ASPECTS OF M5-BRANE ACTION IN SEN FORMALISM

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ABSTRACT

In this dissertation, the M5-brane action is constructed from the inspiration of the Sen formalism which imposes the 2-form and 3-form pseudo-field. They are called pseudo due to these fields transform in non-usual standard way. To evaluate this obtained M5-brane action which is too complicated to understand so we simplify by using the double dimensional reduction. The final result we obtain the D4-brane and dual D4-brane action which depends on choice fields of dual frame to be integrated out then field redefinition. In particular, this clearly appears that the unphysical sector decouples from physical one. Furthermore to ensure this decoupling, the Hamiltonian analysis is applied to consider the constraint. By using field redefinition of first-order Lagrangian, the unphysical part explicitly decouples from the physical part at the level of Lagrangian.

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CHAPTER I

INTRODUCTION

Theoretical physicists have developed several methods for building the low-energy effective of M5-brane action that coupled to the background 11-dimensional supergravity. These methods typically begin by focusing on a theory of chiral 2-form on 6-dimensional spacetime. A chiral 2-form is a special kind of field whose field strength which is self-duality. The self-duality of 3-form field strength make it impossible to directly construct the action in 6 dimensions. The solution of this problem begins from the duality symmetric action which loses the manifest diffeomorphism covariance [1, 2]. The following attempts for the approach are somehow not satisfied manifestly diffeomorphism invariance as well [3, 4, 5, 6]. The extension to non-linear duality symmetric action is considered in [7, 8, 9]. Alternatively, a chiral 2-form can be constructed by introducing auxiliary fields which has no dynamics. This approach is called the PST (Pasto, Sorokin, Tonin) formalism [10, 11, 12, 13, 14, 15]. Particularly, once the action of the chiral field is constructed, it is usually relatively simpler to extend to the complete M5-brane action.

A recent approach is called the Sen formalism [16, 17]. This approach is inspired from string field theory. Originally, this method is applied to self-dual 4-form fields in 10 dimensions type IIB supergravity theory. This method is applied to describe chiral 2p-form for 4p+2 dimensions. In the original construction, the chiral 2-form fields has linear self-dual field strength. In our case, we impose $p = 1$ since it is related to M5-brane. In this case, there are 2-form field P and a 3-form field Q which are the independent fields. In particular, these fields transform in non-standard way under diffeomorphism transformation. Moreover, the Q field is linear self-dual with respect to flat 6-dimensional metric even through this theory depends on curved spacetime. Nevertheless the information about curved metric is not lost. The Q field is combined with g metric to give rising for 3-form field strength H. This H field is off-shell self-dual with respect to curved metric and is closed on-shell. Thus H is exact on-shell in the spacetime with trivial topology. Note that this is difference from PST formalism in which the chiral 2-form is independent field. In the Sen formalism, the chiral 2-form field is not independent field.

The further explorations and extensions of the Sen formalism are given as examples as [18, 19, 20, 21, 22, 23]. We focus on the extension in [22] for achieving a complete M5-brane action which is coupled to the background 11-dimensional supergravity in the Green-Schwarz (GS) formalism. This GS formalism has the supertarget spacetime which is manifestly supersymmetry. The symmetries which are gauge symmetries, kappa-symmetry, and diffeomorphism, is required for the construction. Even if M5-brane Sen acion has been constructed, there are still many points to further study and evaluate for more understanding. Especially, this procedure of Sen formalism is difficult because these independent fields P and Q are non-standard way under diffeomorphism transformation. Furthermore, even for the linear self-duality case the coupling of P and Q to the gravity is complicated. In order to better understand the Sen formalism, we apply the dimensional reduction to the action

According to [17, 18], the dimensional reduction are possible applied to various space. However, this study are restricted only in the case of uncompact flat space. We are interesting in the case that if we extend a simply case of quadratic action to the more complicate case of a complete M5-brane action. What is the result from applying the dimensional reduction. This is the majority of task we work in this dissertation. To evaluate the Sen formalism, we provide the double dimensional reduction to the complete M5-brane action. Furthermore, the Hamiltonian analysis is applied to the action for constraint consideration where its result confirms with the Lagrangian formalism that unphysical sector decouples. However the combination of pseudo-form with unphysical or physical part to become the field strength is still the interesting subject to be evaluated.

Furthermore, applying a double dimensional reduction of the complete M5-brane action on a circle give rise to D4-brane action. In the case of PST action [6, 10, 11], the M5-brane is dimensionally reduced to the dual D4-brane action which can be dualise to the D4-brane action [24]. While in [25, 15], if one starts with the complete dual M5-brane action then applies a double dimensional reduction on a circle. The result is the D4-brane action.

Outline of the Dissertation

This outline of dissertation is given as following. In chapter 2, the basic detail about mathematics is provided including with the definition of p-form and the exterior derivative where we usually encounter for calculation. Note that in this dissertation, we use the right-hand definition. In particular, the Hodge duality is also given as simply version and extended for the general manifold. This duality plays the important role in the most part of calculations. Moreover, we also offer the vielbein formalism which is the connection between coordinate and non-coordinate basis. This vielbein is very important part for the embedding method which is the essential one in the case of superstring. We point out the Green-Schwarz formalism which is useful in the case of M5-brane evaluation.

In chapter 3 after finish preparing mathematical basic, we introduce approaches of chiral p-form action from early works about the dual symmertric action. Including with the Sen formalism is inspired from string field theory. As well as, the M5-brane and its construction are given here then it is extended to a complete M5-brane action. In particular, the mappings of $\mathcal{M}$ is necessary to be evaluated in difference pictures from [17] to [19] and [22]. However, we are interesting to consider a perspective in [22]. Then we evaluate $\mathcal{M}$ for useful in the dimensional reduction in this chapter.

The dimension reduction is provided in the chapter 4. Firstly, we simply consider the Sen quadratic action that the result is the 5-dimensionl Maxwell's theory. According to Sen formalism, it appears unphysical and physical part which combined from the pseudo 2-form and 3-form field. Moreover, an unphysical part decouples from a physical one. When we consider a complete M5-brane action from Green-Schwarz formalism. Then applying the double dimensional reduction thus the reduced action is D4-brane and dual D4-brane which depends on which field is integrated out in the case of dual frame and field redefinition.

To consider the constraint of this action, we apply the Hamiltonian analysis into the quadratic Sen action, in the chapter 5. This result is confirm that the unphysical decouples from physical part. When we transform back to Lagrangian, we obtain the first-order Lagrangian. Moreover, after we apply the field redefinition. We obtain the action that analogues to Henneaux-Teitelboim action for both physical and unphysical sector. And the last chapter, we conclude the all corresponding result and suggest the direction of following research or the possible future works.

CHAPTER II

BASIC MATHEMATICS AND SUPERSTRING

2.1 Differential form and Vielbein formalism

We will give the mathematics detail that is the important for further calculation in following chapter. As well as the physics method in the case of superstring theory is provided.

2.1.1 Differential form

A differential p-form or a "p-form" and is a totally antisymmetric tensor of rank $(0, p)$. It is written as

$$w_p = \frac{1}{p!} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p} w_{\mu_p \cdots \mu_2 \mu_1} \tag{2.1}$$

where $\wedge$ is the wedge product. Moreover, $w_{\mu_p \cdots \mu_2 \mu_1}$ is totally anti-symmetric. Corresponding to this convention, we define interior product and exterior derivative to act from the right. The wedge product has to satisfy

- $dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}$=0 if there is a repeated index.
- $dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}$ is linear in each dx^{μ}

There are n choices from p, the set thus the independent basis is given as

$$\begin{pmatrix} n \\ p \end{pmatrix} = \frac{n!}{(n-p)!p!} \tag{2.2}$$

When we look at $\mathbb{R}^n$. Let $x = (x_1, x_2, \ldots, x_n)$ be the coordinates in that space.

- 0-form is just a function $f(x)$ (scalar field)
- 1-form is $\alpha = \alpha_\mu(x) dx^\mu$ (covector field)
- 2-form is $w = \frac{1}{2} dx^\mu \wedge dx^\nu w_{\nu\mu}(x)$ (area form)

$\vdots$ $\qquad$ $\vdots$

- p-form is $w_p = \frac{1}{p!} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p} w_{\mu_p \cdot \mu_2 \mu_1}$

In this dissertation, the most of our calculation usually uses p-form.

2.1.2 Exterior derivative operator

An exterior derivative operator is applied to increase a rank of vector or covector. An simple example, this operator applies the derivative to a scalar function f (0-form) then this scalar becomes a covector field. It is shown as

$$d(p\text{-form}) \rightarrow ((p+1)\text{-form})$$

So we simply said that this operator increases the rank of p-form by 1.
Suppose we have $f(x)$ where $x = (x_1, x_2, \ldots, x_n)$ so

$$\alpha = df = dx^\mu \alpha_\mu(x) \tag{2.3}$$

where $\alpha_\mu \equiv \partial f / \partial x^\mu$. Certainly, it is defined by (1-form)=d(0-form).
When acting on 1-form, we clearly have

$$\omega = d\alpha = d(dx^\mu \alpha_\mu(x)) = dx^\mu \wedge dx^\nu \partial_\nu(\alpha_\mu(x)) \tag{2.4}$$

then (2-form)=d(1-form). So, we obtain the new expression of this 2-form as

$$\omega = dx^\mu \wedge dx^\nu \omega_{\nu\mu} \tag{2.5}$$

where $\omega_{\nu\mu} \equiv \partial\alpha_\mu / \partial x_\nu$ and $\mu < \nu$.
Then if we have p-form $\alpha = dx^I \alpha_I$ where $I = i_1, i_2, \ldots, i_p$, we will have

$$d\alpha = dx^I \wedge dx^\nu \frac{\partial \alpha_I}{\partial x_\nu} \tag{2.6}$$

where $dx^I \equiv dx^{i_1} \wedge dx^{i_2} \cdots \wedge dx^{i_p}$ and $\alpha_I \equiv \alpha_{i_p \cdots i_2 i_1}$.
For p-form is satisfied *Theorem*:

1. If α is p-form and β is k-form then

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + \alpha \wedge (d\beta) \tag{2.7}$$

2. $d(d\alpha) = 0$, this is called "closed".

These theorem is important when we consider the fields of theory in the chapter 4. For our case, the exterior derivative is defined to act from the right

$$dw_p = \frac{1}{p!} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p} \wedge dx^{\nu} \partial_\nu (w_p)_{\mu_p \cdots \mu_2 \mu_1}. \tag{2.8}$$

2.1.3 Dual transformation

In this section we illustrate the definition and properties of Hodge star operator $\tilde{*}$ in flat spacetime. For an example in Riemannian, let us look at $\mathbb{R}^n$. When we apply this operator on the p-form, we will obtain the $(n-p)$-form where $n > r$. For clear this state, it is demonstrated in $\mathbb{R}^3$, $x_i = (x, y, z)$. This Hodge duality operates on dx, we have

$$\begin{aligned} \tilde{*}dx &= \frac{1}{2}\epsilon_{1ij} dx^i \wedge dx^j \\ &= \frac{1}{2}(dy \wedge dz - dz \wedge dy) \\ &= dy \wedge dz \end{aligned} \tag{2.9}$$

where $\wedge$ wedge product is totally antisymmetric. This ϵ is the totally antisymmetric tensor which is defined as

$$\epsilon_{\mu_1\mu_2\ldots\mu_n} = \begin{cases} +1 & \text{if } (\mu_1\mu_2\ldots\mu_n) \text{ is an even permutaion of } (12\ldots n) \\ -1 & \text{if } (\mu_1\mu_2\ldots\mu_n) \text{ is an odd permutaion of } (12\ldots n) \\ 0 & \text{otherwise.} \end{cases} \tag{2.10}$$

where the index is $\mu = \{1, 2, \ldots, n\}$.

While in the case of Lorentzian, we can define this operation on the space $\mathbb{R}^n$ when acting on wedge product. The index is given as $\mu = \{0, 1, \ldots, n\}$. In the case of flat spacetime $\eta_{\mu\nu}$, it is given as

$$*' dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} = \frac{(-1)^{p+1}}{(n-p)!\sqrt{-\eta}} dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_n} \eta_{\mu_{p+1}\nu_{p+1}} \cdots \eta_{\mu_n\nu_n} \epsilon^{\nu_{p+1}\cdots\nu_n\mu_1\cdots\mu_p}. \tag{2.11}$$

Furthermore in a case depends on metric $g_{\mu\nu}$ of curve spacetime, we have

$$* dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} = \frac{(-1)^{p+1}}{(n-p)!\sqrt{-g}} dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_n} g_{\mu_{p+1}\nu_{p+1}} \cdots g_{\mu_n\nu_n} \epsilon^{\nu_{p+1}\cdots\nu_n\mu_1\cdots\mu_p}. \tag{2.12}$$

When we apply the Hodge duality to p-form, we have

$$\begin{aligned} *w_p &= \frac{1}{p!} * (dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}) w_{\mu_p\cdots\mu_2\mu_1} \\ &= \frac{1}{p!} \frac{(-1)^{p+1}}{(n-p)!\sqrt{-g}} dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_n} g_{\mu_{p+1}\nu_{p+1}} \cdots g_{\mu_n\nu_n} \epsilon^{\nu_{p+1}\cdots\nu_n\mu_1\cdots\mu_p} w_{\mu_p\cdots\mu_2\mu_1} \end{aligned} \tag{2.13}$$

so only an element, it is

$$*w_{\mu_1\cdots\mu_p} = \frac{(-1)^{p+1}}{(n-p)!\sqrt{-g}} g_{\mu_1\nu_1} \cdots g_{\mu_p\nu_p} \epsilon^{\nu_1\cdots\nu_p\mu_{p+1}\cdots\mu_n} w_{\mu_{p+1}\cdots\mu_n} \tag{2.14}$$

where it is swapped the index of w.
For the manifold is Riemannian, there is

$$\tilde{*}\tilde{*}\, \omega = (-1)^{p(n-p)} \omega \tag{2.15}$$

while the manifold is Lorentzian, we have

$$* * \omega = (-1)^{1+p(n-p)} \omega. \tag{2.16}$$

Our calculation will use the Hodge duality of Lorentzian manifold for the dimensional reduction method in chapter 4.

2.1.4 Exterior derivative and Hodge operator relation

In the simplest explanation of this relation, let us consider more calculation between the exterior derivative and Hodge operator in $\mathbb{R}^3$, we start with defining forms as

$$\omega = \frac{1}{2} dx^i \wedge dx^j \omega_{ji} \tag{2.17}$$

$$\Omega = \frac{1}{2} dx^i \wedge dx^j \Omega_{ji}. \tag{2.18}$$

Firstly, we have

$$\omega \wedge *\Omega = (\omega \cdot \Omega)dV \tag{2.19}$$

where $*\Omega$ is transformed then becomes 1-form. Explicitly, the wedge product is nothing but the volume.

Next we consider the operation $* \, d \, *$. Suppose we have a covector field as $\alpha = Pdx^1 + Qdx^2 + Rdx^3$. So, the calculation is expressed as

$$\begin{aligned} *d*\alpha &= * \, d * (Pdx^1 + Qdx^2 + Rdx^3) \\ &= * \, d(Pdx^2 \wedge dx^3 + Qdx^3 \wedge dx^1 + Rdx^1 \wedge dx^2). \end{aligned} \tag{2.20}$$

Let's consider some term in bracket, we have

$$d(Pdx^2 \wedge dx^3) = (dP)(dx^2 \wedge dx^3) + \cancelto{0}{Pd(dx^2 \wedge dx^3)} \tag{2.21}$$

where

$$dP = \frac{\partial P}{\partial x_1}dx^1 + \frac{\partial P}{\partial x_2}dx^2 + \frac{\partial P}{\partial x_3}dx^3 \tag{2.22}$$

then we finally obtain

$$d(Pdx^2 \wedge dx^3) = \frac{\partial P}{\partial x_1}dx^1 \wedge dx^2 \wedge dx^3. \tag{2.23}$$

Repeating this trick to remain terms, we have

$$* \, d * \alpha = * \left(\frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2} + \frac{\partial R}{\partial x_3} \right) dV = \frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2} + \frac{\partial R}{\partial x_3} \tag{2.24}$$

Suppose there exists a vector field is $\vec{F} = P\vec{e}_1 + Q\vec{e}_2 + R\vec{e}_3$ then

$$* \, d * \alpha = \mathrm{Div}\vec{F}. \tag{2.25}$$

Moreover, we also have

$$* \, d\alpha = (\vec{\nabla} \times \vec{F}) \cdot d\vec{x} \tag{2.26}$$

where $d\vec{x} = (dx^1)\vec{e}_1 + (dx^2)\vec{e}_2 + (dx^3)\vec{e}_3$.

2.1.5 Vielbein formalism

Vielbein formalism is also called tetrad formalism. It provides the relationship between of the coordinate basis and non-coordinate basis. Before we begin, many definitions must be provided. Firstly, the metric is defined in term of vielbein or tetrad form as

$$\vec{g}(\hat{e}_a, \hat{e}_b) = \eta_{ab} \tag{2.27}$$

where the latin latter is in non-coordinate basis index. And the vector and coverter from the coordinate basis is given in the non-coordinate basis as

$$\partial_\mu = e_\mu{}^a \hat{e}_a \tag{2.28}$$

$$dx^\mu = e^\mu{}_a \hat{\theta}^a. \tag{2.29}$$

In addition, if each non-coordinate basis contract to each other we obtain the kronecker delta as

$$e^\mu{}_a e_\nu{}^a = \delta^\mu{}_\nu \ , \qquad e^\mu{}_a e_\mu{}^b = \delta_a{}^b \tag{2.30}$$

Furthermore, the invert form of eq.(2.27) is illustrated as

$$\begin{aligned}\eta_{ab} &= \vec{g}(\vec{e}_a, \vec{e}_b)\\ &= \vec{g}(e^\mu{}_a \partial_\mu, e^\nu{}_b \partial_\nu)\\ \eta_{ab} &= e^\mu{}_a e^\nu{}_b g_{\mu\nu}.\end{aligned} \tag{2.31}$$

The vector which is transformed from coordinate basis to non-coordinate basis, is

$$\vec{V} = V^\mu \partial_\mu = V^\mu e_\mu{}^a \hat{e}_a = V^a \hat{e}_a \tag{2.32}$$

and the vector is transformed from the coordinate basis to non-coordinate basis is given as

$$V^a = e_\mu{}^a V^\mu \tag{2.33}$$

In general, for the tensor in n-dimension is

$$T^{\mu_1 \ldots \mu_u}{}_{\nu_1 \ldots \nu_l} = e^{\mu_1}_{\ a_1} \cdots e^{\mu_u}_{\ a_u} e^{b_1}_{\ \nu_1} \cdots e^{b_l}_{\ \nu_l} T^{a_1 \ldots a_u}{}_{b_1 \ldots b_l} \tag{2.34}$$

These tetrad basis transform under Lorentz transformation are given as

$$\tilde{\eta}_{ab} = \Lambda_a{}^c \Lambda_b{}^d \eta_{cd} \tag{2.35}$$

and the induced mapping from flat to curve spacetime is

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab} \tag{2.36}$$

The metric is obviously invariant $\tilde{g}_{\mu\nu} = g_{\mu\nu}$ in which the tetrad basis transform as

$$\tilde{e}_\mu{}^a = \Lambda^a{}_b e_\mu{}^b. \tag{2.37}$$

In the general coordinate transformation, they are expressed like

$$\tilde{e}_\mu{}^a = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \Lambda^a{}_b e_\nu{}^b. \tag{2.38}$$

and for the tensor type, it is given as

$$\tilde{T}^\mu{}_a = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \Lambda_a{}^b T^\nu{}_b. \tag{2.39}$$

Next, let us discuss about to apply differential operator on V^a, like $\partial_\mu V^a$. It is invariant under general coordinate transformation but not under local Lorentz transformation. This V^a is a scalar in coordinate basis but in non-coordinate basis, it is a vector. So we have to express with some extra term like the connection term in curve space. Then we obtain the covariant derivative as

$$\nabla_\mu V^a = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b \tag{2.40}$$

where $\omega_\mu{}^a{}_b$ is called spin connection which is a mixed basis.

