ENTROPIES AND THERMODYNAMIC STABILITY OF BLACK HOLES

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ABSTRACT

In this thesis, we investigates thermodynamic stability of black holes, including to the Schwarzschild-de Sitter black hole and Schwarzschild-like black holes in the Horndeski theory. We study thermodynamics of the vector-tensor Horndeski black holes by using a proper black hole's entropy, that is the Wald entropy. The vector-tensor Horndeski black hole is classified into two types, viz. magnetic-Horndeski black hole (MHBH) and electric-Horndeski black hole (EHBH). Since the solution of MHBH is the analytic one, it allows us to investigate thermodynamic stability together with the linear stability. For the EHBH, the solution is obtained from numerical method. The thermodynamic stability can be considered solely. MHBH and EHBH are thermodynamic stable with the similar conditions for value of coupling constant. In case of the Schwarzschild-de Sitter black hole, we consider the black hole with other statistics' representation. The Rényi entropy is one that we have chosen, because it can serve the zeroth law compatibility of thermodynamics. By extending the phase space, the Schwarzschild-de Sitter black hole can be thermodynamically stable in this representation.

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CHAPTER I

INTRODUCTION

1.1 OVERVIEW

On the gravity, in the past we would be reminded of Newton's gravity which concerns the force between objects that have mass. After Einstein proposed though experiment such as Einstein's equivalence principle, it led to a new concept for gravity such that gravitation is caused by curvature of spacetime, then Einstein's field equation was formulated. This equation describes the relation between curvature and matter (including properties of matter e.g. pressure and momentum, etc.) and is given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}\,,$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $g_{\mu\nu}$ is the metric tensor, $T_{\mu\nu}$ is the energy momentum tensor and G is Newton's universal gravitational constant.

The first solution was proposed by Karl Schwarzschild called the "Schwarzschild solution". He applied spherical symmetry and static condition to Einstein's field equation. This solution implies mysterious objects which have intensive gravitation that even the light cannot escape. We call these objects as "black holes".

There are several investigations on the behavior of black holes. One of the most simple features is the No-hair theorem which tells us black holes are described by mass, charge, and angular momentum. Besides, black holes have a specific behavior obeying the so called "black hole's mechanical laws". Coincidentally, the black hole's mechanical laws are similar to thermodynamic laws.

Bekenstein and Hawking proposed the temperature and entropy of black holes based on Gibbs-Boltzmann statistics [1, 2, 3]. Temperature and entropy are given by

$$S_{BH} = \frac{A}{4G} \quad \text{and} \quad T_H = \frac{\kappa}{2\pi},$$

where A is area of event horizon and κ is surface gravity. So, we can say that black hole is thermodynamic objects. To study about existence of black holes via a thermodynamics point of view, we need to consider on thermodynamic stability of black holes. The stability analysis can be performed by considering the thermal properties of a black hole, for example, heat capacity and compressibility, etc.

Nowadays, there are many unsolvable physical problems, such as accelerated expanding of the Universe, the Inflation, the coincidence problem, etc. These are not explained by General Relativity. Whereupon, physicists propose the new theories for solving them, one of those is “modified gravity theories”. There are several models of the modification theory of GR, and then black hole’s solutions are possible to be found for a theory. Thereafter, the doubt was arisen, are they (black hole’s solutions of modified theories of gravity) employ the Bekenstein-Hawking entropy? The answer is “no”, because the derivation of S_{BH} is based on only the General Relativity [2, 4]. The formula of entropy in general theory of gravity might be derived, and then Wald achieved that purpose [5]. Such entropy is known as “Wald entropy” and given by

$$S_W = -2\pi \int_H \underline{X}^{\mu\nu} \underline{b}_{\mu\nu}, \tag{1.1}$$

where $\underline{b}_{\mu\nu} = 2N_{[\mu}K_{\nu]}$ is a binormal vector which N_μ and K_μ are normal vectors on surface H. Here, the tensor $\underline{X}^{\mu\nu}$ is defined by

$$\underline{X}^{\mu\nu} \equiv \varepsilon_{\lambda_1\lambda_2\mu_3\cdots\mu_n} E_R^{\mu\nu\lambda_1\lambda_2}, \tag{1.2}$$

where the equation of motion tensor $E_R^{\mu\nu\rho\sigma}$ is written by

$$E_R^{\mu\nu\rho\sigma} = \Bigg[\frac{\partial\mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} - \nabla_{\mu_1}\frac{\partial\mathcal{L}}{\partial\left(\nabla_{\mu_1}R_{\mu\nu\rho\sigma}\right)} + \cdots$$

$$+(-1)^k \nabla_{\mu_1} \cdots \nabla_{\mu_k} \frac{\partial \mathcal{L}}{\partial \left(\nabla_{\mu_1} \cdots \nabla_k R_{\mu\nu\rho\sigma} \right)} \Bigg], \tag{1.3}$$

where $\mathcal{L}$ is the Lagrangian density and $R_{\mu\nu\rho\sigma}$ is the Riemann tensor. Roughly speaking, Wald entropy depends on the Riemann tensor which appears in the Lagrangian density. Consequently, S_{BH} and S_W are different in the non-minimal coupling of gravity theory (there exits the interactions between gravity an others). For example, the vector-tensor Horndeski theory which is given by following action

$$\mathcal{S} = \int d^4x \sqrt{-g} \left(\frac{1}{16\pi G} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \beta L^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \right) \tag{1.4}$$

where g is the determinant of $g_{\mu\nu}$, $F_{\mu\nu}$ is the field strength tensor and β is the coupling constant. Here, $L^{\mu\nu\alpha\beta}$ is the double dual Riemann tensor,

$$L^{\mu\nu\alpha\beta} = \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} R_{\rho\sigma\gamma\delta}. \tag{1.5}$$

This theory is expected whether it would be done for the cosmological unsolvable problem [6, 7, 8, 9, 10]. Moreover, it is the most general theory for second-order equation of motion [11, 12, 13, 14]. According to reference [15], the black hole in such theory, so-called “vector-tensor Horndeski black hole”, is linearly stabilities. Combining with thermodynamic stability, we will be able to conclude that the Horndeski black hole can exist by itself in the nature. Furthermore, the black hole image from the Event Horizon Telescope [16, 17, 18, 19, 20, 21] might be the Horndeski black hole. For that reason, the vector-tensor Horndeski theory might be the proper theory for solving the unsolvable one.

Nevertheless, the Bekenstein-Hawking entropy obeys the Gibbs-Boltzmann statistics, in other words

$$S_{BH} = S_{GB}, \tag{1.6}$$

where S_{GB} is the Gibbs-Boltzmann entropy. According to eq.(1.1), black hole's entropy depends on the horizon area, and then it is not an additive entropy,

$$S_{12} \neq S_1 + S_2. \tag{1.7}$$

The consequence is that such an entropy is not the extensive one. Meanwhile, Gibbs-Boltzmann entropy in thermodynamics is both additivite and extensive. Furthermore, the zeroth law of thermodynamics requires the property of additivity [22]. Thus, the entropy Gibbs-Boltzmann statistics is quite not suitable to describe Bekenstein-Hawking entropy. We need to interpret black hole's entropy with another type of statistics. One chooses the Tsallis entropy for a representation of the Bekenstein-Hawking entropy [23, 24, 25, 26, 27]. It is reasonable because both of them are the non-extensive and non-additive entropy. To recover the zeroth law of thermodynamics, the Tsallis entropy is not enough. The formal logarithm map gives us the additive version of the Tsallis entropy, that is "Rényi entropy" [28]. The black hole's entropy in this representation is defined by

$$S_{bh} = S_R = \frac{1}{\lambda}\ln\left(1 + \lambda S_{BH}\right), \tag{1.8}$$

where λ is the non-extensive parameter. It is called "black hole's Rényi entropy". In this representation, the Rényi entropy can stabilize black holes, for example, Schwarzschild black hole as shown in reference [23].

In the present, the most successful model for explanation of accelerated expanding Universe is the ΛCDM model. The action of this theory is given by

$$\mathcal{S} = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi G}R - \Lambda\right), \tag{1.9}$$

where Λ is the cosmological constant. The expansion of Universe is described by the positive cosmological constant, corresponding to de Sitter spacetime. However, the Schwarzschild-de Sitter black hole is thermodynamically unstable in Bekenstein-Hawking entropy [29]. It is possible that the Schwarzschild-de Sitter black hole in the Rényi's representation is thermodynamically stable.

According to title of the thesis, entropies and thermodynamic stability of black holes, it relates to black hole thermodynamics. The background of this area includes general relativity and thermodynamics, and then they will be presented in chapter 2. After solving the solutions of Einstein's field equation in various situation, a black hole and its mechanical laws would be presented in chapter 3. To studying thermodynamics of black hole, we will talk about this topic through the black hole's entropies in chapter 4. Thermodynamics of black holes in a modified gravity theory which is the vector-tensor Horndeski theory is presented in chapter 5. Thermodynamics of Schwarzchild-de Sitter black hole with black hole's Rényi entropy is studied in chapter 6. In the end of thesis, chapter 7, we will summarize what we found in this research.

1.2 OBJECTIVE

1. Investigation of thermodynamic stability of the vector-tensor Horndeski black holes and comparison with linear stability.
2. Investigation of thermodynamic stability in Rényi's representation of the Schwarzschild-de Sitter black hole along with formulating the first law of thermodynamics of such a black hole.

CHAPTER II

THEORIES

2.1 GENERAL RELATIVITY

2.1.1 Accelerated Frame

Special Relativity (SR) describes motion of an object in the different frames where those frames have relatively uniform velocity to each other. From SR, we found some strange features, for example, length contraction and time dilation. What about a relatively accelerated frame?

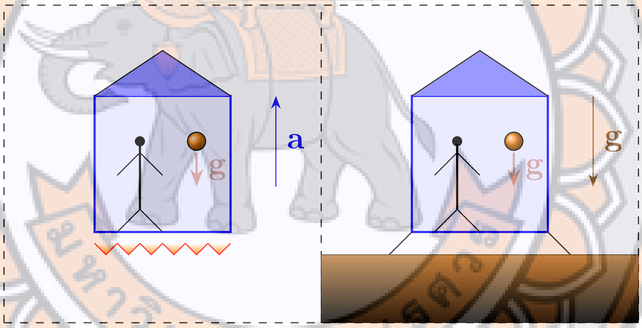


Figure 1 Weak Equivalent Principle.

From Figure 1, the observer in the rocket would see the ball moving downward with acceleration. Incidentally, it is the same as the observer dropping the ball in a rest frame on the gravitation field. Therefore, the acceleration and gravity are equivalent. Moreover, when we talk about the rest frame, we cannot distinguish that we are in the freely falling frame in a gravitational field or absolutely empty space, as illustrated in Figure 2.

Note that, the above equivalence valids only in small enough regions. For the case of two particles that are far away enough, they are falling by the influence of the gravitational field, an observer in this frame will see the particles move toward each other. This is the effect of tidal force because the direction of gravi-

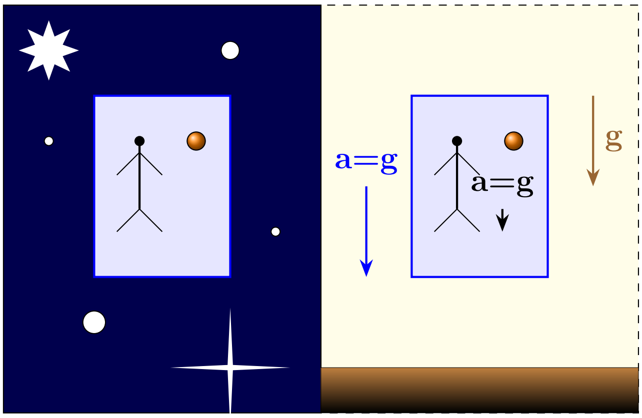


Figure 2 Einstein's elevator thought experiment.

tational force direct to center of mass of source, as shown in figure 3.

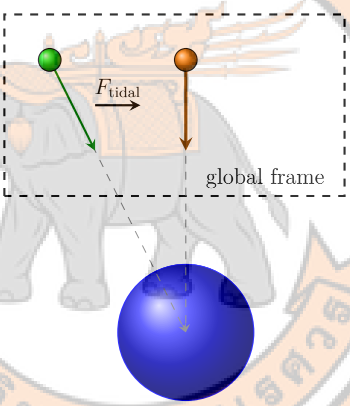


Figure 3 The effect of tidal in the global frame.

The above equivalence is called the "Weak Equivalence Principle" or WEP which states that "The motion of freely-falling particles are the same in a gravitational field and uniformly accelerated frame, in small enough regions of spacetime". Another description of WEP is realized by considering dropping particles. A free falling object described by Newton equation as follows

$$\vec{F} = m_i\vec{a} = -m_g\vec{\nabla}\Phi.$$

where $\vec{F}$ is the exerted force on the object, m_i is inertial mass, $\vec{a}$ is the acceleration of the object, m_g is gravitational mass and Φ is the gravitational potential. Then,

$$\vec{a} = -\frac{m_g}{m_i}\vec{\nabla}\Phi.$$

WEP implies that

$$\vec{a} = -\vec{\nabla}\Phi.$$

Therefore, we obtain

$$m_i = m_g.$$

This is another definition of WEP. Moreover, the Pisa experiment states that "The motion of a gravitational test particle in a gravitational field is independent of its mass and composition".

Einstein generalized the idea of WEP to be Einstein's Equivalence Principle (EEP) which states that "Laws of physics reduce to those of SR since it is impossible to detect the existence of a gravitational field by means of local experiments". According to EEP together with Figure 2, in freely falling frame we cannot claim that we are in a place with gravitational field or not. We need to redefine the definition of the inertial frame, that is the inertial frame is an "unaccelerated" frame where accelerated means free-falling by influence of only gravity. In this sense, the definition of the initial frame can be generalized as "freely-falling frame".

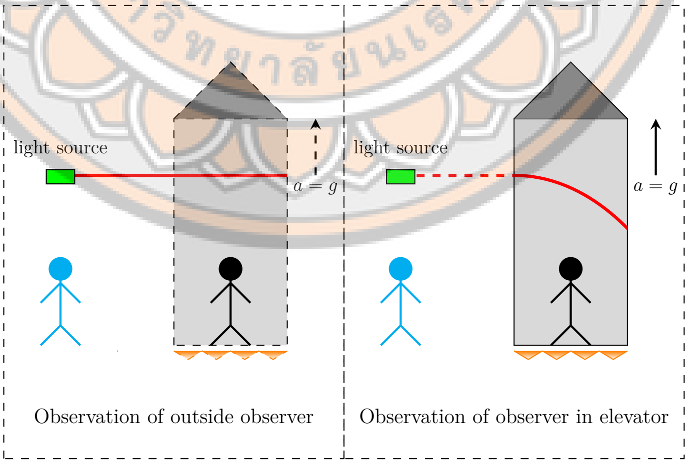


Figure 4 Curved trajectory of light.

Moreover, Einstein proposed a new concept of gravity. To show this concept, let us consider the thought experiment as shown in Figure 5.

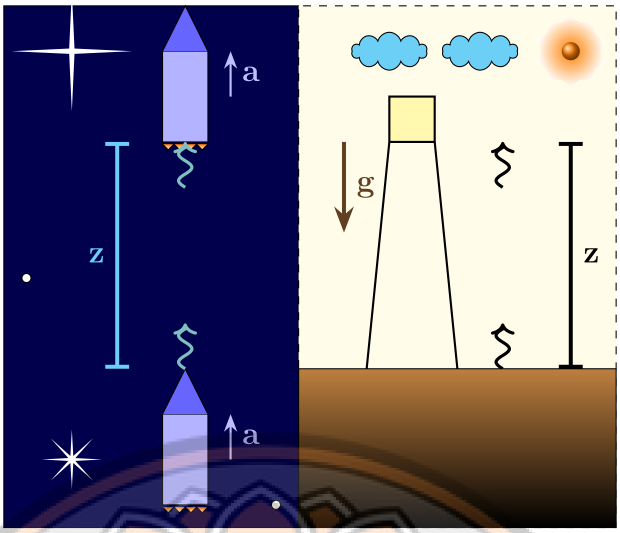


Figure 5 **Light wave in accelerated frame and gravitational field.**

From this figure, the observer in an elevator see the trajectory of light as curved line. And the outside observer sees the trajectory of the light as a straight line. We can think that spacetime in the accelerated frame (or gravitation) may be curved and causes the curved trajectory of light. As the WEP, we cannot measure a gravity in freely-falling frame even if it exists. Thus, gravity might be described by the concept of curved spacetime, because everything cannot avoid influence of gravity which is like the existence of curvature of spacetime.

To visualize gravity as the curvature of spacetime, we will consider the gravitational redshift as illustrated in Figure 4. From the left panel of Figure 4, the first rocket sends a photon to the second one where the time for photon travelling is $\Delta t = \frac{z}{c}$ where z and c are the distance between rockets and c is the speed of light, respectively. The second rocket has increasing of velocity as $\Delta v = a\Delta t = \frac{az}{c}$ where a is the acceleration of rockets. Thus the photon will be redshifted by Doppler effect

$$\frac{\Delta\lambda}{\lambda_0} = \frac{\Delta v}{c} = \frac{az}{c^2},$$

where λ_0 is the wavelength of photon which observed by the observer in the first rocket. The different of the wavelengths which observed by the observers in the first and second rockets is denoted by $\Delta\lambda$.

The right panel of Figure 4 should have the same phenomena because they obey WEP (gravity is acceleration locally),

$$\frac{\Delta\lambda}{\lambda_0} = \frac{\Delta v}{c} = \frac{gz}{c^2}.$$

This implies that even the rest frame in gravitational field, there is redshift effect. From the left panel of Figure 4, if the source emits one photon with a certain wavelength, the time which is observed by the tower's observer is $\Delta t = \frac{\lambda}{c}$. We found that the period of photon which is observed by the tower's observer is longer than one in the source as follows

$$\Delta t = \frac{\lambda}{c} = \left(\frac{gz}{c^2} + 1\right)\frac{\lambda_0}{c} > \Delta t_0.$$

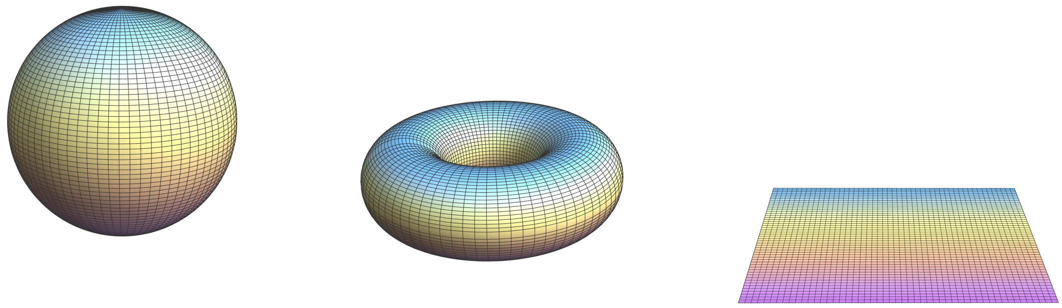
This implies that the time at a different position in gravitational field are different. In other words, the spacetime which photon travels through is curved. Therefore, gravity can be represented by curvature of spacetime.

2.1.2 Manifolds

As previously discussion, the gravity is the curvature of spacetime. In this section, we would like to find out a mathematical description of curvature of spacetime, that is a "manifold". An informal concept of a manifold is a space that may be curved or having complicated topology but in local region looks like Euclidean space $\mathbb{R}^n$. The definition of a manifold is compatible with curved spacetime. Such a spacetime is curved spacetime but locally flat. Some examples of manifold are shown in Figure 6. The sphere and the torus is sewing with Euclidean space $\mathbb{R}^2$. Obviously, a flat plane is also a manifold.

2.1.2.1 Objects on the manifold

Since many physical quantities can be defined via vectors, it is important to investigate how to define the vector and tensor in the manifold. In Euclidean



(a) Sphere **(b) Torus** **(c) Flat plane**

Figure 6 Examples of the manifold.

geometry, the vector is a quantity that has both magnitude and direction. For curved manifolds, the vector is just an object associated with a single point on manifold. Let us consider the curve $\gamma = \gamma(\lambda)$ which is parametrized by λ. This curve passes through a point p in manifold then the vector in this sense is a "tangent vectors to curve at point p" and a (tangent) vector space is the collection of all tangent vectors passing through point p. Considering a scalar function $f(x)$, at point p in some coordinate system x^{μ}, the change of the function $f(x)$ can be written as

$$\frac{d}{d\lambda}f = \frac{d}{d\lambda}f(x^{\mu}(\lambda)) = \frac{dx^{\mu}}{d\lambda}\frac{\partial f}{\partial x^{\mu}}. \tag{2.1}$$

Let us treat $\frac{\partial}{\partial x^{\mu}}$ as a basis known as coordinate basis, then we can write the vector in the form as

$$V = \frac{\partial x^{\mu}}{\partial \lambda}\partial_{\mu} = V^{\mu}\partial_{\mu}, \tag{2.2}$$

where $\frac{\partial x^{\mu}}{\partial \lambda}$ can be treated as a component (after this we will call it as vector) and $\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$ as a basis. Under general coordinate transformation $x^{\mu} \to x^{\mu'}$, the component of vector in a new coordinates $x^{\mu'}$ is written as follows

$$V^{\mu'} = \frac{\partial x^{\mu'}(x)}{\partial \lambda} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}\frac{\partial x^{\mu}}{\partial \lambda} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}V^{\mu}. \tag{2.3}$$

This is the formula of general coordinate transformation of a (component) vector.

For the basis in the coordinate $x^{\mu'}$, it is written by

$$\partial_{\mu'} f(x) = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \partial_{\mu} f(x). \tag{2.4}$$

Thus a vector in the coordinate $x^{\mu'}$ can be expressed by

$$V' = V^{\mu'} \partial_{\mu'} = \left(\frac{\partial x^{\mu'}}{\partial x^{\mu}} V^{\mu} \right) \left(\frac{\partial x^{\mu}}{\partial x^{\mu'}} \partial_{\mu} \right) = V, \tag{2.5}$$

where one used the eq.(2.2), (2.3) and (2.4). This implies that a vector is invariant under general coordinate transformation which is the important property of such a mathematical object.

In order to obtain a product of vector, it is useful to introduce a mapping as follows

$$\omega : T_p \to \mathbb{R}, \tag{2.6}$$

where T_p is the tangent space at point p. This object is in “cotangent space” T_p^*, and called “dual vector” or covariant vector. A dual vector can be written by

$$\omega = \omega_{\mu} dx^{\mu}, \tag{2.7}$$

where dx^{μ} is a basis of the dual vector and ω_{μ} is a component of the dual vector. The example of a dual vector is the gradient of a function $f(x)$ which is expressed as following

$$df = \frac{\partial f}{\partial x^{\mu}} dx^{\mu}, \tag{2.8}$$

where $\frac{\partial f}{\partial x^{\mu}}$ is the component of the dual vector df. The transformation law of a dual vector can be derived by considering the transformation of the gradient of a function f. Under general coordinate transformation $x^{\mu} \to x^{\mu'}$, the gradient of a function f in the new coordinate $x^{\mu'}$ is written by

$$df' = \frac{\partial f(x)}{\partial x^{\mu'}} dx^{\mu'}(x) = \left(\frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial f}{\partial x^{\mu}} \right) \left(\frac{\partial x^{\mu'}}{\partial x^{\mu}} dx^{\mu} \right). \tag{2.9}$$

This can infer that the transformation of the component and basis of a dual vector is written as follows

$$\omega_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\omega_{\mu}, \tag{2.10}$$

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}dx^{\mu}. \tag{2.11}$$

Moreover, the eq.(2.9) implies that a dual vector is also invariant under the general coordinate transformation. Let us consider the mapping in eq.(2.6) of vector and dual vector. We obtain

$$\begin{aligned} \omega(V) &= \omega_{\mu}dx^{\mu}(V^{\nu}\partial_{\nu}) \\ &= \omega_{\mu}V^{\nu}dx^{\mu}(\partial_{\nu}) \\ &= \omega_{\mu}V^{\nu}\delta^{\mu}_{\nu} \\ &= \omega_{\mu}V^{\mu} \in \mathbb{R}, \end{aligned}$$

where $dx^{\mu}(\partial_{\nu}) = \frac{dx^{\mu}}{dx^{\nu}} = \delta^{\mu}_{\nu}$.

The generalization of vectors and dual vectors is the "tensor" which can be written as follows

$$T = T^{\mu_1\mu_2\ldots\mu_n}{}_{\nu_1\nu_2\ldots\nu_m}\,\partial_{\mu_1}\otimes\ldots\otimes\partial_{\mu_n}\otimes dx^{\nu^1}\otimes\ldots\otimes dx^{\nu^m}, \tag{2.12}$$

where $\otimes$ is the tensor product. This is called tensor rank (n,m) where $T^{\mu_1\mu_2\ldots\mu_n}{}_{\nu_1\nu_2\ldots\nu_m}$ is a component. The transformation law of the tensor is given by

$$T^{\mu'_1\mu'_2\ldots\mu'_n}{}_{\nu'_1\nu'_2\ldots\nu'_m} = \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}}\cdots\frac{\partial x^{\mu'_n}}{\partial x^{\mu_n}}\frac{\partial x^{\nu_1}}{\partial x^{\nu'_1}}\cdots\frac{\partial x^{\nu_m}}{\partial x^{\nu'_m}}T^{\mu_1\mu_2\ldots\mu_n}{}_{\nu_1\nu_2\ldots\nu_m}. \tag{2.13}$$

Moreover, there is a special class of a tensor which is totally antisymmetrized tensor rank $(0,m)$,

$$\underline{\underline{\alpha}} \equiv \alpha_{\mu_1\mu_2\cdots\mu_m}dx^{\mu_1}\wedge dx^{m_2}\wedge\cdots\wedge dx^{\mu_m}, \tag{2.14}$$

where $\wedge$ is the wedge product which is the antisymmetrized tensor product. It is called "differential $m-$form", or "$m-$form". Vector, dual vector and tensor are object on a manifold. In the next section, we will present structure of a manifold and an operation on a manifold.

