 1$, then similar Subcase 1.1, we get $E_Y(\gamma) = \{Y, X \setminus Y\}$ or $E_Y(\gamma) = \{X\}$ and similar Subcase 1.2, we have $E_Y(\gamma) = \{A \cap Y : A \in X/E\} \cup \{X \setminus Y\}$. Hence, we can conclude that $E_Y(\gamma)$ is in the form (4) and (5).

Subcase 2.2: $(X \setminus Y)\gamma \not\subseteq A \cap Y$ for all $A \in X/E$. By Lemma 5.3.5, we have $|A \cap Y \cap (X \setminus Y)\gamma| \leq 1$. Furthermore, according to Lemma 5.3.3, $(w\gamma, z\gamma) \notin E$ for all $w \in X, z \in X \setminus Y$ and $z \neq w$, implying that $\{\{x\} : x \in X \setminus Y\} \subseteq E_Y(\gamma)$. Consider $Y\gamma$. If $Y\gamma \subseteq A \cap Y$ for some $A \in X/E$, then $Y = (A \cap Y)\gamma^{-1} \in E_Y(\gamma)$, and hence $E_Y(\gamma)$ is in the form (4). If $Y\gamma \not\subseteq A \cap Y$ for all $A \in X/E$, then, by the same proof as given in Subcase 1.2, we conclude that $\{A \cap Y : A \in X/E\} \subseteq E_Y(\gamma)$. This implies $E_Y(\gamma)$ is in the form (5). $\square$

Theorem 5.3.9. *Let $\gamma \in \overline{S}_E(X, Y)$. Then γ is right-compatible if and only if γ is constant or $\pi(\gamma)$ and $E_Y(\gamma)$ satisfy Lemma 5.3.7 and 5.3.8, respectively.*

Proof. By Lemmas 5.3.7 and 5.3.8, the sufficient part remains to be proven. Let $\alpha, \beta \in \overline{S}_E(X, Y)$ such that $\alpha \leq \beta$ and $|X\gamma| > 1$. It is evident that $X\alpha\gamma \subseteq X\beta\gamma$, and for each $A \in X/E$, there exists $B \in X/E$ such that $(A \cap Y)\alpha\gamma \subseteq (B \cap Y)\alpha\gamma$. To demonstrate that $\alpha\gamma$ and $\beta\gamma$ satisfy the remaining conditions in Theorem 5.1.1, we first establish that $\pi(\beta\gamma)$ refines $\pi(\alpha\gamma)$. Let $P \in \pi(\beta\gamma)$. Then $P = x(\beta\gamma)^{-1}$ for some $x \in X\beta\gamma$. Three cases are considered.

Case 1: $\pi(\gamma) = \{\{x\} : x \in X\}$. Then $P = z\beta^{-1}$, where $z \in X\beta$ and $z\gamma = x$. Since $\pi(\beta)$ refines $\pi(\alpha)$, we have $P = z\beta^{-1} \subseteq z'\alpha^{-1}$ for some $z' \in X\alpha$. Let $z'\gamma = x'$. Then $x' \in X\alpha\gamma$ and $P = z\beta^{-1} \subseteq z'\alpha^{-1} = x'(\alpha\gamma)^{-1}$.

Case 2: $\pi(\gamma) = \{Y\} \cup \{\{z\} : z \in X \setminus Y\}$. Then $P = Y\beta^{-1}$ or $P = z\beta^{-1}$, where $x\beta = z \in X \setminus Y$. If $P = Y\beta^{-1} = \bigcup_{A \in X/E} (A \cap Y)\beta^{-1}$, then since $E_Y(\beta)$ refines $E_Y(\alpha)$, we have

$$\bigcup_{A \in X/E} (A \cap Y)\beta^{-1} \subseteq \bigcup_{B \in X/E} (B \cap Y)\alpha^{-1} = Y\alpha^{-1} = y\gamma^{-1}\alpha^{-1} = y(\alpha\gamma)^{-1},$$

where $y \in Y \cap X\alpha\gamma$. For the case $P = z\beta^{-1}$, where $x\beta = z \in X \setminus Y$, we have $P \in \pi(\beta)$. Since $\pi(\beta)$ refines $\pi(\alpha)$, and $\pi(\alpha)$ also refines $\pi(\alpha\gamma)$, there exists $P' \in \pi(\alpha\gamma)$ such that $P \subseteq P'$.