In the case of tensor, we have

$$\nabla_\mu T^a{}_\nu = \partial_\mu T^a{}_\nu + \omega_\mu{}^a{}_b T^b{}_\nu - \Gamma^\rho{}_{\mu\nu} T^a{}_\rho \tag{2.41}$$

and in the case of covector in non-coordinate basis, they are express as

$$\nabla_\mu W_a = \partial_\mu W_a - \omega_{\mu a}{}^b W_b. \tag{2.42}$$

In particular, the relation between of the affine connection and spin connection can be showed in the form of

$$\Gamma^{\nu}{}_{\mu\rho} = e^{\nu}{}_{a}\partial_{\mu}e_{\rho}{}^{a} + \omega_{\mu}{}^{a}{}_{b}e^{\nu}{}_{a}e_{\rho}{}^{b} \tag{2.43}$$

where we can see the different between coordinate basis and non-coordinate basis in the first term of above equation. When we contract Eq.(2.43) then we have

$$\partial_{\mu}e_{\rho}{}^{c} = e_{\nu}{}^{c}\Gamma^{\nu}{}_{\mu\rho} - \omega_{\mu}{}^{c}{}_{b}e_{\rho}{}^{b} \tag{2.44}$$

thus we obtain the covariant derivative for tetrad basis as

$$\nabla_{\mu}e_{\rho}{}^{c} = \partial_{\mu}e_{\rho}{}^{c} + \omega_{\mu}{}^{c}{}_{b}e_{\rho}{}^{b} - \Gamma^{\lambda}_{\mu\rho}e_{\lambda}{}^{c} = 0, \tag{2.45}$$

where this is called vielbein postulate.
Then the vielbein postulate which is expressed as

$$\nabla_{\rho}g_{\mu\nu} = \nabla_{\rho}(e^{a}{}_{\nu}e^{b}{}_{\nu}\eta_{ab}) = 0, \tag{2.46}$$

which is called metric compatibility.

At last we will be able to apply this vielbein formalism to find the specific variables which is curvature of spacetime.

2.2 Superstring: Embedding methods

According to the bosonic string theory, it only consists the bosonic quantization but, normally in physics, all of these particles are also included with fermions. Therefore, we have to extend our theory to cover all of these fields especially fermions. Consequently, Supersymmetry comes to play the important role in following theory which is called superstring theory.

There are three ways consider superstring where the worldsheet is embedded into target space. Ramond-Neveu-Schwarz formulation (RNS), the worldsheet is supersymmetric but spacetime is not. While the Green-Schwarz formulation (GS), its spacetime is supersymmetric. For superembedding where both the worldsheet and spacetime are manifestly supersymmetric.

To emphasize, for RNS formulation, the world-volume has the superspace that is embedded on the spacetime. From the string quantization, the spectrum is truncated into two sectors which are Ramond sector and Neumann-Schwarz sector when we consider low energy limit. This quantization of spectrum suggests that the target space has supersymmetry.

From the method of GS formalism, the worldvolume is not manifest in the supersymmetry where it is bosonic. They are embedded on the manifest superspacetime where bosonic and feminonic strings are unified in a single Fock space. Consequently, the kappa-symmetry is provided where is a local fermionic symmetry to cancel an extra degree of freedom for fermionic. In particular, this GS action is difficult to be quantized but it can be done in the light-cone gauge.

For the superembedding, superworldsheet is embedded on superspacetime. The extension to the superspace gives the extra degrees of freedom. These are not physical. However, they are be removed by imposing the constraints analysis. Moreover, this formulation normally deals with the relation of supervielbein and superconnection. We will not go in the detail here.

All the formulations are illustrated as Fig.1 as following.

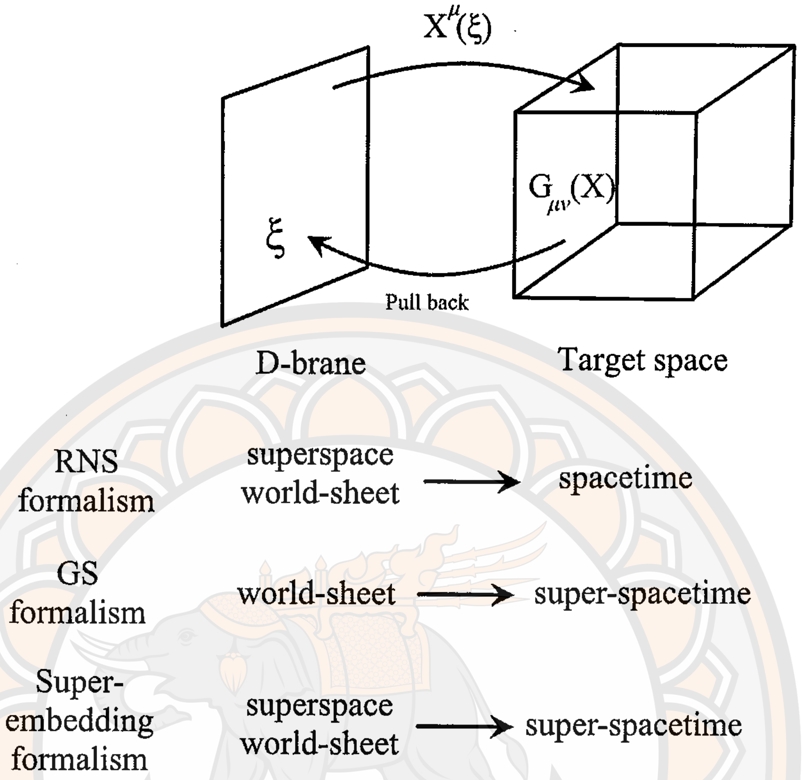

Figure 1 Illustration of embedding methods.

2.2.1 The Green-Schwarz formalism

For this dissertation, the majority for our calculation and evaluation will use the Green-Schwarz formalism. So we will give the information for this formulation only.

The example for the bosonic string action in the GS formalism [26] of the covariant action. Firstly we provide a basis detail of the point particle string. The action is given as

$$S = -m \int d\tau \sqrt{-\dot{X}^m \dot{X}^n g_{mn}(X)}, \tag{2.47}$$

where g_{mn} is a fixed curved spacetime and m is mass for this particle action. The $X^m(\tau)$ are a mapping of embedding the trajectory of particle with affine parameter

τ into spacetime. This $X^m(\tau)$ of the local coordinates x^m is descried as field, has dynamics on the world line.

Next we discuss about the bosonic string action where we focus on the Nambu-Goto action, 1+1 dimension. The action is given as

$$S_1 = -T_f \int d^2\sigma \sqrt{-\det G} \tag{2.48}$$

where T_f is the tension. This action associates worldsheet coordinates σ^μ $\mu = 0, 1$ and the spacetime coordinates X^m by the embedding $X^m(\sigma)$, $m = 0, 1 \ldots, d-1$ with $g_{ab}(X)$ metric,

$$G_{\mu\nu} = \partial_\mu X^m \partial_\nu X^n g_{mn}(X), \tag{2.49}$$

where this is the induced matrix from the supertarget spacetime. Then theory becomes a non-linear interaction. We have the Wess-Zumino term where strings can be charged under 2-forms, as

$$S_2 = Q_f \int \Omega_{(2)}, \tag{2.50}$$

where $\Omega_{(2)}$ is a 2-form. This $\Omega_{(2)}$ is the pull-back from the superspace background. The bosonic string action is

$$S = -T_f \int d^2\sigma \sqrt{-\det G} + Q_f \int \Omega_{(2)}. \tag{2.51}$$

Furthermore, introducing the supersymmetry, so we have the spacetime vector X^m and a scalar fields θ^α which transforms as a spinor. The action is invariant under the supersymmetry transformations which is

$$\delta\theta^A = \epsilon^A, \qquad \delta X^m = \bar{\epsilon}^A (\Gamma^m)_A{}^B \theta_B, \tag{2.52}$$

where ϵ^A is a constant spacetime spinor and $\bar{\epsilon}^A = (\epsilon^t)_B C^{BA}$ which C is the charge conjugation matrix. Moreover A is the independent indices for counting supersymmetries (spinor indices), $A = 1, 2, \ldots \mathcal{N}$. The supersymmetric string action is given as

$$S_1 = -\frac{T_f}{2} \int d^2\sigma \sqrt{h} h^{\mu\nu} \Pi_\mu^m \Pi_\nu^n \eta_{mn}, \tag{2.53}$$

where $h_{\mu\nu}$ is an auxiliary 2-dimensional metric and we define the supersymmetric combination as

$$\Pi^m = dX^m + \bar{\theta}^A (\Gamma^m)_A{}^B d\theta_B. \tag{2.54}$$

However, the number of the spacetime dimension and the spinor must be equaled at on-shell. So to satisfy this requirement, the extra amount degree of freedom of fermions have to be canceled by applying the kappa-symmetry.

The kappa-symmetry gives a modification of the spinor of local parameter $\kappa(\sigma)$ and requires θ to transform by projection operators as

$$\delta_\kappa \theta = (\mathbb{1} \pm \Gamma_\kappa)\kappa, \quad \text{with} \quad \Gamma_\kappa^2 = \mathbb{1}. \tag{2.55}$$

When we focus on the type II, the Wess-Zumino term under this extension is given as

$$S_2 = Q_f \int d^2\sigma \Big[-\epsilon^{\mu\nu}\partial_\mu X^m (\bar{\theta}^1 \Gamma_m \partial_\nu \theta^1 - \bar{\theta}^2 \Gamma_m \partial_\nu \theta^2) + \epsilon^{\mu\nu} \bar{\theta}^1 \Gamma^m \partial_\mu \theta^1 \bar{\theta}^2 \Gamma_m \partial_n \theta^2 \Big]. \tag{2.56}$$

The action is invariant under the global supersymmetry transformations which is

$$\delta \bar{\theta}^A = \kappa^A \delta_\kappa \theta. \tag{2.57}$$

The local fermionic κ symmetry gauges away the half of fermion. Moreover, from its supersymmetric and kappa-symmetry invariant, it requires

$$T_f = Q_f = \frac{1}{2\pi\alpha} = \frac{1}{2\pi l_s^2}, \tag{2.58}$$

where l_s is the length of the string.

In the case of background curved spacetime, we consider the action that becomes

$$S = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-\det G_{\mu\nu}} + \frac{1}{2\pi\alpha'} \int \Omega_{(2)}. \tag{2.59}$$

Actually, the $G_{\mu\nu}$ and $\Omega_{(2)}$ depend on superfields E_M^A supervielbein and Ω_{AB}, respectively, where M stands for curved superspace, $M = \{m, \alpha\}$, and A is for tangent superspace indices $A = \{a, \underline{\alpha}\}$. These superfields E_M^A and Ω_{AB} that string couples, are defined as the pull-back from superspace into the worldsheet as

$$G_{\mu\nu} = \Pi_\mu \cdot \Pi_\nu = \partial_\mu Z^M E_M^a(Z) \partial_n Z^N E_N^b(Z) \eta_{ab},$$
$$\Omega_{(2)} = \partial_\nu Z^M E_M^A(Z) \partial_\nu Z^N E_N^C(Z) \Omega_{AC}(Z), \tag{2.60}$$

where $Z^M = \{X^m, \theta^\alpha\}$ is the local coordinate of worldsheet that embedded on background superspace.

According to this GS formalism, we will apply to construct the complete M5-brane action in the next chapter.

CHAPTER III

M5-BRANE ACTION AND SEN FORMALISM

3.1 Previous approaches of chiral p-form action

The early research, the dual symmetric actions in 4-dimension was proposed for obtaining the local Lagrangian field equations [1, 2, 12, 13]. However, their result are still lose some identities, manifestly Lorentz invariance, to approach the Maxwell's theory in higher dimension. We will discuss some examples of approaching as following.

From Henneaux and Teitelboim [3], they constructed the action for chiral p-forms with self-duality condition. This is local, Lorentz invariant and invariant under the standard gauge transformation as well but the resulting action still loses its manifestly Lorentz invariant.

According to PST research [10, 27, 28, 29, 30], this approach constructs the dual symmetric action by proposing the method to treat this problem. Schwarz and Sen give the original idea for an auxiliary field to the action. This action satisfied the diffeomorphism transformation. But the action still loses the local invariance transformation with that auxiliary field. Hence PST considered the duality symmetric action by using an auxiliary field in extending detail from Schwarz and Sen. Furthermore, this auxiliary field couples to dual gauge field. As a result the low-energy effective action is satisfied the manifestly Lorentz invariant.

These approach of chiral 2-form field theory can be used to describe the M5-brane action which possible to be applied the dimensional reduction for explaining lower dimension action. However our calculation of this dissertation will focus on the Sen formalism that gives the different perspective, provided as following section.

3.2 Sen action inspiration

The Sen formalism inherits from the string field theory aspect, BV formalism [31]. The ghost field are proposed from the one particle irreducible effective action. This ghost field has a wrong sign kinetic term and decouples from the theory at last.

In the type IIB supergravity, the action is in $(9+1)$-dimensions and has 4-form gauge potential and 5-form field strength which has a self-duality constraint. While Sen formalism, the 4-form field and 5-form field are imposed to be the fundamental fields instead of a 4-form gauge potential. Moreover, this formalism has two kinds of Hodge duality. One respects to flat metric while another one respects to curved metric. According to the action, this 4-form field has the wrong sign kinetic term and decouples from the theory at the level of equations of motion. For a 5-form field, this comes with the self-duality condition which respects to the flat metric. These fundamental fields are explained at the local tangent space so this is possible to use a flat metric, even this theory is under the curved spacetime. This formulation preserves the manifest Lorentz invariant, has a finite auxiliary field, and is polynomial. However this fields transform under the general coordinate transformation in an unusual way.

3.2.1 Type IIB supergravity action

According to Sen idea [16] from the string field theory perspective, we consider this concept for type IIB supergravity action in 10-dimensions. The imposed fields are given as the 4-form unconstrained gauge field P_4 while the 5-form field Q_5 obeys a self-dual constraint.

The self-duality condition of Q_5 is given as

$$*' Q_5 = Q_5 \tag{3.1}$$

where $*'$ is Hodge duality with respect to the flat metric which denoted as the internal space or the local tangent space for fields. This idea is come from the view of considering a vielbein.

From Sen [16, 17], the action is

$$S = \int \left(\frac{1}{2}dP_4 \wedge *'dP_4 - dP_4 \wedge Q_5 + \tilde{S}(Q_5, M)\right) \tag{3.2}$$

where $\tilde{S}(Q_5, M)$ is the term that interacts with other fields. This term has been shown that describes type IIB supergravity. The other physical fields are collected in term of M. This P_4 appears in only kinetic terms while Q_5 is in the kinetic term only linearly and in the interacting terms.

The variation of $\tilde{S}$ take form as

$$\delta\tilde{S} = -\frac{1}{2}\int R \wedge \delta Q_5 + \delta_M \tilde{S} \tag{3.3}$$

where δ_M relates to the all other fields. R is anti-self-dual due to a self-duality of δQ_5 and property of M, given as

$$\begin{aligned} R \wedge \delta Q_5 &= R \wedge \frac{(1+*')}{2}\delta Q_5 \\ &= R \wedge \frac{1}{2}\delta Q_5 + R \wedge \frac{1}{2}\delta *' Q_5 \\ &= \frac{1}{2}R \wedge \delta Q_5 + \frac{1}{2}\delta Q_5 \wedge *'R \\ &= \frac{1}{2}R \wedge \delta Q_5 - \frac{1}{2} *' R \wedge \delta Q_5 \end{aligned} \tag{3.4}$$

which forces $*'R = -R$ for keeping this equation corrects.

The equations of motion from eq.(3.2) are given as

$$P_4 : \qquad d(*'dP_4 - Q) = 0 \tag{3.5}$$

$$Q_5 : \qquad dP_4 - *'dP_4 + R = 0 \tag{3.6}$$

$$M : \qquad \delta_M \tilde{S} = 0. \tag{3.7}$$

Applying the exterior derivative on eq.(3.6) and using eq.(3.5), we have

$$d(Q_5 - R) = 0. \tag{3.8}$$

These eq.(3.7) and (3.8) can be used as the equations of motion for physical fields in type IIB supergravity. While eq.(3.6) is the equation for P_4. The different solution to eq.(3.6) for term of R which given by Q_5 and M so the equation becomes

$$d(\Delta P_4) - *' d(\Delta P_4) = 0. \tag{3.9}$$

From this result, the given differ solution is not effect eq.(3.7) and (3.8). Thus this P_4 decouples from theory and the interacting term does not depend on it.

From Sen's invention, this theory has been extended further in [17, 18, 19, 20, 32]. We will consider in the detail in next section. In particular, we consider R in terms of a mapping $\mathcal{M}$ which encodes the interaction term corresponding to external source and gravity.

3.3 M5-brane action

M5-brane is the one of objects in M-theory. This brane is magnetically charged under C_3 of background 11-dimensions supergravity. Field content on M5-brane, this contains a chiral 2-form field with non-linear self-dual 3-form field strength. The main motivation of this theory is analogue to Maxwell's theory which has field strength contains with vector potential in 4-dimensions. This is given as

$$\mathcal{L} = -\frac{1}{4} F \wedge *F. \tag{3.10}$$

Likewise, we attempt try to write chiral 2-form action directly like

$$\mathcal{L} \approx -H \wedge *H, \tag{3.11}$$

where H is 3-form field strength in 6-dimensions. However, this attempt fails because the property of H that is self-duality, $*H = H$ where $*$ is Hodge duality with respect to the curved space-time. From this self-duality condition leads to eq.(3.11) to be zero.

3.4 M5-brane action construction

The brane is considered its fields content where this depends on which theory we interest. These fields have to satisfy the equation of motion where the self-duality condition is naturally proposed. The action is evaluated and obtained from a known solution of the equations of motion. Then instead of linear self-duality, applying the non-linear self-duality condition which calculated from the result of superembedding. Adding the background fields g, C_3, C_6 which pullback from the 11-dimension supergravity target space to the action from the Green-Schwarz formalism then the action becomes the complete M5-brane action. It is important to evaluate the symmetries of the action such as kappa-symmetry, gauge symmetry, and diffeomorophism. Particularly, kappa-symmetry is used to cancel the abundant degree of freedom of fermions. These steps can be summarized below.