2.1.2.2 Structure of manifolds

The gravity can be described by the curvature of spacetime. In order to see the structure of curved spacetime, it is useful to introduce "metric tensor" , $g_{\mu\nu}$, via the distance in the manifold or spacetime interval as follows

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu. \tag{2.15}$$

In flat Minkowski spacetime, the interval for this spacetime can be written by

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = -(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2. \tag{2.16}$$

For spherical coordinates, it is given by

$$ds^2 = -(dt)^2 + (dr)^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2. \tag{2.17}$$

They are metrics for the flat spacetime but difference coordinate systems. For curved spacetimes, the simplest example is 2-sphere and the metric can be found by

$$ds^2 = (d\theta)^2 + (\sin\theta d\phi)^2. \tag{2.18}$$

Furthermore, the metric can be used to define the dot product (or inner product) as follows

$$W \cdot V = g_{\mu\nu}W^\mu V^\nu, \tag{2.19}$$

where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$. We can use the metric for "raising" and "lowering" index as

$$W_\nu = g_{\mu\nu}W^\mu \quad \text{and} \quad W^\nu = g^{\mu\nu}W_\mu.$$

Another property of the metric tensor is

$$g_{\nu\rho}g^{\rho\mu} = g_{\sigma\nu}g^{\sigma\mu} = \delta^\mu_\nu.$$

Moreover, the metric is symmetric tensor,

$$g_{\mu\nu} = g_{\nu\mu} \quad \text{and} \quad g^{\mu\nu} = g^{\nu\mu}.$$

2.1.2.3 Derivative

Since many physical quantities are rate of change, for example, acceleration is the rate of change of velocity with respect to time, the derivative operator must be properly defined. Let us start with considering the partial derivative of a vector V,

$$\frac{\partial V}{\partial x^\nu} = \frac{\partial}{\partial x^\nu}(V^\mu \partial_\mu).$$

The concept of derivative is the comparison between two vectors in different points in manifold. The basis of such two vectors are in different tangent spaces. Therefore it is useful to define "connection", $\Gamma^\rho_{\mu\nu}$, between two bases as follows

$$\frac{\partial}{\partial x^\nu}(V^\mu \partial_\mu) = \left(\frac{\partial V^\mu}{\partial x^\nu}\right)\partial_\mu + V^\mu\left(\frac{\partial}{\partial x^\nu}\partial_\mu\right) = \left(\frac{\partial V^\mu}{\partial x^\nu}\right)\partial_\mu + V^\mu\left(\Gamma^\rho_{\mu\nu}\partial_\rho\right). \quad (2.20)$$

Note that this object is not tensor because it does not transform like a tensor which is given by

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\Gamma^\nu_{\mu\lambda} - \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}. \quad (2.21)$$

Then eq.(2.20) can be rearranged

$$\frac{\partial}{\partial x^\nu}(V^\mu \partial_\mu) = \left(\frac{\partial V^\mu}{\partial x^\nu}\right)\partial_\mu + \left(\Gamma^\rho_{\mu\nu}V^\mu\right)\partial_\rho = \left(\frac{\partial V^\nu}{\partial x^\mu} + \Gamma^\mu_{\rho\nu}V^\rho\right)\partial_\mu = (\nabla_\mu V^\nu)\partial_\mu.$$

As a result, we can define the "covariant derivative", ∇_μ,

$$\nabla_\mu V^\nu \equiv \frac{\partial V^\nu}{\partial x^\mu} + \Gamma^\mu_{\rho\nu}V^\rho. \quad (2.22)$$

The covariant derivative is invariant under general coordinate transformation. The operation $\nabla_\mu V^\nu$ transforms as (1,1) tensor ,

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\nabla_\mu V^\nu. \quad (2.23)$$

Since this quantity transforms as a tensor, the covariant derivative of a tensor is invariant under general coordinate transformation which it is the important property in GR. Meanwhile, the partial derivative cannot serve such a property. To obtain the expression of covariant derivative of a dual vector, we consider the covariant derivative of a scalar, $\phi = A^\mu B_\mu$. One have

$$\begin{aligned} \nabla_\nu \phi &= \partial_\nu \phi, \\ \nabla_\nu (A^\mu B_\mu) &= \partial_\nu (A^\mu B_\mu) \\ &= (\partial_\nu A^\mu) B_\mu + A^\mu (\partial_\nu B_\mu). \end{aligned} \tag{2.24}$$

Substituting equation eq.(2.22) into eq.(2.24), we obtained

$$\begin{aligned} \nabla_\nu (A^\mu B_\mu) &= (\partial_\nu A^\mu) B_\mu + A^\mu (\partial_\nu B_\mu) \\ &= (\nabla_\nu A^\mu - \Gamma^\mu_{\rho\nu} A^\rho) B_\mu + A^\mu (\partial_\nu B_\mu), \\ A^\mu (\nabla_\nu B_\mu) + (\nabla_\nu A^\mu) B_\mu &= \left[A^\mu (\partial_\nu B_\mu) - \Gamma^\mu_{\rho\nu} A^\rho B_\mu \right] + B_\mu \nabla_\nu A^\mu, \\ A^\mu (\nabla_\nu B_\mu) &= A^\mu (\partial_\nu B_\mu) - \Gamma^\mu_{\rho\nu} A^\rho B_\mu \\ &= A^\mu (\partial_\nu B_\mu) - \Gamma^\rho_{\mu\nu} A^\mu B_\rho. \end{aligned}$$

As a result, we obtain the covariant of dual vector as follows

$$\nabla_\nu B_\mu = \partial_\nu B_\mu - \Gamma^\rho_{\mu\nu} B_\rho.$$

We can generalize the covariant derivative to the (k, l) tensor as

$$\begin{aligned} \nabla_\sigma T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} &= \partial_\sigma T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} \\ &\quad + \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1\lambda\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} + \ldots \\ &\quad - \Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1\mu_2\ldots\mu_k}{}_{\lambda\nu_2\ldots\nu_l} - \Gamma^{\lambda}_{\sigma\nu_2} T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\lambda\ldots\nu_l} - \ldots. \end{aligned} \tag{2.25}$$

In GR, there are assumptions for connection and metric tensor following

- Spacetime is “torsion free” : $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$.
- The metric obeys “metric compatibility” : $\nabla_\rho g_{\mu\nu} = 0$.

According to metric compatibility, we have

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - g_{\lambda\nu}\Gamma^\lambda_{\rho\mu} - g_{\mu\lambda}\Gamma^\lambda_{\rho\nu} = 0, \tag{2.26a}$$

$$\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - g_{\lambda\rho}\Gamma^\lambda_{\mu\nu} - g_{\nu\lambda}\Gamma^\lambda_{\nu\rho} = 0, \tag{2.26b}$$

$$\nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - g_{\lambda\mu}\Gamma^\lambda_{\nu\rho} - g_{\rho\lambda}\Gamma^\lambda_{\mu\nu} = 0. \tag{2.26c}$$

Considering eq.(2.26a) - eq.(2.26b) - eq.(2.26c), we obtained

$$\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2g_{\rho\lambda}\Gamma^\lambda_{\mu\nu} = 0,$$

where the torsion-free condition is also used. Then contracting with $g^{\rho\alpha}$,the connection can be written in terms of metric tensor as

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\alpha}\left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right). \tag{2.27}$$

This connection is called "Christoffel symbols" which is special kind of connection.

2.1.2.4 Integration

The key issue of the General Relativity is that physics does not change under general coordinate transformation. The appropriated objects such as tensors and covariant derivative are well defined in this theory. It is worthwhile to investigate the proper integral form which is invariant under general coordinate transformation in this theory.

The volume element d^nx which is not invariant under general coordinate transformation can be expressed by following

$$d^nx' = \left|\frac{\partial x^{\mu'}}{\partial x^\mu}\right| d^nx, \tag{2.28}$$

where $\left|\partial x^{\mu'}/\partial x^\mu\right|$ is the determinant of the Jacobian matrix $\partial x^{\mu'}/\partial x^\mu$. In order to redefine the proper volume element, one can construct the element d^nx from a vector element dx^μ with using a wedge product. The wedge product can be defined as antisymmetrized tensor product mapping $p-$form and $q-$form to $(p+q)-$form. For example, the wedge of $A_{\mu_1\cdots\mu_p}$ and $B_{\mu_1\cdots\mu_q}$ is written by

$$(A\wedge B)_{\mu_1\cdots\mu_{p+q}} = \frac{(p+q)!}{p!q!}A_{[\mu_1\cdots\mu_p}B_{\mu_{p+1}\cdots\mu_{p+q}]}. \tag{2.29}$$

Figure 7 visualizes the geometrical interpretation of the wedge product of vectors. For the wedge products of 2 and 3 vectors, they represent an area and a volume,

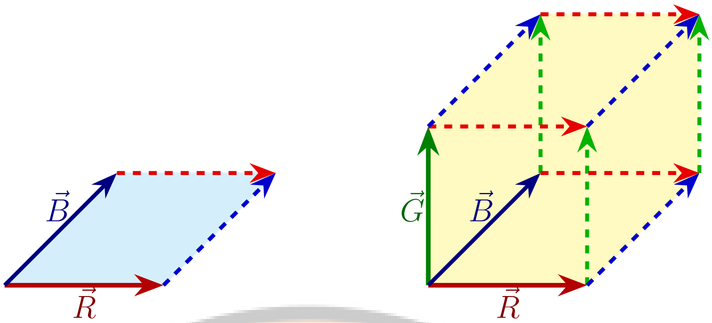


Figure 7 **Examples of the result of the wedge product.**

respectively. Then, the volume element can be defined by

$$d^n x \equiv dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{n-1}. \tag{2.30}$$

Since the product of wedge operator is antisymmetric, then we can write the volume element $d^n x$ as following

$$dx^0 \wedge \cdots \wedge dx^{n-1} = \frac{1}{n!}\tilde{\epsilon}_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}, \tag{2.31}$$

where $\tilde{\epsilon}_{\mu_1 \cdots \mu_n}$ is the Levi-Civita *symbol* defined by

$$\tilde{\epsilon}_{\mu_1 \cdots \mu_n} = \begin{cases} 1 & \text{if } \mu_1 \cdots \mu_n \text{ is an even permutation of } 01 \cdots (n-1) \\ -1 & \text{if } \mu_1 \cdots \mu_n \text{ is an odd permutation of } 01 \cdots (n-1) \\ 0 & \text{otherwise.} \end{cases} \tag{2.32}$$

The factor $1/n!$ exists for over counting by summation over permutation of the indices. The Levi-Civita *symbol* transforms under coordinate transformation as follows [30]

$$\tilde{\epsilon}_{\mu'_1 \cdots \mu'_n} = \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right| \frac{\partial x^{\mu_1}}{\partial x^{\mu'_1}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu'_n}} \tilde{\epsilon}_{\mu_1 \cdots \mu_n}. \tag{2.33}$$

The $\tilde{\epsilon}_{\mu_1 \cdots \mu_n}$ does not transform like a tensor in eq.(2.13), this is a reason why it is called “symbol”. Then, the coordinate transformation of eq.(2.31) can be written by

$$\begin{aligned}\tilde{\epsilon}_{\mu_1\cdots\mu_n} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n} &= \left(\left|\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right| \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}}\cdots\frac{\partial x^{\mu'_n}}{\partial x^{\mu_n}}\tilde{\epsilon}_{\mu'_1\cdots\mu'_n}\right) \\ &\qquad \left(\frac{\partial x^{\mu_1}}{\partial x^{\mu'_1}}\cdots\frac{\partial x^{\mu_n}}{\partial x^{\mu'_n}} dx^{\mu'_1}\wedge\cdots\wedge dx^{\mu'_n}\right) \\ &= \left|\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right| \tilde{\epsilon}_{\mu'_1\cdots\mu'_n} dx^{\mu'_1}\wedge\cdots\wedge dx^{\mu'_n}, \end{aligned} \tag{2.34}$$

where we have used $\left|\partial x^{\mu'}/\partial x^{\mu}\right|^{-1} = \left|\partial x^{\mu}/\partial x^{\mu'}\right|$. This implies that the volume element $d^n x$ is not invariant under general coordinate transformation. We can construct the proper volume element ε which is invariant under coordinate transformation by introducing the metric's determinant. Let's us consider the determinant of the following equation,

$$g_{\mu\nu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}\frac{\partial x^{\nu'}}{\partial x^{\nu}} g_{\mu'\nu'}. \tag{2.35}$$

Then the determinant of metric $g_{\mu\nu}$ transforms as

$$g(x) = \left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right|^2 g(x'). \tag{2.36}$$

Since the determinant of metric transforms by square of Jacobian, one can use the square root of metric's determinant to obtain the proper volume as follows

$$\begin{aligned}\sqrt{-g(x)}\tilde{\epsilon}_{\mu_1\cdots\mu_n} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n} &= \left(\left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right| \sqrt{-g(x')}\right) \\ &\qquad \left(\left|\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right| \tilde{\epsilon}_{\mu'_1\cdots\mu'_n} dx^{\mu'_1}\wedge\cdots\wedge dx^{\mu'_n}\right) \\ &= \sqrt{-g(x')}\tilde{\epsilon}_{\mu'_1\cdots\mu'_n} dx^{\mu'_1}\wedge\cdots\wedge dx^{\mu'_n}. \end{aligned} \tag{2.37}$$

Indeed, one can define the Levi-Civita *tensor* from the Levi-Civita *symbol* by using square root of the metric's determinant as follows

$$\epsilon_{\mu_1\cdots\mu_n} = \sqrt{-g}\tilde{\epsilon}_{\mu_1\cdots\mu_n} \tag{2.38}$$

Thus, the proper volume element ε is defined by

$$
\begin{aligned}
\varepsilon &\equiv \frac{1}{n!}\epsilon_{\mu_1\cdots\mu_n} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n}\\
&= \sqrt{-g}\,\tilde{\epsilon}_{01\cdots n-1} dx^0\wedge dx^1\wedge\cdots\wedge dx^{n-1}\\
&= \sqrt{-g}\,d^n x, \qquad (2.39)
\end{aligned}
$$

where eq,(2.30) and (2.32) are used. The integral I of scalar function $\phi(x)$ is written as

$$
\begin{aligned}
I &= \int \phi(x)\varepsilon\\
&= \int \phi(x)\sqrt{-g}\,d^n x. \qquad (2.40)
\end{aligned}
$$

In the language of differential form, the integral of $n-$form $\underline{\underline{\omega}} = \omega_{\mu_1\cdots\mu_n} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n}$ over $n-$dimensional region Σ which is a submanifold of manifold $\mathcal{M}$, is mapping from $n-$form to $(n-m)-$form $\underline{\underline{\Omega}} = \Omega_{\mu_1\cdots\mu_{n-m}} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{n-m}}$,

$$
\int_\Sigma : \underline{\underline{\omega}} \rightarrow \underline{\underline{\Omega}}. \qquad (2.41)
$$

The integration of $\omega_{\mu_1\cdots\mu_n}$ over $m-$dimensional region Σ is written by

$$
\begin{aligned}
\underline{\underline{\Omega}} &= \int_\Sigma \underline{\underline{\omega}} \qquad (2.42)\\
&= \int_\Sigma \omega_{\mu_1\cdots\mu_n} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_m}. \qquad (2.43)
\end{aligned}
$$

From the integration in eq.(2.40), one can treat the scalar function $\phi(x)$ as the $n-$form by absorbing the Levi-Civita tensor as follows

$$
\begin{aligned}
\underline{\underline{\phi}} &\equiv \phi(x)_{\mu_1\cdots\mu_n} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n}\\
&= \phi(x)\epsilon_{\mu_1\cdots\mu_n} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n}. \qquad (2.44)
\end{aligned}
$$

By using this language, the integrand in eq.(2.40) can be obtained as follows

$$\begin{aligned}\int_\Sigma \underset{=}{\phi} &= \int_\Sigma \phi_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \\ &= \int_\Sigma \phi(x) \epsilon_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \\ &= \int_\Sigma \phi(x) \sqrt{-g} \tilde{\epsilon}_{01 \cdots n-1} dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{n-1} \\ &= \int_\Sigma \phi(x) \sqrt{-g} d^n x. \qquad (2.45)\end{aligned}$$

This expression corresponds to the integration in eq.(2.40).

2.1.2.5 Geodesics

Since the covariant derivative is the change of the vector from one point to other point in curved spacetime, it is useful to define the moving of the vector between such two points via covariant derivative. If the vector is moved to some point without a change with respect to the vector at the origin, such moving is called “parallel transport”, illustrated in Figure 8.

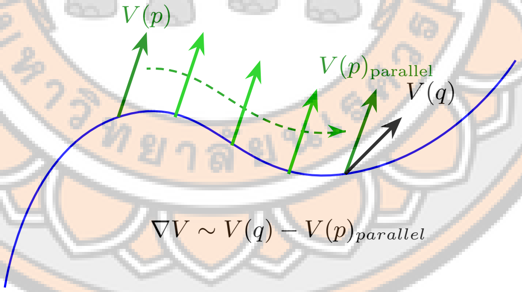


Figure 8 Difference between the vector and its parallel transport.

The parallel transported is transportation of a vector which keeps the vector constant, this depends on the path that we move. Since the parallel transport depends on path parameterized by λ, one can define the “directional covariant derivative” as

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda} \nabla_\mu. \qquad (2.46)$$

As a result, the parallel transport can be defined via the directional derivative as follows

$$\frac{D}{d\lambda}V^{\mu} = \frac{dx^{\rho}}{d\lambda}\nabla_{\rho}V^{\mu} = 0. \tag{2.47}$$

Then eq.(2.47) can be rearranged by

$$\begin{aligned}\frac{dx^{\rho}}{d\lambda}\nabla_{\rho}V^{\mu} &= \frac{dx^{\rho}}{d\lambda}\left(\frac{\partial V^{\mu}}{\partial x^{\rho}} + \Gamma^{\mu}_{\sigma\rho}V^{\sigma}\right)\\ 0 &= \frac{d}{d\lambda}V^{\mu} + \Gamma^{\mu}_{\sigma\rho}\frac{dx^{\rho}}{d\lambda}V^{\sigma}. \end{aligned}\tag{2.48}$$

This is called “equation of parallel transport”. In flat spacetime, $\Gamma^{\rho}_{\mu\nu} = 0$, eq.(2.48) reduces to

$$\frac{d}{d\lambda}V^{\mu} = 0. \tag{2.49}$$

By treating the vector as the tangent vector, $V^{\mu} = \frac{dx^{\mu}}{d\lambda}$, we have

$$\frac{d}{d\lambda}\frac{dx^{\mu}}{d\lambda} = 0. \tag{2.50}$$

Along a straight line, the tangent vector will unchange, as illustrated in Figure 9, so that parallel transport of the tangent vector can give us the shortest path in the flat spacetime. In the same manner, for the curved spacetime, the shortest path can be defined as parallel transport of the tangent vector as follows

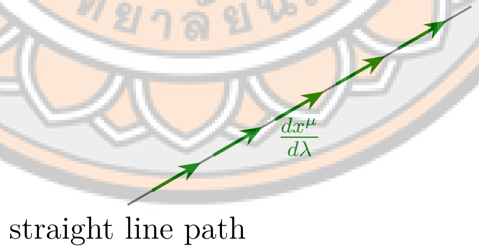


Figure 9 Shortest path in flat space.

$$\frac{d}{d\lambda}\frac{dx^{\mu}}{d\lambda} + \Gamma^{\mu}_{\sigma\rho}\frac{dx^{\rho}}{d\lambda}\frac{dx^{\sigma}}{d\lambda} = 0. \tag{2.51}$$

This generalization of the shortest path for curved spacetime is called “geodesic”. Furthermore, The parameter λ which satisfies the geodesics equation is called affine parameter. On other words, the geodesics is parametrized by affine parameter.

2.1.2.6 Symmetry

It is interesting to investigate the symmetry of spacetime, and it is useful to consider the **Lie derivative** of a vector A along the direction of a vector u defined by

$$\hat{L}_u A^{\mu} \equiv u^{\rho}\nabla_{\rho}A^{\alpha} - A^{\rho}\nabla_{\rho}u^{\mu}. \tag{2.52}$$

For the dual vector, the Lie derivative is given by

$$\hat{L}_u B_{\mu} = u^{\rho}\nabla_{\rho}B_{\mu} + B_{\rho}\nabla_{\mu}u^{\rho}. \tag{2.53}$$

The Lie derivative of the (m,n) tensor $T^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_n}$ is then

$$\begin{aligned}\hat{L}_u T^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_n} = u^{\rho}\nabla_{\rho}T^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_n} - T^{\rho\cdots\mu_m}{}_{\nu_1\cdots\nu_n}\nabla_{\rho}u^{\mu_1} - \cdots \\ + T^{\mu_1\cdots\mu_m}{}_{\rho\cdots\nu_n}\nabla_{\nu_1}u^{\rho} + \cdots .\end{aligned} \tag{2.54}$$

If the Lie derivative of metric tensor $g_{\mu\nu}$ vanishes, spacetime will admit symmetry infer from the vector u. We can find the spacetime symmetry via the metric tenor by using the Lie derivative as follows

$$\hat{L}_K g_{\mu\nu} = \nabla_{\mu}K_{\nu} + \nabla_{\nu}K_{\mu} = 0. \tag{2.55}$$

The reference vector K obeyed in eq.(2.55) is called “Killing vector”. The existence of the Killing vector implies the existence of spacetime symmetry. Moreover, the Killing vectors can infer the conserved quantities along the geodesic, suppose that v^{μ} is the tangent vector along the curve parametrized by λ, then

$$\frac{d}{d\lambda}(v^{\nu}K_{\nu}) = \nabla_{\rho}(v^{\nu}K_{\nu})\frac{\partial}{\partial\lambda}x^{\rho} = K_{\nu}v^{\rho}\nabla_{\rho}v^{\nu} + v^{\rho}v^{\nu}\nabla_{\rho}K_{\nu}.$$

For the first term, it corresponds to geodesic equation (2.51). According to eq.(2.55), $\nabla_{\nu}K_{\nu}$ is antisymmetric, the second term vanishes because $u^{\mu}u^{\nu}$ is symmetric. Thus

$$\frac{d}{d\lambda}(v^{\nu}K_{\nu}) = 0. \tag{2.56}$$

Moreover, the Killing vectors can be refered to conserved quantities of particle in such spacetime. For example, spacetime which admits the spherical symmetry and static condition is described by the metric as follows

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2,$$

where A and B are a function of radius r. Obviously, them metric is independent, so that the Killing vectors which satisfy eq.(2.55) can be written as

$$K^{\mu}_{(t)} = \frac{\partial x^{\mu}}{\partial t} \quad \text{and} \quad K^{\mu}_{(\phi)} = \frac{\partial x^{\mu}}{\partial \phi}. \tag{2.57}$$

As a result, the conserved quantities for such Killing vectors are expressed as

$$E = -v_{\nu} K^{\nu}_{(t)} \quad \text{and} \quad l = v_{\nu} K^{\nu}_{(\phi)}, \tag{2.58}$$

where v^{μ} is the tangent vector. If v^{μ} is the four-velocity of particle on geodesic, then E and l are energy and angular momentum per mass, respectively.

The existence of the timelike Killing vector, $K^{\mu}_{(t)}$, is infered from the static condition corresponding to time translation invariance. For the rotational invariance, there exists the spacelike Killing vector, $K^{\mu}_{(\phi)}$, corresponding to spherical symmetry of spacetime.

2.1.3 Curvature

We are dealing a "curved" manifold, then our next task is to measure the curvature of the manifold. To visualize, let us consider a vector moving along a different paths to the same final point as shown in Figure 10. From this figure, one can see that the initial vector and the final vector are different. Therefore, it is possible to measure the curvature by measuring the different of the vectors at the final point. This corresponds to the commutator of the covariant derivative because the covariant derivative of tensor in certain direction measures how much the vector changes relative to what it would have been if it had been parallel transported, Then the commutator measures the difference between parallel transporting the vector

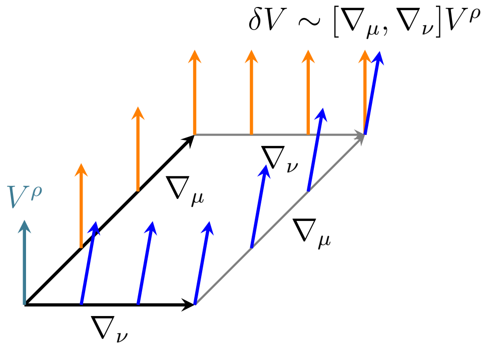


Figure 10 **Difference between parallel transport vectors along different paths.**

first one way and then the other, versus the opposite ordering. As a result, the commutator of the covariant derivative to a vector can be written as

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu] V^\rho &= \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho \\
&= \partial_\mu(\nabla_\nu V^\rho) - \Gamma^\lambda_{\mu\nu}(\nabla_\lambda V^\rho) + \Gamma^\rho_{\lambda\mu}(\nabla_\nu V^\lambda) \\
&\quad - \partial_\nu(\nabla_\mu V^\rho) + \Gamma^\lambda_{\nu\mu}(\nabla_\lambda V^\rho) - \Gamma^\rho_{\lambda\nu}(\nabla_\mu V^\lambda) \\
&= \partial_\mu(\nabla_\nu V^\rho) - \partial_\nu(\nabla_\mu V^\rho) + \Gamma^\rho_{\lambda\mu}(\nabla_\nu V^\lambda) - \Gamma^\rho_{\lambda\nu}(\nabla_\mu V^\lambda) \\
&= \partial_\mu \partial_\nu V^\rho + \partial_\mu(\Gamma^\rho_{\lambda\nu} V^\lambda) - \partial_\nu \partial_\mu V^\rho - \partial_\nu(\Gamma^\rho_{\lambda\mu} V^\lambda) \\
&\quad + \Gamma^\rho_{\lambda\mu} \partial_\nu V^\lambda + \Gamma^\rho_{\lambda\mu} \Gamma^\lambda_{\sigma\nu} V^\sigma - \Gamma^\rho_{\lambda\nu} \partial_\mu V^\lambda - \Gamma^\rho_{\lambda\nu} \Gamma^\lambda_{\sigma\mu} V^\sigma \\
&= (\partial_\mu \Gamma^\rho_{\lambda\nu}) V^\lambda - (\partial_\nu \Gamma^\rho_{\lambda\mu}) V^\lambda + \Gamma^\rho_{\lambda\mu} \Gamma^\lambda_{\sigma\nu} V^\sigma - \Gamma^\rho_{\lambda\nu} \Gamma^\lambda_{\sigma\mu} V^\sigma \\
&= \left(\partial_\mu \Gamma^\rho_{\lambda\nu} - \partial_\nu \Gamma^\rho_{\lambda\mu} + \Gamma^\rho_{\sigma\mu} \Gamma^\sigma_{\lambda\nu} - \Gamma^\rho_{\sigma\nu} \Gamma^\sigma_{\lambda\mu} \right) V^\lambda. \qquad (2.59)
\end{aligned}$$

Thus the quantity in parentheses can describe the spacetime curvature. It is called “Riemann tensor” and given by

$$R^\rho{}_{\lambda\mu\nu} \equiv \partial_\mu \Gamma^\rho_{\lambda\nu} - \partial_\nu \Gamma^\rho_{\lambda\mu} + \Gamma^\rho_{\sigma\mu} \Gamma^\sigma_{\lambda\nu} - \Gamma^\rho_{\sigma\nu} \Gamma^\sigma_{\lambda\mu}. \qquad (2.60)$$

Clearly, in flat spacetime, $\Gamma^\rho_{\lambda\nu} = 0$, then Riemann tensor vanishes. Furthermore, Riemann tensor has properties as follows

1. $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$: Antisymmetric under first two indices,
2. $R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}$: Antisymmetric under last two indices,

3. $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$: Symmetric under first and second pairs of indices,

4. $R_{\rho\sigma\mu\nu} + R_{\rho\nu\sigma\mu} + R_{\rho\mu\nu\sigma} = 0$: Cyclic permutations of last three indices vanishes.