Case 3: $\pi(\gamma) = \{A \cap Y : A \in X/E\} \cup \{\{z\} : z \in X \setminus Y\}$. Then $P = (A \cap Y)\beta^{-1}$ with $x\beta \in A \cap Y$, or $P = z\beta^{-1}$ with $x\beta = z \in X \setminus Y$. If $P = (A \cap Y)\beta^{-1}$, due to the fact that $E_Y(\beta)$ refines $E_Y(\alpha)$, we have $(A \cap Y)\beta^{-1} \subseteq (B \cap Y)\alpha^{-1}$ for some $B \in X/E$ with $(B \cap Y) \cap X\alpha \neq \emptyset$. Since $\pi(\gamma) = \{A \cap Y : A \in X/E\} \cup \{\{z\} : z \in X \setminus Y\}$, it implies that $B \cap Y = x'\gamma^{-1}$ for some $x' \in X\gamma$. Since $(B \cap Y) \cap X\alpha \neq \emptyset$, we have $x' \in X\alpha\gamma$, and thus, $P \subseteq (B \cap Y)\alpha^{-1} = x'\gamma^{-1}\alpha^{-1} = x'(\alpha\gamma)^{-1}$. The remaining case can be proven similarly to Case 2.

Considering the above three cases, we can conclude that $\pi(\beta\gamma)$ refines $\pi(\alpha\gamma)$.

To demonstrate that $E_Y(\beta\gamma)$ refines $E_Y(\alpha\gamma)$, consider $U \in E_Y(\beta\gamma)$. Then $U = (A \cap Y)(\beta\gamma)^{-1}$ for some $A \in X/E$. There are five cases to consider.

Case 1: $E_Y(\gamma) = \{Y\}$. Since $E_Y(\beta)$ refines $E_Y(\alpha)$, we have $U = Y\beta^{-1} = \bigcup_{A \in X/E} (A \cap Y)\beta^{-1} \subseteq \bigcup_{B \in X/E} (B \cap Y)\alpha^{-1}$. Therefore, $U \subseteq \bigcup_{B \in X/E} (B \cap Y)\alpha^{-1} = Y\alpha^{-1} = (A \cap Y)\gamma^{-1}\alpha^{-1} = (A \cap Y)(\alpha\gamma)^{-1}$.

Case 2: $E_Y(\gamma) = \{A \cap Y : A \in X/E\}$. Then $U = (A' \cap Y)\beta^{-1}$, where $(A \cap Y)\gamma^{-1} = A' \cap Y$. Since $E_Y(\beta)$ refines $E_Y(\alpha)$, we have $(A' \cap Y)\beta^{-1} \subseteq (B \cap Y)\alpha^{-1}$ for some $B \in X/E$ with $(B \cap Y) \cap X\alpha \neq \emptyset$. Furthermore, $B \cap Y = (C \cap Y)\gamma^{-1}$ for some $C \in X/E$. This implies that $U \subseteq (B \cap Y)\alpha^{-1} = (C \cap Y)\gamma^{-1}\alpha^{-1} = (C \cap Y)(\alpha\gamma)^{-1}$.

Case 3: $E_Y(\gamma) = \{X\}$. Then $U = (A \cap Y)(\beta\gamma)^{-1} = X = (A \cap Y)(\alpha\gamma)^{-1}$.

Case 4: $E_Y(\gamma) = \{Y\} \cup \{\{z\} : z \in X \setminus Y\}$. Then $U = Y\beta^{-1}$ or $U = z\beta^{-1}$, where $z \in X\beta \setminus Y$. The case $U = Y\beta^{-1}$ can be proven similar to Case 1. For the case $U = z\beta^{-1}$, as $\pi(\beta)$ refines $\pi(\alpha)$, we have $z\beta^{-1} \subseteq w\alpha^{-1}$ for some $w \in X\alpha$. If $w \in Y$, then $w\alpha^{-1} \subseteq Y\alpha^{-1} = (B \cap Y)\gamma^{-1}\alpha^{-1}$ for some $B \in X/E$. This implies that $U = z\beta^{-1} \subseteq (B \cap Y)(\alpha\gamma)^{-1}$. If $w \in X \setminus Y$, then $\{w\} = (B \cap Y)\gamma^{-1}$ for some $B \in X/E$, implying that $U \subseteq (B \cap Y)\gamma^{-1}\alpha^{-1} = (B \cap Y)(\alpha\gamma)^{-1}$.