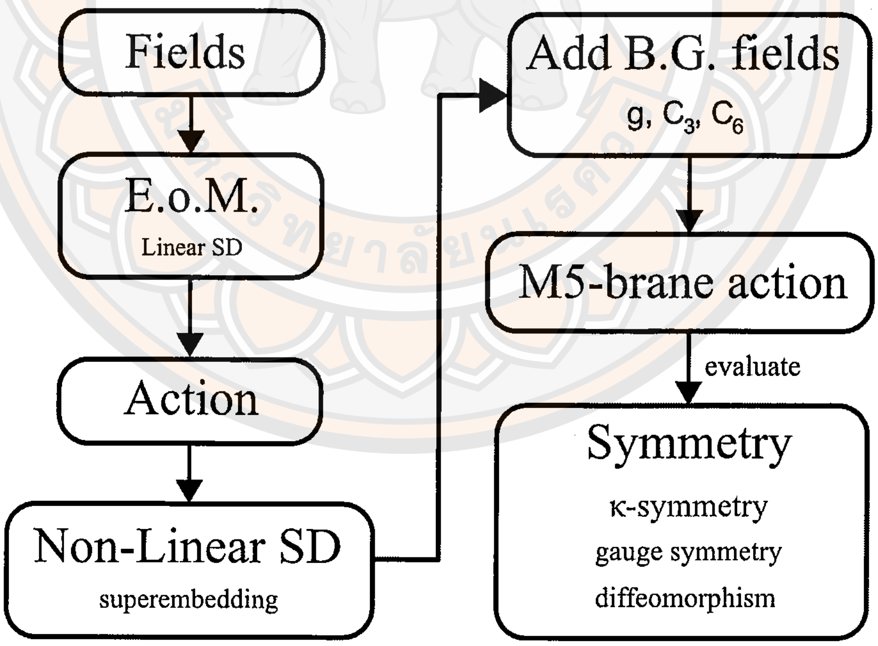

Figure 2 M5-brane construction

3.5 Complete M5-brane action

According to the work [16] then extended in [17], the action is discussed in the general form. The simplified version in [19] when focusing the chiral 2-form field in 6-dimension, the action is given as

$$S = \int \Big(\frac{1}{2}dP \wedge *'dP - 2Q \wedge dP + Q \wedge \widetilde{\mathcal{M}}(Q, g, J)\Big) \tag{3.12}$$

where P is a 2-form gauge field and Q is a 3-form field strength under self-dual constraint, $*'Q = Q$. The interaction term is collected in $\widetilde{\mathcal{M}}(Q, g, J)$ coupling with the background field. It is the function of Q, metric g, and the external source J.

For the simple function $\mathcal{M}(Q)$, it is normally the linear mapping for the self-duality to anti-self-duality condition. However, this mapping can be modified by extending a domain and its acting on basis property where these are discuss in [19, 22]. For the detail will give in the next section. This $\mathcal{M}$ function is given in the index notation as

$$\mathcal{M}(Q)_{\mu\nu\rho} = \frac{1}{3!}\mathcal{M}^{\alpha\beta\gamma}_{\mu\nu\rho}Q_{\alpha\beta\gamma}, \tag{3.13}$$

which obeys anti-self-dual, is

$$\mathcal{M}(Q) = - *' \mathcal{M}(Q). \tag{3.14}$$

Its symmetric property is

$$Q_1 \wedge \mathcal{M}(Q_2) = Q_2 \wedge \mathcal{M}(Q_1). \tag{3.15}$$

Taking the variation for the action in eq.(3.12) where the source term is $J = 0$ and using the properties of $*'Q = Q$ and symmemtric of $\mathcal{M}$, we have

$$\delta S = \int \Bigg(\delta P \wedge (d *' dP + 2dQ) + \delta Q \wedge (*'dP - dP + 2\mathcal{M}(Q))\Bigg). \tag{3.16}$$

Thus the following result for the equations of motion are expressed as

$$\delta P : \qquad d *' dP + 2dQ = 0 \tag{3.17}$$

$$\delta Q : \qquad dP - *'dP - 2\mathcal{M}(Q) = 0, \tag{3.18}$$

and the corresponding calculation are

$$d(\frac{1}{2} *' dP + Q) = 0 \tag{3.19}$$

$$dP - \mathcal{M}(Q) = *'(dP - \mathcal{M}(Q)). \tag{3.20}$$

By observing the equations of motions, we define $H_{(s)}$ and H as

$$H^{(s)} = \frac{1}{2}(dP + *'dP) + Q \tag{3.21}$$

$$H = Q - \mathcal{M}(Q). \tag{3.22}$$

The first equation (3.21) is defined as the unphysical part because the kinetic term of P field has wrong sign where P field is unphysical field or pseudo-field. However this term is not effect the whole theory due to it decouples from the physical sector, H. In particular, this H is given in analogy of eq.(3.11) so $H = *H$ where it is off-shell self-duality with respects to curve spacetime.
The corresponding equation of motion of H and $H^{(s)}$ are

$$dH = 0, \qquad dH^{(s)} = 0, \tag{3.23}$$

which imply that these $H^{(s)}$ and H are closed on-shell.

We now consider the action from [22]. We start by using the action is given as

$$S = \int \left(\frac{1}{4} dP \wedge *'dP - Q \wedge dP + \mathcal{L}_I(Q, g, J) \right) \tag{3.24}$$

where this is rescaled from the original [16, 17, 19]. From the variation of the action, $\mathcal{L}_I$ with respect to Q is

$$\delta_Q \mathcal{L}_I = \delta Q \wedge R(Q, g, J). \tag{3.25}$$

Recalling this R is satisfied the anti-self-duality condition as $R = - *' R$.

From Q is self-duality thus directly applying the external source J into the equation of motion of physical part H and redefinition H to H^J, we have

$$H^J \equiv Q - R + J, \tag{3.26}$$

which should be $*$-self-dual off-shell. For linear theory, $H^J = *H^J$. From the calculation of the equation of motion, we have

$$dH^J = dJ, \qquad dH^{(s)} = 0. \tag{3.27}$$

This $H^J - J$ is exact on-shell, obtained from

$$\begin{aligned} dH^J &= dJ \\ d(H^J - J) &= 0. \end{aligned} \tag{3.28}$$

According to [22], we promote this action to non-linear theory where only term of $*$-self-duality becomes

$$* H^J = \mathcal{V}(H^J, g). \tag{3.29}$$

For a complete M5-brane action which is proposed in the Green-Schwarz formalism, it is given in the term of eq.(3.24) as

$$S_{M5} = \frac{1}{2}\int \left(\frac{1}{2}dP \wedge *'dP - 2Q \wedge dP - \frac{1}{12} * U(F, g) + Q \wedge +2C_6 + F \wedge C_3\right). \tag{3.30}$$

This action describes 6-dimensional worlvolume embedded in the 11-dimensional background target superspace. For C_3 and C_6, these fields is just the induction from the background target superspace as well as the metric g. They are no dynamics themselves.

Consider by comparing with eq.(3.24), we replace J with C_3 while H^J with F. Thus we have

$$F = Q - R + C_3. \tag{3.31}$$

This eq.(3.30) is rescaled to be match with other M5-brane in previous works. Using from [33], we define

$$C_6 \to \frac{C_6}{2}, \qquad S_{M5} \to \frac{S_{M5}}{2}. \tag{3.32}$$

In particular the form of $\mathcal{V}$ is given in U_{M5} which is simply shown in the equation as U. This function eq.(3.29) will be described in an index notation as the Greek indices μ, ν, ρ.

For the M5-brane action, the non-linear $*$-self-duality condtion in eq.(3.29) is

$$(*F)_{\mu\nu\rho} = \big(-\frac{U}{12} + \frac{24}{U}\big) F_{\mu\nu\rho} + \frac{6}{U}(F^3)_{\mu\nu\rho}, \tag{3.33}$$

where

$$(F^3)_{\mu\nu\rho} \equiv F_{[\mu|\nu'\rho'} F^{\mu'\nu'\rho'} F_{\mu'|\nu\rho]}. \tag{3.34}$$

For U in eq.(3.30), it is given as

$$U = -24\sqrt{1 + \frac{F_{\mu\nu\rho}F^{\mu\nu\rho}}{24}}. \tag{3.35}$$

Indices in these terms are raised and lowered by $g^{\mu\nu}$ and $g_{\mu\nu}$, respectively.

As the requirement of symmetries such as diffeomorphism, gauge symmetries, and kappa-symmetry, the relation of transformation rules for P and Q is

$$\delta Q = -\frac{1}{2}(1 + *')d\delta P, \tag{3.36}$$

where this equation is the same for all these symmetries. From eq.(3.36), this $H^{(s)}$ is invariant under all symmetry transformations. For diffeomorphism, the transformations on P and Q by $x^\mu \mapsto x^\mu + \xi^\mu$ are

$$\begin{aligned} \delta_\xi P &= i_\xi (F - C_3), \\ \delta_\xi Q &= -\frac{1}{2}(1 + *')d i_\xi (F - C_3). \end{aligned} \tag{3.37}$$

We note that these transformations are non-standard for 2-form and 3-form fields, respectively. Hence these P and Q are called pseudo-forms.

We have been presenting and considering the basics of Sen M5-brane action. Furthermore, this action will be calculated for double dimensional reduction in chapter 4. For the next section, we will evaluate in the deep detail for $\mathcal{M}$.

3.6 The form of $\mathcal{M}$ mapping

From the mapping of Sen's concept, the notation is $\mathcal{M}$ (it is M in figure). Recall the mapping $\mathcal{M}$, this is self-duality to anti-self-duality. We consider the $\mathcal{M}$ form with different mapping in [31]. From [19] and [22] concept of these $\mathcal{M}$ constructions are illustrated as

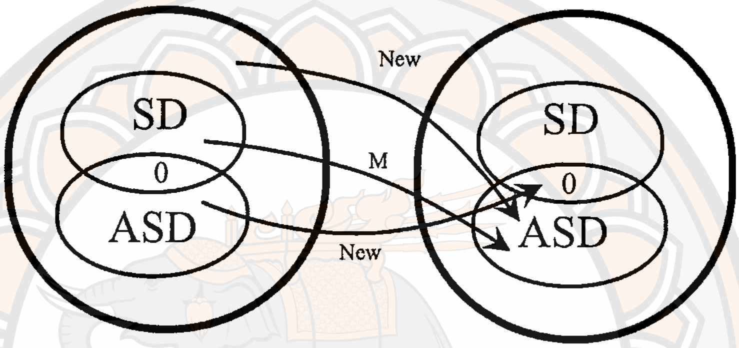

Figure 3 Asymmetric perspective mapping

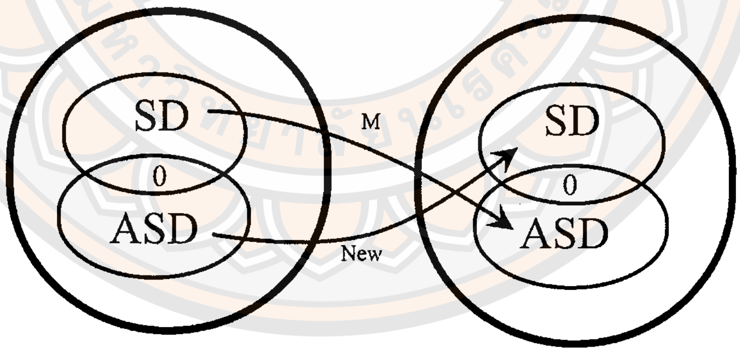

Figure 4 Symmetric perspective mapping

where we define "$*'$-SD" as self-dual, "$*'$-ASD" as anti-self-dual, and $\mathcal{M}$ as an explicit mapping. From [19], it is extended the domain of mapping not only self-duality to anti-self-duality but from anti-self-duality to be zero and universe set is the same as $\mathcal{M}$. While mapping in [22], it is extended for anti-self-duality to self-duality. We will follow the mapping from [22] but we show the idea and calculation from [19] in the next subsection.

3.6.1 Asymmetric mapping: Lambert's perspective

We will consider and give the detail about the idea of mapping $\mathcal{M}$ [19] in Fig.3.6. The definitions of self-duality condition for H with respect to g metric are given as

$$* \left(Q - \mathcal{M}(Q)\right) = Q - \mathcal{M}(Q) \tag{3.38}$$

where $*$ respect to g curve spacetime matrix, and $Q = *'Q$. When defining, $H := Q - \mathcal{M}(Q)$, we have

$$H = *H, \tag{3.39}$$

$$dH = 0. \tag{3.40}$$

Where eq.(3.39) are constructed by imposing, self-duality off-shell and normally given by hand, while eq.(3.40) is obtained by the equation of motion. Let $\mathcal{M}$ is a linear mapping of self-dual 3-forms to anti-self-dual 3-forms by $*'$ while vanishing on anti-self-dual 3-forms. It is given as

$$\mathcal{M}(\text{ASD}) = 0. \tag{3.41}$$

Then extending his action to an arbitrary 3-form, it need a symmetry property as

$$Q_1 \wedge \mathcal{M}(Q_2) = Q_2 \wedge \mathcal{M}(Q_1), \tag{3.42}$$

which holds for any 3-form.

Considering, $\mathcal{M}$ is rewritten as

$$\mathcal{M} \to \frac{1}{2}(1 - *')\mathcal{M}, \tag{3.43}$$

where $\mathcal{M} = - *' \mathcal{M}$. Hence,

$$\mathcal{M} \mapsto \frac{1}{4}(1 - *')\mathcal{M}(1 + *'), \tag{3.44}$$

$$Q = \frac{1}{2}(1 + *')Q. \tag{3.45}$$

According to eq.(3.38), we calculate

$$
\begin{aligned}
*H &= H \\
*(Q-\mathcal{M}(Q)) &= Q-\mathcal{M}(Q) \\
*Q - *\mathcal{M}(Q) &= Q-\mathcal{M}(Q) \\
\mathcal{M}(Q) - *\mathcal{M}(Q) &= Q - *Q \\
(1-*)\mathcal{M}(Q) &= (1-*)Q
\end{aligned}
\tag{3.46}
$$

then using eq.(3.44) and (3.45), we have

$$
(1-*)\frac{1}{4}(1-*')\mathcal{M}(1+*')Q = \frac{1}{2}(1-*)(1+*')Q \tag{3.47}
$$

This $\mathcal{M}$ acts as the linear operator on 3-form. It is expressed as

$$
\frac{1}{4}(1-*)(1-*')\mathcal{M}(1+*') = \frac{1}{2}(1-*)(1+*'). \tag{3.48}
$$

To solve this symmetry, defining a basis of 3-forms as ω_+^A, ω_{-A} with $A = 1,\ldots,10$. $\pm$ indicates their eigenvalues under $*'$. They are

$$
\omega_+^A = *'\omega_+^A, \qquad \omega_{-A} = -*'\omega_{-A},
$$

where a upper index A is for self-duality basis while a lower index is for anti-self-duality basis. The matrix $\mathcal{M}^{AB}$ act on this basis as

$$
\mathcal{M}(\omega_{-A}) = 0, \tag{3.49}
$$

$$
\mathcal{M}(\omega_+^A) = \mathcal{M}^{AB}\omega_{-B}. \tag{3.50}
$$

If we choose a basis where

$$
\omega_+^A \wedge \omega_{-B} = 2\delta_B^A dx^{0\ldots5}, \tag{3.51}
$$

then we consider eq.(3.42)

$$
\begin{aligned}
H_A\omega_+^A \wedge \mathcal{M}(H_B\omega_+^B) &= H_B\omega_+^B \wedge \mathcal{M}(H_A\omega_+^A) \\
H_AH_B\omega_+^A \wedge \mathcal{M}(\omega_+^B) &= H_BH_A\omega_+^B \wedge \mathcal{M}(\omega_+^A) \\
\mathcal{M}^{AB}2\delta_B^A dx^{0\ldots5} &= \mathcal{M}^{BA}2\delta_B^A dx^{0\ldots5}.
\end{aligned}
\tag{3.52}
$$

Hence, it is proof a symmetry as

$$\mathcal{M}^{AB} = \mathcal{M}^{BA}. \tag{3.53}$$

When eq.(3.48) acts on ω_{-A}, it becomes 0 (trivial). While acting on ω_+^A, we have the left-hand-side is

$$\begin{aligned}\frac{1}{4}(1-*)(1-*')\mathcal{M}(1+*')\omega_+^A &= \frac{1}{2}(1-*)\mathcal{M}(1+*')\omega_+^A \\ &= (1-*)\mathcal{M}(\omega_+^A) \\ &= (1-*)\mathcal{M}^{AB}\omega_{-B},\end{aligned} \tag{3.54}$$

and the right-hand-side is

$$\frac{1}{2}(1-*)(1+*')\omega_+^A = (1-*)\omega_+^A. \tag{3.55}$$

So, it becomes

$$\begin{aligned}(1-*)\mathcal{M}^{AB}\omega_{-B} &= (1-*)\omega_+^A \\ (1-*)(\omega_+^A - \mathcal{M}^{AB}\omega_{-B}) &= 0.\end{aligned} \tag{3.56}$$

Especially, we have eq.(3.56), is SD with respect to $*$.

Constructing φ^A basis of SD 3-form with respect to $*$. Any given point is

$$\varphi^A = \mathcal{N}^A{}_B\omega_+^B + \mathcal{K}^{AB}\omega_{-B} \tag{3.57}$$

where it is extended to arbitrary 3-forms. From the condition that $\omega_+^A - \mathcal{M}^{AB}\omega_{-B}$ is SD with respect to $*$, we define $\Theta^A{}_B$ through

$$\begin{aligned}\omega_+^A - \mathcal{M}^{AB}\omega_{-B} &= \Theta^A{}_B\varphi^B \\ &= \Theta^A{}_B(\mathcal{N}^B{}_C\omega_+^C + \mathcal{K}^{BC}\omega_{-C}) \\ &= \Theta^A{}_B\mathcal{N}^B{}_C\omega_+^C + \Theta^A{}_B\mathcal{K}^{BC}\omega_{-C}\end{aligned} \tag{3.58}$$

where ω_+^A, ω_{-A} are a basis of 3-forms. So, we compare 1^{st} term as

$$\Theta^A{}_B\mathcal{N}^B{}_C = 1, \tag{3.59}$$

hence, it is

$$\Theta^{A}{}_{B} = (\mathcal{N}^{-1})^{A}{}_{B}. \tag{3.60}$$

While 2^{nd} term is

$$-\mathcal{M}^{AB} = \Theta^{A}{}_{C}\mathcal{K}^{BC}$$
$$\mathcal{M}^{AB} = -\Theta^{A}{}_{C}\mathcal{K}^{BC}$$
$$\mathcal{M}^{AB} = -(\mathcal{N}^{-1})^{A}{}_{C}\mathcal{K}^{CB} \tag{3.61}$$

where we use eq.(3.60) on the last step.

These operators are considered in the local and valid at general in space-time. $\mathcal{N}^{A}{}_{B}$ and $\mathcal{K}^{AB}$ are defined only at local, but $\mathcal{N}^{A}{}_{B}$ may not be invertible every where in the space-time. So if $\mathcal{N}$ is not invertible at any point so it exists self-duality 3-forms with respect to $*$ which anti-self-duality 3-forms with respect to $*'$. If the spacetime is *orientable*, this property is impossible.

To emphasize, "orientability" means that it is a property of some topological spaces such as real vector spaces, Euclidean spaces, surfaces, and more generally manifolds that allows a consistent definition of "clockwise" and "counterclockwise". For example, real vector spaces, Euclidean space, and sphere are orientable where an object is transported along these space, is not changed. While non-orientable space when an object is moved around, is changed where it is clockwise at start and after moving it is anticlockwise. Its example is Möbius strip.