These reduce the number of independent components of the Riemann tensor from n^4 to $\frac{1}{12}n^2(n^2-1)$, so for four-dimensional spacetime, it has 20 independent components. Another important property is the "Bianchi identity" which is written as

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0.$$

It can be written in another expression as follows

$$\begin{aligned}
\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} &= \frac{1}{6}\left(\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu}\right. \\
&\quad \left. - \nabla_\lambda R_{\sigma\rho\mu\nu} - \nabla_\sigma R_{\rho\lambda\mu\nu} - \nabla_\rho R_{\lambda\sigma\mu\nu}\right) \\
&= \frac{1}{3}\left(\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu}\right),
\end{aligned}$$

$$\nabla_\rho R_{\mu\nu\lambda\sigma} + \nabla_\sigma R_{\mu\nu\rho\lambda} + \nabla_\lambda R_{\mu\nu\sigma\rho} = 0, \tag{2.61}$$

where we used $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$. It is possible to define other tensors by contracting the indices of the Riemann curvature tensor as follows

$$R^{\sigma}{}_{\mu\sigma\nu} \equiv R_{\mu\nu} \quad \text{and} \quad R = g^{\mu\nu} R_{\mu\nu}, \tag{2.62}$$

where $R_{\mu\nu}$ is a symmetric tensor called "**Ricci tensor**" and R is called **"Ricci scalar"**. Contracting $g^{\mu\lambda}$ to the Bianchi equation, (2.61), we obtain

$$\begin{aligned}
g^{\mu\lambda}\left(\nabla_\rho R_{\mu\nu\lambda\sigma} + \nabla_\sigma R_{\mu\nu\rho\lambda} + \nabla_\lambda R_{\mu\nu\sigma\rho}\right) &= 0, \\
\nabla_\rho R^{\lambda}{}_{\nu\lambda\sigma} + \nabla_\sigma R^{\lambda}{}_{\nu\rho\lambda} + \nabla_\lambda R^{\lambda}{}_{\nu\sigma\rho} &= 0.
\end{aligned} \tag{2.63}$$

Contracting it again with $g^{\nu\sigma}$, one obtains

$$\begin{aligned}
g^{\nu\sigma}\left(\nabla_\rho R_{\nu\sigma} - \nabla_\sigma R_{\nu\rho} + \nabla_\lambda R^{\lambda}{}_{\nu\sigma\rho}\right) &= 0, \\
\nabla_\rho R - \nabla^\nu R_{\nu\rho} + \nabla_\lambda \left(g^{\nu\sigma} g^{\lambda\alpha} R_{\alpha\nu\sigma\rho}\right) &= 0.
\end{aligned} \tag{2.64}$$

The third term of the above equation can be expressed as

$$g^{\nu\sigma}g^{\lambda\alpha}\nabla_\lambda R_{\alpha\nu\sigma\rho} = -g^{\lambda\alpha}\nabla_\lambda R^{\sigma}{}_{\alpha\sigma\rho} = -\nabla^\alpha R_{\alpha\rho}.$$

Substituting the above equation to eq.(2.64), we obtain

$$\begin{aligned} \nabla_\rho R - 2\nabla^\nu R_{\nu\rho} = \nabla^\nu g_{\nu\rho} R - 2\nabla^\nu R_{\nu\rho} &= 0, \\ \nabla^\nu \left(R_{\nu\rho} - \frac{1}{2} R g_{\rho\nu} \right) &= 0. \end{aligned} \tag{2.65}$$

The term in the bracket is defined as “Einstein tensor”, then we have

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \quad \text{and} \quad \nabla^\mu G_{\mu\nu} = 0. \tag{2.66}$$

2.1.3.1 Curvature and Matter

As we mentioned before, gravity can be described by the curvature of spacetime, and matter causes gravity. So our equation should manifest the relation of matter and the curvature of spacetime. The energy-momentum tensor is a tensorial quantity that sufficiently describes properties of matter. We have many choices to describe spacetime curvature. The equation can be simply written as an arbitrary function of $R^{\rho}{}_{\sigma\mu\nu}, R_{\mu\nu}, R$ as

$$f(R^{\rho}{}_{\sigma\mu\nu}, R_{\mu\nu}, R) = \kappa T_{\mu\nu}.$$

Since $T_{\mu\nu}$ is symmetric and conserved,

$$\nabla_\mu T^{\mu\nu} = 0,$$

the left-hand side should have the same property. Fortunately, the tensor satisfying such the property is Einstein tensor as defined in eq.(2.66). Therefore, the equation can be written as

$$G_{\mu\nu} = \kappa T_{\mu\nu},$$

where κ is a constant. The Einstein gravity must be reduced to Newtonian gravity. This leads to $\kappa = 8\pi G$. Then we obtain

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.67}$$

which is called the "Einstein's field equations".

2.1.4 Geodesics Congruences

This section, we will talk about mathematical techiniques to consider the system of geodesics. The system of geodesics can be viewed as a deformable fluid. In the view point of fluid [31], we can define a small displacement ξ^a between two neighborhoods of the fluid elements as shown in Figure 11. From this figure, the fluid element can evolve with time and then the evolution of the displacement can generally take the forms as

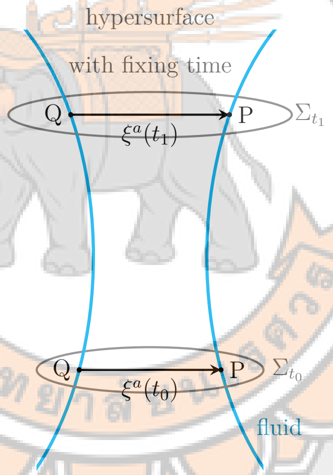


Figure 11 Deformation of fluid.

$$\frac{d\xi^a}{dt} = B^a{}_b(t)\xi^b + O(\xi^2), \tag{2.68}$$

where a and b are indices on the surface and $B^a{}_b$ is deformation tensor which describes how the surface changes in time. In general, the deformation tensor can be written as

$$B_{ab} = \frac{1}{2}\theta\delta_{ab} + \sigma_{ab} + \omega_{ab}, \tag{2.69}$$

where

- $\theta = B^a{}_a$ is the **expansion scalar** or trace part.
- $\sigma_{ab} = B_{(ab)} - \frac{1}{2}\theta\delta_{ab}$ is the **shear** tensor or the symmetric part.
- $\omega_{ab} = B_{[ab]}$ is the **rotation** tensor or antisymmetric part.

In GR, a congruence is a family of curves passing through the cross-section. We are interested in two cases, namely, timelike and null geodesics. For timelike case, we want to investigate the deviation vector ξ^μ between two neighbouring geodesics in the congruence as shown in Figure 11. Therefore, each cross-section can be parametrized by the proper time τ. In this sense, we can introduce the deformation tensor as

$$B_{\mu\nu} = \nabla_\mu t_\nu, \tag{2.70}$$

where t^μ is the tangent vector. It is important to know that the deformation tensor is purely transverse to the tangent vector. Since the deviation vector ξ^μ and the tangent vector are orthogonal to each other, we have $\hat{L}_\xi t^\nu = \hat{L}_t \xi^\mu = 0$. This leads to the relation $t^\nu \nabla_\nu \xi^\mu = \xi^\nu \nabla_\nu t^\mu$. By using these properties, we obtain

$$t^\nu \nabla_\nu \xi^\nu = B^\mu{}_\nu \xi^\nu. \tag{2.71}$$

This implies that the deformation tensor $B_{\mu\nu}$ measures the failure of ξ^μ to be parallel transported along the congruence. By using the same manner for the fluid, the deformation tensor can be expressed as

$$B_{\mu\nu} = \frac{1}{3}\theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}, \tag{2.72}$$

where $h_{\mu\nu}$ is a projection defined by $h_{\mu\nu} = g_{\mu\nu} + t_\mu t_\nu$. Actually, we consider the change of the deviation vector at each hypersurface of fixing proper time. In other words, the deformation tensor is a tensor on hypersurface which is orthogonal to the tangent vector.

2.1.4.1 Frobenius theorem

This theorem states that if each family of geodesics is orthogonal to everywhere of hypersurface, so-called the hypersurface orthogonal, then the rotation tensor $\omega_{\mu\nu} = 0$. We would show explicitly for only the case of timelike.

If the congruence is hypersurface orthogonal, then the tangent vector is proportional to the normal vector which can be expressed as

$$t_\mu = N\nabla_\mu \Phi,$$

where Φ is a function of hypersurface and N is a normalization factor. Then we obtain

$$t_{[\mu}\nabla_\nu t_{\alpha]} = \frac{1}{3!}(t_\mu\nabla_\nu t_\alpha + t_\alpha\nabla_\mu t_\nu + t_\nu\nabla_\alpha t_\mu - t_\nu\nabla_\mu t_\alpha - t_\mu\nabla_\alpha t_\mu - t_\alpha\nabla_\nu t_\mu) = 0.$$

Note that we used $\nabla_\mu t_\nu \propto \nabla_\mu\nabla_\nu\Phi = \nabla_\nu\nabla_\mu\Phi$. Using property of rotation tensor, we obtain

$$\begin{aligned} 3! t_{[\mu}\nabla_\nu t_{\alpha]} &= 2(t_{[\mu}\nabla_{\nu]} t_\alpha + t_{[\alpha}\nabla_{\mu]} t_\nu + t_{[\nu}\nabla_{\alpha]} t_\mu) \\ &= 2(B_{[\mu\nu]} t_\alpha + B_{[\alpha\mu]} t_\nu + B_{[\nu\alpha]} t_\mu), \\ 0 &= 2(\omega_{\mu\nu} t_\alpha + \omega_{\alpha\mu} t_\nu + \omega_{\nu\alpha} t_\mu). \end{aligned} \tag{2.73}$$

Contracting the above equation with t^α and using $\omega_{\alpha\mu} t^\alpha = \omega_{\nu\alpha} t^\alpha = 0$, we have

$$\omega_{\mu\nu} = 0. \tag{2.74}$$

Therefore, we have proven the **Frobenius's theorem**.

2.1.4.2 Raychaudhuri's equation

There is an evolution equation for expansion scalar. In order to obtain such a equation, we begin with the evolution equation of the deformation tensor in a direction of the tangent vector,

$$\begin{aligned}
t^{\rho}\nabla_{\rho}B_{\mu\nu} &= t^{\rho}\nabla_{\rho}\nabla_{\mu}t_{\nu} \\
&= (\nabla_{\mu}\nabla_{\rho}t_{\nu} - R_{\mu\sigma\nu\rho}t^{\sigma})t^{\rho} \\
&= \nabla_{\mu}(t^{\rho}\nabla_{\rho}t_{\nu}) - \nabla_{\mu}t^{\rho}\nabla_{\rho}t_{\nu} - R_{\mu\sigma\nu\rho}t^{\sigma}t^{\rho} \\
&= -B_{\mu}{}^{\rho}B_{\rho\nu} - R_{\mu\sigma\nu\rho}t^{\sigma}t^{\rho}. \qquad (2.75)
\end{aligned}$$

Since the tangent vector satisfies geodesic equation, $t^{\rho}\nabla_{\rho}t_{\nu} = 0$. Contracting the above equation with $g^{\mu\nu}$, we obtain

$$t^{\rho}\nabla_{\rho}B^{\nu}{}_{\nu} = -B^{\nu\rho}B_{\rho\nu} - R_{\sigma\rho}t^{\sigma}t^{\rho}.$$

From the relations as follows $t^{\mu} = \frac{dx^{\mu}}{d\tau}$, $\nabla_{\mu}B^{\nu}{}_{\nu} = \partial_{\mu}\theta$, and $B^{\mu\nu}B_{\nu\mu} = \frac{1}{3}\theta^{2} + \sigma^{\mu\nu}\sigma_{\mu\nu} - \omega^{\mu\nu}\omega_{\mu\nu}$, the equation for the evolution of expansion scalar can be written as

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^{2} - \sigma^{\mu\nu}\sigma_{\mu\nu} + \omega^{\mu\nu}\omega_{\mu\nu} - R_{\mu\nu}t^{\mu}t^{\nu}. \qquad (2.76)$$

This equation is known as “**Raychaudhuri’s equation**” for the congruence of timelike geodesics.

For the null case, there is tangent vector k^{μ} with $k^{\mu}k_{\mu} = 0$, the deformation tensor can be expressed as

$$B_{\mu\nu} = \nabla_{\mu}k_{\nu}.$$

The relation between deviation vector and deformation tensor can be obtbtained in similar way for timelike case as

$$(k^{\nu}\nabla_{\nu}\tilde{\xi}^{\mu})^{\sim} = \tilde{B}^{\mu}{}_{\nu}\tilde{\xi}^{\nu}, \qquad (2.77)$$

where $(k^{\nu}\nabla_{\nu}\tilde{\xi}^{\mu})^{\sim} = h^{\mu}{}_{\nu}(k^{\nu}\nabla_{\nu}\tilde{\xi}^{\mu})$, $\tilde{\xi}^{\nu} = h^{\mu}{}_{\nu}\xi^{\nu}$ and $\tilde{B}_{\mu\nu} = h^{\rho}{}_{\mu}h^{\sigma}{}_{\nu}B_{\rho\sigma}$. Note that for the null case, we need to project on the deformation tensor on two the surface which is orthogonal to both tangent vector k^{μ} and auxiliary vector N^{μ}. By performing in the same way as done in the timelike case, we obtain

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^{2} - \sigma^{\mu\nu}\sigma_{\mu\nu} + \omega^{\mu\nu}\omega_{\mu\nu} - R_{\mu\nu}k^{\mu}k^{\nu}, \qquad (2.78)$$

where the deformation tensor can be expressed as

$$\tilde{B}_{\mu\nu} = \frac{1}{2}\theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}. \tag{2.79}$$

Another feature of expansion scalar for the timelike case is the fractional rate of change of the congruence's cross-sectional volume δV which can be expressed as

$$\theta = \frac{1}{\delta V}\frac{d}{d\tau}\delta V.$$

For the null case, the expansion scalar is the fractional rate of change of the congruence's cross-sectional area δA as follows

$$\theta = \frac{1}{\delta A}\frac{d}{d\lambda}\delta A. \tag{2.80}$$

2.1.5 Solution of Einstein's Field Equation

It is very hard to solve the Einstein's field equation for the general solution because it composes of coupled nonlinear second-order differential equations as illustrated in Figure 12. In order to solve for an exact solution, one may need to impose some conditions. By applying static condition and spherical symmetry, the first solution was discovered by Karl Schwarzschild in 1916.

$$\boxed{\Gamma^{\rho}_{\mu\nu} = \tfrac{1}{2}g^{\rho\alpha}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)}$$

$$\Downarrow$$

$$\boxed{R^{\rho}{}_{\lambda\mu\nu} = \partial_\mu \Gamma^{\rho}_{\lambda\nu} - \partial_\nu \Gamma^{\rho}_{\lambda\mu} + \Gamma^{\rho}_{\sigma\mu}\Gamma^{\sigma}_{\lambda\nu} - \Gamma^{\rho}_{\sigma\nu}\Gamma^{\sigma}_{\lambda\mu}}$$

$$\Downarrow$$

$$\boxed{R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}}$$

Figure 12 Solving the Einstein's field equation.

2.1.5.1 Schwarzschild solution

Since in our Universe, it composes of stars (and etc.) and shape of mostly star is spherical, it is useful for solving the curvature around massive spherical object.

The first condition that we apply is the spherical symmetry. For a general metric in coordinates $x^\mu = (t, r, \theta, \phi)$ is given by

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -g_{tt} dt^2 + 2g_{tr} dt dr + 2g_{t\theta} dt d\theta + 2g_{t\phi} dt d\phi + g_{rr} dr^2 \\ &\quad + 2g_{r\theta} dr d\theta + 2g_{r\phi} dr d\phi + 2g_{\theta\phi} d\theta d\phi + g_{\theta\theta} d\theta^2 + g_{\phi\phi} d\phi^2. \end{aligned} \tag{2.81}$$

For the spherical symmetry, the metric tensor must be a function of radial and time $g_{\mu\nu} = f(t, r)$, and invariant under angular reversion

$$d\theta \rightarrow -d\theta, \quad d\phi \rightarrow -d\phi. \tag{2.82}$$

As a result, some components of metric tensor must be vanished as follows

$$g_{t\theta} = g_{t\phi} = g_{r\theta} = g_{r\phi} = 0.$$

By rewritting the components of the metric as

$$g_{tt} = A(t, r),\ g_{tr} = B(t, r),\ g_{rr} = C(t, r),\ g_{\theta\phi} = D(t, r),\ g_{\theta\theta} = E(t, r),\ g_{\phi\phi} = F(t, r),$$

the metric becomes

$$\begin{aligned} ds^2 = -A(t, r) dt^2 + B(t, r) dt dr + C(t, r) dr^2 \\ + D(t, r) d\theta d\phi + E(t, r) d\theta^2 + F(t, r) d\phi^2. \end{aligned} \tag{2.83}$$

For the last three terms, $D(t, r) d\theta d\phi + E(t, r) d\theta^2 + F(t, r) d\phi^2$, is a two-dimensional spatial metric, and there is a theorem which states that “any two dimensional Riemannian manifold is conformally flat” or

$$g_{ab} = \Omega^2 \eta_{ab}, \tag{2.84}$$

where the indices a and b run over 2 to 3, and Ω is a conformal function. Therefore, we obtain

$$D(t, r) d\theta d\phi + E(t, r) d\theta^2 + F(t, r) d\phi^2 = H(t, r)(r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2). \tag{2.85}$$

By defining a new timelike coordinate as

$$dt' = -A(t,r)dt + \frac{B(t,r)}{2}dr$$

and using

$$dt'^2 = A^2(t,r)dt^2 - A(t,r)B(t,r)dtdr + \frac{B^2(t,r)}{4}dr^2,$$

we obtain

$$\begin{aligned} A(t,r)dt^2 - B(t,r)dtdr &= A^{-1}(t,r)dt'^2 - \frac{B^2(t,r)}{4}dr^2, \\ &= A'(t',r)dt'^2 - B'(t',r)dr^2. \end{aligned} \tag{2.86}$$

As a result, the metric can be written as

$$ds^2 = -A'(t',r)dt'^2 + B'(t',r)dr^2 + H(t,r)(r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2). \tag{2.87}$$

By defining the radial coordinate as $r^2 H(t,r) = r'^2$, we obtain

$$ds^2 = -A''(t',r')dt'^2 + B''(t',r')dr^2 + r'^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.88}$$

By rewritting $A''(t',r') = e^{2\alpha(t',r')}$, $B''(t',r') = e^{2\beta(t',r')}$ and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, and relabelling $t' \to t$ and $r' \to r$, the metric for the spherical symmetry can be written as

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + r^2 d\Omega^2. \tag{2.89}$$

Next, we will solve the Einstein's field equation for obtaining the solution of $e^{\alpha(t,r)}$ and $e^{\beta(t,r)}$. By considering the spacetime around a spherical object, we have to solve the vacuum Einstein's field equation as follows

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0. \tag{2.90}$$

Contracting with $g^{\mu\nu}$, we found that $R - \frac{4}{2}R = 0$ and then

$$R = 0. \tag{2.91}$$

Substituting this into the vacuum Einstein's field equation (2.90), we obtain

$$R_{\mu\nu} = 0. \tag{2.92}$$

The Ricci tensor can be expressed in terms of the Christoffel symbol as

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} = \partial_\rho \Gamma^{\rho}_{\mu\nu} - \partial_\nu \Gamma^{\rho}_{\mu\rho} + \Gamma^{\rho}_{\sigma\rho}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\rho}_{\sigma\nu}\Gamma^{\sigma}_{\mu\rho}, \tag{2.93}$$

while the Christoffel symbol can be written in terms of the metric tensor as

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha}\left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right).$$

By substuting the metric from eq.(2.89) into the above equation, the non-vanishing components of the Christoffel symbol are

$$\begin{aligned}
&\Gamma^{t}_{tt} = \dot{\alpha}, && \Gamma^{t}_{tr} = \alpha', && \Gamma^{t}_{rr} = \dot{\beta} e^{2(\alpha-\beta)}, \\
&\Gamma^{r}_{tt} = \alpha' e^{2(\alpha-\beta)}, && \Gamma^{r}_{tr} = \dot{\beta}, && \Gamma^{r}_{rr} = \beta', \\
&\Gamma^{\theta}_{r\theta} = \frac{1}{r}, && \Gamma^{r}_{\theta\theta} = -re^{-2\beta}, && \Gamma^{\phi}_{r\phi} = \frac{1}{r}, \\
&\Gamma^{r}_{\phi\phi} = -re^{-2\beta}\sin^2\theta, && \Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta, && \Gamma^{\phi}_{\theta\phi} = \frac{\cos\theta}{\sin\theta},
\end{aligned} \tag{2.94}$$

where the dot denotes time derivative and the prime denotes radial derivative. Substituting above quantities into eq.(2.93), the nonvanishing components of the Ricci tensor are

$$\begin{aligned}
R_{tt} &= (\ddot{\beta} + \dot{\beta}^2 - \dot{\alpha}\dot{\beta}) + e^{2(\alpha-\beta)}(\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{2}{r}\alpha'), \\
R_{rr} &= -(\alpha'' + \alpha'^2 - \alpha'\beta' - \frac{2}{r}\beta') + e^{2(\beta-\alpha)}(\ddot{\beta} + \dot{\beta}^2 - \dot{\alpha}\dot{\beta}), \\
R_{tr} &= \frac{2}{r}\dot{\beta}, \\
R_{\theta\theta} &= e^{-2\beta}[r(\beta' - \alpha') - 1] + 1, \\
R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta,
\end{aligned} \tag{2.95}$$

For $R_{tr} = 0$, it corresponds to $\dot{\beta} = 0$, one gets

$$\beta = \beta(r). \tag{2.96}$$

By taking time derivative to $R_{\theta\theta} = 0$, it corresponds to $(\dot{\alpha'}) = 0$, we get

$$\alpha = \alpha_r(r) + \alpha_t(t), \tag{2.97}$$

where α_r and α_t respectively are the functions of r and t solely. Considering $e^{-2(\alpha-\beta)}R_{tt} + R_{rr} = 0$, it is equivalent to

$$\alpha' = -\beta'. \tag{2.98}$$

Integrating this equation over r, we obtain

$$\begin{aligned} -\beta &= \alpha + f(t) \\ &= \alpha_r(r) + \alpha_t(t) + f(t) \end{aligned} \tag{2.99}$$

Since $\beta = \beta(r)$, we have $\alpha_t(t) + f(t) = 0$. As a result, and then

$$\beta = -\alpha_r(r). \tag{2.100}$$

Now, the metric becomes

$$ds^2 = -e^{2\alpha_r(r)}e^{2\alpha_t(t)}dt^2 + e^{-2\alpha_r(r)}dr^2 + r^2 d\Omega^2.$$

Defining the new time coordinate such as $dt' = e^{\alpha_t(t)}dt$, we obtain

$$ds^2 = -e^{2\alpha_r(r)}dt'^2 + e^{-2\alpha_r(r)}dr^2 + r^2 d\Omega^2.$$

Relabeling $t \to t'$ and $\alpha_r \to \alpha$, the spherical symmetric metric is written as follows

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{-2\alpha(r)}dr^2 + r^2 d\Omega^2. \tag{2.101}$$

Notice that $g_{\mu\nu}$ is independent on time and $g_{ti} = 0$, they correspond to the **static** spacetime. Moreover, even if we start with time dependence metric, we still obtain time-independent one. This is **Birkhoff's theorem** which states that "Any spherically symmetric vacuum solution is static". Now, let us solve eq.(2.92) by considering $R_{\theta\theta} = 0$. Applying eq.(2.100), we obtain

$$\begin{aligned} 2r\alpha'(r)e^{2\alpha(r)} + e^{2\alpha(r)} &= 1, \\ \partial_r[re^{2\alpha(r)}] &= 1, \\ re^{2\alpha(r)} &= r + C, \\ e^{2\alpha(r)} &= 1 + \frac{C}{r}. \end{aligned} \tag{2.102}$$

where C is an integration constant. Since the theory should be reduced to the Newtonian theory in the weak field limit, the intregation constant must be

$$C = -2mG, \tag{2.103}$$

where m is a mass of that object and G is the universal gravitational constant. Finally, the metric for spherical symmetry can be written as

$$ds^2 = -\left(1 - \frac{2mG}{r}\right)dt^2 + \left(1 - \frac{2mG}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{2.104}$$

This metric is known as “**Schwarzschild metric**”.

2.1.5.2 Reissner-Nordström solution

In the previous section, we solved the Einstein’s field equation in vacuum. The solution of Einstein’s field equation can be solved by relaxing the vacuum condition. One considers spacetime which possesses electric charge, and then the electric field would propagate through the spacetime. Such a solution was proposed by Hans Reissner and Gunnar Nordström in 1917.

It is impossible to collect charge particles because the Coulumb force is much more stronger than the gravitational force, for example the force between two electrons,

$$\vec{F}_e = \frac{ke^2}{r^2} \approx 2.3 \times 10^{-28} \quad \text{and} \quad \vec{F}_g = \frac{Gm_e^2}{r^2} \approx 5.5 \times 10^{-71}. \tag{2.105}$$

However, the star cannot be formed due to particles with same charges. We will consider a charged solution as a toy model of the solutions of GR in this section. Moreover, we will include the magnetic charge for this consideration.