Case 5: $E_Y(\gamma) = \{A \cap Y : A \in X/E\} \cup \{\{z\} : z \in X \setminus Y\}$. Then $U = (A' \cap Y)\beta^{-1}$, where $(A \cap Y)\gamma^{-1} = A' \cap Y$, or $U = z\beta$, where $z \in X\beta \setminus Y$. Both instances can be demonstrated similarly to Case 2 and Case 4, respectively.

From the five cases outlined above, we can infer that $E_Y(\beta\gamma)$ refines $E_Y(\alpha\gamma)$.

Finally, let $x \in X$ be such that $x\beta\gamma \in X\alpha\gamma$. This implies $x\beta\gamma = z\alpha\gamma$ for some $z \in X$. Hence, $x\beta$ and $z\alpha$ belong to $w\gamma^{-1}$ for some $w \in X\gamma$. If $|w\gamma^{-1}| = 1$, then $x\beta = z\alpha \in X\alpha$, indicating $x\beta\gamma = x\alpha\gamma$. In the case where $w\gamma^{-1} = Y$, then $x\beta \in A \cap Y$ for some $A \in X/E$. This leads to $x \in (A \cap Y)\beta^{-1} \subseteq (B \cap Y)\alpha^{-1}$ for some $B \in X/E$. Thus, $x\alpha \in Y$, and so $x\alpha\gamma = w = x\beta\gamma$. If $w\gamma^{-1} = A \cap Y$ for some $A \in X/E$, then $x\beta, z\alpha \in A \cap Y$. Considering $X\alpha \subseteq X\beta$, we have $z\alpha = z'\beta = z'\alpha$ for some $z' \in X$. Therefore, $x\beta, z'\beta \in A \cap Y$, and so $x, z' \in (A \cap Y)\beta^{-1} \subseteq (B \cap Y)\alpha^{-1}$ for some $B \in X/E$ with $(B \cap Y) \cap X\alpha \neq \emptyset$. Consequently, $x\alpha, z'\alpha \in B \cap Y$. Since $z'\alpha = z\alpha \in A \cap Y$, we obtain $B = A$. Hence, $x\alpha, x\beta \in A \cap Y$, and then $x\alpha\gamma = w = x\beta\gamma$. $\square$

We conclude this section by deriving the necessary and sufficient conditions for elements of $\overline{S}_E(X, Y)$ to be left-compatible when the partition X/E of X is finite.

Lemma 5.3.10. *Let X/E be finite and $\gamma \in \overline{S}_E(X, Y)$. Then, $Y\gamma = Y$ if and only if, for each $A \in X/E$, there exists $B \in X/E$ such that $A \cap Y = (B \cap Y)\gamma$.*

Proof. Suppose $Y\gamma = Y$. Let $A \in X/E$ and $a \in A \cap Y$. As $Y\gamma = Y$, there exists $B \in X/E$ and $b \in B \cap Y$ such that $b\gamma = a$. Clearly, $(B \cap Y)\gamma \subseteq A \cap Y$. To establish the equality, assume, to the contrary, that there exists $c \in (A \cap Y) \setminus (B \cap Y)\gamma$. Again,

since $Y\gamma = Y$, there exists $C \in X/E$ and $c' \in C \cap Y$ such that $c'\gamma = c$. This implies that $C \neq B$, and both $(B \cap Y)\gamma$ and $(C \cap Y)\gamma$ are subsets of $A \cap Y$. As X/E is finite, there exists $D \in X/E$ such that $D \cap Y \neq \emptyset$, but $(D \cap Y) \cap Y\gamma = \emptyset$. Consequently, $Y\gamma \neq Y$, leading to a contradiction.