Let us consider that eq.(3.61) is compatible with symmetry condition in eq.(3.53). First, constructing A and B for $\mathcal{N}$. At start, we set

$$(\mathcal{N}^{-1})^{A}{}_{C}\varphi^{C} = \omega_{+}^{A} - \mathcal{M}^{AC}\omega_{-C} \tag{3.62}$$
$$(\mathcal{N}^{-1})^{B}{}_{D}\varphi^{D} = \omega_{+}^{B} - \mathcal{M}^{BD}\omega_{-D}, \tag{3.63}$$

where

$$(\mathcal{N}^{-1})^{A}{}_{C}\varphi^{C} \wedge (\mathcal{N}^{-1})^{B}{}_{D}\varphi^{D} = 0. \tag{3.64}$$

Then we calculate

$$
\begin{aligned}
(\omega_+^A - \mathcal{M}^{AC}\omega_{-C}) \wedge (\omega_+^B - \mathcal{M}^{BD}\omega_{-D}) &= 0 \\
\cancelto{0}{\omega_+^A \wedge \omega_+^B} - \omega_+^A \wedge \mathcal{M}^{BD}\omega_{-D} - \mathcal{M}^{AB}\omega_{-C} \wedge \omega_+^B + \cancelto{0}{\mathcal{M}^{AC}\omega_{-C} \wedge \mathcal{M}^{BD}\omega_{-D}} &= 0
\end{aligned}
$$

$$
\begin{aligned}
-\omega_+^A \wedge \mathcal{M}^{BD}\omega_{-D} - \mathcal{M}^{AC}\omega_{-C} \wedge \omega_+^B &= 0 \\
-\mathcal{M}^{BD}\omega_+^A \wedge \omega_{-D} + \mathcal{M}^{AC}\omega_+^B \wedge \omega_{-C} &= 0. \qquad (3.65)
\end{aligned}
$$

From eq.(3.51), we have

$$
\begin{aligned}
-\mathcal{M}^{BD} 2\delta_D^A dx^{0\ldots5} + \mathcal{M}^{AC} 2\delta_C^B dx^{0\ldots5} &= 0 \\
2(-\mathcal{M}^{BA} + \mathcal{M}^{AB}) dx^{0\ldots5} &= 0 \\
2(\mathcal{M}^{AB} - \mathcal{M}^{BA}) dx^{0\ldots5} &= 0. \qquad (3.66)
\end{aligned}
$$

So, we have the satisfied condition as

$$
\begin{aligned}
\mathcal{M}^{AB} - \mathcal{M}^{BA} &= 0 \\
\mathcal{M}^{AB} &= \mathcal{M}^{BA}. \qquad (3.67)
\end{aligned}
$$

Further point to consider, it is about $H_{(g)}$ property and the mapping of $\mathcal{M}$ which acts on it. $H_{(g)} = H - \mathcal{M}(H)$ is SD with respect to $*$. We find that

$$
H = \frac{1}{2}(Q + *Q) - \frac{1}{2}(1 + *)\mathcal{M}(Q). \qquad (3.68)
$$

If $Q = Q_A \omega_+^A$ with eq.(3.62) and (3.63) $(\mathcal{N}^{-1})^A{}_B$, basis with respects to $*$,

$$
H = Q_A (\mathcal{N}^{-1})^A{}_B \varphi^B. \qquad (3.69)
$$

Imposing more compact notations for any 3-forms ω, $\mathcal{M}(\mathcal{M}(\omega)) = 0$, we define the mapping as

$$
\begin{aligned}
\mathfrak{m} :&\quad \omega \mapsto \omega - \mathcal{M}(\omega) \\
\mathfrak{m}^{-1} :&\quad \omega \mapsto \omega + \mathcal{M}(\omega)
\end{aligned}
$$

where ω is with respect to $*'$ while $\omega \pm \mathcal{M}(\omega)$ is with respect to $*$. We have

$$\mathfrak{m} : *' \text{ SD} \mapsto * \text{ SD} \tag{3.70}$$

which maps from $*'$-self-duality 3-form to $*$-self-duality 3-form. Mapping acts identity on $*'$ ASD which is not for all 3-forms such that $\mathcal{M}(\mathcal{M}(\omega)) = 0$.

If H is $*$ SD then $\mathfrak{m}$ map is

$$H = \mathfrak{m}\Big(\frac{1}{2}(1 + *')H\Big). \tag{3.71}$$

Due to $\mathcal{M}$ is $*'$ ASD ($\mathcal{M}(Q) = - *' \mathcal{M}(Q)$), H is $*$ SD so there is always on a $*'$ SD Q such that $H = \mathfrak{m}(Q)$ which is given as

$$\begin{aligned}
\underbrace{\frac{1}{2}(1 + *')}_{\text{SD}} H &= \frac{1}{2}(1 + *')(Q - \mathcal{M})(Q) \\
&= \frac{1}{2}(Q + *'Q) - \frac{1}{2}(\mathcal{M} + *'\mathcal{M}) \\
&= \frac{1}{2}(Q + Q) - \frac{1}{2}(\mathcal{M}(Q) - \mathcal{M}(Q)) \\
\frac{1}{2}(1 + *')H &= Q.
\end{aligned} \tag{3.72}$$

So, eq.(3.71) is

$$H = \mathfrak{m}(Q) \tag{3.73}$$

which is

$$H = \mathfrak{m}(Q) = Q - \mathcal{M}(Q). \tag{3.74}$$

Mapping of [19], changing the basis property from self-duality to anti-self-duality eq.(3.50) while for anti-self-duality is forced to be zero eq.(3.49).

3.6.2 Symmetric mapping: PV's perspective

According to [22] and from Fig.3.6, we have

$$H = Q - R, \quad R = \mathcal{M}Q \tag{3.75}$$

which come with its property as

$$H = *H, \quad Q = *'Q, \quad R = -*'R. \tag{3.76}$$

This $\mathcal{M}$ has a symmetry property as $\omega_1 \wedge \mathcal{M}\omega_2 = \omega_2 \wedge \mathcal{M}\omega_1$ where ω_1, ω_2 are $*'$-SD-3-forms. According to [16], this $\mathcal{M}$ term is proposed an operator, while from [19] it is considered as the basis. From [22], the mappings is extended with

$$\mathcal{M} : *' \text{ ASD} \mapsto *' \text{ SD} \tag{3.77}$$

where the conditions have to satisfy eq.(3.76) as

$$*'\mathcal{M}\omega_+ = -\mathcal{M}\omega_+, \tag{3.78}$$

$$(1-*)(1-\mathcal{M})\omega_+ = 0. \tag{3.79}$$

Let us consider

$$\begin{aligned}(1-*)(1-\mathcal{M})\omega_+ &= (1-*)(\omega_+ - \mathcal{M}\omega_+)\\ &= (\omega_+ - \mathcal{M}\omega_+) - *(\omega_+ - \mathcal{M}\omega_+)\\ &= (\omega_+ - \mathcal{M}\omega_+) - (\omega_+ - \mathcal{M}\omega_+)\\ &= 0.\end{aligned} \tag{3.80}$$

Extending domain of $\mathcal{M}$ to any 3-forms (ω), we have from eq.(3.75) and property of self-duality conditions eq.(3.76) as

$$*'\mathcal{M}*' = -\mathcal{M} \tag{3.81}$$

$$**' - *\mathcal{M}*' = 1 - \mathcal{M}. \tag{3.82}$$

Next, to check eq.(3.81) and (3.82) satisfied eq.(3.78) and (3.79), and also symmetries. From eq.(3.81) we have

$$
\begin{aligned}
'\mathcal{M}' &= -\mathcal{M} \\
''\mathcal{M}*' &= -*'\mathcal{M} \\
+\mathcal{M}*' &= -*'\mathcal{M}.
\end{aligned} \tag{3.83}
$$

Hence the solution to eq.(3.81) and (3.82) is

$$
\begin{aligned}
**' - *\mathcal{M}*' &= 1 - \mathcal{M} \\
-*\mathcal{M}*' + \mathcal{M} &= 1 - **' \\
\mathcal{M} - *\mathcal{M}*' &= 1 - **' \\
(1 - *(-1)*')\mathcal{M} &= 1 - **' \\
(1 + **')\mathcal{M} &= 1 - **'
\end{aligned} \tag{3.84}
$$

Then eq.(3.84) becomes

$$
\begin{aligned}
(1+**')^{-1}(1+**')\mathcal{M} &= (1+**')^{-1}(1-**') \\
\mathcal{M} &= \underbrace{(1+**')^{-1}}_{\text{inverse operator}}(1-**').
\end{aligned} \tag{3.85}
$$

This $**'$ is the 3-forms mapping to 3-forms where this operator is represented by square matrix.

When this operator is considered in the view of matrix so we use the Cayley-Hamilton formula to express $(1+**')^{-1}$. This $**'$ series is up to 19^{th} order.

3.7 Applying $\mathcal{M}$ to the dimensional reduction

We will consider the reduced form for $\mathcal{M}$ in this method. At first, we have $\mathcal{M} = (1 + **')^{-1}(1 - **')$. Let us consider only $**'$ acting on the basis of any 3-forms.

The curve and flat metric are given respectively as

$$g = \begin{pmatrix} \eta_5 & 0 \\ 0 & r^2 \end{pmatrix}, \qquad \eta_5 = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & & 1 \end{pmatrix} \tag{3.86}$$

In the case of a dimensional reduction from $6 \to 5$, we fix 6^{th} dimension (dx^5). Let us calculate $* *' \, dx^{abc}$ where $a, b = \{0, 1, \dots, 4\}$.

For $* *' \, dx^{ab5}$,

$$\begin{aligned} *'dx^{a_1a_25} &= \frac{(-1)^{3+1}}{(6-3)!} dx^{a_3a_4a_5} \eta_{a_3b_3}\eta_{a_4b_4}\eta_{a_5b_5}\epsilon^{b_3b_4b_5a_1a_25} \\ &= \frac{1}{3!} dx^{a_3a_4a_5}\eta_{a_3b_3}\eta_{a_4b_4}\eta_{a_5b_5}\epsilon^{b_3b_4b_5a_1a_2}, \end{aligned}$$

where $\epsilon^{b_3b_4b_5a_1a_25} = \epsilon^{b_3b_4b_5a_1a_2}$.

$$\begin{aligned} *'dx^{a_1a_25} &= \frac{1}{3!} dx^{a_3a_4a_5} \overset{-1}{\cancel{(-\det\eta)}} \epsilon_{a_3a_4a_5b_1b_2}\eta^{b_1a_1}\eta^{b_2a_2} \\ &= \frac{1}{3!} dx^{a_3a_4a_5}\epsilon_{a_3a_4a_5b_1b_2}\eta^{b_1a_1}\eta^{b_2a_2}. \end{aligned}$$

Then

$$\begin{aligned} * *' \, dx^{a_1a_25} &= \frac{1}{3!}(*dx^{a_3a_4a_5})\epsilon_{a_3a_4a_5b_1b_2}\eta^{b_1a_1}\eta^{b_2a_2} \\ &= \frac{1}{3!}\Big[\frac{(-1)^{3+1}}{(6-3)!\sqrt{-g}} 3dx^{c_1c_25} g_{c_1d_1}g_{c_2d_2}g_{55}\epsilon^{d_1d_25a_3a_4a_5}\Big]\epsilon_{a_3a_4a_5b_1b_2}\eta^{b_1a_1}\eta^{b_2a_2} \\ &= \frac{3r^2}{(3!)^2 r} dx^{c_1c_25} g_{c_1d_1}g_{c_2d_2}(-1)^3\epsilon^{d_1d_2a_3a_4a_5}\epsilon_{a_3a_4a_5b_4b_5}\eta^{b_1a_1}\eta^{b_2a_2} \\ &= -\frac{3r}{(3!)^2} dx^{c_4c_55} g_{c_1d_1}g_{c_2d_2}(-1)^{3+3}\epsilon^{a_3a_4a_5d_1d_2}\epsilon_{a_3a_4a_5b_1b_2}\eta^{b_1a_1}\eta^{b_2a_2}, \end{aligned}$$

Consider

$$\epsilon^{a_3a_4a_5d_1d_2}\epsilon_{a_3a_4a_5b_1b_2} = \cancel{(\det\eta)}\,\overset{-1}{3!}2!\,\delta^{d_1d_2}_{b_1b_2}$$
$$= -2\cdot 3!\,\delta^{d_1d_2}_{b_1b_2},$$

$$**'\,dx^{a_1a_25} = +\frac{\cancel{3\cdot 2\cdot 3!}}{\cancel{(3!)^2}} r\,dx^{c_1c_25} g_{c_1d_1}g_{c_2d_2}\delta^{d_1d_2}_{b_1b_2}\eta^{b_1a_1}\eta^{b_2a_2}$$
$$= r g_{c_1d_1}g_{c_2d_2}\eta^{b_1a_1}\eta^{b_2a_2}dx^{c_1c_25}.$$

Then flatten down $g \to \eta$,

$$**'\,dx^{a_1a_25} = r\eta_{c_1b_1}\eta_{c_2b_2}\eta^{b_1a_1}\eta^{b_2a_2}dx^{c_1c_25} = r\delta^{a_1a_2}_{c_1c_2}dx^{c_1c_25}$$
$$= r\,dx^{a_1a_25}. \tag{3.87}$$

For $**'\,dx^{abc}$,

$$*'dx^{a_1a_2a_3} = \frac{(-1)^{3+1}}{(6-3)!}3dx^{a_4a_55}\eta_{a_4b_4}\eta_{a_5b_5}\eta_{55}\epsilon^{b_3b_4b_55a_1a_2a_3}$$
$$= \frac{3}{3!}dx^{a_4a_55}\eta_{a_4b_4}\eta_{a_5b_5}(-1)^3\epsilon^{b_4b_5a_1a_2a_35}$$
$$= -\frac{3}{3!}dx^{a_4a_55}\eta_{a_4b_4}\eta_{a_5b_5}\epsilon^{b_4b_5a_1a_2a_3}$$
$$= -\frac{3}{3!}dx^{a_3a_4a_55}(-1)^1(\det\eta)\epsilon_{a_4a_5b_1b_2b_3}\eta^{b_1a_1}\eta^{b_2a_2}\eta^{b_3a_3}$$
$$= -\frac{3}{3!}dx^{a_3a_4a_55}\epsilon_{a_4a_5b_1b_2b_3}\eta^{b_1a_1}\eta^{b_2a_2}\eta^{b_3a_3},$$

then

$$**'\,dx^{a_1a_2a_3} = -\frac{3}{3!}(*dx^{a_4a_55})\epsilon_{a_4a_5b_1b_2b_3}\eta^{b_1a_1}\eta^{b_2a_2}\eta^{b_3a_3}$$
$$= -\frac{3}{3!}\Big[\frac{(-1)^{3+1}}{(6-3)!\sqrt{-g}}dx^{c_1c_2c_3}g_{c_1d_1}g_{c_2d_2}g_{c_3d_3}\epsilon^{d_1d_2d_3a_4a_55}\Big]$$
$$\times\ \epsilon_{a_4a_5b_1b_2b_3}\eta^{b_1a_1}\eta^{b_2a_2}\eta^{b_3a_3}$$
$$= -\frac{3}{(3!)^2}\frac{1}{r}dx^{c_1c_2c_3}g_{c_1d_1}g_{c_2d_2}g_{c_3d_3}\epsilon^{d_1d_2d_3a_4a_5}\epsilon_{a_4a_5b_1b_2b_3}\eta^{b_1a_1}\eta^{b_2a_2}\eta^{b_3a_3}.$$

Consider

$$\epsilon^{d_1d_2d_3a_4a_5}\epsilon_{a_4a_5b_1b_2b_3} = (-1)^{3+3}\epsilon^{a_4a_5d_1d_2d_3}\epsilon_{a_4a_5b_1b_2b_3}$$
$$= \cancel{(\det\eta)}\,\overset{-1}{2!}3!\,\delta^{d_1d_2d_3}_{b_1b_2b_3}$$
$$= -2!3!\,\delta^{d_1d_2d_3}_{b_1b_2b_3}.$$

So, we have

$$
\begin{aligned}
**'\,dx^{a_1a_2a_3} &= +\frac{\cancel{3\cdot 2\cdot 3!}}{\cancel{(3!)^2}r}dx^{c_1c_2c_3}g_{c_1d_1}g_{c_2d_2}g_{c_3d_3}\delta^{d_1d_2d_3}_{b_1b_2b_3}\eta^{b_1a_1}\eta^{b_2a_2}\eta^{b_3a_3} \\
&= \frac{1}{r}g_{c_1d_1}g_{c_2d_2}g_{c_3d_3}\eta^{b_1a_1}\eta^{b_2a_2}\eta^{b_3a_3}dx^{c_1c_2c_3},
\end{aligned}
$$

then flatten down $g \to \eta$,

$$
\begin{aligned}
**'\,dx^{a_1a_2a_3} &= \frac{1}{r}\eta_{c_1b_1}\eta_{c_2b_2}\eta_{c_3b_3}\eta^{b_1a_1}\eta^{b_2a_2}\eta^{b_3a_3}dx^{c_1c_2c_3} \\
&= \frac{1}{r}\delta^{a_1a_2a_3}_{c_1c_2c_3}dx^{c_1c_2c_3} \\
**'\,dx^{a_1a_2a_3} &= \frac{1}{r}dx^{a_1a_2a_3}. \qquad (3.88)
\end{aligned}
$$

As a result, we have

$$**'\,dx^{a_1a_2 5} = r dx^{a_1a_2 5} \qquad (3.89)$$

$$**'\,dx^{a_1a_2a_3} = \frac{1}{r}dx^{a_1a_2a_3} \qquad (3.90)$$

To obtain the $\mathcal{M}$ in the matrix form, earlier we have

$$**'(dx^{a_1a_2 5}) = r dx^{a_1a_2 5}, \text{ and } \quad **'(dx^{a_1a_2a_3}) = \frac{1}{r}dx^{a_1a_2a_3}. \qquad (3.91)$$

In general form, it is $**'\,dx^{\mu\nu\rho} = \mathcal{G}^{\mu\nu\rho}_{\alpha\beta\gamma}dx^{\alpha\beta\gamma}$. Expanding the basis for dx^{ab5} and dx^{abc} as

$$ab5 \in \{015, 025, 035, 045, 125, 135, 145, 235, 245, 345\}$$

$$abc \in \{012, 013, 014, 023, 024, 034, 123, 124, 134, 234\}$$

which are twenty of them. For the matrix form, it is

$$
**'\begin{bmatrix} 012 \\ 013 \\ \vdots \\ 245 \\ 345 \end{bmatrix} = \begin{bmatrix} \frac{1}{r} & & & & \cdots & 0 \\ & \ddots & & & & \vdots \\ & & \frac{1}{r} & & & \\ & & & r & & \\ \vdots & & & & \ddots & \\ 0 & \cdots & & & & r \end{bmatrix}\begin{bmatrix} 012 \\ 013 \\ \vdots \\ 245 \\ 345 \end{bmatrix}. \qquad (3.92)
$$

Thus we have

$$\mathbb{1} = I \oplus I$$
$$**' = \frac{1}{r} I \oplus rI, \tag{3.93}$$

where I is 10×10 identity matrix.