We start with the static spherical symmetric metric (2.101). The charged solution differs from the Schwarzschild solution by the existence of the energy momentum tensor of the gauge field while Schwarzschild one is vacuum solution (no energy-momentum tensor). The energy-momentum tensor is given by

$$T_{\mu\nu} = \frac{1}{\mu_0}\left(F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \tag{2.106}$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the field stress tensor and A_μ is the four-vector potential. The components of field strength tensor can be expressed in terms of the electric and magnetic fields as follows

$$F_{0i} = -E_i \quad \text{and} \quad F_{ij} = \frac{1}{2}\epsilon_{ijk}B^k = \frac{1}{2}\sqrt{g}\tilde{\epsilon}_{ijk}B^k, \tag{2.107}$$

where g is the metric's determinant. By considering the static condition and spherically symmetric metric, we have $E_\theta = E_\phi = B_\theta = B_\phi = 0$, and $E_r(r) = E(r) \neq 0$ and $B_r(r) = B(r) \neq 0$. The the field strenght tensor can be reexpressed as

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E(r) & 0 & 0 \\ E(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2\sin\theta B(r) \\ 0 & 0 & -r^2\sin\theta B(r) & 0 \end{pmatrix}. \tag{2.108}$$

The equations of motion of the gauge field can be written as

$$\nabla_\mu F^{\mu\nu} = 0, \tag{2.109}$$

$$\nabla_\rho F_{\mu\nu} + \nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} = 0. \tag{2.110}$$

In curved spacetime, the covariant divergence can be written in terms of the partial derivative as

$$\nabla_\rho T^{\rho\nu_2\nu_3\cdots} = \frac{1}{\sqrt{-g}}\partial_\rho\left(\sqrt{-g}T^{\rho\nu_2\nu_3\cdots}\right). \tag{2.111}$$

The derivation will be shown in section 2.2.2.1. Hence eq.(2.109) can be written as

$$\nabla_\rho F^{\rho\nu} = \frac{1}{\sqrt{-g}}\partial_\rho\left(\sqrt{-g}F^{\rho\nu}\right). \tag{2.112}$$

Considering components $\nu = 0$, we have

$$\begin{aligned}
\nabla_\rho F^{\rho 0} &= \nabla_i F^{i0}, \\
&= \frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}F^{i0}) \\
&= \frac{1}{e^{\alpha+\beta}r^2\sin\theta}\partial_r\left(e^{\alpha+\beta}r^2\sin\theta F^{10}\right) \\
&= \frac{1}{e^{\alpha+\beta}r^2\sin\theta}\partial_r\left(e^{\alpha+\beta}r^2\sin\theta g^{00}g^{11}F_{10}\right) \\
&= \frac{1}{e^{\alpha+\beta}r^2}\partial_r\left(e^{\alpha+\beta}r^2 E\right) = 0. \qquad (2.113)
\end{aligned}$$

Integrating it, the electric field is written as

$$e^{-(\alpha+\beta)}r^2 E = C_1 \quad \rightarrow \quad E = \frac{Ce^{(\alpha+\beta)}}{r^2}, \qquad (2.114)$$

where C_1 is an integration constant. Since the solution must reduce to one for the flat spacetime at large r, or $e^\alpha|_{r\rightarrow\infty} = e^\beta|_{r\rightarrow\infty} \approx 1$, the integration constant can be obtained as

$$E|_{r\rightarrow\infty} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}, \qquad (2.115)$$

where Q is an electric charge and ϵ_0 is the electric permittivity in vacuum. Thus the integration constant C_1 is interpreted as

$$C_1 = \frac{Q}{4\pi\epsilon_0}. \qquad (2.116)$$

Then the electric field can be written as

$$E(r) = e^{(\alpha+\beta)}\frac{Q}{4\pi\epsilon_0 r^2}. \qquad (2.117)$$

To find the magnetic field, we have to consider eq.(2.110). Considering the torsion-free spacetime and the antisymmetric property of $F_{\mu\nu}$, one obtains

$$\begin{aligned}\nabla_\rho F_{\mu\nu} + \nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} &= \partial_\rho F_{\mu\nu} - \Gamma^\lambda_{\rho\mu} F_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} F_{\mu\lambda} \\ &\quad + \partial_\mu F_{\nu\rho} - \Gamma^\lambda_{\mu\nu} F_{\lambda\rho} - \Gamma^\lambda_{\mu\rho} F_{\nu\lambda} \\ &\quad + \partial_\nu F_{\rho\mu} - \Gamma^\lambda_{\nu\rho} F_{\lambda\mu} - \Gamma^\lambda_{\nu\mu} F_{\rho\lambda} \\ &= \partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu}. \qquad (2.118)\end{aligned}$$

Next, considering the component $\rho = r$, $\mu = \theta$, and $\nu = \phi$, eq.(2.118) becomes

$$\begin{aligned}\partial_r F_{\theta\phi} + \partial_\theta F_{\phi r} + \partial_\phi F_{r\phi} &= \partial_r F_{\theta\phi}, \\ 0 &= \partial_r [r^2 \sin\theta B(r)]. \qquad (2.119)\end{aligned}$$

We obtain

$$B(r) = \frac{C_2}{r^2}, \qquad (2.120)$$

where C_2 is integration constant. Suppose that there exists the magnetic monopole in the flat spacetime. In the similar way, the integration constant can obtained as

$$B|_{r\to\infty} = \frac{\mu_0}{4\pi}\frac{P}{r^2} \quad \Rightarrow \quad C_2 = \frac{\mu_0 P}{4\pi}, \qquad (2.121)$$

where P is a magnetic charge and μ_0 is the magnetic permittivity in vacuum. The magnetic field can be written as

$$B(r) = \frac{\mu_0 P}{4\pi r^2}. \qquad (2.122)$$

Let us compute a term contributed to energy momentum tensor as follows

$$\begin{aligned}F_{\rho\sigma} F^{\rho\sigma} &= g^{\rho\rho'} g^{\sigma\sigma'} F_{\rho'\sigma'} F_{\rho\sigma} \\ &= 2 g^{rr} g^{tt} (F_{rt})^2 + 2 g^{\theta\theta} g^{\phi\phi} (F_{\theta\phi})^2 \\ &= 2\left[-e^{-2(\alpha+\beta)} E^2 + B^2\right]. \qquad (2.123)\end{aligned}$$

Then non-vanishing components of the energy momentum tensor are

$$T_{tt} = \frac{1}{\mu_0}\left(g^{\rho\sigma}F_{t\rho}F_{t\sigma} - \frac{1}{4}g_{tt}F^{\rho\sigma}F_{\rho\sigma}\right) = \frac{1}{2\mu_0}e^{2\alpha}\left[e^{-2(\alpha+\beta)}E^2 + B^2\right], \tag{2.124}$$

$$T_{rr} = \frac{1}{\mu_0}\left(g^{\rho\sigma}F_{r\rho}F_{r\sigma} - \frac{1}{4}g_{rr}F^{\rho\sigma}F_{\rho\sigma}\right) = -\frac{1}{2\mu_0}e^{2\beta}\left[e^{-2(\alpha+\beta)}E^2 + B^2\right], \tag{2.125}$$

$$T_{\theta\theta} = \frac{1}{\mu_0}\left(g^{\rho\sigma}F_{\theta\rho}F_{\theta\sigma} - \frac{1}{4}g_{\theta\theta}F^{\rho\sigma}F_{\rho\sigma}\right) = \frac{r^2}{2\mu_0}\left[e^{-2(\alpha+\beta)}E^2 + B^2\right], \tag{2.126}$$

$$T_{\phi\phi} = \frac{1}{\mu_0}\left(g^{\rho\sigma}F_{\phi\rho}F_{\phi\sigma} - \frac{1}{4}g_{\phi\phi}F^{\rho\sigma}F_{\rho\sigma}\right) = \frac{r^2\sin^2\theta}{2\mu_0}\left[e^{-2(\alpha+\beta)}E^2 + B^2\right]. \tag{2.127}$$

By using the spherically symmetric metric (2.101). The Ricci scalar can be computed as

$$\begin{aligned} R = R^t{}_t + R^r{}_r + R^\theta{}_\theta + R^\phi{}_\phi = -2e^{-2\beta}\Bigg[&\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{2}{r}\left(\alpha' - \beta'\right) \\ &+\frac{1}{r^2}\left(1 - e^{2\beta}\right)\Bigg]. \end{aligned} \tag{2.128}$$

Substituting Ricci scalar into Einstein tensor, we obtain

$$G_{tt} = e^{2(\alpha-\beta)}\left[\frac{2\beta'}{r} - \frac{1}{r^2}\left(1 - e^{2\beta}\right)\right], \tag{2.129}$$

$$G_{rr} = \frac{2\alpha'}{r} + \frac{1}{r^2}\left(1 - e^{2\beta}\right), \tag{2.130}$$

$$G_{\theta\theta} = e^{-2\beta}\left[r^2\left(\alpha'' + \alpha'^2 - \alpha'\beta'\right) + r(\alpha' - \beta')\right], \tag{2.131}$$

$$G_{\phi\phi} = \sin^2\theta G_{\theta\theta}. \tag{2.132}$$

Then the nonvanishing components of Einstein's field equation are

$$\begin{aligned} \text{(tt)} : \text{(2.129) and (2.124)} ;\quad & e^{-2\beta}\left[\frac{2\beta'}{r} - \frac{1}{r^2}\left(1 - e^{2\beta}\right)\right] \\ &= \frac{4\pi G}{\mu_0}\left[e^{-2(\alpha+\beta)}E^2 + B^2\right], \end{aligned} \tag{2.133}$$

$$\begin{aligned} \text{(rr)} : \text{(2.130) and (2.125)} ;\quad & e^{-2\beta}\left[\frac{2\alpha'}{r} + \frac{1}{r^2}\left(1 - e^{2\beta}\right)\right] \\ &= -\frac{4\pi G}{\mu_0}\left[e^{-2(\alpha+\beta)}E^2 + B^2\right], \end{aligned} \tag{2.134}$$

$$\begin{aligned} (\theta\theta) : \text{(2.131) and (2.126)} ;\quad & e^{-2\beta}\left[r^2\left(\alpha'' + \alpha'^2 - \alpha'\beta'\right) + r(\alpha' - \beta')\right] \\ &= \frac{4\pi G r^2}{\mu_0}\left[e^{-2(\alpha+\beta)}E^2 + B^2\right]. \end{aligned} \tag{2.135}$$

Combining eq.(2.133) and eq.(2.134) , we obtain

$$\alpha' + \beta' = 0 \quad \Rightarrow \quad \beta = -\alpha. \tag{2.136}$$

By redefining electric and magnetic charge as follows

$$\frac{4\pi}{\mu_0}\left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2 \equiv \frac{q^2}{r^4} \quad \text{and} \quad \frac{4\pi}{\mu_0}\left(\frac{\mu_0 P}{4\pi r^2}\right)^2 \equiv \frac{p^2}{r^4}. \tag{2.137}$$

We can obtain the tt component of the metric from eq.(2.134) as follows

$$\begin{aligned} e^{2\alpha}\left[\frac{2\alpha'}{r} + \frac{1}{r^2}\left(1 - e^{-2\alpha}\right)\right] &= -\left(\frac{q^2}{r^4} + \frac{p^2}{r^4}\right) G, \\ 2\alpha' r e^{2\alpha} + e^{2\alpha} - 1 &= -\frac{1}{r^2}\left(q^2 + p^2\right) G, \\ \partial_r\left(r e^{2\alpha}\right) - 1 &= -\frac{1}{r^2}(q^2 + p^2)G, \\ r e^{2\alpha} &= \frac{1}{r}\left(q^2 + p^2\right) G + r + C_3, \end{aligned}$$

where C_3 is the integration constant. We obtain the component (tt) of the metric as

$$-g_{tt} = e^{2\alpha} = \frac{(q^2 + p^2)G}{r^2} + 1 + \frac{C_3}{r}. \tag{2.138}$$

Therefore, the metric for the charge solution (2.101), can be written as

$$\begin{aligned} ds^2 = -&\left[1 + \frac{(q^2 + p^2)G}{r^2} + \frac{C_3}{r}\right] dt^2 \\ &+ \left[1 + \frac{(q^2 + p^2)G}{r^2} + \frac{C_3}{r}\right]^{-1} dr^2 + r^2 d\Omega^2. \end{aligned} \tag{2.139}$$

If we get rid of the electric and magnetic charges, it should recover the Schwarzschild metric (2.104). Therefore, the integration constant C_3 is

$$C_3 = -2mG. \tag{2.140}$$

As a result, the charged solution is given by

$$ds^2 = -\left[1+\frac{(q^2+p^2)G}{r^2}-\frac{2mG}{r}\right]dt^2 + \left[1+\frac{(q^2+p^2)G}{r^2}-\frac{2mG}{r}\right]^{-1}dr^2 + r^2 d\Omega^2. \qquad (2.141)$$

This is the **Reissner-Nordström metric** which describes the static and spherically symmetric spacetime with radial electromagnetic field.

2.1.5.3 Kerr solution

Many objects in our Universe, especially stars, are spinning. Therefore, it is more useful to find a rotating solution. Since there exists a rotational axis, we have to relax the spherical symmetry to be an **axial symmetry**. The rotating one cannot be treated as a static object. It is useful to consider the spacetime with **stationary** instead of one with static.

We assume that the object is rotating around the z-axis and use the spherical coordinates. By applying the axial symmetry and the stationary condition,

$$t \rightarrow -t \quad \text{and} \quad \phi \rightarrow -\phi, \qquad (2.142)$$

to eq.(2.81), the metric becomes

$$ds^2 = -g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + g_{rr}dr^2 + 2g_{r\theta}drd\theta + g_{\theta\theta}d\theta^2, \qquad (2.143)$$

where $g_{\mu\nu}$ does not depend on t and ϕ, $g_{\mu\nu} = g_{\mu\nu}(r,\theta)$. We can conformally transform the last three terms similar to ones in eq.(2.84) as follows

$$g_{rr}dr^2 + 2g_{r\theta}drd\theta + g_{\theta\theta}d\theta^2 = Ddr^2 + Ed\theta^2, \qquad (2.144)$$

where D and E are arbitrary functions. As a result, the metric becomes

$$ds^2 = -Adt^2 + 2Bdtd\phi + Cd\phi^2 + Ddr^2 + Ed\theta^2, \qquad (2.145)$$

where A, B and C are arbitrary function. It is complicated to find the solution of Einstein's field equation directly, like the Schwarzschild and Reissner-Nordström solutions. There is a trick to obtain the rotating solution from non-rotating one called "**Newman-Janis trick**" [32, 31].

Let us use this trick for the Schwarzschild metric which can be written as

$$ds^2 = -fdt^2 + f^{-1}dr^2 + r^2 d\Omega^2, \tag{2.146}$$

where $f = 1 - \frac{2mG}{r}$. Next, we transform the coordinates to be the advanced Eddington-Finkelstein coordinate which is

$$p = t + \tilde{r}, \tag{2.147}$$

$$dp = dt + f^{-1}dr. \tag{2.148}$$

The metric becomes

$$ds^2 = -fdp^2 + 2dpdr + r^2 d\Omega^2. \tag{2.149}$$

The metric tensor and its inverse are

$$g_{\mu\nu} = \begin{pmatrix} -f & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \tag{2.150}$$

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & f & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}. \tag{2.151}$$

Next step is finding the null bases, for the first one we can choose

$$l^\mu = (0, 1, 0, 0), \tag{2.152}$$

with

$$l^\mu l_\mu = g_{11} l^1 l^1 = 0. \tag{2.153}$$

The second one should satisfy with $l^{\mu}n_{\mu} = -1$ and $n^{\mu}n_{\mu} = 0$. As a result, the second null basis can be written as

$$n^{\mu} = \left(-1, -\frac{f}{2}, 0, 0\right). \tag{2.154}$$

Another two are a pair of complex conjugate which can be obtained as follows

$$m^{\mu} = \frac{1}{\sqrt{2}r}\left(0, 0, 1, \frac{i}{\sin\theta}\right) \quad \text{and} \quad m^{*\mu} = \frac{1}{\sqrt{2}r}\left(0, 0, 1, \frac{-i}{\sin\theta}\right). \tag{2.155}$$

Next, we have to transform coordinate to be complex as follows

$$p \to p' = p + ia\cos\theta, \quad r \to r' = r + ia\cos\theta, \quad \theta \to \theta', \quad \phi \to \phi', \tag{2.156}$$

where a is the rotation parameter. The function $f(r) = 1 - \frac{2mG}{r}$ can be written in terms of the complex coordinate as

$$f(r') = 1 - \frac{2mGr'}{\rho^2} \quad \text{with} \quad r^n = \frac{Re[r]^{n+2}}{|r|^2}, \tag{2.157}$$

where $\rho^2 = r'^2 + a^2\cos^2\theta$. Accordingly, the null bases are transformed due to the coordinate transformation as follows

$$\begin{aligned} l^{\mu'} &= (0, 1, 0, 0), \quad n^{\mu'} = \left(-1, -\frac{f(r')}{2}, 0, 0\right), \\ m^{\mu'} &= \frac{1}{\sqrt{2}(r' - ia\cos\theta)}\left(-ia\sin\theta, -ia\sin\theta, 1, \frac{i}{\sin\theta}\right), \\ m^{*\mu'} &= \frac{1}{\sqrt{2}(r' + ia\cos\theta)}\left(ia\sin\theta, ia\sin\theta, 1, \frac{-i}{\sin\theta}\right). \end{aligned} \tag{2.158}$$

Note that a vector transforms according to general coordinate transformation as $u^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} u^{\mu}$. For example, the zeroth component of basis $m^{\mu'}$ can be computed as

$$m^{0'} = \frac{\partial x^{0'}}{\partial x^2} m^2 + \frac{\partial x^{0'}}{\partial x^3} m^3 = \frac{\partial}{\partial\theta}(p + ia\cos\theta)(1) = ia\sin\theta. \tag{2.159}$$

The inverse metric tensor can be written in terms of the transformed basis as

$$g^{\mu'\nu'} = -l^{\mu'}n^{\nu'} - l^{\nu'}n^{\mu'} + m^{\mu'}n^{\nu'} + m^{\nu'}n^{\mu'}. \tag{2.160}$$

The explicit form of the inverse metric can be expressed as

$$g^{\mu\nu} = \frac{1}{\rho^2}\begin{pmatrix} a^2\sin^2\theta & a^2+r^2 & 0 & -a \\ a^2+r^2 & \Delta & 0 & -a \\ 0 & 0 & 1 & 0 \\ -a & -a & 0 & \csc^2\theta \end{pmatrix}. \tag{2.161}$$

The metric tensor can be written as

$$g_{\mu\nu} = \begin{pmatrix} -\frac{\Delta-a^2\sin^2\theta}{\rho^2} & 1 & 0 & \frac{a\sin^2\theta(r^2+a^2-\Delta)}{\rho^2} \\ 1 & 0 & 0 & a\sin^2\theta \\ 0 & 0 & \rho^2 & 0 \\ \frac{a\sin^2\theta(r^2+a^2-\Delta)}{\rho^2} & a\sin^2\theta & 0 & \frac{\Sigma^2\sin^2\theta}{\rho^2} \end{pmatrix}, \tag{2.162}$$

where $\Delta = r^2 + a^2 - 2mGr$ and $\Sigma^2 = (r^2+a^2)^2 - a^2\Delta\sin^2\theta$. As a result, the line element is

$$\begin{aligned} ds^2 = -\left(\frac{\Delta - a^2\sin^2\theta}{\rho^2}\right)dp^2 + 2dpdr + 2\left[\frac{a\sin^2\theta(r^2+a^2-\Delta)}{\rho^2}\right]dpd\phi \\ + 2a\sin^2\theta drdp + \rho^2 d\theta^2 + \frac{\Sigma^2\sin^2\theta}{\rho^2}d\phi^2. \end{aligned} \tag{2.163}$$

By using the transformation,

$$dp = dt + \left(\frac{r^2+a^2}{\Delta}\right)dr \quad \text{and} \quad d\phi = d\phi' + \frac{a}{\Delta}dr, \tag{2.164}$$

the metric becomes

$$\begin{aligned} ds^2 = -\left(\frac{\Delta - a^2\sin^2\theta}{\rho^2}\right)dt^2 - 2\left[\frac{a\sin^2\theta(r^2+a^2-\Delta)}{\rho^2}\right]dtd\phi + \frac{\Sigma^2\sin^2\theta}{\rho^2}d\phi^2 \\ + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2. \end{aligned} \tag{2.165}$$

Note that the angular coordinate transformation (2.164) is used in order to eliminate the cross terms, $drd\phi$. This is **Kerr solution**. The Newman-Janis trick can also be used for the charged rotating solution which is called “Kerr-Newman solution”.

Note that, the coordinate is used for Kerr solution (2.165) is called **Boyer-Liquist coordinate**. By taking limit m approaches zero, we have

$$\lim_{m\to 0} ds^2_{Kerr} = -dt^2 + \left(\frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}\right) dr^2 + (r^2 + a^2\cos^2\theta) d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2. \qquad (2.166)$$

It can be transformed to the flat Minkowski metric

$$ds^2_{Minkowski} = -dt^2 + dx^2 + dy^2 + dz^2,$$

with the transformation as

$$x = \sqrt{r^2 + a^2}\sin\theta\cos\phi \ , \ y = \sqrt{r^2 + a^2}\sin\theta\sin\phi \ , \ z = r\cos\theta, \qquad (2.167)$$

where $\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1$ or $\frac{x^2+y^2}{a^2\sin^2\theta} - \frac{z^2}{a^2\cos^2\theta} = 1$. One can see that these coordinates are elliptic-like coordinates illustrated in Figure 13.

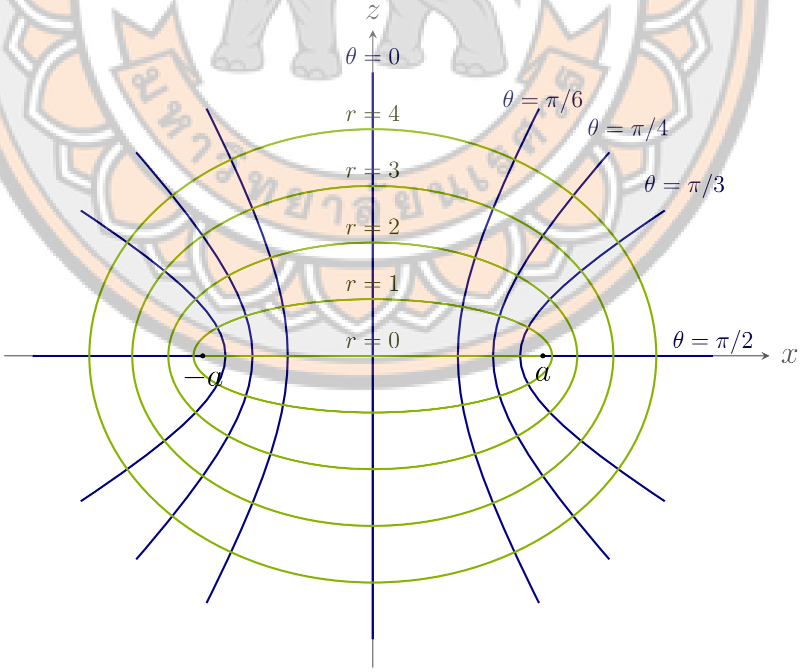


Figure 13 Boyer-Liquist coordinates system.

2.2 CLASSICAL FIELD THEORY

A **field** is a quantity that depends on every point of spacetime $(\vec{x}, t)$. The field can be found in our daily life, viz., the temperature which distributes through space, the pressure is influenced by the fluid, and etc. The field can be a scalar, vector, or higher ranked tensor. For example, the temperature and pressure are the scalar fields, the electric and gravitational field are the vector ones, the Riemann curvature tensor can be taken into account as the tensor field, and so on.

2.2.1 Classical Field Theory

2.2.1.1 Equation of Motion

The field theory can be categorized to classical and quantum ones, this thesis focuses only on the classical field. The generalized coordinates $q(t)$ are treated as dynamical variables of the system, while dynamical variables in field theory can be treated as a field $\phi(\vec{x}, t)$. The behavior of a field is described by the equation of motion of field which can be derived from the **Least Action Principle**. A Lagrangian L depending on fields and their derivatives (with respect to time and spatial), can be written in the forms of Lagrangian density $\mathcal{L}$ as

$$L = \int d^3x \mathcal{L}(\phi_a, \partial_\mu \phi_a), \tag{2.168}$$

where the subscript a denotes the field's species. An action is written as

$$S = \int_{t_1}^{t_2} dt \int d^3x \mathcal{L} = \int d^4x \mathcal{L}. \tag{2.169}$$

Then, the Least Action principle leads us

$$\begin{aligned} \delta S\left[\phi_a, \partial_\mu \phi_a\right] &= \int d^4x \delta \mathcal{L}(\phi_a, \partial_\mu \phi_a) \\ &= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} \delta \phi_a + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)} \delta\left(\partial_\mu \phi_a\right) \right\} \\ &= \int d^4x \left\{ \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)} \right) \right] \delta \phi_a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)} \delta \phi_a \right) \right\}. \end{aligned} \tag{2.170}$$

Using the divergence theorem, the last term can be expressed by

$$\int d^4x\ \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi_a\right)} \delta\phi_a \right) = \int_V d^3x \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_t \phi_a\right)} \delta\phi_a \right) \Bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \int_{\partial V} dS^i \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_i \phi_a\right)} \delta\phi_a \right) \Bigg|_{-\infty}^{\infty} . \tag{2.171}$$

Here, the integration is done over volume V which is bounded by a surface ∂V where its element is written by dS^i. Moreover, the time-boundaries are lower and upper boundaries as t_1 and t_2, respectively. The first term vanishes by the same condition in particle's mechanics, $\delta\phi_a(\vec{x}, t_1) = \delta\phi_a(\vec{x}, t_2) = 0$. We assume that a dynamical fields ϕ_a fall off at spatial infinity, then the variations $\delta\phi_a$ at infinity vanish. Finally, approximating $\delta S \approx 0$, we obtain the equation of motion as follows

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi_a\right)} \right] = 0. \tag{2.172}$$

It is known as the "Euler-Lagrange equation".

2.2.1.2 Hamiltonian-like Equations

For Euler-Lagrange equation provides (2.172) with the Lagrangian depending on the first derivative of the field $\partial_\mu \phi_a$, it gives rise n second-order differential equations, where n is the number of the fields. To find the solution of a system, it is quite hard, because we need to solve second-order differential equations. Actually, there are equivalent equations of motion which are first-order differential equations. These known as "Hamilton-like equations". We will show the conventional Hamiltonian is a subpart of the Hamiltonian-like.

To obtain the Hamilton-like equation, we consider the variation of the Lagrangian density as follows

$$\begin{aligned} \delta\mathcal{L}(\phi_a, \partial_\mu \phi^a) &= \frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi^a\right)} \delta(\partial_\mu \phi^a) & (2.173) \\ &= \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi^a\right)} \right) \right] \delta\phi_a + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi^a\right)} \delta\phi_a \right]. & (2.174) \end{aligned}$$

We would denote this boundary term such as

$$\theta^\mu(\phi_a, \delta\phi_a) \equiv \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi^a\right)} \delta\phi_a. \tag{2.175}$$

The canonical momentum is defined by

$$\pi_a^\mu \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^a\right)}, \tag{2.176}$$

and then the Euler-Lagrange equation (2.172) implies that

$$\partial_\mu \pi_a^\mu = \frac{\partial \mathcal{L}}{\partial \phi_a}. \tag{2.177}$$

Using these equations, the right-hand side of eq.(2.173) is written as follows

$$\delta\mathcal{L} = \left(\partial_\mu \pi_a^\mu\right)\delta\phi_a + \pi_a^\mu \delta\left(\partial_\mu \phi^a\right). \tag{2.178}$$

We would use this equation to derive the Hamilton equation. The **Hamiltonian-like** density $\mathcal{H}$ is given by the Legendre transformation of the Lagrangian as follows

$$\mathcal{H}(\phi_a, \pi_a^\mu) = \pi_a^\mu \partial_\mu \phi^a - \mathcal{L}(\phi_a, \partial_\mu \phi^a). \tag{2.179}$$

Note that this is not conventional Hamiltonian density in classical filed theory, since such the Hamiltonian is the Legendre transformation which includes partial derivative on spatial coordinates. To recover such a conventional Hamiltonian, the Legendre transformation should consider only time derivative. Its variations lead us the following equation of motion,

$$\delta\mathcal{H}(\phi_a, \pi_a^\mu) = \frac{\partial \mathcal{H}}{\partial \phi_a}\delta\phi_a + \frac{\partial \mathcal{H}}{\partial \pi_a^\mu}\delta\pi_a^\mu, \tag{2.180}$$

and the variation of eq.(2.179) gives

$$\begin{aligned}\delta\mathcal{H}(\phi_a, \pi_a^\mu) &= \left(\delta\pi_a^\mu\right)\partial_\mu\phi^a + \pi_a^\mu\delta\left(\partial_\mu\phi^a\right) - \delta\mathcal{L}, \\ &= \left(\partial_\mu\phi^a\right)\delta\pi_a^\mu - \left(\partial_\mu\pi_a^\mu\right)\delta\phi_a,\end{aligned} \tag{2.181}$$

where eq.(2.178) is used. Then the equations of motion are

$$-\partial_\mu \pi_a^\mu = \frac{\partial \mathcal{H}}{\partial \phi_a} \quad , \quad \partial_\mu \phi_a = \frac{\partial \mathcal{H}}{\partial \pi_a^\mu}. \tag{2.182}$$

These are “Hamilton-like equations”. Now, we have the first-order differential equations of motion, but there are $2n$ equations rather than n.