Conversely, assume the condition holds. Then, for each $A \in X/E$, there exists $B_A \in X/E$ such that $A \cap Y = (B_A \cap Y)\gamma$. This implies that

$$Y = \bigcup_{A \in X/E} (A \cap Y) = \bigcup_{A \in X/E} (B_A \cap Y)\gamma \subseteq Y\gamma \subseteq Y.$$

This completes the proof. $\square$

Theorem 5.3.11. *Let X/E be finite and $\gamma \in \overline{S}_E(X, Y)$. Then γ is left-compatible if and only if $Y\gamma = Y$ and either one of the following conditions hold:*

- (1) $(X \setminus Y)\gamma \subseteq Y$.
- (2) $X \setminus Y \subseteq X\gamma$.

Proof. Suppose that $Y\gamma \neq Y$ or that neither $(X \setminus Y)\gamma \subseteq Y$ nor $X \setminus Y \subseteq X\gamma$.

Case 1: $Y\gamma \neq Y$. Then there exists $a \in Y \setminus Y\gamma$ such that $a \in A \cap Y$ for some $A \in X/E$.

Subcase 1.1: $(A \cap Y) \cap Y\gamma \neq \emptyset$. In this instance, there exists $B \in X/E$ such that $B \cap Y \neq \emptyset$ and $(B \cap Y)\gamma \subseteq A \cap Y$. Choose $b \in (B \cap Y)\gamma$ and define $\alpha \in \overline{S}_E(X, Y)$ as follows:

$$x\alpha = \begin{cases} b & \text{if } x = a, \\ a & \text{otherwise.} \end{cases}$$

Clearly, $\overline{Y}_a \leq \alpha$ and $(A \cap Y)\gamma\overline{Y}_a = \{b\}$. However, for any $C \in X/E$, we have $(C \cap Y)\gamma\alpha \subseteq (Y \setminus \{a\})\alpha = \{a\}$. Therefore, $\gamma\overline{Y}_a \not\leq \gamma\alpha$.

Subcase 1.2: $(A \cap Y) \cap Y\gamma = \emptyset$. Then $(A \cap Y)\gamma \subseteq B \cap Y$ for some $B \in X/E$. Choose $b \in (A \cap Y)\gamma \subseteq Y\gamma$ and define $\alpha \in \overline{S}_E(X, Y)$ as follows:

$$x\alpha = \begin{cases} b & \text{if } x \in A \cap Y, \\ a & \text{otherwise.} \end{cases}$$

Clearly $\overline{Y}_a \leq \alpha$ and $(A \cap Y)\gamma \overline{Y}_a = \{b\}$. Again, for any $C \in X/E$, we have $(C \cap Y)\gamma \alpha \subseteq (Y \setminus (A \cap Y))\alpha = \{a\}$, implying that $\gamma \overline{Y}_a \not\leq \gamma \alpha$.

Case 2: $(X \setminus Y)\gamma \not\subseteq Y$ and $X \setminus Y \not\subseteq X\gamma$. Consequently, there exist $w \in (X \setminus Y)\gamma \setminus Y$ and $z \in (X \setminus Y) \setminus X\gamma$. Choose $y \in Y$ and let $\alpha, \beta \in \overline{S}_E(X, Y)$ be defined as follow:

$$x\alpha = \begin{cases} y & \text{if } x \in Y, \\ z & \text{otherwise,} \end{cases} \quad \text{and } x\beta = \begin{cases} y & \text{if } x \in Y, \\ z & \text{if } x = z, \\ w & \text{otherwise.} \end{cases}$$

It is evident that $\alpha \leq \beta$. However, $z \in X\gamma\alpha$ while $z \notin X\gamma\beta$, resulting in $X\gamma\alpha \not\subseteq X\gamma\beta$. Therefore, $\gamma\alpha \not\leq \gamma\beta$.

Considering the above two cases, we can conclude that γ is not left-compatible.