Hence in order to compute $\mathcal{M} = (1 + **')^{-1}(1 - **')$, we consider

$$\begin{aligned}(1 - **')dx^{\mu\nu\rho} &= \left[\left(1 - \frac{1}{r}\right) I \oplus \left(1 - r\right) I\right] dx^{\mu\nu\rho} \\ &= \left[\left(\frac{r-1}{r}\right) I \oplus \left(-(r-1)\right) I\right] dx^{\mu\nu\rho}\end{aligned} \tag{3.94}$$

then let us express the above equation as

$$(1 - **')dx^{\mu\nu\rho} = \mathbf{B} dx^{\mu\nu\rho}. \tag{3.95}$$

We also need the expression for $(1 + **')^{-1}$. So let us first consider

$$\begin{aligned}(1 + **')dx^{\mu\nu\rho} &= \left[\left(1 + \frac{1}{r}\right) I \oplus \left(1 + r\right) I\right] dx^{\mu\nu\rho} \\ &= \left[\left(\frac{r+1}{r}\right) I \oplus \left(r + 1\right) I\right] dx^{\mu\nu\rho},\end{aligned} \tag{3.96}$$

then it becomes

$$(1 + **')dx^{\mu\nu\rho} = \mathbf{A} dx^{\mu\nu\rho}. \tag{3.97}$$

Let us consider $(1 + **')^{-1} = \mathbf{A}^{-1}$, we have

$$\mathbf{A} = \left(\frac{r+1}{r}\right) I \oplus \left(r + 1\right) I. \tag{3.98}$$

Where this matrix A is already diagonal so the corresponding result is

$$\mathbf{A}^{-1} = \left(\frac{r}{r+1}\right) I \oplus \left(\frac{1}{r+1}\right) I. \tag{3.99}$$

As a result, the form of $\mathcal{M}$ is

$$\begin{aligned}\mathcal{M} &= \mathbf{A}^{-1}\mathbf{B} \\ &= \left[\left(\frac{r}{r+1}\right) I \oplus \left(\frac{1}{r+1}\right) I\right] \times \left[\left(\frac{r-1}{r}\right) I \oplus \left(-(r-1)\right) I\right] \\ \mathcal{M} &= \left[\frac{(r-1)}{r+1}\right] I \oplus \left[-\left(\frac{r-1}{r+1}\right)\right] I\end{aligned} \tag{3.100}$$

According to a result, we know how $\mathcal{M}$ changes basis which are

$$\mathcal{M}dx^{abc} = \Big(\frac{r-1}{r+1}\Big)dx^{abc}, \tag{3.101}$$

$$\mathcal{M}dx^{ab5} = -\Big(\frac{r-1}{r+1}\Big)dx^{ab5} \tag{3.102}$$

where they are confirmed as same as [17] in the case of dimensional reduction. This result is also obtain from [17, 19]. They are not the same mapping of basis but the physics is still the same. However, the mappings are difference but the result of this $\mathcal{M}$ is the same when we consider the dimensional reduction. This eq.(3.101) is useful for the following chapter that is about the dimensional reduction.

CHAPTER IV

DIMENSIONAL REDUCTION

The dimensional reduction is an original idea from Kaluza-Klein. In this chapter, we will continue consider Sen's action which is applied the this method on specific case.

4.1 Reduction on S^1

Firstly, we set up by giving the necessary convention. The Hodge star for curve spacetime are given as

$$* \, dx^{\mu_1 \cdots \mu_p} = \frac{(-1)^{p+1}}{(6-p)!\sqrt{-g}} dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_6} g_{\mu_{p+1}\nu_{p+1}} \cdots g_{\mu_6 \nu_6} \epsilon^{\nu_{p+1} \cdots \nu_6 \mu_1 \cdots \mu_p}. \tag{4.1}$$

While in the case of a flat space-time, the Hodge star is

$$*' \, dx^{\mu_1 \cdots \mu_p} = \frac{(-1)^{p+1}}{(6-p)!\sqrt{-\eta}} dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_6} \eta_{\mu_{p+1}\nu_{p+1}} \cdots \eta_{\mu_6 \nu_6} \epsilon^{\nu_{p+1} \cdots \nu_6 \mu_1 \cdots \mu_p}. \tag{4.2}$$

where $-\epsilon_{\mu_1 \mu_2 \cdots \mu_p} = \epsilon^{\mu_1 \mu_2 \cdots \mu_p} = 1$.

For the dimensional reduction, the important first step is imposing the metric g which is proposed as

$$g = \begin{pmatrix} \eta_5 & 0 \\ 0 & r^2 \end{pmatrix}, \tag{4.3}$$

which η_5 is a flat metric in 5-dimension. While the coordinate x^5 is given in the term of r, is a positive real number and a radial of circle S^1. This x^5 is compactified as $x^5 \sim x^5 + l$ where l is parameter with the length dimension. Hence every fields are independent from x^5. We use the Roman alphabets for 5-dimensional spacetime such as $a, b, c, i, j, k, l, m, n$. We give the convention as

$$\epsilon_{ijkmn} \equiv \epsilon_{ijkmn5}, \qquad \epsilon^{ijkmn} \equiv \epsilon^{ijkmn5}. \tag{4.4}$$

These ϵ_{ijkmn} and ϵ^{ijkmn} are Levi-Civita symbols on the 5-dimensional spacetime. Because of $*'$-self-duality of Q and $*'$-anti-self-duality of R, we obtain

$$Q_{abc} = \frac{1}{2}\epsilon_{abcij}\eta^{im}\eta^{jn}q_{mn}, \qquad R_{abc} = -\frac{1}{2}\epsilon_{abcij}\eta^{im}\eta^{jn}R_{mn}, \tag{4.5}$$

where Q and R are given in the term of x^5 independent quantities

$$q_{ab} \equiv Q_{ab5}, \qquad R_{ab} \equiv R_{ab5}. \tag{4.6}$$

After all definitions are given here, the next step we will consider about the dimensional reduction of quadratic action and following by a complete M5-brane action.

4.1.1 Dimensional reduction of quadratic action

According to the previous chapter, Sen's action is provided as

$$S = \int \left(\frac{1}{4}dP \wedge *'dP - Q \wedge dP + \frac{1}{2}Q \wedge R\right) \tag{4.7}$$

where we recall the duality property for Q and R which are satisfied $*'Q = Q$ and $*'R = -R$, respectively. This R is given as the function of Q where is defined $R = \mathcal{M}Q$.

In the index notation, dP and $*'dP$ are expressed as

$$dP = -dx^{ab5}\partial_{[a}P_{b]5} - \frac{1}{2}dx^{abc}\partial_a P_{bc} \tag{4.8}$$

$$*'dP = -\frac{1}{3!}dx^{ijk}\epsilon_{ijklm}\partial^{[l}P^{m]5} + \frac{1}{4}dx^{ij5}\epsilon_{ijlmn}\partial^{[l}P^{mn]}. \tag{4.9}$$

According to eq.(3.101), we have R as

$$R_{ab5} = -\frac{(r-1)}{(r+1)}q_{ab}, \tag{4.10}$$

where q_{ab} is defined in eq.(4.6). By using eq.(4.5) and (4.10), we obtain the dimensional reduced action as following as

$$\begin{aligned} S =& \frac{l}{2}\int d^5x\Big\{\frac{3}{4}\partial_a P_{bc}\partial^{[a}P^{bc]} + \partial_a P_{b5}\partial^{[a}P^{b]}{}_5 + 2q^{ab}\partial_a P_{b5} + \frac{1}{2}q_{ab}\partial_m P_{np}\epsilon^{mnpab} \\ &+ \frac{(r-1)}{(r+1)}q^{ab}q_{ab}\Big\}. \end{aligned} \tag{4.11}$$

We define

$$X_{abc} = 3\partial_{[a}P_{bc]} \tag{4.12}$$

then replace into the action eq.(4.7) and introduce Lagrange multiplier K_m for $\partial_{[a}X_{bcd]} = 0$. So the action becomes

$$\begin{aligned} S =& \frac{l}{2}\int d^5x\Big\{\frac{1}{12}X_{abc}X^{[abc]} + \frac{1}{6}q_{ab}\epsilon^{abdef}X_{def} + \frac{1}{6}X_{abc}\partial_m K_n\epsilon^{abcmn} + \partial_a P_{b5}\partial^{[a}P^{b]}{}_5 \\ &+ 2q^{ab}\partial_a P_{b5} + \frac{(r-1)}{(r+1)}q^{ab}q_{ab}\Big\}. \end{aligned} \tag{4.13}$$

And solving for the equations of motion of X, we obtain

$$X^{abc} = -\epsilon^{mnabc}(q_{mn} + \partial_{[m}K_{n]}). \tag{4.14}$$

Then integrating out X thus the action is

$$\begin{aligned} S =& \frac{l}{2}\int d^5x\Big\{(q_{mn} + \partial_{[m}K_{n]} + \partial_{[m}P_{n]})(q^{mn} + \partial^m K^n + \partial^m P^n) \\ &- 2\partial_m K_n\partial^m P^n + \frac{(r-1)}{(r+1)}q^{ab}q_{ab}\Big\}. \end{aligned} \tag{4.15}$$

Following up step, we find the equation of motion of q which is given as

$$q^{mn} = -\frac{(r+1)}{2r}\Big(\partial^{[m}K^{n]} + \partial^{[m}P^{n]}\Big). \tag{4.16}$$

Applying into eq.(4.15), thus the action is

$$S = \frac{l}{2}\int d^5x\Big\{\frac{(r-1)}{2r}\partial_m K_n\partial^{[m}K^{n]} - \frac{(r+1)}{r}\partial_m K_n\partial^{[m}P^{n]} + \frac{(r-1)}{2r}\partial_m P_n\partial^{[m}P^{n]}\Big\}. \tag{4.17}$$

Observing the action, we can obtain a quadratic form. This is given in the matrix form as

$$\frac{1}{2r}\begin{pmatrix} r-1 & -(r+1) \\ -(r+1) & r-1 \end{pmatrix} \tag{4.18}$$

Eigenvalues for this matrix are $-1/r, 1$. Their eigenvectors are $(1,1)^T$ and $(1,-1)^T$, respectively.

We define

$$f_{mn} = l(\partial_{[m}K_{n]} + \partial_{[m}P_{n]}), \qquad f^{(s)}_{mn} = l(\partial_{[m}K_{n]} - \partial_{[m}P_{n]}), \tag{4.19}$$

where they have mass dimension two and in $f^{(s)}$ this "(s)" stand for unphysical sector. So these terms are field strength of following redefined fields which are given as

$$A_m = \frac{l}{2}(K_m + P_m), \qquad A^{(s)}_m = \frac{l}{2}(K_m - P_m). \tag{4.20}$$

Corresponding K_m and P_m, are given as

$$K_m = \frac{1}{l}(A_m + A^{(s)}_m), \qquad P_m = \frac{1}{l}(A_m - A^{(s)}_m) \tag{4.21}$$

and

$$\partial_m K_n = \frac{1}{l}(\partial_m A_n + \partial_m A^{(s)}_n), \qquad \partial_m P_n = \frac{1}{l}(\partial_m A_n - \partial_m A^{(s)}_n). \tag{4.22}$$

Then we replace into eq.(4.17) so we obtain

$$S = \int d^5x \Big\{ -\frac{1}{rl}\partial_{[m}A_{n]}\partial^m A^n + \frac{1}{l}\partial_{[m}A^{(s)}_{n]}\partial^m A^{(s)n} \Big\} \tag{4.23}$$

From eq.(4.19), we have

$$f_{mn} = 2\partial_{[m}A_{n]}, \qquad f^{(s)}_{mn} = 2\partial_{[m}A^{(s)}_{n]}. \tag{4.24}$$

As a result the action becomes

$$S = \int d^5x \Big\{ -\frac{1}{4rl} f_{mn}f^{mn} + \frac{1}{4l} f^{(s)}_{mn} f^{(s)mn} \Big\}. \tag{4.25}$$

We can see that we have the two field strengths where the wrong-sign kinetic term of $A^{(s)}_m$ of $f^{(s)}_{mn}$ is the unphysical decouples from the physical sector. This action describes the 5-dimension Maxwell theory. For the first term with A_m of f_{mn}, it is scaled by $1/r$ where $r \in \mathbb{R}^+$. This scaling is required from the conformal symmetry [34]. Moreover, this $1/r$ confirms with the PST formalism result for dimensional reduction on circle.

4.1.2 Double dimensional reduction of the full M5-brane action

In this section, we extend from a quadratic Sen's action to the Sen M5-brane action which is non-linear with source terms. Let continue same process as previous section.

After expressing R in term of Q, the non-linear term is too difficult so we simplify some steps to achieve the D4-brane in type IIA supergravity. The double dimensional reduction is that the background target space X^{10} is compactified on a circle and x^5 coordinate of worldvolume warps around that X^{10}. The length of x^5 is given as

$$\int dx^5 = 2\pi g_s l_s. \tag{4.26}$$

If the T_{M5} tension of the M5-brane action (4.7) is recovered, then a scaling

$$S_{M5} \mapsto T_{M5} S_{M5}, \qquad \text{and} \qquad T_{M5} 2\pi g_s l_s = T_{D4}. \tag{4.27}$$

After this, we set up this to be simplified by defining

$$\int dx^5 = 1, \qquad T_{M5} = 1 = T_{D4}. \tag{4.28}$$

The metric on the worldvolume for the double dimensional reduction is provided as

$$g_{\mu\nu} = \begin{pmatrix} e^{-2\phi/3}\gamma_{ab} + e^{4\phi/3} C_a C_b & e^{4\phi/3} C_a \\ e^{4\phi/3} C_b & e^{4\phi/3} \end{pmatrix}, \tag{4.29}$$

where the fields ϕ and C_a are considered as pullbacks of the background superfield to 5-dimensional worldvolume.

Let us consider the pullbacks for C_3 and C_6 of the target space superfields that is reduced. The components of C_3 are separated as

$$C_3 = \frac{1}{2} dx^a \wedge dx^b \wedge dx^5 C_{ba5} + \frac{1}{3!} dx^i \wedge dx^j \wedge dx^k C_{kij}. \tag{4.30}$$

Defining the corresponding,

$$b_{ab} \equiv C_{ab5}, \qquad \tilde{C}^{ab} \equiv \frac{1}{3!} \epsilon^{abijk} C_{ijk}. \tag{4.31}$$

We have that $\tilde{C}^{ab}$ is dual of C_{ijk}, raising and lowering by η. As well as the indices of B. Where we have B as

$$B = \frac{1}{2} dx^a \wedge dx^b b_{ba}. \tag{4.32}$$

Next for C_6, it can be expressed as

$$C_6 = -C_5 \wedge dx^5. \tag{4.33}$$

We summarize the pullbacks of the target space superfields of type IIA supergravity to 5-dimensional worldvoulme are $\gamma_{ab}, b_2, C_1, C_3, C_5$ for this dimensional reduced theory.

4.1.3 D4-brane action

The corresponding worldvolume action is given as

$$\begin{aligned} S = \frac{1}{2}\int d^5x \left(\frac{3}{4}\partial_a P_{bc}\partial^{[a}P^{bc]} + \partial_a P_{b5}\partial^{[a}P^{b]}{}_5 + 2q^{ab}\partial_a P_{b5} + \frac{1}{2}q_{ab}\partial_m P_{np}\epsilon^{mnpab}\right) \\ + \frac{1}{2}\int d^5x \left(\frac{1}{12}\sqrt{-g}U + [q(R+b)] - \frac{1}{2}[(q-R)(b-\tilde{C})]\right) - \int C_5, \end{aligned} \tag{4.34}$$

where we define the trace $(\cdot)^a{}_b$ as

$$[MN] = M^a{}_b N^b{}_a. \tag{4.35}$$

But we have to be careful for using this because the 5-dimensional of worldvolume is not flat. It is not like the previous section. Normally the given 5-dimensional matrix M will be specific for its components to be raised or lowered by η or γ. However the components of $q, R, B, \tilde{C}$ that we are discussing here are raised and lowered by η.

Next we follow the same process as the previous section but keep the term of R to be calculated later. Then we simply calculate by dualising P_{ab} by using the definition from eq.(4.12) and input into the Lagrange multiplier. Thus the action becomes

$$S = \frac{1}{2}\int d^5x\Big(\frac{1}{12}X_{abc}X^{abc} + \partial_a P_{b5}\partial^{[a}P^{b]}{}_5 + 2q^{ab}\partial_a P_{b5} + \frac{1}{6}q_{ab}X_{mnp}\epsilon^{mnpab} + \frac{1}{3!}X_{abc}\partial_m K_n\epsilon^{abcmn}\Big) + \frac{1}{2}\int d^5x\left(\frac{1}{12}\sqrt{-g}U + [q(R+b)] - \frac{1}{2}[(q-R)(b-\tilde{C})]\right) - \int C_5. \tag{4.36}$$

Integrating out X, we have

$$S = \frac{1}{2}\int d^5x\left(-\left[(\partial K + \partial P + q)^2\right] + 2\left[\partial K \partial P\right]\right) \tag{4.37}$$

$$+ \frac{1}{2}\int d^5x\left(\frac{1}{12}\sqrt{-g}U + [q(R+b)] - \frac{1}{2}[(q-R)(b-\tilde{C})]\right) - \int C_5, \tag{4.38}$$

where ∂K and ∂P are matrices with components respectively as

$$(\partial K)_{mn} \equiv \partial_{[m}K_{n]}, \qquad (\partial P)_{mn} \equiv \partial_{[m}P_{n]5}, \tag{4.39}$$

The components of ∂K and ∂P are raised or lowered by η.

Let us define

$$\psi \equiv \partial K + \partial P, \qquad \psi^{(s)} \equiv \partial K - \partial P, \tag{4.40}$$

whose components are also raised or lowered by η.

We give the redefinitions for the fields as

$$K_m + P_m \equiv -\frac{A_m}{2}, \qquad K_m - P_m \equiv \frac{A_m^{(s)}}{2}. \tag{4.41}$$

So we have that $-\psi$ and $\psi^{(s)}$ are field strengths of A and $A^{(s)}$, respectively.

Thus the action becomes

$$S = \frac{1}{2}\int d^5x\left(-\left[(\psi+q)^2\right] + \frac{1}{2}[\psi^2] - \frac{1}{2}[(\psi^{(s)})^2]\right) + \frac{1}{2}\int d^5x\left(\frac{1}{12}\sqrt{-g}U + [q(R+b)] - \frac{1}{2}[(q-R)(b-\tilde{C})]\right) - \int C_5. \tag{4.42}$$

Solving for Euler-Lagrange equation of q gives

$$R = \psi + q. \tag{4.43}$$

The term of R is eliminated by using eq.(4.43) so the action becomes

$$S = \int d^5x \left(-\frac{1}{4}[(\psi^{(s)})^2] - \frac{1}{4}[\psi^2] + \frac{1}{2}[q(-\psi + b)] + \frac{1}{24}\sqrt{-g}U + \frac{1}{4}[\psi b] - \frac{1}{4}[\psi \tilde{C}] \right) - \int C_5. \tag{4.44}$$

After that we simplify U in eq.(3.35), let us define matrices F and $\tilde{F}$ whose components are

$$F^a{}_b \equiv \frac{1}{r} F^a{}_{b5}, \qquad \tilde{F}^a{}_b \equiv \frac{1}{r} \tilde{F}^a{}_{b5}. \tag{4.45}$$

In this case the indices of F_{ab} and $\tilde{F}_{ab}$ are raised and lowered by γ. So we have

$$F_{\mu\nu\rho} F^{\mu\nu\rho} = -3[F^2] + 3[\tilde{F}^2]. \tag{4.46}$$

For $F_{\mu\nu\rho}$, this term has twenty components which are related to each other by the non-linear self-duality condition. Where only ten independent components are required. Thus in our case, $\tilde{F}_{ab5}$ will be eliminated.