Moreover, there exists the formula for the Hamilton-like equations which are expressed in terms of a boundary term θ^μ. We would define the partial derivative ∂_μ as the variation along a vector $V_\mu = (\partial_t, \partial_i)$,

$$\delta_V \equiv \partial_\mu. \tag{2.183}$$

Substituting eq.(2.176) into eq.(2.175), we have

$$\theta^\mu(\phi_a, \delta\phi_a) = \pi^\mu_a \delta\phi_a. \tag{2.184}$$

After that, the variation of Hamiltonian-like in density eq.(2.181) can be rewritten as followz

$$\begin{aligned}\delta\mathcal{H} &= (\delta\pi)(\delta_V\phi) - (\delta_V\pi)(\delta\phi)\\ &= \delta\left[\pi(\delta_V\phi)\right] - \delta_V\left[\pi(\delta\phi)\right]\\ &= \delta\theta(\phi, \delta_V\phi) - \delta_V\theta(\phi, \delta\phi)\\ &= \omega(\phi, \delta_V\phi, \delta\phi),\end{aligned} \tag{2.185}$$

where one has defined

$$\omega(\phi, \delta_V\phi, \delta\phi) \equiv \delta\theta(\phi, \delta_V\phi) - \delta_V\theta(\phi, \delta\phi). \tag{2.186}$$

For the value of Hamiltonian-like is calculated by integrating Hamiltonian-like density over spatial volume V,

$$H = \int_V \mathcal{H}. \tag{2.187}$$

2.2.1.3 Noether's Theorem

In classical field theory, the Noether's theorem said that "every continuous symmetry of the Lagrangian gives a conserved current in which the equation of motion is satisfied". Such a conserved current called "Noether's current", denoted by J^μ, satisfies the condition below

$$\partial_\mu J^\mu = 0. \tag{2.188}$$

To obtain the expression of the Noether's current, we would consider the symmetry of a theory. Let us consider an infinitesimal transformation of the fields,

$$\phi_a(x) \to \phi_a(x) + \delta\phi_a(x). \tag{2.189}$$

Then the Lagrangian density transforms as follows

$$\mathcal{L} \to \mathcal{L} + \delta_\phi \mathcal{L}, \tag{2.190}$$

where $\delta_\phi \mathcal{L}$ is calculated as

$$\begin{aligned} \delta_\phi \mathcal{L} &= \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial_\mu \phi_a}\right)\right] \delta\phi_a + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)} \delta\phi_a\right] \\ &= \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)} \delta\phi_a\right]. \end{aligned} \tag{2.191}$$

Here, we recalled eq.(2.173) and then used eq.(2.172). In fact, the Lagrangian density in eq.(2.169) is not unique. The equations of motion are invariant under the transformation [33, 34] such that

$$\mathcal{L} \;\to\; \mathcal{L} + \partial_\mu F^\mu = \mathcal{L} + \delta_F \mathcal{L}, \tag{2.192}$$

where F^μ is a function of fields ϕ_a. Equating eqs.(2.190) and (2.192), one obtains

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)} \delta\phi_a - F^\mu\right] = 0. \tag{2.193}$$

Consequently, the conserved quantity or "Noether's current" can be defined by

$$J^\mu = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)} \delta\phi_a - F^\mu. \tag{2.194}$$

It obviously satisfies eq.(2.188). To specify the vector F^μ, one considers the infinitesimal transformation of spacetime coordinates as follows

$$x^\mu \to x^\mu - \xi^\mu, \tag{2.195}$$

where ξ^μ is a constant vector. By using Taylor's expansion, the dynamical fields would change as follows

$$\phi_a(x) \to \phi_a + \xi^\mu \partial_\mu \phi_a(x). \tag{2.196}$$

Then, the variation of Lagrangian density in eq.(2.191) is expressed by

$$\delta_\phi \mathcal{L}(x) = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)} \xi^\nu \partial_\nu \phi_a(x) \right]. \tag{2.197}$$

Under transformation (2.195), the Lagrangian density changes

$$\mathcal{L}(x) \ \rightarrow \ \mathcal{L}(x) + \delta\mathcal{L}(x) = \mathcal{L}(x) + \xi^\mu \partial_\mu \mathcal{L}(x). \tag{2.198}$$

So, we have

$$\delta_F \mathcal{L}(x) = \xi^\mu \partial_\mu \mathcal{L}(x). \tag{2.199}$$

Applying eqs.(2.197) and (2.199) to eq.(2.194), the Noether's current can be expressed as

$$J^\mu = \xi^\nu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)} \partial_\nu \phi_a - \xi^\mu \mathcal{L}. \tag{2.200}$$

Moreover, the conserved current gives another conserved quantity, that is a "conserved charge". To obtain the conserved charge, we would express eq.(2.188) as follows

$$\frac{\partial J^0}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0. \tag{2.201}$$

This is nothing but the continuity equation. Integrating this equation over all spatial volume V and using the divergence theorem, we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \int_V d^3x \left(J^0\right) &= - \int_V d^3x \left(\vec{\nabla} \cdot \vec{J}\right) \\ &= - \oint_{\partial V} d\vec{S} \cdot \vec{J} \\ &= 0, \end{aligned} \tag{2.202}$$

where ∂V denotes a boundary surface of V, and $d\vec{S}$ is the vector surface element. By choosing the boundary surface that $\vec{J}$ can be approximated to zero, e.g., $\vec{J}$ is supposed to be zero at spatial infinity, then the equality is obtained. Thus, we can define a "globally conserved charge" as follows

$$Q \equiv \int_V d^3x J^0. \tag{2.203}$$

2.2.2 Gravitational Field

2.2.2.1 Einstein-Hilbert Action

The behavior of a system is described by the equation of motion which is derived from the principle that variation of the action vanishes. We know that the gravitation, or curvature of spacetime, is explained by the Einstein's field equation. So, the variation of the gravitational action must give the Einstein's field equation. Such an action is the "Einstein-Hilbert action" which is simply given by the Ricci scalar R and the metric tensor $g_{\mu\nu}$ is treated as a dynamical field [35]. The crucial feature of this action is that the equation of motion is second-order differential equation, even if the action contains second-derivative in the metric, ($\Gamma \sim \partial g$ and then $R \sim \partial\Gamma \sim \partial^2 g$.) According to the integration in eq.(2.40), the (vacuum) Einstein-Hilbert action S_{EH} is written by

$$S_{EH}[g_{\mu\nu}] = \int d^4x\sqrt{-g}R. \tag{2.204}$$

The variation of S_{EH} with respect to the inverse metric $g^{\mu\nu}$ leads to

$$\delta S_{EH} = \int d^4x \left[\left(\delta\sqrt{-g}\right) g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}\left(\delta g^{\mu\nu}\right)R_{\mu\nu} + \sqrt{-g}g^{\mu\nu}\left(\delta R_{\mu\nu}\right)\right]. \tag{2.205}$$

Let us consider the term of variation $\delta\sqrt{-g}$. We use the identity of the matrix, such that, any diagonalizable matrix $\mathbf{A}$ satisfies the identity

$$\log\left[\det \mathbf{A}\right] = \mathrm{tr}\left[\log \mathbf{A}\right]. \tag{2.206}$$

After taking variation with respect to $\mathbf{A}$, we obtain

$$\frac{1}{\det\left[\mathbf{A}\right]}\delta\left(\det\left[\mathbf{A}\right]\right) = \mathrm{tr}\big[\mathbf{A}^{-1}\delta\mathbf{A}\big]. \tag{2.207}$$

Let $\mathbf{A}$ be the metric $g_{\mu\nu}$. From eq.(2.207), the variation of g can be expressed in terms of $\delta g_{\mu\nu}$. We can convert the result to be $\delta g^{\mu\nu}$ by varying the identity

$$g^{\mu\lambda}g_{\lambda\nu} = \delta^{\mu}_{\nu}.$$

Thereafter, one obtains

$$\left(\delta g^{\mu\lambda}\right) g_{\lambda\nu} + g^{\mu\lambda}\left(\delta g_{\lambda\nu}\right) = 0. \tag{2.208}$$

Contracting both sides with $g_{\mu\sigma}$, the following relation is obtained as

$$\delta g_{\sigma\nu} = -g_{\lambda\nu} g_{\mu\sigma} \delta g^{\mu\lambda}. \tag{2.209}$$

Applying to the variation of $\sqrt{-g}$, then we attain

$$\begin{aligned} \delta\sqrt{-g} &= \frac{1}{2}\frac{1}{\sqrt{-g}}\delta(-g), \\ &= \frac{1}{2}\frac{(-g)}{\sqrt{-g}}\left(g^{\rho\lambda}\delta g_{\rho\lambda}\right), \\ &= -\frac{1}{2}\sqrt{-g}g^{\rho\lambda}g_{\rho\nu}g_{\mu\lambda}\left(\delta g^{\mu\nu}\right), \\ &= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\left(\delta g^{\mu\nu}\right), \end{aligned} \tag{2.210}$$

where eqs.(2.207) and (2.209) are used in the second and third lines, respectively. Substituting eq.(2.210) into eq.(2.205), we obtain

$$\delta S_{EH} = \int d^4x\sqrt{-g}\left[\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu} - g^{\mu\nu}\left(\delta R_{\mu\nu}\right)\right]. \tag{2.211}$$

Actually, the Einstein's field equation has been achieved in the first term, but the final term still alive. There are several approaches for manipulating the term $\delta R_{\mu\nu}$, for example, the additional term in the Gravitational action, and the Palatini formalism. In this thesis, one chooses the method that imposing an extra condition, that is, the first covariant derivative of the dynamical field vanishes at boundary,

$$\nabla_\rho\left(\delta g^{\mu\nu}\right)\big|_{\mathscr{B}} = 0, \tag{2.212}$$

where $\mathscr{B}$ stands for the boundary surface. Such a condition is reasonable, because any theory second-derivative of the field needs more than $\delta\phi|_{\mathscr{B}} = 0$ for obtaining the equation of motion. The condition $\partial_\mu(\delta\phi)|_{\mathscr{B}} = 0$ is applied too [36]. Let us move to the variation of the Ricci tensor. We found that the variation of the

Riemann tensor can be written in terms of the covariant derivative of the variation of the Christoffel symbol as following

$$\begin{aligned}\delta R^{\rho}{}_{\mu\lambda\nu} &= \delta\left(\partial_\lambda \Gamma^{\rho}_{\nu\mu} + \Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} - \partial_\nu \Gamma^{\rho}_{\lambda\mu} - \Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu}\right) \\ &= \nabla_\lambda\left(\delta\Gamma^{\rho}_{\nu\mu}\right) - \nabla_\nu\left(\delta\Gamma^{\rho}_{\lambda\mu}\right), \qquad (2.213)\end{aligned}$$

where one has the relations

$$\begin{aligned}\nabla_\lambda\left(\delta\Gamma^{\rho}_{\nu\mu}\right) &= \partial_\lambda\left(\delta\Gamma^{\rho}_{\nu\mu}\right) + \Gamma^{\rho}_{\lambda\sigma}\left(\delta\Gamma^{\sigma}_{\nu\mu}\right) - \Gamma^{\sigma}_{\lambda\nu}\left(\delta\Gamma^{\rho}_{\sigma\mu}\right) - \Gamma^{\sigma}_{\lambda\mu}\left(\delta\Gamma^{\rho}_{\nu\sigma}\right), \\ \nabla_\nu\left(\delta\Gamma^{\rho}_{\lambda\mu}\right) &= \partial_\nu\left(\delta\Gamma^{\rho}_{\lambda\mu}\right) + \Gamma^{\rho}_{\nu\sigma}\left(\delta\Gamma^{\sigma}_{\lambda\mu}\right) - \Gamma^{\sigma}_{\nu\lambda}\left(\delta\Gamma^{\rho}_{\sigma\mu}\right) - \Gamma^{\sigma}_{\nu\mu}\left(\delta\Gamma^{\rho}_{\lambda\sigma}\right).\end{aligned}$$

After that the quantity $g^{\mu\nu}\delta R_{\mu\nu}$ can be written by

$$\begin{aligned}g^{\mu\nu}\left(\delta R_{\mu\nu}\right) &= g^{\mu\nu}\left(\delta R^{\lambda}{}_{\mu\lambda\nu}\right) \\ &= g^{\mu\nu}\left[\nabla_\lambda\left(\delta\Gamma^{\lambda}_{\nu\mu}\right) - \nabla_\nu\left(\delta\Gamma^{\lambda}_{\lambda\mu}\right)\right] \\ &= \nabla_\lambda\left[g^{\mu\nu}\left(\delta\Gamma^{\lambda}_{\nu\mu}\right) - g^{\mu\lambda}\left(\delta\Gamma^{\nu}_{\nu\mu}\right)\right]. \qquad (2.214)\end{aligned}$$

We can express the Christoffel's variation $\delta\Gamma^{\rho}_{\mu\nu}$ in terms of the metric's variation $\delta g_{\mu\nu}$. According to references [37, 35], the variation of the Christoffel symbol is given by

$$\delta\Gamma^{\lambda}_{\nu\mu} = -\frac{1}{2}\left[g_{\rho\mu}\nabla_\nu\left(\delta g^{\rho\lambda}\right) + g_{\rho\nu}\nabla_\mu\left(\delta g^{\rho\lambda}\right) - g_{\mu\alpha}g_{\nu\beta}\nabla^{\lambda}\left(\delta g^{\alpha\beta}\right)\right], \qquad (2.215)$$

$$\delta\Gamma^{\nu}_{\nu\mu} = -\frac{1}{2}\left[g_{\rho\mu}\nabla_\nu\left(\delta g^{\rho\nu}\right) + g_{\rho\nu}\nabla_\mu\left(\delta g^{\rho\nu}\right) - g_{\mu\alpha}g_{\nu\beta}\nabla^{\nu}\left(\delta g^{\alpha\beta}\right)\right]. \qquad (2.216)$$

The first and second terms of eq.(2.214) become

$$g^{\mu\nu}\left(\delta\Gamma^{\lambda}_{\nu\mu}\right) = -\frac{1}{2}\left[\nabla_\rho\left(\delta g^{\rho\lambda}\right) + \nabla_\rho\left(\delta g^{\rho\lambda}\right) - g_{\alpha\beta}\nabla^{\lambda}\left(\delta g^{\alpha\beta}\right)\right], \qquad (2.217)$$

$$g^{\mu\lambda}\left(\Gamma^{\nu}_{\nu\mu}\right) = -\frac{1}{2}\left[\nabla_\nu\left(\delta g^{\lambda\nu}\right) + g_{\rho\nu}\nabla^{\lambda}\left(\delta g^{\rho\nu}\right) - \nabla_\beta\left(\delta g^{\lambda\beta}\right)\right]. \qquad (2.218)$$

Substituting them into eq.(2.214), we obtain

$$\begin{aligned}g^{\mu\nu}\left(\delta R_{\mu\nu}\right) &= \nabla_\lambda\left[g_{\alpha\beta}\nabla^{\lambda}\left(\delta g^{\alpha\beta}\right) - \nabla_\rho\left(\delta g^{\rho\lambda}\right)\right] \\ &= \nabla_\lambda B^{\lambda}, \qquad (2.219)\end{aligned}$$

where one defines

$$B^{\mu} \equiv g_{\alpha\beta}\nabla^{\mu}\left(\delta g^{\alpha\beta}\right) - \nabla_{\rho}\left(\delta g^{\rho\mu}\right).$$

We would like to consider the expression of $\nabla_{\mu}B^{\mu}$,

$$\nabla_{\nu}B^{\nu} = \partial_{\nu}B^{\nu} + \Gamma^{\nu}_{\nu\mu}B^{\mu}. \tag{2.220}$$

Then, the expression of $\Gamma^{\nu}_{\nu\mu}$ would be found. According to the expression of the Christoffel symbol in eq.(2.27) and then applying the identity for a diagonalizable matrix $\mathbf{A}$ as follows

$$\mathrm{tr}[\log\mathbf{A}] = \log\left[\det\mathbf{A}\right], \tag{2.221}$$

the expression $\Gamma^{\mu}_{\mu\nu} = \frac{1}{2}g^{\rho\mu}\partial_{\nu}g_{\rho\mu} = \frac{1}{2}\mathrm{tr}[\mathbf{g}^{-1}\partial_{\nu}\mathbf{g}]$ is written by

$$\begin{aligned}\Gamma^{\mu}_{\mu\nu} &= \frac{1}{2}\mathrm{tr}[\partial_{\nu}\left(\log\mathbf{g}\right)]\\ &= \frac{1}{2}\partial_{\nu}\left(\mathrm{tr}[\log\mathbf{g}]\right)\\ &= \frac{1}{2}\partial_{\nu}\left(\log g\right)\\ &= \frac{1}{2g}\partial_{\nu}g\\ &= \frac{1}{\sqrt{-g}}\partial_{\nu}\sqrt{-g},\end{aligned} \tag{2.222}$$

where $\mathbf{g}$ represents $g_{\mu\nu}$. By using the above equation, eq.(2.220) is expressed by

$$\begin{aligned}\nabla_{\nu}B^{\nu} &= \partial_{\nu}B^{\nu} + \left(\frac{1}{\sqrt{-g}}\partial_{\nu}\sqrt{-g}\right)B^{\nu}\\ &= \frac{1}{\sqrt{-g}}\partial_{\nu}\left(\sqrt{-g}B^{\nu}\right).\end{aligned} \tag{2.223}$$

This is known as the “divergence formula”, the same expression in eq.(2.111). Using the divergence formula, the variation of the action in eq.(2.211) becomes to

$$\delta S_{EH} = \int d^4x\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu} + \int d^4x\left(\partial_{\nu}\sqrt{-g}B^{\nu}\right). \tag{2.224}$$

By analogy to the divergence theorem, the last term is evaluated at the boundary as

$$\int d^4x\left(\partial_{\nu}\sqrt{-g}B^{\nu}\right) \sim \left.\sqrt{-g}B^{\nu}\right|_{B}.$$

Since the additional condition eq.(2.212) is imposed to this variation, and B^{μ} is proportional to $\nabla_{\nu}\left(\delta g^{\alpha\beta}\right)$, the last term of the variation in eq.(2.224) vanishes,

$$\sqrt{-g}B^{\nu}\big|_{B} \approx \nabla_{\nu}\left(\delta g^{\alpha\beta}\right)\big|_{B} = 0.$$

Thus, the first term in eq.(2.224) gives us the equation of motion which is written as follows

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0, \tag{2.225}$$

which is the (vacuum) Einstein's field equation.

In order to include a matter in the theory, we simply add the "matter's Lagrangian $\mathcal{L}_m$". Now, the action is written as

$$S = \frac{1}{\kappa}S_{EH} + S_m, \tag{2.226}$$

where κ is arbitrary constant, and the matter's action S_m is given by

$$S_m = \int d^4x\sqrt{-g}\mathcal{L}_m. \tag{2.227}$$

Then, the variation of S_m with respect to the dynamical field $g^{\mu\nu}$ can be expressed as

$$\delta_g S_m = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g_{\mu\nu}\mathcal{L}_m + \frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}, \tag{2.228}$$

where we have used eq.(2.210). We can define the energy-momentum tensor as follows

$$T_{\mu\nu} \equiv -2\left(-\frac{1}{2}g_{\mu\nu}\mathcal{L}_m + \frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}}\right). \tag{2.229}$$

Eventually, the $g^{\mu\nu}$'s variation of the action (2.226) is given by

$$\delta S = \int d^4x\sqrt{-g}\left[\frac{1}{\kappa}G_{\mu\nu} - \frac{1}{2}T_{\mu\nu}\right]\delta g^{\mu\nu}. \tag{2.230}$$

This leads to the equation of motion as follows

$$G_{\mu\nu} = \frac{\kappa}{2}T_{\mu\nu}. \tag{2.231}$$

Comparing the above equation to the Einstein's equation (2.67), the constant κ is identified by

$$\kappa = 16\pi G.$$

Therefore, the Einstein-Hilbert action is written by

$$S_{EH} = \int d^4x\sqrt{-g}\frac{R}{16\pi G}. \tag{2.232}$$

For example, the Einstein-Maxwell theory, which is coupled between the gravitation and electromagnetic, is described by the following action

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right], \tag{2.233}$$

where $F_{\mu\nu}$ is the field strength tensor and then the energy-momentum tensor is given by the eq.(2.106).

2.2.3 Equation of Motion

In section 2.1.5, we solved the vacuum-spherically symmetric Einstein's field equation by considering $R_{\mu\nu} = 0$, and then the key equations are that $R_{\theta\theta} = 0$ and $R_{tr} = 0$. Actually, we can found these key equations directly from the variation of the action.

We would derive the key equation of the Schwarzschild metric as an example. Using the Birkhoff's theorem, the spherically symmetric metric in eq.(2.89) can be written by

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2d\Omega^2. \tag{2.234}$$

Then, the Ricci scalar and the metric's determinant are evaluated as

$$R = -2e^{-2\beta}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - (\partial_r\alpha)(\partial_r\beta) + \frac{2}{r}(\partial_r\alpha - \partial_r\beta) + \frac{1}{r^2}\left(1 - e^{2\beta}\right)\right], \quad (2.235)$$

$$\sqrt{-g} = e^{\alpha}e^{\beta}r^2\sin\theta. \quad (2.236)$$

Then, we would treat the arbitrary functions α and β as dynamical fields. The action can be expressed as follows

$$S_{EH} = \int d^4x\left\{2\sin\theta e^{\alpha-\beta}\left[-r^2\alpha'' + r^2\left(-\alpha'^2\right) + r\alpha'\left(r\beta' - 2\right) + 2r\beta' + e^{2\beta} - 1\right]\right\}, \quad (2.237)$$

where α and β are understood as the functions of coordinate r. The equation of motion would be achieved from the Euler-Lagrange equations. Since it is the second-order theory, according to reference [36], the Euler-Lagrange's equation for the second-order Lagrangian $\mathcal{L}(\phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi)$ is expressed by

$$E^{(\phi)} = \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\right] + \partial_\mu\partial_\nu\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\partial_\nu\phi\right)}\right], \quad (2.238)$$

where $E^{(\phi)} = 0$. After calculating, the equation of motion with respect to the dynamical fields are given by

$$E^{(\alpha)} = 2\sin\theta e^{\alpha(r)-\beta(r)}\left(2r\beta'(r) + e^{2\beta(r)} - 1\right), \quad (2.239)$$

$$E^{(\beta)} = 2\sin\theta e^{\alpha(r)-\beta(r)}\left(-2r\alpha'(r) + e^{2\beta(r)} - 1\right). \quad (2.240)$$

Since $E^{(\alpha)} = E^{(\beta)} = 0$, the above equations are equivalent to

$$2r\beta'(r) = -e^{2\beta(r)} + 1, \quad (2.241)$$

$$-2r\alpha'(r) = -e^{2\beta(r)} + 1. \quad (2.242)$$

The right-hand side of eqs.(2.241) and (2.242) are identical, then this gives the same result (2.99).

Another example is the Einstein-Maxwell theory. The field strength tensor in terms of the four-vector potential is given by

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu . \tag{2.243}$$

In this case, we consider only the electric field distributes in spacetime. Such a situation, the four-vector potential is written by

$$A_\mu = (V(r), 0, 0, 0) , \tag{2.244}$$

where $V(r)$ is the electric potential. Then the quantity $F_{\mu\nu}F^{\mu\nu}$ is calculated by

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}e^{-2\alpha(r)-2\beta(r)}V'(r)^2 . \tag{2.245}$$

The equation of motion for the electric field, is derived by the variation of the action (2.233),

$$S = \int d^4x \left[\sqrt{-g}R(\alpha,\beta) + \frac{1}{2}\left(r^2 \sin\theta\right) e^{-\alpha(r)-\beta(r)}V'(r)^2\right] , \tag{2.246}$$

with respect to the dynamical field $V(r)$. Putting this Lagrangian into the Euler-Lagrange equation in eq.(2.172), we obtain

$$-\partial_r \left[\left(r^2 \sin\theta\right) e^{-\alpha(r)-\beta(r)}V'(r)\right] = 0. \tag{2.247}$$

Since the definition of the electric field is given by

$$\vec{E} = -\vec{\nabla}V(\vec{r}), \tag{2.248}$$

then eq.(2.247) coincides to the equation of motion (2.113).

Therefore, this kind of method is quite useful and convenient to obtain the significant equations of motion in the complicated theory.

2.3 THERMODYNAMICS

The many-body system is described by statistical mechanics in microscopic scale. For the macroscopic scale of many-body system, the system is described by **thermodynamics**. Thermodynamics is related to a concept of heat, work and temperature, and their relations to energy, radiation and physical properties of matter, all of this governed by four law of thermodynamics.

2.3.1 Thermodynamic state

In thermodynamic system is the system that contains large number of particles (or atoms, molecules). Such a system can be described by "thermodynamic variables" which are "measurable" macroscopic physical quantities including to volume (V), pressure (P), and temperature (T). In thermodynamic system, the state can be specified by given values of a set of thermodynamic variables. There always exist three suitable thermodynamic parameters to specify the state of the system.

2.3.2 Laws of Thermodynamics

2.3.2.1 Temperature

Thermodynamics has many subbranches, each using a different fundamental model as a theoretical or experimental basis, or applying the principles to various types of systems. The subbranches include

- **Classical thermodynamics** is the description of the states of thermodynamics systems at near-equilibrium by using macroscopic and measurable properties,
- **Statistical mechanics** is supplement of classical thermodynamics with an interpretation of the microscopic interactions between individual particles or quantum-mechanical states,

- **Chemical thermodynamics** is the study about interrelation of energy with chemical reaction or with a physical change of state within the confines of the law of thermodynamics,
- **Equilibrium thermodynamics** is the study of transfers of matter and energy in systems by agencies in their surroundings. The systems can be driven from one state of thermodynamic equilibrium to others,
- **Non-equilibrium thermodynamics** is thermodynamics which deals with systems that are not in equilibrium. Actually most systems in nature are not in thermodynamic equilibrium.

For this situation, we use equilibrium thermodynamics for studying black hole thermodynamics. There are three catagories of **equilibrium** including

- Mechanical equilibrium,
- Chemical equilibrium,
- Thermal equilibrium.

If the system achieves all types of equilibrium, the system is in "thermodynamic equilibrium" state.

The zeroth law of thermodynamics is associated with a quantity which can describe thermal equilibrium. Let us consider two contacted systems as shown in Figure 14. In the left panel of Figure 14, there are two separated systems. In the middle panel of Figure 14, the systems thermally contact together. There exists

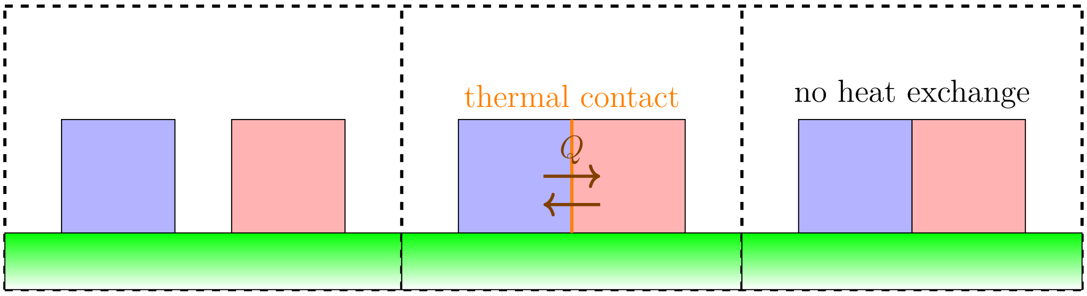


Figure 14 Thermal equilibrium.

heat transfer between them. In the right panel of Figure 14, there is no more heat transfer. The **zeroth law of thermodynamics** states that “If objects A and B are separate in thermal equilibrium with an object C, then A and B are in thermal equilibrium with each other.” A quantity which is used to represent thermal equilibrium is **temperature**. Another definition of the zeroth law is “at thermodynamic equilibrium state, temperature of entire system is constant”.