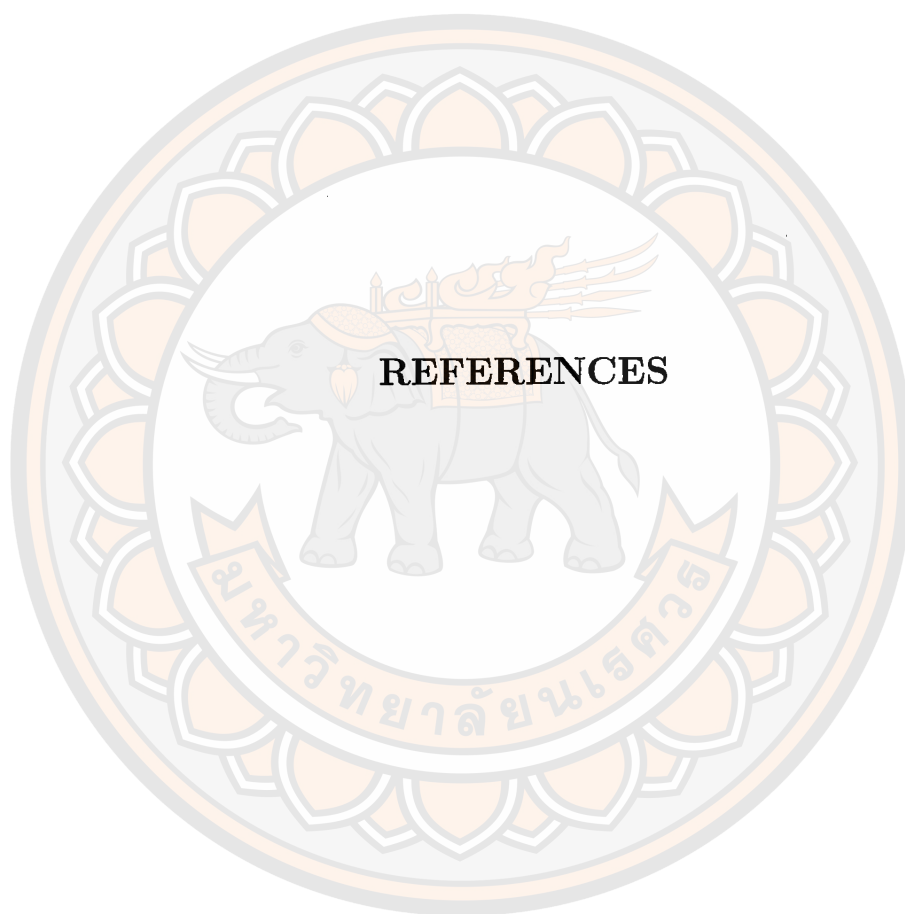
Conversely, assume that $Y\gamma = Y$ and $(X \setminus Y)\gamma \subseteq Y$ or $X \setminus Y \subseteq (X \setminus Y)\gamma$. Let $\alpha, \beta \in \overline{S}_E(X, Y)$ be such that $\alpha \leq \beta$. To demonstrate $\gamma\alpha \leq \gamma\beta$, we first verify that $\pi(\gamma\beta)$ refines $\pi(\gamma\alpha)$. Let $P \in \pi(\gamma\beta)$. Then $P = z(\gamma\beta)^{-1} = (z\beta^{-1})\gamma^{-1}$ for some $z \in X\gamma\beta$. Since $z = z'\gamma\beta$ for some $z' \in X$, we have $z'\gamma \in z\beta^{-1} \subseteq w\alpha^{-1}$ because $\pi(\beta)$ refines $\pi(\alpha)$. Hence, $w = z'\gamma\alpha \in X\gamma\alpha$ and $P = z(\gamma\beta)^{-1} = (z\beta^{-1})\gamma^{-1} \subseteq (w\alpha^{-1})\gamma^{-1} = w(\gamma\alpha)^{-1}$. To show that $E_Y(\gamma\beta)$ refines $E_Y(\gamma\alpha)$, consider $U \in E_Y(\gamma\beta)$. This implies $U = (A \cap Y)(\gamma\beta)^{-1} = (A \cap Y)\beta^{-1}\gamma^{-1}$ for some $A \in X/E$ and $(A \cap Y) \cap X\gamma\beta \neq \emptyset$. Since $E_Y(\beta)$ refines $E_Y(\alpha)$, it follows that $(A \cap Y)\beta^{-1} \subseteq (B \cap Y)\alpha^{-1}$ for some $B \in X/E$ and $B \cap Y \cap X\alpha \neq \emptyset$. Consequently, $U = (A \cap Y)\beta^{-1}\gamma^{-1} \subseteq (B \cap Y)\alpha^{-1}\gamma^{-1} = (B \cap Y)(\gamma\alpha)^{-1}$. To confirm that $X\gamma\alpha$ is a subset of $X\gamma\beta$, we first consider the case where $(X \setminus Y)\gamma \subseteq Y$. In this situation, we observe that $X\gamma\alpha = Y\alpha \subseteq Y\beta = X\gamma\beta$. For the case where $X \setminus Y \subseteq (X \setminus Y)\gamma$, we have $X\gamma = X$, so $X\gamma\alpha = X\alpha \subseteq X\beta = X\gamma\beta$. Now, considering an element $x \in X$ such that $x\beta \in X\alpha$, we find that $(x\gamma)\beta \in X\gamma\alpha \subseteq X\alpha$, implying that $x\gamma\beta = x\gamma\alpha$. Finally, consider $A \in X/E$. Then, $(A \cap Y)\gamma \subseteq B \cap Y$ for some $B \in X/E$. Since $\alpha \leq \beta$, there exists $C \in X/E$ such that $(B \cap Y)\alpha \subseteq (C \cap Y)\beta$. According to Lemma 5.3.10, there exists $D \in X/E$ such that $C \cap Y = (D \cap Y)\gamma$. This implies that $(A \cap Y)\gamma\alpha \subseteq (B \cap Y)\alpha \subseteq (C \cap Y)\beta = (D \cap Y)\gamma\beta$. Therefore, $\gamma\alpha \leq \gamma\beta$, indicating that γ is left-compatible. $\square$