The relevant non-linear self-duality condition is given as

$$\tilde{F}_{ab} = \frac{(1-y_1)F_{ab} + (F^3)_{ab}}{\sqrt{1 - y_1 + y_1^2 - y_2}}, \tag{4.47}$$

where

$$y_1 \equiv \frac{1}{2}[F^2], \qquad y_2 \equiv \frac{1}{4}[F^4]. \tag{4.48}$$

Therefore, after substituting this result into eq.(4.46) then calculating U, we obtain

$$U = -12 \frac{2 - y_1}{\sqrt{1 - y_1 + \frac{1}{2}y_1^2 - y_2}}. \tag{4.49}$$

Integrating out q. By considering the ijk component of $H = Q - R$, with eq.(4.43), we have

$$q^{ab} = \frac{1}{2}\left(\sqrt{-g}\tilde{F}^{ab} + \tilde{C}^{ab} - \frac{1}{2}\epsilon^{abijk} F_{ij} C_k + \eta^{ac}\eta^{bd} F_{cd} - b^{ab} \right). \tag{4.50}$$

This Q is completely determined in terms of other fields because of eq.(4.47) and

$$F_{ab} = -\psi_{ab} + b_{ab}, \tag{4.51}$$

where ψ_{ab} is a field strength of a 1-form field, from eq.(4.40)-(4.41). By substituting eq.(4.50) into eq.(4.44), we obtain the action as

$$S = \int d^5x \left(\frac{1}{4}\psi^{(s)}_{ab}\psi^{(s)}_{ij}\eta^{ai}\eta^{bj} - e^{-\phi}\sqrt{-\det(\gamma_{ab}+F_{ab})} \right) - \int (C_1 + C_3 + C_5') \wedge e^F, \tag{4.52}$$

where

$$F_{ab} = \partial_a A_b - \partial_b A_a + b_{ab}, \tag{4.53}$$

$$C_5' = C_5 - \frac{1}{2}B \wedge C_3, \tag{4.54}$$

$$\psi^{(s)}_{ab} = \partial_a A^{(s)}_b - \partial_b A^{(s)}_a. \tag{4.55}$$

The action (4.52) describes the complete D4-brane decoupled with ghost field [24].

4.1.4 Dual D4-brane action

According to the previous subsection, we slightly change the procedure to obtain the dual D4-brane action from Sen M5-brane action. To modify this process, we simply switch the role of P components. Instead of dualise P_{a5} components, we apply to P_{ab} components. As the same process, the Lagrange multiplier is also introduced to the action. These terms will be combined to form the gauge and ghost field.

Let us consider by following the steps as those of the previous. So we skip to eq.(4.34). Then giving the definition as

$$Y_{ab} = 2\partial_{[a}P_{b]5} \tag{4.56}$$

and introducing Lagrange multiplier term, the action becomes

$$\begin{aligned}\tilde{S} = \frac{1}{2}\int d^5x \Big(&\frac{3}{4}\partial_a P_{bc}\partial^{[a}P^{bc]} + \frac{1}{4}Y_{ab}Y^{ab} + q^{ab}Y_{ab} \\ &+ \frac{1}{2}q_{ab}\partial_m P_{np}\epsilon^{mnpab} + \frac{1}{4}Y_{ab}\partial_m K_{np}\epsilon^{abmnp}\Big) \\ + \frac{1}{2}\int d^5x &\left(\frac{1}{12}\sqrt{-g}U + [q(R+\tilde{C})] + \frac{1}{2}[(q+R)(b-\tilde{C})] \right) - \int C_5. \end{aligned} \tag{4.57}$$

Integrating out Y, we obtain

$$\tilde{S} = \frac{1}{2}\int d^5x\left(\left[(\widetilde{\partial K} - \widetilde{\partial P} + q)^2\right] + 2\left[\widetilde{\partial K}\widetilde{\partial P}\right]\right)$$
$$+ \frac{1}{2}\int d^5x\left(\frac{1}{12}\sqrt{-g}U + [q(R+\tilde{C})] + \frac{1}{2}[(q+R)(b-\tilde{C})]\right) - \int C_5, \quad (4.58)$$

where we define $\widetilde{\partial K}$ and $\widetilde{\partial P}$ as

$$\widetilde{\partial K}^{mn} \equiv \frac{1}{4}\epsilon^{mnijk}\partial_i K_{jk}, \qquad \widetilde{\partial P}^{mn} \equiv \frac{1}{4}\epsilon^{mnijk}\partial_i P_{jk}. \quad (4.59)$$

Indices for matrices $\widetilde{\partial K}$ and $\widetilde{\partial P}$ are raised and lowered by the 5-dimensional flat metric η. Let us define

$$\chi \equiv \widetilde{\partial P} - \widetilde{\partial K}, \qquad \chi^{(s)} \equiv \widetilde{\partial P} + \widetilde{\partial K}. \quad (4.60)$$

Then we impose the field redefinitions which

$$P_{ij} - K_{ij} \equiv 2B_{ij}, \qquad P_{ij} + K_{ij} \equiv 2B^{(s)}_{ij}. \quad (4.61)$$

So χ and $\chi^{(s)}$ are dual of field strengths of B and $B^{(s)}$, respectively.

Thus the action becomes

$$\tilde{S} = \frac{1}{2}\int d^5x\left([(-\chi+q)^2] - \frac{1}{2}[\chi^2] + \frac{1}{2}[(\chi^{(s)})^2]\right)$$
$$+ \frac{1}{2}\int d^5x\left(\frac{1}{12}\sqrt{-g}U + [q(R+\tilde{C})] + \frac{1}{2}[(q+R)(b-\tilde{C})]\right) - \int C_5, \quad (4.62)$$

A result from variation with respect to q gives

$$R = \chi - q. \quad (4.63)$$

Eliminating R from the action, we have

$$\tilde{S} = \int d^5x\left(\frac{1}{4}[(\chi^{(s)})^2] + \frac{1}{4}[\chi^2] + \frac{1}{2}[q(-\chi+\tilde{C})] + \frac{1}{24}\sqrt{-g}U + \frac{1}{4}[\chi b] - \frac{1}{4}[\chi\tilde{C}]\right) - \int C_5. \quad (4.64)$$

The next step, U is reexpressed by giving firstly the definition as

$$\hat{F}^a{}_b \equiv \frac{F^a{}_b{}^5}{\sqrt{g^{55}}}, \qquad \hat{\tilde{F}}^a{}_b \equiv \frac{\tilde{F}^a{}_b{}^5}{\sqrt{g^{55}}}. \quad (4.65)$$

Let the indices of $\hat{F}^{ab}$ and $\hat{\tilde{F}}^{ab}$ be lowered by γ_{ab} while the indices of $F^{\mu\nu\rho}$ and $\tilde{F}^{\mu\nu\rho}$ are lowered by $g_{\mu\nu}$. So we obtain the functional term of U as

$$F_{\mu\nu\rho}F^{\mu\nu\rho} = -3[\hat{F}^2] + 3[\hat{\tilde{F}}^2]. \tag{4.66}$$

By using nonlinear self-duality condition to eliminate $\hat{F}_{ab}$ which is

$$\hat{F}^a{}_b = \frac{(1+z_1)\hat{\tilde{F}}^a{}_b - (\hat{\tilde{F}}^3)^a{}_b}{\sqrt{1+z_1+\frac{1}{2}z_1^2 - z_2}}, \tag{4.67}$$

where

$$z_1 \equiv \frac{1}{2}[\hat{\tilde{F}}^2], \qquad z_2 \equiv \frac{1}{4}[\hat{\tilde{F}}^4]. \tag{4.68}$$

As a result of calculation, we have

$$U = -\frac{12(2+z_1)}{\sqrt{1+z_1+\frac{1}{2}z_1^2 - z_2}}. \tag{4.69}$$

Eliminating q from the action (4.62) by considering the $ab5$ component of $H = Q - R$, we obtain

$$q_{ab} - R_{ab} = \frac{1}{\sqrt{g^{55}}}\left(g_{ac}\hat{F}^c{}_b - \frac{1}{2}\sqrt{-g}\epsilon_{abcmn}g^{c5}\hat{\tilde{F}}^m{}_p g^{pn}\right) - b_{ab} \tag{4.70}$$

So along with eq.(4.63), we obtain

$$\begin{aligned} q_{ab} = \frac{1}{2}\Big(&\frac{1}{\sqrt{g^{55}}}\left(g_{ac}\hat{F}^c{}_b - \frac{1}{2}\sqrt{-g}\epsilon_{abcmn}g^{c5}\hat{\tilde{F}}^m{}_p g^{pn}\right) \\ &- b_{ab} + \sqrt{-g}\sqrt{g^{55}}\hat{\tilde{F}}^m{}_p\tilde{g}^{pn}\eta_{ma}\eta_{nb} + \tilde{C}_{ab}\Big), \end{aligned} \tag{4.71}$$

where this Q is completely determined in terms of other fields due to eq.(4.47). We have

$$\tilde{g}^{ab} \equiv g^{ab} - \frac{g^{a5}g^{b5}}{g^{55}}, \tag{4.72}$$

and

$$\chi^{ab} = \sqrt{-g}\sqrt{g^{55}}\hat{\tilde{F}}^a{}_c\tilde{g}^{cb} + \tilde{C}^{ab}. \tag{4.73}$$

Let us define

$$\begin{aligned} W^{ab} &\equiv \sqrt{-g}\sqrt{g^{55}}\hat{\tilde{F}}^a{}_d\tilde{g}^{db} \\ &= -\frac{1}{3!}\epsilon^{abijk}F_{ijk} \end{aligned} \tag{4.74}$$

as the dualisation of F_{ijk} with respect to five-dimensional flat metric η. While the components of W are raised and lowered by η.

So the action becomes

$$\begin{aligned}\tilde{S} = &\int d^5x \bigg(\frac{3}{4}\partial_{[a}b^{(s)}_{bc]}\partial_{[i}b^{(s)}_{jk]}\eta^{ai}\eta^{bj}\eta^{ck} - e^{-\phi}\sqrt{-\gamma}\sqrt{1+z_1+\frac{1}{2}z_1^2 - z_2} \\ &+ \frac{1}{2}[Wb] + \frac{1}{4}[\tilde{C}b] - \frac{1}{8}\frac{1}{g^{55}}\epsilon_{abcmn}g^{c5}W^{mn}W^{ab}\bigg) - \int C_5 \\ = &\int d^5x \bigg(\frac{3}{4}\partial_{[a}b^{(s)}_{bc]}\partial_{[i}b^{(s)}_{jk]}\eta^{ai}\eta^{bj}\eta^{ck} - e^{-\phi}\sqrt{-\det G} \\ &- \frac{1}{24}\frac{1}{e^{-4\phi/3} + e^{2\phi/3}C_i\gamma^{ij}C_j}\gamma^{cd}C_d\epsilon^{abmnp}F_{mnp}F_{abc}\bigg) \\ &+ \int (b\wedge(F - C_3) - C_5'),\end{aligned} \tag{4.75}$$

where

$$G_{ab} \equiv \gamma_{ab} - \frac{i}{6}e^{\phi}\frac{\gamma_{ac}\epsilon^{cdijk}F_{ijk}(\gamma_{db} + e^{2\phi}C_dC_b)}{\sqrt{-\gamma}\sqrt{1+e^{2\phi}C_a\gamma^{ab}C_b}} \tag{4.76}$$

Due to eq.(4.60), (4.73)-(4.74), we have

$$F_{ijk} = \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij} + C_{ijk}. \tag{4.77}$$

The action (4.75) describes the complete dual D4-brane action [24] decoupled with ghost field.

4.1.5 Discussion

From previous two subsections, when the complete M5-brane action is applied the double dimensional reduction on circle. As a result we obtain the complete D4-brane action or the complete dual D4-brane action which depend on the chosen components of P to be integrated out. It has shown that we only simply swapping choose the components of pseudo-form P. This swapping gives the relation of duality transformation.

CHAPTER V

DECOUPLING BETWEEN PHYSICAL AND UNPHYSICAL SECTORS

In this chapter, we apply the Hamiltonian analysis to evaluate the constraint of Sen formalism. The result of this method also provides previously in [17, 19]. Thens, we consider in the Lagrangian analysis to obtain the first-order form Lagrangian. Moreover, we use the field redefinition then at the level of Lagrangian, the unphysical and physical sectors couple from each other.

5.1 Quadratic Sen action

Firstly, we define the collective index as $A = (a_1 a_2)$ and $I = (i_1 i_2 i_3)$. We also define

$$g^{ab}g^{cd} \equiv [g^2]^{(ac|bd)} = [g^2]^{BA}, \tag{5.1}$$

$$g^{ij}g^{kl}g^{mn} \equiv [g^3]^{(ikm|jln)} = [g^3]^{IJ}, \tag{5.2}$$

as well as η. Moreover the Levi-Civita symbol $\epsilon^{\mu_1\mu_2\cdots\mu_{2p+1}}$ such that $\epsilon^{01\cdots(4p+1)} = 1$ and we define

$$\epsilon_{0ijklm} \equiv -\epsilon_{ijklm}, \qquad \epsilon^{0ijklm} \equiv \epsilon^{ijklm}. \tag{5.3}$$

It is also important to define the Levi-Civita tensor

$$\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_{4p+2}} = \sqrt{-g}\epsilon_{\mu_1\mu_2\cdots\mu_{4p+2}}, \tag{5.4}$$

$$\bar{\epsilon}_{\mu_1\mu_2\cdots\mu_{4p+2}} = \sqrt{-\eta}\epsilon_{\mu_1\mu_2\cdots\mu_{4p+2}}. \tag{5.5}$$

Then we consider the $*'$ Hodge duality as

$$
\begin{aligned}
*'(dx^0 \wedge dx^{ab}) &= \frac{(-1)^{(3+1)}}{(6-3)!} dx^{\mu\nu\rho} \epsilon_{\mu\nu\rho 0lm} \eta^{00} \eta^{la} \eta^{mb} \\
&= \frac{1}{3!} \left[dx^{ijk} \epsilon_{0ijklm} \eta^{la} \eta^{mb} \right] \\
*'(dx^t \wedge dx^A) &= -\frac{1}{3!} dx^I \epsilon_{IB} [\eta^2]^{BA}
\end{aligned} \tag{5.6}
$$

while

$$
\begin{aligned}
*'(dx^{ijk}) &= \frac{(-1)^{(3+1)}}{(6-3)!} dx^{\mu\nu\rho} \epsilon_{\mu\nu\rho lmn} \eta^{li} \eta^{mj} \eta^{nk} \\
&= \frac{1}{3!} dx^{\mu\nu\rho} \epsilon_{\mu\nu\rho lmn} \eta^{li} \eta^{mj} \eta^{nk} \\
&= \frac{3}{3!} dx^{0ab} \epsilon_{0ablmn} \eta^{li} \eta^{mj} \eta^{nk} \\
&= -\frac{1}{2} dx^{0ab} \epsilon_{ablmn} \eta^{li} \eta^{mj} \eta^{nk}
\end{aligned}
$$

$$
*'(dx^I) = -\frac{1}{2} dt \wedge dx^A \epsilon_{AJ} [\eta^3]^{JI} \tag{5.7}
$$

and

$$
dx^I = -\frac{1}{2} *' (dt \wedge dx^A) \epsilon_{AJ} [\eta^3]^{JI}. \tag{5.8}
$$

Thus we have

$$
*'(dt \wedge dx^A) = -\frac{1}{3!} dx^I \epsilon_{IB} [\eta^2]^{BA} \tag{5.9}
$$

$$
*'(dx^I) = -\frac{1}{2} (dt \wedge dx^A) \epsilon_{AJ} [\eta^3]^{JI}. \tag{5.10}
$$

While $*$ Hodge duality is given as

$$
*(dt \wedge dx^A) = \frac{\sqrt{-g}}{3!} g^{00} dx^I \epsilon_{IB} [g^2]^{BA} \tag{5.11}
$$

$$
*(dx^I) = -\frac{\sqrt{-g}}{2} (dt \wedge dx^B) \epsilon_{BJ} [g^2]^{JI}. \tag{5.12}
$$

We also prepare an identity for following calculation

$$\begin{aligned}
(dt \wedge dx^A) \wedge *'(dt \wedge dx^B) &= (dt \wedge dx^A) \wedge (\frac{1}{3!}dx^I \epsilon^B{}_I) \\
&= -\frac{1}{3!}dt \wedge dx^A \wedge dx^I \epsilon^B{}_I \\
&= -\frac{1}{3!}dt \wedge d^5x \epsilon^{AI} \epsilon^B{}_I \\
&= -\frac{1}{3!}dt \wedge d^5x \epsilon^{AI} \epsilon_{CI} \eta^{CB} \\
&= -\frac{1}{3!}dt \wedge d^5x (2!3!\delta^A_C) \eta^{CB} \\
&= -2\delta^{AB} dt \wedge d^5x, \qquad (5.13)
\end{aligned}$$

where $\epsilon^{AI}\epsilon_{CI} = (2!3!\delta^A_C)$ for flat spacetime. And we define the inner product for any 2-form $\mathcal{A}, \mathcal{B}$ as

$$\mathcal{B}^A \mathcal{B}_A \equiv (\mathcal{B})^2, \qquad \text{and} \qquad \mathcal{A}^A \mathcal{B}_A \equiv \mathcal{A}\mathcal{B}. \qquad (5.14)$$

We begin evaluation for Hamiltonian analysis. We recall the action

$$S = \int (\frac{1}{4} dP \wedge *' dP - Q \wedge dP + \frac{1}{2} Q \wedge R). \qquad (5.15)$$

We have dP which is defined as

$$P = \frac{1}{2!} dx^{\mu\nu} P_{\nu\mu}, \qquad (5.16)$$

$$dP = \frac{1}{2!} dx^{\mu\nu\rho} \partial_\rho P_{\nu\mu}. \qquad (5.17)$$

Let us setup an index notation, we define

$$(\partial P)_A = (\partial P)_{0ab} \equiv \partial_0 P_{ab} + 2\partial_{[a} P_{b]0}, \qquad (5.18)$$

$$(\partial P)_I = (\partial P)_{abc} \equiv 3\partial_{[a} P_{bc]}, \qquad (5.19)$$

From the action (5.15), we have dP and $*'dP$ from the previous identities as

$$dP = -\frac{1}{2} dt \wedge dx^A (\partial P)_A - \frac{1}{2} *' (dt \wedge dx^A)(\widetilde{\partial P})_A, \qquad (5.20)$$

$$*' dP = -\frac{1}{2} *' (dt \wedge dx^A)(\partial P)_A - \frac{1}{2} dt \wedge dx^A (\widetilde{\partial P})_A, \qquad (5.21)$$

where we define

$$(\widetilde{\partial P})_A \equiv -\frac{1}{3!}\epsilon_A{}^I(\partial P)_I. \tag{5.22}$$

As well as, Q and R are expressed as

$$Q = -\frac{1}{2}dt \wedge dx^A Q_A - \frac{1}{2} *' (dt \wedge dx^A)Q_A, \tag{5.23}$$

$$R = \frac{1}{2} *' (dt \wedge dx^A)R_A - \frac{1}{2}dt \wedge dx^A R_A, \tag{5.24}$$

respectively.