2.3.2.2 Transfer variables

In order to describe thermodynamic system, we can use thermodynamic variables, including pressure, volume, and temperaure. Also, a thermodynamic system can be described by **transfer variables**. The transfer variables can be positive or negative depending on the direction of transfer from the system. Basically, heat, work and mass action are transfering variables.

Heat is defined as process of transferring energy across the boundary of a system caused by temperature difference between the system and surrounding as illustrated in figure 15. In order to determine how the heat transfer is caused by temperature difference of a system, it is important to introduce “heat capacity”. Specifically, heat capacity (C) is the energy that system requires to rise its temperature up to 1 unit. For example,

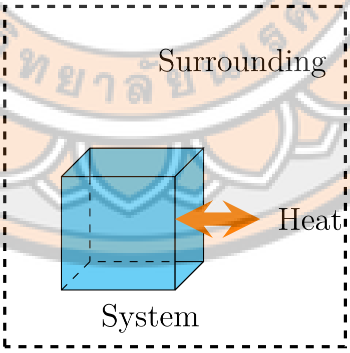


Figure 15 Heat transferring between the system and the surrounding.

$$\begin{aligned} C &\quad \text{can rise temperature up to} \quad 1\ K, \\ \Delta Q &\quad \text{can rise temperature up to} \quad \Delta T = \frac{\Delta Q}{C}\ K, \end{aligned}$$

where the unit of temperature K is called "Kevin". Thus, the heat can be written as

$$\Delta Q = C\Delta T \quad \text{or} \quad \Delta Q = mc\Delta T, \tag{2.249}$$

where $c = \frac{C}{m}$ is the specific heat or heat capacity per unit of mass. Moreover, if substance in a system changes phase, i.e., ice becomes to water. The heat can be expressed by

$$\delta Q = L\Delta m, \tag{2.250}$$

where L is the latent heat and Δm is the change of the mass of substance during the phase changes.

Considering a cylinder of gas which is contained in a piston as shown in Figure 16, one exertes the force $\vec{F}$ on piston and then the height decreases as δr. By using $\vec{F} = P\vec{A} = PA(-\hat{\jmath})$ and $\vec{\delta r} = \delta r(-\hat{\jmath})$, the work done on the system can be written as

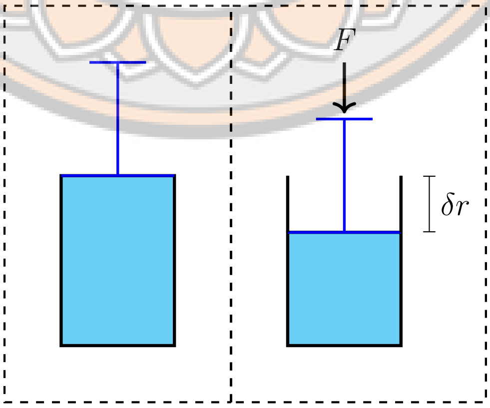


Figure 16 Work done by force on the piston.

$$\delta W = \vec{F} \cdot \vec{\delta r} = F dy = PA dy = P dV. \tag{2.251}$$

Since the volume of the system decreases $dV = -|dV|$, the work can be written as

$$\delta W = -PdV. \tag{2.252}$$

This implies that

→ Positive work is the system gains the work from surrounding, it matches with dV is negative or system is compresses.

→ Negative work is the system loses or do the work to surrounding, it matches with dV is positive or system expands.

For an open system shown in figure 17, the energy that is associated with exchanging particles between the system and surrounding is **mass action**. The mass action can be written as

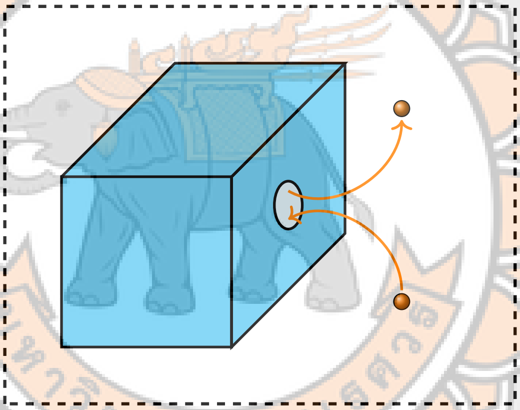


Figure 17 Exchanging of particles between the system and the surrounding.

$$\delta Z = \mu_i dN^i, \tag{2.253}$$

where μ_i is **chemical potential** of exchanging of particles and dN^i is the number of change of type-i particles. The chemical potential is the energy that can be absorbed or released due to the change of number of particles.

In summary, the change of internal energy (U) of the system can be expressed as follows

$$\Delta U = Q + W + Z. \tag{2.254}$$

It is called the "**first law of thermodynamics**". The system from state 1 to state 2 is independent of how its performing. The infinitesimal change of the internal energy is written as

$$dU = \delta Q + \delta W + \delta Z = \delta Q - PdV + \mu_i dN^i, \tag{2.255}$$

which is another definition of the first law.

2.3.2.3 Entropy

An important quantity which is widely used in many fields of physics is **entropy**. For the thermodynamic case, entropy implies the **disorder** of a system, illustrated in Figure 18. The system in the left part of this figure is represented orderness, while the right one is represented randomness (or disorder). And the change of entropy is related to heat transfer. In order to show that, we consider the system of ideal gas which obeys the so-called "ideal gas law" as

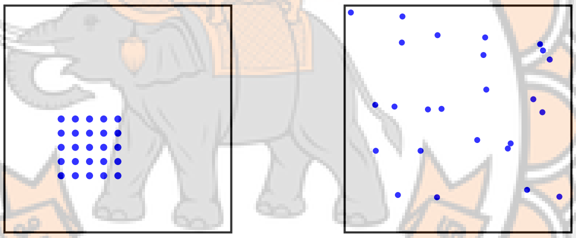


Figure 18 Idea of an entropy through the randomness of particles.

$$PV = Nk_BT, \tag{2.256}$$

where N is number of molecules, k_B is the Boltzmann constant. The internal energy of the system is expressed as follows

$$U = \frac{3}{2}Nk_BT. \tag{2.257}$$

Considering an isothermal expansion process, the first law becomes $\Delta U = 0$ and then heat can be written as

$$\delta Q = -\delta W. \tag{2.258}$$

According to eqs.(2.252) and (2.256), we obtain

$$\delta Q = PdV = Nk_BT\frac{dV}{V},$$
$$\frac{\delta Q}{T} \sim \frac{dV}{V}. \tag{2.259}$$

The fractional volume change $\frac{dV}{V}$ implies the increase (or decrease) of disorder, or randomness, of a system. If the system expands, dV is positive, molecules of the gas of the system has more freedom to move in the container. It is proportional to quantity $\frac{\delta Q}{T}$. Thus, the entropy S can be defined by

$$dS \equiv \frac{\delta Q}{T}. \tag{2.260}$$

Therefore, the first law can be written in terms of the entropy as

$$dU = TdS - PdV + \mu_i dN^i. \tag{2.261}$$

Moreover, we can use the entropy to classify thermodynamic process. There are two catagories including to reversible and irreversible process. For the reversible (irreversible) process of the isolated system, the change of entropy along process is greater than (or equal to) zero. It is the **second law of thermodynamics** which can be expressed as

$$\Delta S \geq 0. \tag{2.262}$$

The **third law of thermodynamics** is still related to the entropy. That is "it is impossible to reduce the entropy to absolute-zero value in finite number of operations". In other words, it is impossible to obtain the absolute-zero temperature, $T = 0\ K$, which can be expressed as

$$\lim_{N\to\infty} T = 0, \tag{2.263}$$

where N is the number of operations. Thermodynamic entropy must obey the second and third laws and this is an important feature of the entropy in thermodynamics. Another crucial feature related to the entropy is that a system proceeds according to maximum entropy called "maximum entropy principle".

2.3.3 Thermodynamic Potential

Apart from the internal energy, there are other functions of state which can describe the system. They are known as **thermodynamic potentials**. In order to obtain thermodynamic potentials, let us consider the first law. According to the first law (2.261), the internal energy can be treated as the function $U = U(S, V, N^1, N^2, ...)$. The derivative form of the internal energy function is

$$dU = \left(\frac{\partial U}{\partial S}\right) dS + \left(\frac{\partial U}{\partial V}\right) P + \left(\frac{\partial U}{\partial N^i}\right) dN^i. \tag{2.264}$$

By comparing the derivative form of the internal energy to the first law (2.261), we obtain

$$\left(\frac{\partial U}{\partial S}\right) = T, \quad \left(\frac{\partial U}{\partial V}\right) = -P, \quad \left(\frac{\partial U}{\partial N^i}\right) = \mu_i. \tag{2.265}$$

To obtain a thermodynamic function which depends on S, P, N^i, the first law (2.261) can be written as follows

$$\begin{aligned} dU &= TdS - PdV + \mu_i dN^i + (VdP - VdP) \\ &= TdS + VdP + \mu_i dN^i - d(PV) \\ d(U + PV) &= TdS + VdP + \mu_i dN^i. \end{aligned} \tag{2.266}$$

As a result, we can define the thermodynamic potential as follows

$$H \equiv U + PV. \tag{2.267}$$

This thermodynamic potential is known as **enthalpy**. Since the enthalpy is defined as $H = H(S, P, N^1, N^2, ...)$, its derivative form can be expressed as $dH = \left(\frac{\partial H}{\partial S}\right) dS + \left(\frac{\partial H}{\partial P}\right) dP + \left(\frac{\partial H}{\partial N^i}\right) dN^i$. By comparing the derivative form to eq.(2.266), we obtain

$$\left(\frac{\partial H}{\partial S}\right) = \left(\frac{\partial U}{\partial S}\right) = T, \quad \left(\frac{\partial H}{\partial P}\right) = V, \quad \left(\frac{\partial H}{\partial N^i}\right) = \left(\frac{\partial U}{\partial N^i}\right) = \mu_i.$$

Using the same procedure, we can have other thermodynamic potential. The first law eq.(2.261) can be written as follows

$$\begin{aligned} dU &= TdS - PdV + \mu_i dN^i + (SdT - SdT) \\ &= SdT + PdV + \mu_i dN^i + d(TS) \\ d(U - TS) &= SdT - PdV + \mu_i dN^i. \end{aligned} \tag{2.268}$$

We can define a thermodynamic potential as follows

$$F \equiv U - TS. \tag{2.269}$$

This thermodynamic potential is known as **Helmholtz free energy**. Since the Helmholtz free energy is defined as $F = F(T, V, N^1, N^2, ...)$, its derivative form is $dF = \left(\frac{\partial F}{\partial T}\right) dT + \left(\frac{\partial F}{\partial V}\right) dV + \left(\frac{\partial F}{\partial N^i}\right) dN^i$. By comparing the derivative form to eq.(2.268), we obtain

$$\left(\frac{\partial F}{\partial T}\right) = S, \quad \left(\frac{\partial F}{\partial P}\right) = \left(\frac{\partial U}{\partial V}\right) = -P, \quad \left(\frac{\partial F}{\partial N^i}\right) = \left(\frac{\partial H}{\partial N^i}\right) = \left(\frac{\partial U}{\partial N^i}\right) = \mu_i.$$

There is another important thermodynamic potential called **Gibbs free energy**. By using the same procedure, The Gibbs free energy is written by

$$G \equiv U - TS + PV. \tag{2.270}$$

Since the Gibbs free energy is defined as $G = G(T, P, \mu_1, \mu_2, ...)$, its derivative form is $dG = \left(\frac{\partial G}{\partial T}\right) dT + \left(\frac{\partial G}{\partial P}\right) dP + \left(\frac{\partial G}{\partial N^i}\right) dN^i$. We still obtain the relations as follows

$$\left(\frac{\partial G}{\partial T}\right) = \left(\frac{\partial F}{\partial T}\right) = S \quad , \quad \left(\frac{\partial G}{\partial P}\right) = \left(\frac{\partial H}{\partial P}\right) = V \quad , \quad \left(\frac{\partial G}{\partial N^i}\right) = \mu_i.$$

These thermodynamic potentials can be used to describe the system as well as the internal energy.

Note that one can obtain the thermodynamic potentials by using the Legendre transformation. The Legendre transformation is another expression of a function $f(x^1, x^2, ...)$ in terms of its slope $m_i = \frac{\partial f(x)}{\partial x^i}$, the transformation is given by

$$g(m_1, m_2, ...) = f(x^1, x^2, ...) - m_i x^i. \tag{2.271}$$

For example, one can use the Legendre transformation to obtain the Helmholtz free energy as follows

$$F = U - \left(\frac{\partial U}{\partial S}\right) S = U - TS, \tag{2.272}$$

where we have used $\frac{\partial U}{\partial S} = T$. It is consistent with the definition of Helholtz free energy (2.269).

From the maximum entropy principle, it is possible to write this principle in terms of the thermodynamic potentials. In order to obtain these conditions, we consider the system which is not isolated. Recalling the second law, the change of entropy can be written by

$$\delta Q < TdS. \tag{2.273}$$

Adding δW and δZ to both sides of eq.(2.273), we obtain

$$\delta Q + \delta W + \delta Z < TdS + \delta W + \delta Z. \tag{2.274}$$

The left hand side of eq.(2.274) is the change of internal energy dU. And then substituting the definition of work eq.(2.252) and mass action eq.(2.253) into the above equation, we obtain

$$dU < TdS - PdV + \mu_i dN^i. \tag{2.275}$$

At thermodynamic equilibrium, S, V and N are fixed or $dS = dV = dN = 0$, the change of internal energy satisfies

$$dU < 0. \tag{2.276}$$

This implies that the system would proceed the internal energy to its minimum. Doing the same procedure, the thermodynamic potentials of non-isolated system satisfy conditions as follows

$$dH < TdS + VdP + \mu_i dN^i,\ dF < SdT - PdV + \mu_i dN^i,\ dG < SdT + VdP + \mu_i dN^i.$$

At thermodynamic equilibrium, the above conditions become

$$dH < 0, \qquad dF < 0, \qquad dG < 0.$$

They are called the “**Energy Minimum principle**”.

2.3.4 Stability

Thermodynamically stable equilibrium can be divided into two catagories, namely, global and local stabilities. Thermodynamic stability can be characterized by thermodynamic quantities. In order to determine a condition for stability, we consider a system contacted with a reservoir with temperature T_0 and pressure P_0 as shown in Figure 19. According to the second law of thermodynamics, the entropy increases while the system evolves to the equilibrium state. Thus, the change of entropy of the entire system is written by

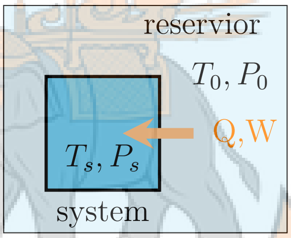


Figure 19 The system contacted with the reservoir.

$$\Delta S_t = \Delta S_s + \Delta S_r \geq 0, \tag{2.277}$$

where S_t, S_s and S_r are the entropies with subscript t, s and r denote total systems, system and reservoir, respectively. The amount of heat transfer, Q, from the reservoir to the system can be written in terms of entropy's change as follows

$$\Delta S_r = \frac{\Delta Q_r}{T_0} = \frac{(-Q)}{T_0}. \tag{2.278}$$

By using the first law (2.261), the heat transfer can be written in terms of the change of internal energy and work done as follows

$$Q = -\Delta U + P_0 \Delta V_s. \tag{2.279}$$

Substituting the heat transfer in eq.(2.279) into the change of reservoir's entropy in eq.(2.278), and then a change of entropy of the entire system in eq.(2.277) can be written as

$$\begin{aligned}\Delta S_t = \Delta S_s + \left(\frac{\Delta U + P_0 \Delta V_s}{T_0}\right) &= \frac{T_0 \Delta S_s - \Delta U - P_0 \Delta V_s}{T_0} \\ &= -\frac{1}{T_0}\Delta(U - T_0 S_s + P_0 V_s). \qquad (2.280)\end{aligned}$$

According to the definition of the Gibbs free energy (2.270), we obtain

$$\Delta S_t = -\frac{\Delta G_s}{T_0} \geq 0. \qquad (2.281)$$

If there are two stable equilibrium states, a thermodynamic system prefers to be a state with lower Gibbs free energy. Since the system contacts with the heat reservoir, there exists the minimum Gibbs free energy. Then, the condition for "**global stability**" is

$$G_s = \text{minimum}. \qquad (2.282)$$

In order to investigate the condition of the local stability, we consider an infinitesimal change of Gibbs free energy of the sysytem with fixing volume. Using the Gibbs free energy of the system which is expressed in eq.(2.280), we have a infinitesimal change of Gibbs free energy $G_s = U - T_0 S_s + P_0 V_s$ as follows

$$dG_s = dU - T_0 dS_s. \qquad (2.283)$$

Let us consider the change of G_s with respect to temperature. From eq.(2.283), we obtain

$$\left(\frac{\partial G_s}{\partial T_s}\right)_V = \left(\frac{\partial U}{\partial T_s}\right)_V - T_0 \left(\frac{\partial S}{\partial T_s}\right)_V. \qquad (2.284)$$

The change of internal energy of the system is $dU = T_s dS_s$. Eq.(2.284) becomes

$$\left(\frac{\partial G_s}{\partial T_s}\right)_V = T_s \left(\frac{\partial S_s}{\partial T_s}\right)_V - T_0 \left(\frac{\partial S_s}{\partial T_s}\right)_V. \qquad (2.285)$$

At the state with minimum Gibbs free energy, we have

$$\left(\frac{\partial G_s}{\partial T_s}\right)_V = 0. \tag{2.286}$$

Then the temperature of the system must be equal to one of the reservoir, $T_s = T_0$, implying the equilibrium state. Taking derivative eq.(2.285) with respect to temperature T, we obtain

$$\left(\frac{\partial^2 G_s}{\partial T_s^2}\right)_V = \left(\frac{\partial S_s}{\partial T_s}\right)_V + T_s\left(\frac{\partial^2 S_s}{\partial T_s^2}\right)_V - T_0\left(\frac{\partial^2 S_s}{\partial T_s^2}\right)_V. \tag{2.287}$$

Since the equilibrium state, the Gibbs free energy is minimum, the second derivative of the Gibbs free energy is greater than zero

$$\left(\frac{\partial^2 G_s}{\partial T_s^2}\right)_V = \left(\frac{\partial S_s}{\partial T_s}\right)_V \geq 0. \tag{2.288}$$

This equation is referred to "**local stability**" condition. The local stability condition can be expressed in terms of the heat capacity as follows

$$C \equiv \frac{\delta Q}{\delta T} = T\left(\frac{\delta S}{\delta T}\right) \geq 0. \tag{2.289}$$

Therefore, the system with positive heat capacity is locally stable under the heat transfer with fixing volume process.

2.3.5 Statistical Entropy

A thermodynamic system can be described by the view points of both macroscopic and microscopic scales. For the microscopic scale, the system is described by statistical mechanics. In this section, we will derived a formula of entropy which is used in statistical mechanics.

An isolated system with constant total energy E is divided into N subsystems. Each subsystem is labelled by $n = 1, 2, .., N$. The total energy of the system is then written as

$$E = \sum_{n=1}^{N} E_n = \text{constant}. \tag{2.290}$$

Microstates of the system can be identified by a set of the energy of subsystems as microstate $= \{E_1, E_2, ..., E_N\}$, and the number of all possible microstates of the system is Ω_N. The energy of subsystems is assumed to be discrete up to ω levels as follows

$$E_n \in \{\epsilon_0, \epsilon_1, ..., \epsilon_{\omega-1}\} \quad \text{with} \quad \epsilon_j = j\epsilon. \tag{2.291}$$

The probability of a given subsystems n has energy ϵ_j is

$$\mathbb{P}(E_n = \epsilon_j) = p_n(\epsilon_j). \tag{2.292}$$

Considering two given subsystems n and m with energy E_n and E_m, respectively. The probability that any two subsystems has their combine energy ϵ can be computed as

$$\mathbb{P}(\varepsilon) = \sum_{j=0}^{\omega-1} p_n(\epsilon_j) p_m(\varepsilon - \epsilon_j), \tag{2.293}$$

For each j, the system is divided into three parts including subsystem n with energy ϵ_j, subsystem m with energy $\varepsilon - \epsilon_j$, and the rest with energy $E - \varepsilon$. It is useful to introduce the fundamental postulate of equilibrium statistical mechanics. This postulate states that “At the equilibrium, all microstates are equally probable, so the probabitlity is $1/\Omega_N$ where Ω_N is the total number of microstates”. By using the fundamental postulate, the probability of subsystems eq.(2.293) can be written by

$$\frac{1}{\Omega_N} = p_n(\epsilon_j) p_m(\varepsilon - \epsilon_j) \frac{1}{\Omega_{N-2}}. \tag{2.294}$$

We obtain

$$p_n(\epsilon_j) p_m(\varepsilon - \epsilon_j) = \frac{\Omega_{N-2}}{\Omega_N}. \tag{2.295}$$

As a result, the probability in eq.(2.293) can be written as follows

$$\mathbb{P}(\varepsilon) = \omega \frac{\Omega_{N-2}}{\Omega_N} = \omega p_n(\epsilon_j) p_m(\varepsilon - \epsilon_j). \tag{2.296}$$

The $\epsilon_j = 0$ case, $\mathbb{P}(\varepsilon) = \omega p_n(0)p_m(\varepsilon)$, and the $\epsilon_j = \varepsilon$ case, $\mathbb{P}(\varepsilon) = \omega p_n(\varepsilon)p_m(0)$, are equivalent. According to this result, we obtain

$$\frac{p_n(\varepsilon)}{p_n(0)} = \frac{p_m(\varepsilon)}{p_m(0)},$$

and then it infers to

$$p_n = p_m = p. \tag{2.297}$$

This implies that the probabilities of the subsystems with the same energy are identical. Thus, for subsystem n with energy ϵ_i and subsystem m with energy ϵ_j, we can write the probability as follows

$$\begin{aligned} \mathbb{P}(\varepsilon = \epsilon_i + \epsilon_j) = \omega p(\epsilon_i)p(\epsilon_j) = \omega p(0)p(\epsilon_i + \epsilon_j), \\ p(\epsilon_i)p(\epsilon_j) = p(0)p(\epsilon_j + \epsilon_j). \end{aligned} \tag{2.298}$$

From eq.(2.298), we can guess the formula of probability as follows

$$p(\epsilon_j) \propto e^{-B\epsilon_j}, \tag{2.299}$$

where B is a constant. We know that any probability must be normalized. Thus, the probability should be expressed as

$$p_j \equiv p(\epsilon_j) = \frac{1}{Z}e^{-B\epsilon_j}, \tag{2.300}$$

where $Z = \sum_{i=0}^{\omega-1} e^{-B\epsilon_j}$ is known as "partition function". Here, p_j stands for the probability of finding a particle with energy ϵ_j. Such a probability corresponds to **Boltzmann distribution**. This is one of the keys ingredient of **Maxwell-Boltzmann statistics**. We would like to apply this Boltzmann distribution to the internal energy which is expectation value of energy of the system. The internal energy can be given by

$$U = \sum_j p_j\epsilon_j. \tag{2.301}$$

Substituting the probability in eq.(2.300) into the internal energy in eq.(2.301), we obtain

$$U = \sum_j \epsilon_j \frac{e^{B\epsilon_j}}{Z} = -\frac{\partial \ln Z}{\partial B}. \tag{2.302}$$

Moreover, we have another formula of the internal energy. Substituting $S = -\frac{\partial F}{\partial T}$ into $F = U - TS$, we obtain

$$U = F + TS = F - T\frac{\partial F}{\partial T} = F + \frac{1}{T}\frac{\partial F}{\partial(1/T)} = \frac{\partial(F/T)}{\partial(1/T)}. \tag{2.303}$$

Comparing the internal energy in eqs.(2.302) and (2.303), we get

$$F = -k_B T \ln Z \quad \text{and} \quad B = \frac{1}{k_B T}. \tag{2.304}$$

Note that the Boltzmann constant k_B is added in order to give the right unit of Helmholtz free energy. Thus, the probability and the energy in eq.(2.300) are written by

$$p_j = \exp[\frac{F}{k_B T} - \frac{1}{k_B T}\epsilon_j], \tag{2.305}$$

$$\epsilon_j = F - k_B T \ln p_j. \tag{2.306}$$

Substituting the energy in eq.(2.306) into the internal energy in eq.(2.301), we obtain

$$U = \sum_{j=1}^{\omega-1} p_j (F - T \ln p_j) = F - k_B T \left(\sum_{j=1}^{\omega-1} p_j \ln p_j\right). \tag{2.307}$$

By comparing the internal energy in eq.(2.307) to $U = F - TS$, the entropy can be expressed as follows

$$S = -k_B \sum_{j=1}^{\omega-1} p_j \ln p_j. \tag{2.308}$$

This is known as **Gibbs entropy** which denoted by S_G [38]. By assigning the total number of microstates as Ω and applying the fundamental postulate $p_i = \frac{1}{\Omega}$

to the formula of the Gibbs entropy in eq.(2.308) and summing over all number of microstates Ω, we obtain

$$S_{GB} = -k_B \sum_{i=1}^{\Omega} \frac{1}{\Omega} \ln \Omega^{-1} = k_B \ln \Omega. \tag{2.309}$$

This is **Gibbs-Boltzmann entropy** and denoted by S_{GB}.

CHAPTER III

BLACK HOLE

3.1 BLACK HOLE

Black hole is “A region of space having a gravitational field so intense that no matter or radiation can escape”. We will show that how objects cannot escape from the black hole.

3.1.1 Light Cone

The point in spacetime is called “event”. The infinitesimal interval between two event is metric which written by eq.(2.15). The trajectory of particle that move in spacetime is called “worldline” as shown in Figure 20. The slope of worldline implies to speed of paricle,

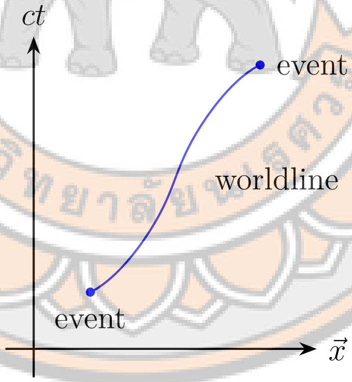


Figure 20 Spacetime diagram.

$$\text{slope } m = \frac{c\Delta t}{\Delta x} = \frac{c}{v}. \tag{3.1}$$

Note that we will show explicitly with speed of light c in this section for convenience. We classify worldline, or any path in spacetime, by metric as eq.(2.16). Let’s consider Minkowski spacetime (flat spacetime)

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = -c^2dt^2 + d\vec{x}^2. \tag{3.2}$$

If we set $ds^2 = 0$, we have

$$
\begin{aligned}
0 &= -c^2 dt^2 + d\vec{x}^2, \\
c &= \pm \left| \frac{d\vec{x}}{dt} \right| . \qquad (3.3)
\end{aligned}
$$

We call this trajectory as “null path”. This corresponds to trajectory of light or massless particles. Furthermore, if we plot trajectory of the light, it is just straight line with slope eqaul to 1,

$$
\text{slope } m = \frac{c}{v_{\text{light}}} = 1. \qquad (3.4)
$$

Then we can classify other two type of trajectory by using the null path as broaden line, namely,

1. $ds^2 < 0$: time-like path,
2. $ds^2 = 0$: null path,
3. $ds^2 > 0$: space-like path.