CHAPTER VI

CONCLUSION

In this chapter, we summarize the thesis by characterizing certain properties of the transformation semigroups preserving an equivalence relation on an invariant set, denoted as $\overline{S}_E(X, Y)$. In Chapter 2, we explore the relationships between $\overline{S}_E(X, Y)$, $S(X, Y)$ and $T_E(X)$, respectively. Specifically, $\overline{S}_E(X, Y)$ and $S(X, Y)$ are equal when $Y \times Y \subseteq E$ or $(Y \times Y) \cap E = id_Y$. Also, $\overline{S}_E(X, Y)$ and $T_E(X)$ are equal when $Y = X$. Chapter 3 delves into the characterization of regular, left regular and right regular elements of $\overline{S}_E(X, Y)$. Additionally, we describe elements of $T(X, Y)$ and $S(X, Y)$ that are both left regular and right regular. We also provide the necessary and sufficient conditions for $LReg(T(X, Y))$ and $RReg(T(X, Y))$ to form a semigroup. Furthermore, we prove that every left regular element of $T(X, Y)$ is regular. Similarly, $S(X, Y)$ constitutes a semigroup that is both left and right regular when $|X| \leq 2$. Moreover, both $LReg(S(X, Y))$ and $RReg(S(X, Y))$ also qualify as semigroups when $|X| \leq 2$. Chapter 4 is dedicated to an investigation of Green's relations on the semigroup $\overline{S}_E(X, Y)$. In Chapter 5, we provide a characterization of the natural partial order on the semigroup $\overline{S}_E(X, Y)$. Additionally, we identify elements that are right-compatible with this order and offer left-compatible elements of $\overline{S}_E(X, Y)$ when X/E is finite. Then, we focus on minimal and maximal elements. Specifically, in $\overline{S}_E(X, Y)$, if an element is surjective, it is maximal, and if an element is injective, it is minimal.

As we observe, $\overline{S}_E(X, Y)$ generalizes the semigroups $S(X, Y)$ and $T_E(X)$. In other words, $\overline{S}_E(X, X) = T_E(X)$ and $\overline{S}_E(X, Y) = S(X, Y)$ when $E = X \times X$. Therefore, our results can be applied to these specific semigroups.



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