Corresponding the wedge product in eq.(5.15), we have

$$dP \wedge *'dP = -\frac{1}{2}dt \wedge d^5x(\partial P)^2 + \frac{1}{2}dt \wedge d^5x(\widetilde{\partial P})^2, \tag{5.25}$$

$$Q \wedge dP = -\frac{1}{2}dt \wedge d^5xQ^A(\widetilde{\partial P})_A + \frac{1}{2}dt \wedge d^5xQ^A(\partial P)_A, \tag{5.26}$$

$$Q \wedge R = dt \wedge d^5xQ^A R_A. \tag{5.27}$$

Thus the quadratic Sen action becomes

$$S = \int \Bigg[\frac{1}{4}\Big(-\frac{1}{2}dt \wedge d^5x(\partial P)^2 + \frac{1}{2}dt \wedge d^5x(\widetilde{\partial P})^2\Big) - \Big(-\frac{1}{2}dt \wedge d^5xQ^A(\widetilde{\partial P})_A + \frac{1}{2}dt \wedge d^5xQ^A(\partial P)_A\Big) + \frac{1}{2}dt \wedge d^5xQ^A R_A\Bigg]. \tag{5.28}$$

Then we rearrange

$$S \approx -\frac{1}{8}dt \wedge d^5x(\partial P)^2 + \frac{1}{8}dt \wedge d^5x(\widetilde{\partial P})^2 + \frac{1}{2}dt \wedge d^5xQ^A(\widetilde{\partial P})_A - \frac{1}{2}dt \wedge d^5xQ^A(\partial P)_A + \frac{1}{2}dt \wedge d^5xQ^A R_A, \tag{5.29}$$

and simplify the Lagrangian density as

$$\mathcal{L} = -\frac{1}{8}(\partial P)^2 + \frac{1}{8}(\widetilde{\partial P})^2 + \frac{1}{2}Q_A(\widetilde{\partial P})_A - \frac{1}{2}Q_A(\partial P)_A + \frac{1}{2}Q_A R_A. \tag{5.30}$$

Then we have to find the conjugate momentum which is defined as

$$\Pi_P^A = \frac{\delta\mathcal{L}}{\delta\dot{P}_A} = \frac{\delta\mathcal{L}}{\delta(\partial_0 P_A)}. \tag{5.31}$$

Let us calculate the conjugate momentum as

$$\begin{aligned}\Pi_P^A &= \frac{\delta\mathcal{L}}{\delta\partial_0 P_A}\\ &= \frac{\delta}{\delta\partial_0 P_A}\Big[-\frac{1}{8}(\partial P)^2+\frac{1}{8}(\widetilde{\partial P})^2+\frac{1}{2}Q_A(\widetilde{\partial P})_A-\frac{1}{2}Q_A(\partial P)_A+\frac{1}{2}Q_AR_A\Big]\\ \Pi_P^A &= -\frac{1}{4}(\partial P)^A-\frac{1}{2}Q^A. \end{aligned} \tag{5.32}$$

We also have the conjugate momentum for Q and P_{a0}

$$\Pi_Q^A = \frac{\delta\mathcal{L}}{\delta\partial_0 Q} = 0, \tag{5.33}$$

$$\pi^a = \frac{\delta\mathcal{L}}{\delta\partial_0 P_{a0}} = 0. \tag{5.34}$$

We have the primary constraint equations which are

$$\Pi_Q^A \approx 0, \qquad \pi^a \approx 0. \tag{5.35}$$

As a result, the Hamiltonian density is

$$\begin{aligned}\mathcal{H} &= \Pi_P^A\partial_0 P_A - \mathcal{L}\\ &= \Pi_P^A\Big[(\partial P)_A - 2\partial_{[a}P_{b]0}\Big]\\ &\quad+\frac{1}{8}(\partial P)^2-\frac{1}{8}(\widetilde{\partial P})^2-\frac{1}{2}Q_A(\widetilde{\partial P})_A-\frac{1}{2}Q_A(\partial P)_A-\frac{1}{2}Q_AR_A\\ &= \Pi_P^A(\partial P)_A - 2\Pi_P^A\partial_{[a}P_{b]0}+\frac{1}{8}(\partial P)^2-\frac{1}{8}(\widetilde{\partial P})^2-\frac{1}{2}Q^A(\widetilde{\partial P})_A-\frac{1}{2}Q^AR_A. \end{aligned} \tag{5.36}$$

Then we trade ∂P from eq.(5.32) so the Hamiltonian density as

$$\mathcal{H} = -2(\Pi_P+\frac{1}{2}Q)^2-\frac{1}{8}(\widetilde{\partial P})^2-\frac{1}{2}Q(\widetilde{\partial P})-\frac{1}{2}QR-2\pi^{ab}\partial_{[a}P_{b]0}. \tag{5.37}$$

So the total Hamiltonian density is given as

$$\mathcal{H}_T = \mathcal{H} + u_A\Pi_Q^A + v_a\pi^a. \tag{5.38}$$

We find the time derivative of primary constraint as

$$\dot{\Pi}_Q^A = \left[\Pi_Q^A, H\right]. \tag{5.39}$$

Let us calculate

$$[\Pi_Q^A(\vec{x}), H(\vec{y})] = \int d\vec{y} \int d\vec{z} \left(\frac{\partial \Pi_Q^A(\vec{x})}{\partial Q^C(\vec{z})} \frac{\partial \mathcal{H}(\vec{y})}{\partial \Pi_Q^C(\vec{z})} - \frac{\Pi_Q^A(\vec{x})}{\partial \Pi_Q^C(\vec{z})} \frac{\partial \mathcal{H}(\vec{y})}{\partial Q^C(\vec{z})} \right)$$
$$= \int d\vec{y} \int d\vec{z} \Big(- \delta_C^A \delta(\vec{x} - \vec{z}) \frac{\partial \mathcal{H}(\vec{y})}{\partial Q^C(\vec{z})} \Big). \quad (5.40)$$

Considering only the partial part in bracket as

$$\frac{\partial \mathcal{H}(\vec{y})}{\partial Q^C(\vec{z})} = \frac{\partial}{\partial Q^C(\vec{z})} \left[- 2(\Pi_P + \frac{1}{2}Q)^2 - \frac{1}{8}(\widetilde{P})^2 - \frac{1}{2}Q(\widetilde{P}) - \frac{1}{2}QR - 2\pi^{ab}\partial_{[a}P_{b]0} \right](\vec{y}), \quad (5.41)$$

then

$$\frac{\partial \mathcal{H}(\vec{y})}{\partial Q^C(\vec{z})} = -2 \cdot 2(\Pi_P^B + \frac{1}{2}Q^B)(\vec{y}) \frac{\partial}{\partial Q^C(\vec{z})} (\Pi_{B(P)} + \frac{1}{2}Q_B)(\vec{y})$$
$$+ \frac{\partial Q_B(\vec{y})}{\partial Q^C(\vec{z})} \Big[- \frac{1}{2}(\partial \widetilde{P})^B \Big](\vec{y}) - \frac{1}{2} \Big[Q^B \frac{\partial R_B(\vec{y})}{\partial Q^C(\vec{z})} + \frac{\partial Q_B(\vec{y})}{\partial Q^C(\vec{z})} R^B(\vec{y}) \Big]. \quad (5.42)$$

From the identity eq.(3.75), we have $\delta_Q R = \mathcal{M}$ so

$$\delta_Q \Big[- \frac{1}{2}QR \Big] = -\frac{1}{2}(Q\delta_Q R + \delta_Q Q R)$$
$$= -\frac{1}{2}(Q\mathcal{M} + R)$$
$$= -\frac{1}{2}(2R) = -R. \quad (5.43)$$

Thus eq.(5.42) becomes

$$\frac{\partial \mathcal{H}(\vec{y})}{\partial Q^C(\vec{z})} = -4(\Pi_P^B + \frac{1}{2}Q^B)(\vec{y}) \frac{1}{2} \delta_C^B \delta(\vec{y} - \vec{z}) + \delta_C^B \delta(\vec{y} - \vec{z}) \Big[- \frac{1}{2}(\partial \widetilde{P})^B \Big](\vec{y})$$
$$- \frac{1}{2} \Big[Q_A \mathcal{M}_B^A + R^B \Big](\vec{y}) \delta_C^B \delta(\vec{y} - \vec{z})$$
$$= -2(\Pi_P^C + \frac{1}{2}Q^C)(\vec{y}) \delta(\vec{y} - \vec{z}) + \delta(\vec{y} - \vec{z}) \Big[- \frac{1}{2}(\partial \widetilde{P})^C \Big](\vec{y})$$
$$- \frac{1}{2} \Big[R^B + R^B \Big](\vec{y}) \delta_C^B \delta(\vec{y} - \vec{z})$$
$$= -2(\Pi_P^C + \frac{1}{2}Q^C)(\vec{y}) \delta(\vec{y} - \vec{z}) + \delta(\vec{y} - \vec{z}) \Big[- \frac{1}{2}(\partial \widetilde{P})^C \Big](\vec{y})$$
$$- \frac{1}{2}(2R^C)(\vec{y}) \delta(\vec{y} - \vec{z})$$
$$= -2(\Pi_P^C + \frac{1}{2}Q^C)(\vec{y}) \delta(\vec{y} - \vec{z}) + \delta(\vec{y} - \vec{z}) \Big[- \frac{1}{2}(\partial \widetilde{P})^C \Big](\vec{y})$$
$$- R^C(\vec{y}) \delta(\vec{y} - \vec{z})$$
$$= -2(\Pi_P^C + \frac{1}{2}Q^C)(\vec{y}) \delta(\vec{y} - \vec{z}) + \delta(\vec{y} - \vec{z}) \Big(- \frac{1}{2}(\partial \widetilde{P})^C - R^C \Big)(\vec{y}). \quad (5.44)$$

Thus eq.(5.40) becomes

$$[\Pi_Q^A(\vec{x}), H(\vec{y})] = \int d\vec{y} \int d\vec{z} \Big[-\delta_C^A \delta(\vec{x}-\vec{z})\delta(\vec{y}-\vec{z}) \Big(-2(\Pi_P^C + \frac{1}{2}Q^C)(\vec{y}) + \Big(-\frac{1}{2}(\partial\widetilde{P})^C - R^C\Big)(\vec{y})\Big)\Big]$$
$$= \int d\vec{y}\delta(\vec{x}-\vec{y}) \left[2(\Pi_P^A + \frac{1}{2}Q^A) + \frac{1}{2}(\partial\widetilde{P})^A + R^A \right](\vec{y}), \qquad (5.45)$$

so we have

$$\dot{\Pi}_Q^A = 2\Pi_P^A + Q^A + \frac{1}{2}(\widetilde{\partial P})^A + R^A. \qquad (5.46)$$

While the time derivative of π_P^a, noted $\pi_P^a \equiv \pi^a$, is

$$\dot{\pi}_P^a = [\pi_P^a, H] \qquad (5.47)$$

so

$$[\pi^a(\vec{x}), H(\vec{y})] = \int d\vec{z} \int d\vec{y} \left(\frac{\partial \pi_p^a(\vec{x})}{\partial P_{c0}(\vec{z})} \frac{\partial \mathcal{H}(\vec{y})}{\partial \pi_p^c(\vec{z})} - \frac{\partial \pi^a(\vec{x})}{\partial \pi^c(\vec{z})} \frac{\partial \mathcal{H}(\vec{y})}{\partial P_{c0}(\vec{z})} \right)$$
$$= \int d\vec{z} \int d\vec{y} \Big(-\delta_c^a \delta(\vec{x}-\vec{z}) \frac{\partial \mathcal{H}(\vec{y})}{\partial P_{c0}(\vec{z})} \Big) \qquad (5.48)$$

and only this term survives,

$$\frac{\partial \mathcal{H}(\vec{y})}{\partial P_{c0}(\vec{z})} = \frac{\partial}{\partial P_{c0}(\vec{z})} \left[-2(\Pi_P + \frac{1}{2}Q)^2 - \frac{1}{8}(\widetilde{\partial P})^2 - \frac{1}{2}Q(\widetilde{\partial P}) - \frac{1}{2}QR - 2\pi^{ab}\partial_{[a}P_{b]0} \right](\vec{y})$$
$$= -2\pi^{ij}\partial(\partial_i P_{j0}(\vec{y}))\partial P_{c0}(\vec{z}). \qquad (5.49)$$

So we have

$$[\pi^a(\vec{x}), H(\vec{y})] = \int d\vec{z}\delta_c^a \delta(\vec{x}-\vec{z}) \int d\vec{y} \Big(2\pi^{ij}(\vec{y})\partial_i \Big[\frac{\partial(P_{j0}(\vec{y}))}{\partial P_{c0}(\vec{z})} \Big] \Big)$$
$$= \int d\vec{z}\delta_c^a \delta(\vec{x}-\vec{z}) \int d\vec{y} \left(2\pi^{ij}(\vec{y})\partial_i \delta_c^j \delta(\vec{y}-\vec{z}) \right) \qquad (5.50)$$

after using by-part integration

$$\int d\vec{y}\pi^{ij}(\vec{y})\partial_i \delta_c^j \delta(\vec{y}-\vec{z}) = -\int d\vec{y}\delta_c^j \delta(\vec{y}-\vec{z})\partial_i \pi^{ij}(\vec{y}) \qquad (5.51)$$

then

$$
\begin{aligned}
[\pi^a(\vec{x}), H(\vec{y})] &= \int d\vec{z} \int d\vec{y} \delta^a_c \delta(\vec{x}-\vec{z}) \Big[-2\delta^j_c \delta(\vec{y}-\vec{z}) \partial_i \pi^{ij}(\vec{y}) \Big] \\
\dot{\pi}^a(\vec{x}) &= -2 \int d\vec{z} \delta^j_a \delta(\vec{x}-\vec{z}) \partial_i \pi^{ij}_P(\vec{z}) \\
0 &= -2\partial_i \pi^{ia}(\vec{x}) \\
0 &= -2\partial_i \pi^{ij}, \qquad (5.52)
\end{aligned}
$$

so

$$\partial_i \pi^{ij} \approx 0. \qquad (5.53)$$

We summarize the corresponding result of time derivative which are

$$2\Pi^A_P + Q^A + \frac{1}{2}(\widetilde{\partial P})^A + R^A \approx 0 \qquad (5.54)$$

$$\partial_i \pi^{ij} \approx 0. \qquad (5.55)$$

To further evaluate the Hamiltonian, we define $\Pi^\pm_A$ as

$$\Pi^\pm_A = \Pi^P_A \pm \frac{1}{4}(\widetilde{\partial P})_A. \qquad (5.56)$$

We further explore its identities and properties for $\Pi^\pm_A$. Firstly, we discuss about its definition. It is given by giving that Π^P_A spans on tangent spaces. From eq.(5.53), Π^P_A is a solution and $\Pi^\pm_A = \Pi^P_A \pm \alpha(\widetilde{\partial P})_A$ is also a solution. This $\Pi^\pm_A = \Pi^P_A \pm \alpha(\widetilde{\partial P})_A$ has 10 components for each

- 5 components for $\Pi^\pm_A = \Pi^P_A + \alpha(\widetilde{\partial P})_A$,
- 5 components for $\Pi^\pm_A = \Pi^P_A - \alpha(\widetilde{\partial P})_A$,

where Π^P_A is antisymmetric with 5×5 matrix so i, j of collective index A has $(5 \times 4)/2$ components that valid. We need to prove that Π^+ and Π^- are linearly independent. Let us define

$$\alpha^A \Pi^+_A = \beta^A \Pi^-_A \qquad ; \alpha^A, \beta^A \in \mathbb{R}, \qquad (5.57)$$

and calculate

$$\alpha^A\Big(\Pi^P_A + \frac{1}{4}(\widetilde{\partial P})_A\Big) = \beta^A\Big(\Pi^P_{ij} - \frac{1}{4}(\widetilde{\partial P})_A\Big)$$
$$(\alpha^A - \beta^A)\Pi^P_A = -(\alpha^A + \beta^A)\frac{1}{4}(\widetilde{\partial P})_A.$$

If and only if, we have

$$\alpha^A - \beta^A = 0 \tag{5.58}$$

$$-(\alpha^A + \beta^A) = 0. \tag{5.59}$$

Let us calculate

$$\alpha^A - \beta^A = 0$$
$$\alpha^A = \beta^A. \tag{5.60}$$

Then applying into eq.(5.59) we obtain

$$2\beta^A = 0, \tag{5.61}$$

so $\alpha^A = \beta^A = 0$. From Π^P and P are linearly independent so eq.(5.56) is also linearly independent.

Moreover, this $\partial_i\Pi^\pm_A$ is also a constraint. We consider by applying derivative, we have

$$\partial_i\Pi^\pm_A = \partial_i\Pi^P_A \pm \frac{1}{4}\partial_i(\widetilde{\partial P})_A$$
$$= 0 \tag{5.62}$$

because of eq.(5.53) and the derivative on ∂P_A is zero.

In the case of Π^+, this form relates to H field strength of this theory. We consider from eq.(5.54), expressed as

$$\begin{aligned}
\Pi_P^A &= -\frac{1}{2}Q^A - \frac{1}{2}R^A - \frac{1}{4}(\widetilde{\partial P})^A \\
Q^A + R^A &= -2(\Pi_P^A + \frac{1}{4}(\widetilde{\partial P})^A) \\
Q^A + R^A &= -2\Pi^{+\,A} \\
*'(Q^A + R^A) &= -2 *' \Pi^{+\,A} \\
Q_I - R_I &= -2\Pi_I^+, \\
H_I &= -2\Pi_I^+,
\end{aligned} \tag{5.63}$$

so this Π^+ gives rise for H field strength and is also defined as the physical part of conjugate momentum. While Π^- relates to $H^{(s)}$ by starting with eq.(5.56) then applying with eq.(5.32), we have

$$\begin{aligned}
\Pi_A^- &= \Pi_A^P - \frac{1}{4}(\widetilde{\partial P})_A \\
&= -\frac{1}{4}(\partial P)_A - \frac{1}{2}Q_A - \frac{1}{4}(\widetilde{\partial P})_A \\
-2\Pi_A^- &= Q_A + \frac{1}{2}\Big[(\partial P)_A + (\widetilde{\partial P})_A\Big].
\end{aligned} \tag{5.64}$$

So from eq.(3.21), an above equation is expressed as

$$H_A^{(s)} = -2\Pi_A^-, \tag{5.65}$$

where this Π^- give rise to $H^{(s)}$.