Note that we include null path in the second type. Considering an event in space-time, there exists a **light cone** at which it is bounded by null trajectory, as shown in Figure 21. Thus, if we want to consider ligh cone, we just set metric to be zero and find the relation between time and spatial coordinate. According to special relativity, everything in vacuum cannot travel faster than speed of light. Then the

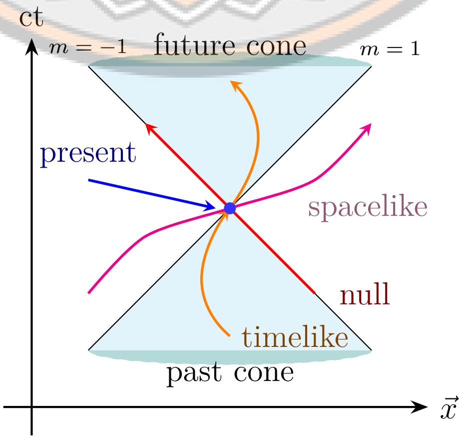


Figure 21 Light cone and worldlines.

slope for massive particle trajectory can be written by

$$m_{\text{mass}} = \frac{c}{v_{\text{mass}}} > 1 = m_{\text{light}}, \tag{3.5}$$

where $v_{\text{mass}} < c$. So, our trajectory is time-like path. Since the time never decrease, massive particle trajectory is "time-like path and always go to upper part of light cone" which is called future light cone, as found in Figure 21. If spacetime is curved, the structure of light cone is changed. Therefore, structure of light cone is an important issue in order to investigate particle motion in curved spacetime around black holes.

3.1.2 Schwarzschild Black Hole

Since the structure of light cone can be investigated by using the null path. For Schwarzschild metric defined in eq.(2.104), the null path can be obtained by

$$\begin{aligned} 0 &= -\left(1-\frac{2mG}{r}\right)dt^2+\left(1-\frac{2mG}{r}\right)^{-1}dr^2, \\ \frac{dt}{dr} &= \pm\left(1-\frac{2mG}{r}\right)^{-1}. \end{aligned} \tag{3.6}$$

Note that we fixed angle coordinates, θ and ϕ. Solving eq.(3.6), we obtain time as a function of radial coordinate as follows

$$t = \pm\int dr\left(1-\frac{2mG}{r}\right)^{-1} = \pm\left[r+2mG\ln|r-2mG|+C\right], \tag{3.7}$$

where C is integration constant. The trajectory of null path can be plotted in order find the structure of light cone as shown in Figure 22. The region of spacetime diagram can be divide into two parts as according to the existence of $r = 2mG$. This specific radius is called "**event horizon**". For the right one the future light cone is upward. While the left part, the future light cone will flip to be leftward. The massive particle trajectory in spacetime diagram cannot be outside the light cone. After it passes the event horizon, it cannot come out to $r > 2mG$. This feature corresponds to definition of black hole. Moreover, time and radial coordinate are

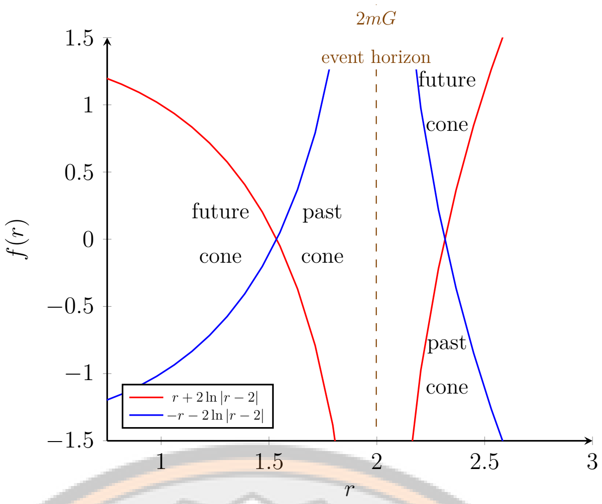


Figure 22 Light-cone structure in Schwarzschild spacetime.

changed their role when passing event horizon. Specifically, r will change to be time-like coordinate and t will change to be space-like coordinate. Inside the black hole, there exists a frame reference such that $dt = 0$. This means that the massive particle can move from some event to another event with the same time. We called this object as **Schwarzschild black hole** and it is one of static black holes.

Notice that, the boundaries of light cone can be identified by 2 null paths, namely, ingoing and outgoing radial null trajectory. For example, the boundaries of light cone in flat spacetime with slope $m = -1$ and $m = 1$ in the Figure 20 are ingoing and outgoing null path, respectively. The red and blue lines in the Figure 21 are another example, where the red (blue) is outgoing (ingoing) trajectory.

3.1.3 Reissner-Nordstorm Black Hole

3.1.3.1 Coordinate singularity

Notice that, at Schwarzschild radius, $r = 2mG$, the Schwarzschild metric blow up. If we transform the radial coordinate to be "tortoise coordinate r^*". This coordinate is defined as follows

$$\frac{dr^*}{dr} \equiv \left(1 - \frac{2mG}{r}\right)^{-1}, \tag{3.8}$$

$$r^* = \int \left(1 - \frac{2mG}{r}\right)^{-1} dr = r + 2mG \ln\left(\left|\frac{r}{2mG} - 1\right|\right), \tag{3.9}$$

Thereafter, the Schwarzschild metric in eq.(2.104) becomes

$$ds^2 = \left(1 - \frac{2mG}{r}\right)(-dt^2 + dr^{*2}) + r^2 d\Omega^2. \tag{3.10}$$

Now, at event horizon, the metric is not diverged anymore. Since the singularity, at event horizon, can be removed by using coordinate transformation, it is called "**coordinate singurarity**". For non-removable singularity or **real singularity**, we check it from **Kretschmann scalar** [37]

$$K = R^{\mu\nu\sigma\rho} R_{\mu\nu\sigma\rho}. \tag{3.11}$$

For Schwarzschild case, the Krestmann scalar can be written as

$$K_{sch} = \frac{48m^2G^2}{r^6}. \tag{3.12}$$

Obviously, at event horizon, this quantity is still finite. But this quantity is infinite at $r = 0$ which infer to the real singularity. Therefore, the event horizon can be found by a point that coordinate singularity.

It is interesting to investigate event horizon of Reissner-Nordstorm metric in eq.(2.141). Firstly, we identify real singularity via the Krestmann scalar as follows [39]

$$K_{rs} = \frac{48m^2}{r^6}\left[1 - \frac{2q^2}{mr} + \frac{7q^4}{6m^2r^2}\right]. \tag{3.13}$$

One can see that $r = 0$ is real singularity. Therefore, the coordinte singularity can be found by solving $g_{rr} = \infty$ as follows

$$\begin{aligned} 1 + \frac{(q^2+p^2)G}{r_H^2} - \frac{2mG}{r_H} &= 0, \\ r_H^2 - 2mGr_H + (q^2+p^2)G &= 0, \\ r_H &= \left[m \pm \sqrt{m^2 - (q^2+p^2)}\right] G. \end{aligned} \tag{3.14}$$

The event horizon can be classified into three cases as follows

1. $m^2 > q^2 + p^2$: there are two event horizons,
2. $m^2 = q^2 + p^2$: there is one event horizon which is called “extreme Reissner-Nordstrom black hole”,
3. $m^2 < q^2 + p^2$: there is no event horizon, the real singularity is said to be “**naked singularity**”.

The physical quantities can be observed from singularity at which physical quantities cannot be well defined. Thus, such physical quantities should not be observed by physical process. As a result, it is a good that describe of singularity cannot be formed in nature. This leads to the cosmic censorship conjecture stated that “Singularities that arise in the solutions of Einstein’s equations are typically hidden within event horizons, and therefore cannot be observed from the rest of spacetime” [37].

3.1.4 Kerr Black Hole

3.1.4.1 Event horizons as null hypersurface

The metric for Schwarzschild and Reissner-Nordstrom are in the same form which can be written as

$$\begin{aligned} ds^2 &= -f(r)ft^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2, \\ \text{Schwarzschild}: \quad f(r) &= 1 - \frac{2mG}{r}, \\ \text{Reissner-Nordstrom}: \quad f(r) &= 1 - \frac{2mG}{r} + \frac{(q^2+p^2)G}{r^2}. \end{aligned}$$

The event horizon for both cases can be found by solving $f(r_H) = 0$. However, Kerr metric is quite complicated and it is not easy to find event horizon in the same way as Schwarzschild and Reissner-Nordstrom case. Formally, the event horizon can be defined as "the hypersurface which its normal vector is null vector everywhere". Let's consider the hypersurface which described by $\Phi(r) = C$ where C is constant. The normal vector is $n_\mu = \frac{\partial \Phi(r)}{\partial x^\mu} = \partial_\mu r$. From null condition for normal vector, the component of metric tensor must be satisfied the condition as follows

$$\begin{aligned} n^\mu n_\mu = g^{\mu\nu}(\partial_\mu r)(\partial_\nu r) = g^{rr}, \\ g^{rr}(r_H) = 0. \end{aligned} \tag{3.15}$$

Notice that this condition still valid for Schwarzschild and Reissner-Nordstrom case, $g^{rr} = f(r) = 0$. Moreover, if these null normal vectors are Killing vector, the event horizon can be said as "**Killing horizon**".

We will find event horizons from the condition in eq.(3.15). According to Kerr metric found in eq.(2.167), $g^{rr} = \frac{\Delta}{\rho^2}$. Therefore, the event horizon can be found by

$$\begin{aligned} \Delta(r_H) = r_H^2 + a^2 - 2mGr_H = 0, \\ r_{H\pm} = \left[m \pm \sqrt{m^2 - (a/G)^2}\right] G. \end{aligned} \tag{3.16}$$

There are two event horizons similar to Reissner-Nordstorm black hole. For real singularity, the Krestmann scalar for Kerr black hole can be expressed as [40]

$$K_{kerr} = \frac{48m^2}{r^2 + a^2\cos^2\theta}(r^6 - 15a^2r^4\cos^2\theta + 15a^4r^2\cos^4\theta - a^6\cos^6\theta). \tag{3.17}$$

Thus, the real singularity is given by

$$\begin{aligned} r_0^2 + a^2\cos^2\theta_0 = 0, \\ r_0 = 0 \text{ and } \theta_0 = \frac{\pi}{2}. \end{aligned} \tag{3.18}$$

As a result, from figure 13 the real singularity correspond to circle of radius a and called "ring singularity".

3.1.4.2 Stationary limit and Infinite redshift surface

Stationary limit surface is the surface that the particle cannot stay at rest. For Schwarzschild black hole, this kind of surface coincides with event horizon. For Kerr black hole, stationary limit surface and event horizon are not coincided. In order to see this, let us consider trajectory of photon with fixing r and θ as follows

$$\begin{aligned} 0 &= g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2, \\ 0 &= g_{tt} + 2g_{t\phi}\frac{d\phi}{dt} + g_{\phi\phi}\left(\frac{d\phi}{dt}\right)^2, \\ \frac{d\phi}{dt} &= -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}. \end{aligned} \tag{3.19}$$

The minus(plus) sign corresponds to trjectory of photon in opposite(same) direction with rotating of source. If $g_{tt} = 0$, eq.(3.19) becomes

$$\frac{d\phi}{dt} = 0 \text{ and } -2\frac{g_{t\phi}}{g_{\phi\phi}}, \tag{3.20}$$

where $\frac{d\phi}{dt} \equiv \omega$ is angular velocity of photon. Thus, the photon moving in opposite direction will stay at rest at $g_{tt} = 0$. If $g_{tt} < 0$, photon will be dragged in the same direction of the source. The stationary limit surface is defined as follows

$$\begin{aligned} g_{tt}(r_s) &= 0, \\ \Delta - a^2\sin^2\theta &= 0, \\ r_s^2 - 2mGr_s + a^2\cos^2\theta &= 0, \\ r_{\pm s} &= \left[m \pm \sqrt{m^2 - (a/G)^2\cos^2\theta}\right]G. \end{aligned} \tag{3.21}$$

The condition $g_{tt} = 0$ is also used to define “infinite redshift surface”. Let us consider the light sending from r_1 to $r_2 > r_1$ with one pulse. Time interval of a pulse of observers at r_1 and r_2 are equal as shown in figure 23. Thus the proper time interval for these two observer are

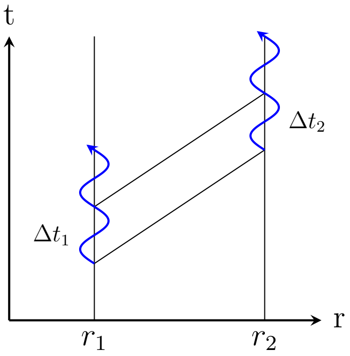


Figure 23 Sending a photon from position r_1 to position r_2.

$$\Delta\tau^2 = g_{\mu\nu}\Delta x^\mu \Delta x^\nu, \tag{3.22}$$

$$\Delta\tau_1 = \sqrt{-g_{tt}(r_1)}\Delta t_1 \quad \text{and} \quad \Delta\tau_2 = \sqrt{-g_{tt}(r_2)}\Delta t_2. \tag{3.23}$$

The ratio of time interval between two observers can be written in terms of frequency and wavelenght of the light as

$$\frac{\Delta\tau_1}{\Delta\tau_2} = \frac{\nu_2}{\nu_1} = \frac{\lambda_1}{\lambda_2} = \frac{\sqrt{-g_{tt}(r_1)}}{\sqrt{-g_{tt}(r_2)}}. \tag{3.24}$$

For $r_1 = r_s$ and $r_2 = r > r_s$, the wavelenght observed by observer at r is given by

$$\lambda_r = \lambda_s \frac{\sqrt{-g_{tt}(r)}}{\sqrt{-g_{tt}(r_s)}} = \lambda_s \infty, \tag{3.25}$$

where $g_{tt}(r_s) = 0$. This means that if we send light signal from surface that $g_{tt} = 0$, the observer at $r > r_s$ will observe the light signal with infinite wavelength. As a result, all surfaces of Kerr black hole can be illustrated in figure 24.

In order to find characteristic behavior of the black hole via the motion of light, let us consider angular velocity of photon in dragging frame and let photon moving in opposite direction. Eq.(3.19) becomes

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} + \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}. \tag{3.26}$$

The maximum angular velocity of the photon can be found by solving equation as follows

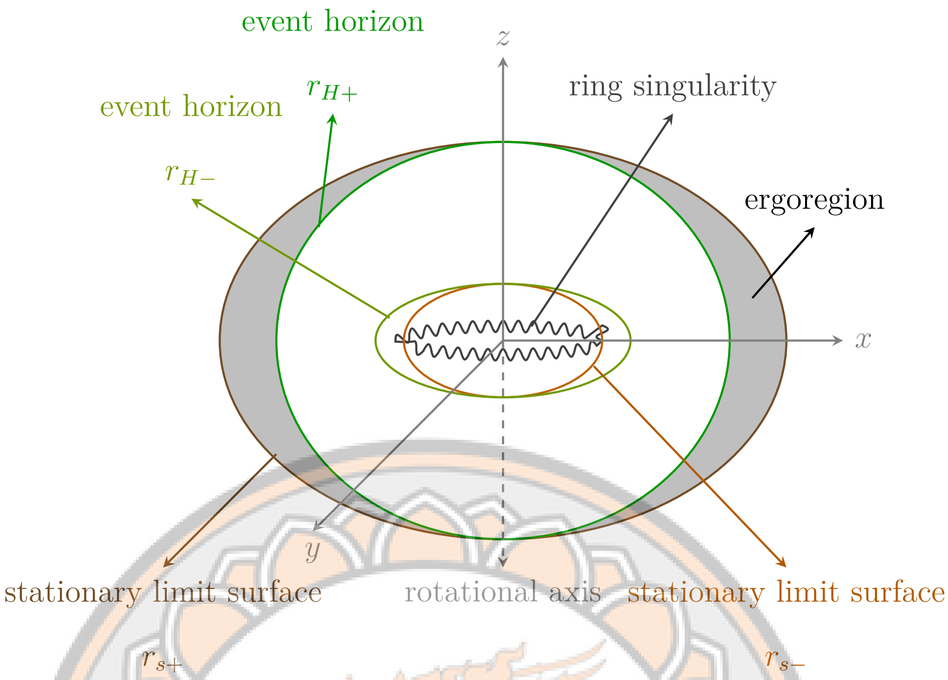


Figure 24 Structure of Kerr black hole.

$$\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}} = 0,$$

$$g_{t\phi}^2 - g_{tt}g_{\phi\phi} = 0. \tag{3.27}$$

The radius that satisfy above equation is found by

$$r_{\max} = r_{H+}. \tag{3.28}$$

One found the angular velocity of photon at event horizon can be written as

$$\left[\frac{d\phi}{dt}\right](r = r_{H+}) = -\frac{g_{t\phi}(r_{H+})}{g_{\phi\phi}(r_{H+})} = -\frac{a}{r_{H+}^2 + a^2} = -\frac{a}{2mGr_{H+}}. \tag{3.29}$$

Notice that the angular velocity depends only on black hole's parameters and does not depend on type of particles. So, this behavior corresponds to characteristic quantity of black hole. As a result, it is convenient to define the angular velocity of the black hole as follows

$$\Omega_H \equiv \frac{g_{t\phi}(r_{H+})}{g_{\phi\phi}(r_{H+})} = \frac{a}{r_{H+}^2 + a^2} = \frac{a}{2mGr_{H+}}. \tag{3.30}$$

3.1.5 Maximally Extended Solution

In the case of Schwarzschild spacetime (2.104), the black hole can be determined by the existence of **coordinate singularity**. There is a coordinates transformation which gets rid of coordinate singularity. The metric in the tortoise coordinate, in eq.(2.104), is an example of the metric without coordinate singularity. The light cone of this coordinate is described by

$$\left(1-\frac{2mG}{r}\right)\left(-dt^2+dr^{*2}\right)=0. \tag{3.31}$$

This implies that the slope of light cone is ±1 at inner and outer horizon's region, $r<2mG$ and $r>2mG$. However, the light cone vanishes at $r=2mG$. So, it is possible to find the coordinates of with non-vanishing light cone as follows

$$v=t+r^* \quad \text{and} \quad u=t-r^*, \tag{3.32}$$

where v and u are known as the **Eddington-Finkelstein coordinates**. The ingoing null trajectory is characterized by $v=$ constant, meanwhile the outgoing one satisfies the condition $u=$ constant.

3.1.5.1 Ingoing Eddington-Finkelstein coordinate

We will consider the Schwarzschild metric in the coordinate system of (v,r,θ,ϕ). Replacing

$$dt=dv-dr^* \quad \text{and} \quad dr^*=\left(1-\frac{2mG}{r}\right)dr$$

in eq.(2.104), we obtain the metric as following

$$ds^2=-\left(1-\frac{2mG}{r}\right)dv^2+2dvdr+r^2d\Omega^2. \tag{3.33}$$

This is the Schwarzschild metric in the **ingoing** Eddington-Finkelstein coordinate. The light cone is described by setting $ds^2=0$ and then we have

$$\left[-\left(1-\frac{2mG}{r}\right)dv+2dr\right]dv=0. \tag{3.34}$$

The solutions can be obtained by solving these equations

$$dv = 0 \quad \text{and} \quad \frac{dv}{dr} = 2\left(1 - \frac{2mG}{r}\right)^{-1}.$$

As a result, the light cone can be characterized by the ingoing and outgoing null trajectory as follows

$$v = \begin{cases} \text{constant} & ;\text{ingoing} \\ 2r + 4mG\ln|r - 2mG| + \text{constant} & ;\text{outgoing}, \end{cases} \tag{3.35}$$

where the integration (3.7) is used. We can set the ingoing null trajectory travel with slope $m = -1$. One defined a new time coordinate t^* as follows

$$t^* \equiv v - r.$$

Then the null geodesics in coordinate of (t^*, r) is described by

$$t^* = \begin{cases} \text{constant} - r & ;\text{ingoing}, \\ r + 4mG\ln|r - 2mG| + \text{constant} & ;\text{outgoing}. \end{cases} \tag{3.36}$$

The null trajectory in the coordinate (t^*, r) is shown in the **Finkelstein** diagram in Figure 25. The red and blue lines represent the ingoing and outgoing null trajectories, respectively. The future light cone is bounded by these null trajectories. Notice that, the light cone does not vanish at horizon $r = 2mG$. After passing

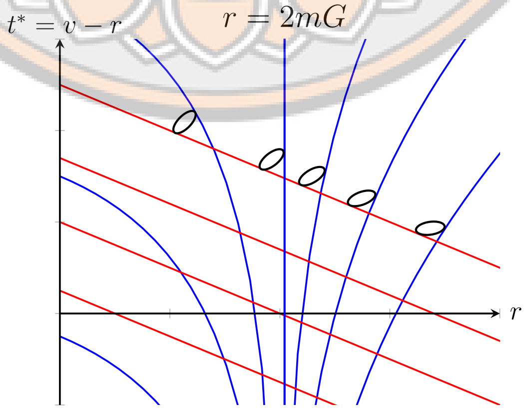


Figure 25 Ingoing Eddington-Finkelstein coordinates.

through the event horizon $r < 2mG$, the light cone tends to $r = 0$. The light cannot go back to $r > 2mG$ after passing the event horizon $r = 2mG$. It corresponds to an important feature of black hole. Therefore, the ingoing Eddington-Finkelstein coordinate also manifests the existence of black hole in such spacetime.

3.1.5.2 Outgoing Eddington-Finkelstein coordinate

For the coordinate u, one can do the same procedure in previous section. Replacing the time coordinate t with

$$dt = du + dr^* = du + \left(1 - \frac{2mG}{r}\right)^{-1} dr, \tag{3.37}$$

the Schwarzschild metric in the **outgoing** Eddington-Finkelstein coordinate is obtained by following

$$ds^2 = -\left(1 - \frac{2mG}{r}\right) du^2 - 2dudr + r^2 d\Omega^2. \tag{3.38}$$

By setting $ds^2 = 0$, the light cone is described by

$$du = 0 \quad \text{and} \quad \frac{du}{dr} = -2\left(1 - \frac{2mG}{r}\right)^{-1}.$$

Then the coordinate u satisfies the following expression,

$$u = \begin{cases} \text{constant} & ;\text{outgoing} \\ -(2r + 4mG \ln|r - 2mG|) + \text{constant} & ;\text{ingoing.} \end{cases} \tag{3.39}$$

By defining a new time coordinate $t_* \equiv u + r$, the null path in the coordinates (t^*, r) is satisfied by

$$t_* = \begin{cases} \text{constant} + r & ;\text{outgoing} \\ -(r + 2mG \ln|r - 2mG|) + \text{constant} & ;\text{ingoing.} \end{cases} \tag{3.40}$$

The Finkelstein diagram of such coordinates is represented in figure 26. The red and blue lines are presented the ingoing and outgoing of light trajectory. The future light cone is bounded by these two lines as shown in Figure 26. In the

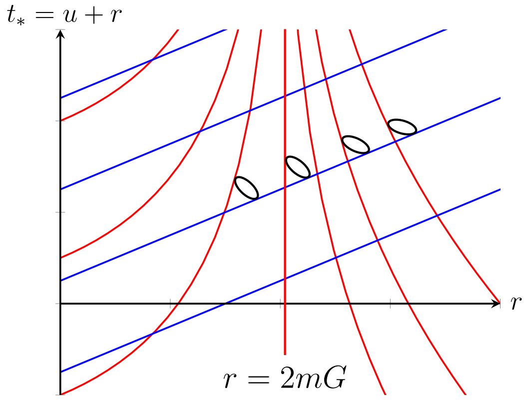


Figure 26 Outgoing Eddington-Finkelstein coordinates.

horizon's region $r < 2mG$, the light cone tends to $r = 2mG$. This implies that whether we begin moving at the inner horizon's region, one can escape from the event horizon. Meanwhile, we cannot get inside through the event horizon. This is obviously different feature from a black hole. Thus, the solution (3.33) is a **white hole** which pushes everything out from the real singularity $r = 0$.

Indeed, the white hole's solution can be obtained by using time reversal transformation $t \to -t$ to the black hole's one. So, the Finkelstein diagram in Figure 26 is merely a black hole with the reversal time coordinate $t' = -t$. According to the Schwarzschild solution, the size of the source must be smaller than $r < 2mG$ which emerged by gravitational collapse. Since the white hole acts a kind of repulsive force, in contrast to gravitational force, the white hole cannot be formed in nature, even if the white hole is the solution of Einstein's equation.

3.1.5.3 Kruskal coordinates

So far, the Schwarzschild metrics in the Eddington-Finkelstein coordinates, eqs.(3.33) and (3.38), are well-defined for all range of r, $r \in (0, \infty)$. However, they still be vanished, or degenerated, at the event horizon $r = 2mG$. In order to obtain proper coordinates in which the metric is well-defined and non-degenerated for $r \in (0, \infty)$, new coordinates should be introduced. The metric with both coordinates $v = t + r^*$ and $u = t - r^*$ can be written by

$$ds^2 = -\left(1 - \frac{2mG}{r}\right) dvdu + r^2 d\Omega^2. \tag{3.41}$$

It also degenerates at horizon radius $r = 2mG$. We will solve the problem by using these coodinates

$$U \equiv -\exp\left(-\frac{u}{4mG}\right) \quad \text{and} \quad V \equiv \exp\left(\frac{v}{4mG}\right). \tag{3.42}$$

The relations between coordinates (U, V) and (t, r) are obtained by

$$UV = -\exp\left(\frac{r^*}{2mG}\right) = \left(1 - \frac{r}{2mG}\right)\exp\left(\frac{r}{2mG}\right), \tag{3.43}$$

$$\frac{U}{V} = -\exp\left(-\frac{t}{2mG}\right), \tag{3.44}$$

where eq.(3.9) is used. Applying these equations and relations

$$dU = -\frac{U}{4mG}du \quad \text{and} \quad dV = \frac{V}{4mG}dv, \tag{3.45}$$

to eq.(3.41), one achieves the metric in coordinates of (U, V, θ, ϕ) as follows

$$ds^2 = -\frac{32(mG)^3}{r}e^{-r/2mG}dUdV + r^2 d\Omega^2. \tag{3.46}$$

Thereafter, the metric which is healthy everywhere in region $r \in (0, \infty)$ is obtained. The spacetime diagram in these coordinates is quite complicated to draw. Nonetheless, there is a coordinate system in which its spacetime diagram is more comfortable to understand. That is **Kruskal coordinates** which are expressed by

$$T \equiv \frac{1}{2}(V + U) \quad \text{and} \quad R \equiv \frac{1}{2}(V - U). \tag{3.47}$$

The metric is written as follows

$$ds^2 = \frac{32(mG)^3}{r}e^{-r/2mG}\left(-dT^2 + dR^2\right) + r^2 d\Omega^2, \tag{3.48}$$

where we substituted for $dV = dT + dR$ and $dU = dT - dR$. The spacetime diagram of this coordinates system, so-called **Kruskal diagram**, is illustrated by

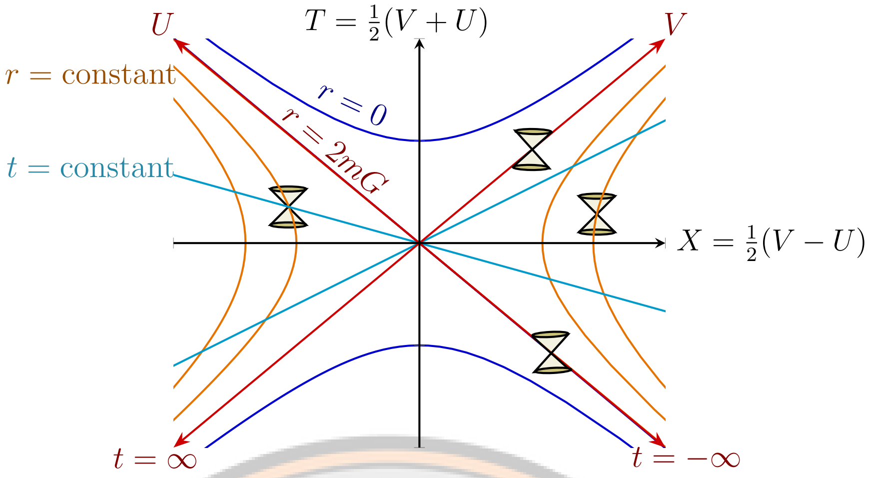


Figure 27 The Kruskal diagram of Schwarzschild spacetime.