We also consider the commutation of $\Pi^\pm$.We have

$$[\Pi_A^+(\vec{x}), \Pi_B^+(\vec{y})] = \int d\vec{z}\left(\frac{\partial\Pi_A^+(\vec{x})}{\partial P_C(\vec{z})}\frac{\partial\Pi_B^+(\vec{y})}{\partial\Pi_C^P(\vec{z})} - \frac{\partial\Pi_A^+(\vec{x})}{\partial\Pi_B^P(\vec{z})}\frac{\partial\Pi_B^+(\vec{y})}{\partial P_C(\vec{z})}\right) \tag{5.66}$$

and calculate

$$\begin{aligned}
\frac{\partial\Pi_A^+(\vec{x})}{\partial P_C(\vec{y})} &= \frac{\partial}{\partial P_C(\vec{y})}\left[\Pi_A^P(\vec{x}) + \frac{1}{4}(\widetilde{\partial P})_A(\vec{x})\right] \\
&= \frac{\partial\Pi_A^P(\vec{x})}{\partial P_C(\vec{y})} + \frac{1}{4}\epsilon_{ADi}\partial_i\left(\frac{\partial P_D(\vec{x})}{\partial P_C(\vec{y})}\right) \\
&= 0 + \frac{1}{4}\epsilon_{ADi}\partial_i\Big(\delta_C^D\delta(\vec{x}-\vec{y})\Big) = \frac{1}{4}\epsilon_{ACi}\partial_i(\delta(\vec{x}-\vec{y})),
\end{aligned} \tag{5.67}$$

and

$$\frac{\partial \Pi_A^+(\vec{x})}{\partial \Pi_C^P(\vec{y})} = \frac{\partial}{\partial \Pi_C^P(\vec{y})}\left[\Pi_A^P(\vec{x}) + \frac{1}{4}(\widetilde{\partial P})_A(\vec{x})\right]$$
$$= \frac{\partial \Pi_A^P(\vec{x})}{\partial \Pi_C^P(\vec{y})} + 0 = \delta_C^A \delta(\vec{x}-\vec{y}). \tag{5.68}$$

Eq.(5.66) becomes

$$[\Pi_A^+(\vec{x}), \Pi_B^+(\vec{y})] = \int d\vec{z}\left[\frac{1}{4}\epsilon_{ACi}\partial_i(\delta(\vec{x}-\vec{z}))\delta_C^B\delta(\vec{y}-\vec{z}) - \delta_C^A\delta(\vec{x}-\vec{z})\frac{1}{4}\epsilon_{BCi}\partial_i(\delta(\vec{y}-\vec{z}))\right]$$
$$= \int d\vec{z}\left[\frac{1}{4}\epsilon_{ABi}\delta(\vec{y}-\vec{z})\partial_i(\delta(\vec{x}-\vec{z})) - \frac{1}{4}\epsilon_{BAi}\delta(\vec{x}-\vec{z})\partial_i(\delta(\vec{y}-\vec{z}))\right]$$
$$= \frac{1}{4}\int d\vec{z}\left[\epsilon_{ABi}\delta(\vec{y}-\vec{z})\frac{\partial}{\partial x^i}(\delta(\vec{x}-\vec{z})) - \epsilon_{BAi}\delta(\vec{x}-\vec{z})\frac{\partial}{\partial y^i}(\delta(\vec{y}-\vec{z}))\right]$$

and by using

$$\frac{\partial}{\partial y^i}\delta(\vec{y}-\vec{z}) = -\frac{\partial}{\partial z^i}\delta(\vec{y}-\vec{z}), \tag{5.69}$$

then

$$[\Pi_A^+(\vec{x}), \Pi_B^+(\vec{y})]$$
$$= \frac{1}{4}\int d\vec{z}\left[-\epsilon_{ABi}\delta(\vec{y}-\vec{z})\frac{\partial}{\partial z^i}(\delta(\vec{x}-\vec{z})) + \epsilon_{BAi}\delta(\vec{x}-\vec{z})\frac{\partial}{\partial z^i}(\delta(\vec{y}-\vec{z}))\right]$$
$$= \frac{1}{4}\int d\vec{z}\left[-\epsilon_{ABi}\delta(\vec{y}-\vec{z})\frac{\partial}{\partial z^u}(\delta(\vec{x}-\vec{z})) + (-1)^{2+2}\epsilon_{ABi}\delta(\vec{x}-\vec{z})\frac{\partial}{\partial z^u}(\delta(\vec{y}-\vec{z}))\right]$$
$$= \frac{1}{4}\epsilon_{ABi}\int d\vec{z}\left[-\delta(\vec{y}-\vec{z})\frac{\partial}{\partial z^i}(\delta(\vec{x}-\vec{z})) + \delta(\vec{x}-\vec{z})\frac{\partial}{\partial z^i}(\delta(\vec{y}-\vec{z}))\right]. \tag{5.70}$$

From by-part integration,

$$\int d\vec{z}\delta(\vec{y}-\vec{z})\frac{\partial}{\partial z}(\delta(\vec{x}-\vec{z})) = \cancel{\int d\vec{z}\frac{\partial}{\partial z}\left(\delta(\vec{y}-\vec{z})\delta(\vec{x}-\vec{z})\right)}\Big| - \int d\vec{z}\delta(\vec{x}-\vec{z})\frac{\partial}{\partial z}(\delta(\vec{y}-\vec{z})), \tag{5.71}$$

so for the first term, we have

$$[\Pi_A^+(\vec{x}), \Pi_B^+(\vec{y})] = \frac{1}{4}\epsilon_{ABi}\int d\vec{z}\left[\delta(\vec{x}-\vec{z})\frac{\partial}{\partial z^i}\delta(\vec{y}-\vec{z}) + \delta(\vec{x}-\vec{z})\frac{\partial}{\partial z^i}\delta(\vec{y}-\vec{z})\right]$$
$$= \frac{2}{4}\epsilon_{ABi}\int d\vec{z}\delta(\vec{x}-\vec{z})\frac{\partial}{\partial z^i}\delta(\vec{y}-\vec{z}) = \frac{1}{2}\epsilon_{ABi}\frac{\partial}{\partial x^i}\delta(\vec{y}-\vec{x})$$
$$= \frac{1}{2}\epsilon_{ABi}\frac{\partial}{\partial x^i}\delta(\vec{x}-\vec{y}). \tag{5.72}$$

As well as $[\Pi_A^-(\vec{x}), \Pi_B^-(\vec{y})]$, we have

$$[\Pi_A^+(\vec{x}), \Pi_B^+(\vec{y})] = -\frac{1}{2}\epsilon_{ABi}\frac{\partial}{\partial x^i}\delta(\vec{x}-\vec{y}). \tag{5.73}$$

And the last one is

$$[\Pi_A^+(\vec{x}), \Pi_B^-(\vec{y})] = \int d\vec{z}\left(\frac{\partial \Pi_A^+(\vec{x})}{\partial P_C(\vec{z})}\frac{\partial \Pi_B^-(\vec{y})}{\partial \Pi_C^P(\vec{z})} - \frac{\partial \Pi_A^+(\vec{x})}{\partial \Pi_C^P(\vec{z})}\frac{\partial \Pi_B^-(\vec{y})}{\partial P_C(\vec{z})}\right), \tag{5.74}$$

we calculate

$$\begin{aligned}
&[\Pi_A^+(\vec{x}), \Pi_B^-(\vec{y})] \\
&= \int d\vec{z}\left[\frac{1}{4}\epsilon_{ACi}\frac{\partial}{\partial x^i}\delta(\vec{x}-\vec{z})\delta_C^B\delta(\vec{y}-\vec{z}) - \delta_C^A\delta(\vec{x}-\vec{z})\left(-\frac{1}{4}\epsilon_{BCi}\frac{\partial}{\partial y^i}\delta(\vec{y}-\vec{z})\right)\right] \\
&= \frac{1}{4}\int d\vec{z}\left[\epsilon_{ABi}\delta(\vec{y}-\vec{z})\frac{\partial}{\partial x^i}\delta(\vec{x}-\vec{z}) + \epsilon_{BAi}\delta(\vec{x}-\vec{z})\frac{\partial}{\partial y^i}\delta(\vec{y}-\vec{z})\right] \\
&= \frac{1}{4}\epsilon_{ABi}\int d\vec{z}\left[-\delta(\vec{y}-\vec{z})\frac{\partial}{\partial z^i}\delta(\vec{x}-\vec{z}) - \delta(\vec{x}-\vec{z})\frac{\partial}{\partial z^i}\delta(\vec{y}-\vec{z})\right].
\end{aligned} \tag{5.75}$$

And for by-part integration which is

$$\int d\vec{z}\delta(\vec{y}-\vec{z})\frac{\partial}{\partial z^u}\delta(\vec{x}-\vec{z}) = \cancel{\frac{\partial}{\partial z^u}\Big(\delta(\vec{y}-\vec{z})\delta(\vec{x}-\vec{z})\Big)}\Bigg| - \int d\vec{z}\delta(\vec{x}-\vec{z})\frac{\partial}{\partial z^u}\delta(\vec{y}-\vec{z}), \tag{5.76}$$

we have

$$[\Pi_A^+(\vec{x}), \Pi_B^-(\vec{y})] = \frac{1}{4}\epsilon_{ABi}\int d\vec{z}\left[\cancel{\delta(\vec{x}-\vec{z})\frac{\partial}{\partial z^i}\delta(\vec{y}-\vec{z})} - \cancel{\delta(\vec{x}-\vec{z})\frac{\partial}{\partial z^i}\delta(\vec{y}-\vec{z})}\right] = 0 \tag{5.77}$$

Finally, we obtain

$$[\Pi_A^\pm(\vec{x}), \Pi_B^\pm(\vec{y})] = \pm\frac{1}{2}\epsilon_{ABi}\frac{\partial}{\partial x^i}\delta(\vec{x}-\vec{y}) \tag{5.78}$$

$$[\Pi_A^+(\vec{x}), \Pi_B^-(\vec{y})] = 0. \tag{5.79}$$

This result is also useful for further calculation but we already finish this exploration for $\Pi_A^\pm$.

Let us continue consider the Hamiltonian density (5.37). Firstly we calculate for trading out Q and R in this Hamiltonian by using $H = *H$. Then we have

$$(Q-R)_A = -\frac{g_{00}}{\sqrt{-g}}[g^2]_{AB}(Q+R)^B. \tag{5.80}$$

By the second class constraint (5.54), we have Q_A and R_A which are expressed as

$$Q_A = \frac{g_{00}}{\sqrt{-g}}[g^2]_{AB}(\Pi^B + \frac{1}{4}(\widetilde{\partial P})^B) - (\Pi_A + \frac{1}{4}(\widetilde{\partial P})_A) \tag{5.81}$$

$$R_A = -\frac{g_0 0}{\sqrt{-g}}[g^2]_{AB}(\Pi^B + \frac{1}{4}(\widetilde{\partial P})^B) - (\Pi_A + \frac{1}{4}(\widetilde{\partial P})_A). \tag{5.82}$$

Trading Q and R from Hamiltonian density, we have

$$\mathcal{H} = -\Pi_-^2 - \frac{g_{00}}{\sqrt{-g}}\Pi_+^A[g^2]_{AB}\Pi_+^B + 2\partial_{[a}\Pi^{ab}P_{b]0} + u_A\Pi_Q^A + v_a\pi^a. \tag{5.83}$$

We use the Legredre transformation. The first-order Lagrangian is given as

$$\mathcal{L} = \Pi_P^A\partial_0 P_A + \Pi_-^2 + \frac{g_{00}}{\sqrt{-g}}\Pi_+^A[g^2]_{AB}\Pi_+^B, \tag{5.84}$$

where $\partial_a\Pi^{ab}$, Π_Q^A, and π^a vanished by their equation of motion from eq.(5.35) and (5.55).

From the second class constraint, it is possible to define

$$\Pi_P^A = \frac{1}{4}(\widetilde{\partial\phi})^A, \tag{5.85}$$

which corresponds to a second class constraint solution. So we have $\Pi_\pm^A$ as

$$\Pi_\pm^A = \frac{1}{4}[(\widetilde{\partial\phi}) \pm (\widetilde{\partial P})]^A. \tag{5.86}$$

Substituting back to the Lagrangian, we obtain

$$\mathcal{L} = \frac{1}{4}(\widetilde{\partial\phi})^A\partial_0 P_A + \frac{1}{16}((\widetilde{\partial\phi}) - (\widetilde{\partial P}))^2 + \frac{1}{16}\frac{g_{00}}{\sqrt{-g}}((\widetilde{\partial\phi})^A + (\widetilde{\partial P})^A)[g^2]_{AB}((\widetilde{\partial\phi})^B + (\widetilde{\partial P})^B). \tag{5.87}$$

Let us gives the field redefinition which is imposed as

$$P_A = \rho_A + \sigma_A, \qquad \phi_A = \rho_A - \sigma_A, \tag{5.88}$$

and

$$\widetilde{\partial\rho}^A \equiv -\frac{1}{2}\partial_i\rho_B\epsilon^{AiB}, \qquad \widetilde{\partial\sigma}^A \equiv -\frac{1}{2}\partial_i\sigma_B\epsilon^{AiB}. \tag{5.89}$$

Applying eq.(5.88) to the Lagrangian density, thus it becomes

$$\mathcal{L} = -\frac{1}{4}(\widetilde{\partial\sigma})^A\dot{\sigma}_A + \frac{1}{4}(\widetilde{\partial\sigma})^2 + \frac{1}{4}(\widetilde{\partial\rho})^A\dot{\rho}_A + \frac{1}{4}(\widetilde{\partial\rho})^A\dot{\sigma}_A + \frac{1}{4}(\widetilde{\partial\rho})^A\dot{\sigma}_A - \frac{1}{4}(\widetilde{\partial\sigma})^A\dot{\rho}_A + \frac{1}{4}\frac{g_{00}}{\sqrt{-g}}(\widetilde{\partial\rho})^A[g^2]_{AB}(\widetilde{\partial\rho})^B. \tag{5.90}$$

Then by using by-part integral to cancel the cross term, we have

$$\mathcal{L} = -\frac{1}{4}(\widetilde{\partial\sigma})^A\dot{\sigma}_A + \frac{1}{4}(\widetilde{\partial\sigma})^2 + \frac{1}{4}(\widetilde{\partial\rho})^A\dot{\rho}_A + \frac{1}{4}\frac{g_{00}}{\sqrt{-g}}(\widetilde{\partial\rho})^A[g^2]_{AB}(\widetilde{\partial\rho})^B, \tag{5.91}$$

where we define

$$\mathcal{L}_- = -\frac{1}{4}(\widetilde{\partial\sigma})\dot{\sigma} + \frac{1}{4}(\widetilde{\partial\sigma})^2 \tag{5.92}$$

$$\mathcal{L}_+ = \frac{1}{4}(\widetilde{\partial\rho})\dot{\rho} + \frac{1}{4\sqrt{-g}}g_{00}(\widetilde{\partial\rho})^{ab}g_{ac}g_{bd}(\widetilde{\partial\rho})^{cd}. \tag{5.93}$$

At this stage, it show us that the decoupling of unphysical and physical sector. This result gives us that the quadratic Sen action relates to Henneaux-Teitelboim action. However the unphysical sector has the wrong sign of kinetic term.

Furthermore, we consider only the sector of physical by giving the definition as

$$F_{ijk} = 3\partial_{[i}\rho_{jk]}, \qquad F_{0ij} = \partial_0\rho_{ij} + 2\partial_{[i}\rho_{j]0}. \tag{5.94}$$

From eq.(5.89), we have

$$\widetilde{F}^{0A} = \frac{1}{3!\sqrt{-g}}\epsilon^{AiB}F_{iB}. \tag{5.95}$$

As we define $F = d\rho$, so eq.(5.95) becomes

$$\widetilde{F}^{0A} = -\frac{1}{\sqrt{-g}}(\widetilde{\partial\rho})^{0A}. \tag{5.96}$$

We calculate and rearrange the physical sector $\mathcal{L}_+$ so we have

$$\mathcal{L}_+ = -\frac{1}{4}\frac{\sqrt{-g}}{g^{00}}\left(F^{0B} - \widetilde{F}^{0B}\right)\widetilde{F}^{0A}[g^2]_{BA}. \tag{5.97}$$

When we return the collective index to normal fashion, it is

$$\mathcal{L}_+ = -\frac{1}{4}\frac{\sqrt{-g}}{g^{00}}\left(F^{0ab} - \widetilde{F}^{0ab}\right)\widetilde{F}^{0cd}g_{ac}g_{bd}, \tag{5.98}$$

where this analogue to Henneaux-Teitelboim action [3] for physical sector.

CHAPTER VI

CONCLUSION

6.1 Summary

To summarize the result from the evaluation of Sen formalism, we have done calculated the dimensional reduction of simply Sen quadratic in chapter 4. A corresponding result, we obtain the 5-dimension Maxwell theory as

$$S = \int d^5x \Big\{ -\frac{1}{4rl} f_{mn} f^{mn} + \frac{1}{4l} f^{(s)}_{mn} f^{(s)mn} \Big\}, \tag{6.1}$$

where we explicitly can see that the unphysical decouples from the physical sector.

In the case of complete M5-brane action where we use the Green-Schwarz formalism to induce the background fields from the supertarget spacetime. The corresponding calculation of D4-brane action where we apply dual frame and field redefinition so we have

$$S = \int d^5x \left(\frac{1}{4} \psi^{(s)}_{ab} \psi^{(s)}_{ij} \eta^{ai} \eta^{bj} - e^{-\phi} \sqrt{-\det(\gamma_{ab} + F_{ab})} \right) - \int (C_1 + C_3 + C_5') \wedge e^F, \tag{6.2}$$

where

$$F_{ab} = \partial_a A_b - \partial_b A_a + b_{ab}, \tag{6.3}$$

$$C_5' = C_5 - \frac{1}{2} B \wedge C_3, \tag{6.4}$$

$$\psi^{(s)}_{ab} = \partial_a A^{(s)}_b - \partial_b A^{(s)}_a. \tag{6.5}$$

It is shown that as same as the 5-dimensional Maxwell theory, the unphysical field decouples from physical one. While the dual D4-brane action, we choose difference components of P to be integrated out. Thus the action is given as

$$\tilde{S} = \int d^5x \bigg(\frac{3}{4} \partial_{[a} b^{(s)}_{bc]} \partial_{[i} b^{(s)}_{jk]} \eta^{ai} \eta^{bj} \eta^{ck} - e^{-\phi}\sqrt{-\det G} \\ - \frac{1}{24} \frac{1}{e^{-4\phi/3} + e^{2\phi/3} C_i \gamma^{ij} C_j} \gamma^{cd} C_d \epsilon^{abmnp} F_{mnp} F_{abc} \bigg) \\ + \int (b \wedge (F - C_3) - C_5'), \tag{6.6}$$

where

$$G_{ab} \equiv \gamma_{ab} - \frac{i}{6} e^{\phi} \frac{\gamma_{ac} \epsilon^{cdijk} F_{ijk} (\gamma_{db} + e^{2\phi} C_d C_b)}{\sqrt{-\gamma}\sqrt{1 + e^{2\phi} C_a \gamma^{ab} C_b}}. \tag{6.7}$$

We can see as well as the D4-brane action above that this dual D4-brane action has the unphysical field decouples from the physical sector.

As a result, the D4-brane and dual D4-brane action can be obtained from the double dimensional reduction on a circle of the complete M5-brane action. This can be realized that the D4-brane relates to the dual D4-brane action via which components of P to be chosen and integrated out.

In the chapter 5, we apply the Hamiltonian analysis to the quadratic Sen action. The final result, we have

$$\mathcal{L} = -\frac{1}{4}(\widetilde{\partial\sigma})^A \dot{\sigma}_A + \frac{1}{4}(\widetilde{\partial\sigma})^2 + \frac{1}{4}(\widetilde{\partial\rho})^A \dot{\rho}_A + \frac{1}{4}\frac{g_{00}}{\sqrt{-g}}(\widetilde{\partial\rho})^A [g^2]_{AB} (\widetilde{\partial\rho})^B, \tag{6.8}$$

where we define

$$\mathcal{L}_- = -\frac{1}{4}(\widetilde{\partial\sigma})\dot{\sigma} + \frac{1}{4}(\widetilde{\partial\sigma})^2 \tag{6.9}$$

$$\mathcal{L}_+ = \frac{1}{4}(\widetilde{\partial\rho})\dot{\rho} + \frac{1}{4\sqrt{-g}} g_{00} (\widetilde{\partial\rho})^{ab} g_{ac} g_{bd} (\widetilde{\partial\rho})^{cd}. \tag{6.10}$$

This result gives us the detail that quadratic Sen action relates to Henneaux-Teitelboim action [3]. We also further consider the physical sector of this Lagrangian $\mathcal{L}_+$. This term is given as

$$\mathcal{L}_+ = -\frac{1}{4}\frac{\sqrt{-g}}{g^{00}} \left(F^{0ab} - \widetilde{F}^{0ab} \right) \widetilde{F}^{0cd} g_{ac} g_{bd} \tag{6.11}$$

This analysis gives us directly that the unphysical field surely separates from the physical one. This confirms the corresponding result from previous chapter 4 that we consider in the Lagrange formalism.

6.2 Future works

Further exploring the Sen formalism, the dimensional reduction is on the other geometry such as T^2, T^3, Riemann surface, and K3. It can be expected that the Sen action also gives rise to other action in string theory and M-theory. Moreover, in the other kind of dimensional reduction, it is the null dimensional reduction which could apply to Sen M5-brane action.

According to [35] where the two kind of metric is provided to discuss as two diffeomorphism. And up coming [36], the diffeomorphism rules, the reduced phase space sinceand shift vectors. Hopefully, to expand this result to full the Hamiltonian is up to linear powers in lapse functions phase space on which the Hamiltonian is a complicated function of lapse functions and shift vectors.

Further investigation whether the Sen formulation is related to the dual of standard gauge-fixed PST formulation as well as the standard PST formulation at the Lagrangian level. In particular, it is also interesting that Sen formalism is also related to the clone field formulation [37].

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