Figure 27. For $r =$ constant $> 2mG$, eq.(3.43) leads us to $UV =$ constant which is presented in the orange lines. At the event horizon $r = 2mG$, eq.(3.43) implies that $U = 0$ and $V = 0$, and then they are shown in the red axes. The blue curves represent the real singularity that satisfies $UV = 1$. For $t =$ constant, it is described by $U/V = -$constant which obtained by eq.(3.44) and shown in the cyan lines. Furthermore, the line $V = 0$ represents $t = -\infty$, since $V \approx 0$ when $v = -\infty$. Consequently, $t = \infty$ is presented in the line $U = 0$, because $U \approx 0$ while $u = \infty$. Lastly, a light trajectory is described by

$$-dT^2 + dR^2 = 0.$$

It implies the light cones are characterized by the lines with slope $m = 1$ and $m = -1$, and the same everywhere in the diagram.

In this diagram, there are 4 regions which are divided by the axes U and V. The right region is corresponding to $r > 2mG$ where the future light cone directs to $t = \mathbb{R}^+$, the asymptotic flat spacetime is in this region. The real singularities are located in the top and bottom regions. One notices the light cone which placed at the line $U = 0$, its right-hand boundary coincides to $r = 2mG$. It implies that a trajectory cannot escape from $r = 2mG$, that is the event horizon. So the top region represents a black hole. The light cone locates at the line $V = 0$, its left-hand boundary lies on $r = 2mG$. This implies that a trajectory cannot pass through $r = 2mG$, and then corresponds to the Figure 26. Thus, the bottom region is represented a white hole. Finally, the left region satisfies $r > 2mG$ but the future light cone directs to $t = \mathbb{R}^-$. This can be interpreted as another asymptotic flat region of spacetime. The connection of the right and left region is placed at the origin of diagram, it is called the Einstein-Rosen bridge, or worm hole.

A black hole, white hole, our universe and parallel universe, they are all possible solutions of Einstein's equation with Schwarzschild's conditions, vacuum, static and spherical symmetry. The Kruskal coordinates assemble them in one diagram. Therefore, the Kruskal diagram represents the **maximal extended solution** of Schwarzschild metric. The maximal extension can be applied in the Reissner-Nordstrom and Kerr black holes. However, it is quite complicated to express them here, we would not shown in this thesis. The reader can found in references [31, 32, 35, 37].

3.1.5.4 Bifurcation sphere and Killing horizon

Each point on the Kruskal diagram is the two sphere $\mathbf{S}^2$ with fixing radius r and time t. The event horizon which is the sphere with radius $r = 2mG$ can be defined on both of the line $U = 0$ and $V = 0$. So the intersected point of the lines $U = 0$ and $V = 0$ is the intersection of 2 spheres with radius $r = 2mG$. In other words, the event horizons are intersected in this point. This kind of point $(U = 0, V = 0)$ is the **bifurcation sphere**. Since the horizon which represented by $U = 0$ $(V = 0)$ is the horizon at $t = -\infty$ $(t = \infty)$, so-called "past (future) horizon", the bifurcation sphere is the point that past and future horizons are intersected.

The interested feature of this point is about Killing vector vanishes. According to the timelike Killing vector (2.57), the Killing vector $K^{\mu}_{(t)} = \partial x^{\mu}/\partial t$ can be written in the coordinates (U, V, θ, ϕ) as follows

$$K^{\mu}_{(t)} = \frac{\partial}{\partial t}(U, V, \theta, \phi) = \frac{1}{4mG}\left(-U, V, 0, 0\right), \tag{3.49}$$

where one used eq.(3.45) and $\partial u/\partial t = \partial v/\partial t = 1$. This Killing vector vanishes at the bifurcation sphere, since $(U, V) = (0, 0)$ there. So, the Killing vector vanishes at the bifurcation sphere. Moreover, this property also holds in the case of Kerr black hole [31].

3.2 LAWS OF BLACK HOLE MECHANICS

In thermodynamics, the behavior of a thermal systems are described by **the four laws of thermodynamics**. As well as black hole's system, there exists a laws of black holes which explain the nature of them. This is called "the laws of black hole's mechanics".

3.2.1 The Zeroth Law

The zeroth law of black hole's mechanics describes characteristic of black hole's quantity, that is "Surface gravity".

3.2.1.1 Surface gravity

To define the work done at event horizon, we should do the work at event horizon. The work done around event horizon depends on the position that observer does the work. However, we should have the same reference. So, it is useful to define the work done where observer do the work at flat space.

Let's consider the work done on particle at event horizon with distant δs., while we are at spatial infinity, or flat space, and do the work via massless string, illustrated by Figure 28. The work done per unit mass at event horizon is $\delta W_H = a_H \delta s$, while one at spatial infinity is $\delta W_\infty = a_\infty \delta s$. In order to well-define the work done of the object in curved spacetime, it is useful to facterize the work done between one in flat spacetime and that curved spacetime. In order to find the proper factor, let's imagine that the work done at horizon will be converted to the radiation and sent to the observer in flat spacetime. Since the energy of radiation

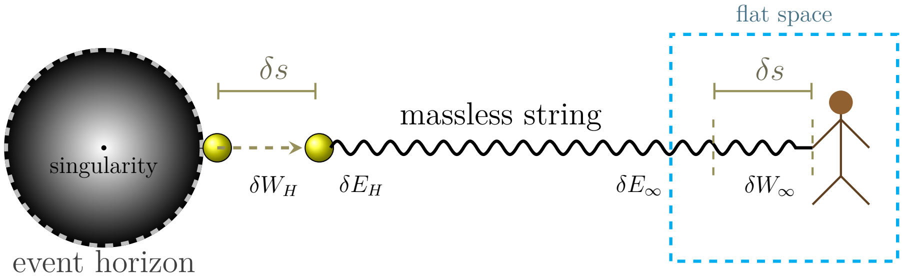


Figure 28 Idea of surface gravity.

is factorized by $\sqrt{|g_{tt}|}$ as shown in eq.(3.25), we have

$$E_\infty = h\nu_\infty = h\frac{\lambda_1}{\lambda_2}\nu_1 = \frac{\lambda_1}{\lambda_2}E_H. \tag{3.50}$$

The work done from the obeserver at infinity to the object at event horizon will be factorized by the $\sqrt{|g_{tt}|}$ as follows

$$\delta E_\infty = \delta W_\infty = \left[\sqrt{\frac{g_{tt}(r=r_H)}{g_{tt}(r=\infty)}}\right]\delta W_H = \sqrt{-g_{tt}(r_H)}a_H\delta s = a_\infty \delta s, \tag{3.51}$$

where $g_{tt}(r=\infty) \approx \eta_{tt} = -1$. As a result, one can define the "**surface gravity**" by the required acceleration of the obeserver at infinity to hold particle at event horizon as follows

$$\kappa \equiv a_\infty = \sqrt{-g_{tt}(r_H)}a_H. \tag{3.52}$$

The four-acceleration can be written in terms of four-velocity as follows

$$a^\mu \equiv u^\nu \nabla_\nu u^\mu. \tag{3.53}$$

The four-velocity of stationary particle can be written as

$$u^\mu_{\text{particle}} = (\dot{t}, 0, 0, 0), \tag{3.54}$$

where $\dot{t} = \sqrt{(-g_{tt})^{-1}}$ obtained from condition $u^\mu u_\mu = -1$. Eq.(3.53) becomes $a^\mu = u^t \nabla_t u^\mu$. The nonvanishing component of the acceleration of particle can be written as

$$\begin{aligned} a^r &= u^t(\partial_t u^r + \Gamma^r_{tt}u^t) \\ &= -\frac{1}{2}(-g_{tt})^{-1}g^{rr}\partial_r g_{tt}. \end{aligned} \tag{3.55}$$

The Christoffel symbol is $\Gamma^r_{tt} = \frac{1}{2}g^{r\alpha}\left(\partial_t g_{t\alpha} + \partial_t g_{\alpha t} - \partial_\alpha g_{tt}\right) = -\frac{1}{2}g^{rr}\partial_r g_{tt}$. Then, norm of a^μ is

$$a = \sqrt{a^\mu a_\mu} = |a^r|\sqrt{g_{rr}} = \frac{1}{2}(-g_{tt})^{-1}g^{rr}|\partial_r g_{tt}|\sqrt{g_{rr}}. \tag{3.56}$$

The surface gravity in eq.(3.52) is given by

$$\kappa = \left[\frac{1}{2}\sqrt{-g_{tt}g_{rr}}g^{rr}(-g_{tt})^{-1}\right]_{r_H}|\partial_r g_{tt}|_{r=r_H}. \tag{3.57}$$

For Schwarzschild case with $f = 1 - \frac{2mG}{r}$, then $g_{\mu\nu} = \mathrm{diag}(-f, f^{-1}, r^2, r^2\sin^2\theta)$ and $g^{\mu\nu} = \mathrm{diag}(-f^{-1}, f, r^{-2}, (r^2\sin^2\theta)^{-1})$, one obtains

$$\kappa_{\mathrm{sch}} = \left[\frac{1}{2}\sqrt{f^{-1}f}ff^{-1}\right]_{r_H}|\partial_r(-f)|_{r=r_H} = \frac{1}{2}\frac{2mG}{r^2}|_{r=r_H} = \frac{1}{4mG}, \tag{3.58}$$

where $r_H = 2mG$. For the earth, the surface gravity can be evaluated by Newton's law of universal gravitation which is

$$\vec{F}_g = G\frac{m_1 m_2}{r^2}\hat{r}. \tag{3.59}$$

According to the definition of surface gravity then we obtain

$$\kappa_e = G\frac{m_E}{r_E^2}, \tag{3.60}$$

where m_E and r_E are mass and radius of the earth, respectively.

Another definition of surface gravity associates with Killing horizon. The Killing horizon is a hypersurface that normal vector is Killing vector, $K_\mu = n_\mu = \partial_\mu\Phi$ where the hypersurface defined by Φ = constant. Recalling the relation of Killing vector in eq.(2.55), we have

$$\begin{aligned} K^{\nu}\nabla_{\nu}K_{\mu} &= -K^{\nu}\nabla_{\mu}K_{\nu} \\ &= -\frac{1}{2}\nabla_{\mu}(K^{\nu}K_{\nu}). \end{aligned} \tag{3.61}$$

Since, we consider the hypersurface Φ = constant, we can choose $\Phi = -K^{\nu}K_{\nu}$, then

$$\begin{aligned} K^{\nu}\nabla_{\nu}K_{\mu} &= \frac{1}{2}\partial_{\mu}\Phi \propto K_{\mu}, \\ K^{\nu}\nabla_{\nu}K_{\mu} &= \kappa K_{\mu}. \end{aligned} \tag{3.62}$$

Note that we have used

$$\partial_{\mu}\Phi = 2\kappa K_{\mu}, \tag{3.63}$$

where κ is the surface gravity. There are proper coordinates for evaluating the surface gravity with eq.(3.63). For example, if we consider Schwarzschild black hole in the coordinates (t, r, θ, ϕ) with the timelike Killing vector $K^{\mu}_{(t)} = \frac{\partial x^{\mu}}{\partial t}$, eq,(3.63) gives nothing because $\partial_t\Phi = \partial_t(-g_{tt}) = 0$. We try to consider the ingoing Eddington-Finkelstein coordinates (v, r, θ, ϕ). In this coordinate, the timelike Killing vector is written as

$$K^{\mu} = \frac{\partial x^{\mu}}{\partial t} = \frac{\partial x^{\mu}}{\partial v}\frac{\partial v}{\partial t} = \frac{\partial x^{\mu}}{\partial v}, \tag{3.64}$$

where we used $\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}(t + r^{*}) = 1$. The dual-vector form of the Killing vector is written as follows

$$K_{\mu} = g_{\mu\nu}K^{\nu} = g_{\mu v}K^{v}. \tag{3.65}$$

According the metric tensor (3.33), the non-vanishing component is $K_v = g_{vv}K^v$ and $K_r = g_{rv}K^v$. The left-hand side of eq.(3.63) is evaluated by

$$\begin{aligned}\partial_\mu \Phi &= \partial_\mu(-K^\rho K_\rho)\\ &= \partial_\mu(-g_{vv})\\ &= \partial_\mu f(r)\\ &= (\partial_\mu r) f'(r),\end{aligned} \tag{3.66}$$

where $g_{vv} = -f(r) = -\left(1 - \frac{2mG}{r}\right)$ is used. Thus, the non-vanishing component of eq.(3.63) is only component of $\mu = r$ as follows

$$(\partial_r r) f'(r) = 2\kappa g_{rv} K^v.$$

Applying $g_{rv} = 1$ and $K^v = 1$, one obtains

$$\kappa = \frac{f'(r)}{2}. \tag{3.67}$$

It is consistent to the surface gravity in eq.(3.58). Actually, we can choose both ingoing and outgoing Eddingtom-Finkelstein coordinates (3.32).

There is other expression of the surface gravity by employing an auxiliary vector. Such a vector relates to the Killing vector as follows

$$N^\mu K_\mu = -1, \tag{3.68}$$

where it is evaluated at the event horizon. By contracting eq.(3.62) with auxiliary null vector, N^μ, the surface gravity can be defined as

$$\kappa = -K^\mu N^\nu \nabla_\mu K_\nu. \tag{3.69}$$

For Kerr black hole, the Killing vector is $K^\mu = (1, 0, 0, \Omega_H)$ then the following surface gravity is

$$\kappa = \frac{\Omega_H}{a}\sqrt{(mG)^2 - a^2} = \frac{\sqrt{(mG)^2 - a^2}}{r_{H+}^2 + a^2}. \tag{3.70}$$

The zeroth law of black hole's mechanics states that "surface gravity constant throughout the event horizon". By treating the event horizon as Killing horizon, the metric can be written in terms of the null bases as $\{K^\mu, N^\mu, m^\mu, \bar{m}^\mu\}$ where K^μ is Killing vector, N^μ is axuiliary null vector, m^μ is null complex vector and $\bar{m}^\mu$ is null complex conjugate. From this setup, m^μ and $\bar{m}^\mu$ lying in event horizon, K^μ and N^ν hypersurface orthogonal to event horizon. By using definition of surface gravity in eq.(3.69), and considering the change of surface gravity along the event horizon, we have

$$\begin{aligned}
m^\mu\nabla_\mu\kappa &= -m^\mu\nabla_\mu(K^\sigma N^\nu\nabla_\sigma K_\nu) \\
&= -(\nabla_\mu\nabla_\sigma K_\nu)m^\mu K^\sigma N^\nu - (\nabla_\sigma K_\nu)(\nabla_\mu K^\sigma)m^\mu N^\nu \\
&\qquad\qquad\qquad\qquad - (\nabla_\sigma K_\nu)(\nabla_\mu N^\nu)m^\mu K^\sigma.
\end{aligned} \tag{3.71}$$

Considering $m^\mu N^\nu\nabla_\mu K_\nu$ and using property $K^\mu N_\nu = -1$, we obtain

$$\begin{aligned}
m^\mu N^\nu\nabla_\mu K_\nu &= \nabla_\mu(m^\mu N^\nu K_\nu) - N^\nu K_\nu\nabla_\mu m^\mu - m^\mu K_\nu\nabla_\mu N^\nu \\
&= -\nabla_\mu m^\mu + \nabla_\mu m^\mu - m^\mu K_\nu\nabla_\mu N^\nu \\
&= -m^\mu K_\nu\nabla_\mu N^\nu, \\
g_{\nu\rho}m^\mu N^\nu\nabla_\mu K^\rho &= -g_{\nu\rho}m^\mu K^\rho\nabla_\mu N^\nu, \\
m^\mu N^\nu\nabla_\mu K^\rho &= -m^\mu K^\rho\nabla_\mu N^\nu.
\end{aligned} \tag{3.72}$$

Note that the second and third term of eq.(3.71) are cancelled out. Using the relation between derivatives of Killing vector and Riemann tensor,

$$\nabla_\mu\nabla_\nu K^\sigma = R^\sigma{}_{\nu\mu\rho}K^\rho, \tag{3.73}$$

together with reuslt from eqs.(3.73) and (3.72), eq.(3.71) becomes

$$m^\mu\nabla_\mu\kappa = -R_{\nu\sigma\mu\rho}K^\rho m^\mu K^\sigma N^\nu. \tag{3.74}$$

It is shown that deformation tensor vanishes on the event horizon, $\tilde{B}_{\mu\nu} = 0$ [41]. Using this agrument together with Raychaudhuri's equation for null congruence in eq.(2.78), we obtain

$$R_{\mu\nu}K^\mu K^\nu = 0. \tag{3.75}$$

Since deformation tensor vanishes on event horizon, we obtain

$$m^\mu \bar{m}^\nu B_{\mu\nu} = m^\mu \bar{m}^\nu \nabla_\mu K_\nu = 0. \tag{3.76}$$

By using $m^\sigma \nabla_\sigma m^\mu = m^\sigma \nabla_\sigma \bar{m}^\mu = 0$, the relation for the second covariant derivative of the Killing vector can be expressed as

$$m^\sigma \nabla_\sigma (m^\mu \bar{m}^\nu \nabla_\mu K_\nu) = m^\sigma m^\mu \bar{m}^\nu \nabla_\sigma \nabla_\mu K_\nu = 0. \tag{3.77}$$

We substitute eq.(3.73) in eq.(3.77), then one obtains

$$m^\sigma m^\mu \bar{m}^\nu R_{\nu\mu\sigma\rho} K^\rho = 0. \tag{3.78}$$

Substituting $g^{\mu\nu} = -K^\mu N^\nu - N^\mu K^\nu + m^\mu \bar{m}^\nu + \bar{m}^\mu m^\nu$ in eq.(3.78), we obtain

$$\begin{aligned} m^\mu m^\sigma \bar{m}^\nu K^\rho R_{\nu\mu\sigma\rho} &= m^\mu K^\rho R_{\nu\mu\sigma\rho}(g^{\sigma\nu} + K^\sigma N^\nu + N^\sigma K^\nu - \bar{m}^\sigma m^\nu), \\ 0 &= m^\mu K^\rho R_{\mu\rho} + m^\mu K^\rho R_{\nu\mu\sigma\rho} N^\sigma K^\nu, \\ m^\mu K^\rho N^\sigma K^\nu R_{\sigma\rho\mu\nu} &= m^\mu K^\rho R_{\mu\rho}. \end{aligned} \tag{3.79}$$

Note that the third and fourth term are vanished because anti-symmetric and symmetric tensor contract together. Substituting eq.(3.79) into eq.(3.74), one obtains

$$m^\mu \nabla_\mu \kappa = -R_{\mu\nu} m^\mu K^\nu. \tag{3.80}$$

Using Einstein's field equation, eq.(2.67), and $K^\nu m_\nu = K^\nu \bar{m}_\nu = 0$, one obtains

$$\begin{aligned} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right) m^\mu K^\nu &= 8\pi G T_{\mu\nu} m^\mu K^\nu, \\ R_{\mu\nu} m^\mu K^\nu &= 8\pi G T_{\mu\nu} m^\mu K^\nu. \end{aligned} \tag{3.81}$$

Then eq.(3.80) becomes

$$m^\mu \nabla_\mu \kappa = -8\pi G T_{\mu\nu} m^\mu K^\nu. \tag{3.82}$$

By contracting Einstein's field equation with $K^\mu K^\nu$ and using the relation in eq.(3.75), one obtains

$$T_{\mu\nu} K^\mu K^\nu = 0. \tag{3.83}$$

Eq.(3.83) implies that $T_{\mu\nu}K^{\nu} \propto K_{\mu}$. Then eq.(3.82) can be expressed as

$$m^{\mu}\nabla_{\mu}\kappa = (T_{\mu\nu}K^{\nu})m^{\mu} \propto K_{\mu}m^{\mu} = 0. \tag{3.84}$$

This implies that surface gravity constant in tranverse direction on event horizon which is refered to the **zeroth law** of black hole's mechanics [40, 41].

3.2.2 The First Law

The first law of black hole's mechanics is the relation between variation of black hole's parameters, such as mass, horizon area, and angular momentum.

3.2.2.1 Smarr's formula

There exists a relation between the mass, m, angular momentum, $J \equiv am$, and horizon area of black hole. The outer horizon area of Kerr black hole is given by

$$A_H = \oint_H d^2\theta\sqrt{\sigma} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta(r_H^2 + a^2)\sin\theta = 4\pi(r_H^2 + a^2), \tag{3.85}$$

where σ is determinant of the metric on two-surface with fixing $t =$ constant and $r = r_+$ as follows

$$\sigma_{AB}d\theta^A d\theta^B = \rho^2 d^2\theta + \frac{\Sigma}{\rho^2}\sin^2\theta d\phi^2.$$

From eqs.(3.16), (3.29) and (3.70), the mass can be written in terms of event horizon as follows

$$\begin{aligned} m = \frac{r_H^2 + a^2}{2Gr_{H+}} &= \frac{r_H}{2G} + \frac{a^2}{2Gr_H} \\ &= \frac{\kappa(r_H^2 + a^2)}{G} + 2\Omega_H J \\ &= \kappa\frac{A_H}{4\pi G} + 2\Omega_H J. \end{aligned} \tag{3.86}$$

This relation is refered to **Smarr's formula**.

From eq.(3.86), the area can be written in terms of mass and angular momentum as follows

$$\frac{A_H}{4\pi G} = m + \sqrt{m^2 - \left(\frac{J}{mG}\right)^2}. \tag{3.87}$$

This correspond to the **no-hair theorem** which states that "stationary, asymptotically flat black hole are fully described by the parameter of mass, electromagnetic charge and angular momentum". As a result, the variation of the area can be written as

$$\frac{\kappa}{8\pi G}\delta A_H(m,J) = \delta m - \Omega_H \delta J, \tag{3.88}$$

$$\delta m = \frac{\kappa}{8\pi G}\delta A_H + \Omega_H \delta J. \tag{3.89}$$

This equation is known as the **first law** of black hole's mechanics.

3.2.3 The Second Law

The second law of black hole's mechanics describes the behavior of the black hole after perturbation is applied.

3.2.3.1 Penrose process

For Kerr black hole, the metric in eq.(2.167) is independent of t and ϕ. Thus, the Killing vectors can be written as

$$K^{\mu}_{(t)} = (1,0,0,0)\ ,\ K^{\mu}_{(\phi)} = (0,0,0,1). \tag{3.90}$$

It is more convenient to construct the linear combination as follows

$$K^{\mu} = K^{\mu}_{(t)} + \Omega K^{\mu}_{(\phi)} = (1,0,0,\Omega), \tag{3.91}$$

wherer Ω is angular frequency of a particle. This form of Killing vector is useful to find the conserved quantities of a particle moving in Kerr geoemtry. Those conserved quantities of a particle can be expressed as

$$E = -p_{\nu}K^{\nu}_{(t)} \quad \text{and} \quad l = p_{\nu}K^{\nu}_{(\phi)}, \tag{3.92}$$

where p^{μ} is four-momentum of particle in spacetime, E and l are energy and angular momentum, respectively. In rest frame of particle in ergoregion, the conserved energy can be expressed in terms of K^{μ} as

$$-K^{\nu}p_{\nu} = E > 0. \tag{3.93}$$

Considering particle A originally outside ergoregion and then moving to the ergoregion, this particle is spilted to particles B and C. Let's particle B move out from the ergoregion, as shown in Figure 29. For this process, we have three step as follows

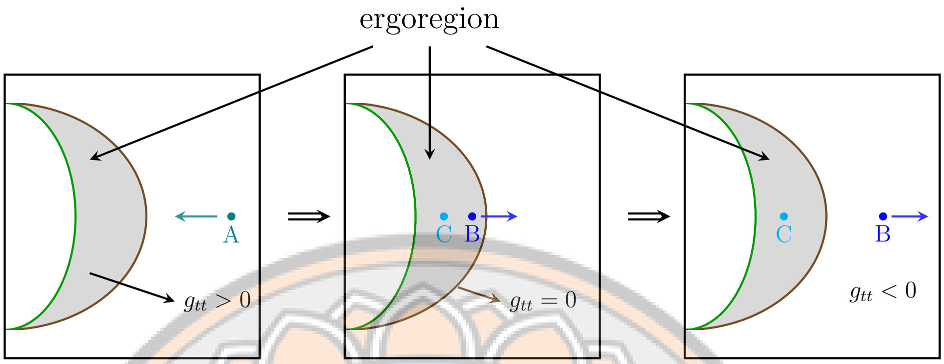


Figure 29 Penrose process.

1. The energy of particle A outside the ergoregion with momentum $p^{\mu} = m(1, 0, 0, 0)$ can be written as

$$E^{(\mathrm{A})} = -g_{\mu\nu} p^{\mu} K^{\nu}_{(t)} = m g_{tt}|_{\text{outside}} > 0, \tag{3.94}$$

2. The particle A in the ergoregion is splilted into B and C,
3. The particle B moves outside the ergoregion where the particle C still be in the ergoregion.

Since the ergoregion $g_{tt} > 0$, the energy of particle C in the ergoregion can be written as

$$E^{(\mathrm{C})} = -m g_{tt}|_{\text{ergoregion}} < 0. \tag{3.95}$$

Therefore, the energy of particle B can be obtained from the conservation of energy as follows

$$\begin{aligned} E^{(\mathrm{A})} &= E^{(\mathrm{C})} + E^{(\mathrm{B})}, \\ E^{(\mathrm{B})} &= E^{(\mathrm{A})} - E^{(\mathrm{C})}, \\ E^{(\mathrm{B})} &> E^{(\mathrm{A})}. \end{aligned} \tag{3.96}$$

All of the mentioned procedure is called "**Penrose process**".

This implies that we can extract energy from black hole. In other words, the mass of black hole decreases due to this process. In addition, the angular momentum of black hole also decrease. Considering the four-momentum of the particle C at rest frame, $p^\mu = m\frac{dt}{d\tau}(1,0,0,\dot{\phi})$, its energy can be expressed as

$$E^{(\mathrm{C})} - \Omega_H l^{(\mathrm{C})} = -K^\nu p_\nu^{(\mathrm{C})} > 0. \tag{3.97}$$

Note that according to eq.(3.91), this quantity corresponds to energy of particle observed by observer in the rest frame and then it must be positive. According to eq.(3.95), the angular momentum of the partilcle in the rest frame is also negative as follows

$$0 > \frac{E^{(\mathrm{C})}}{\Omega_H} > l^{(\mathrm{C})}. \tag{3.98}$$

Therefore, the angular momentum part is also extracted from the black hole more than one from energy part as follows

$$\Omega_H \delta J < \delta m < 0. \tag{3.99}$$

By using this process and applying to the first law of black hole's mechanics in eq.(3.89), we obtain

$$\delta A_H > 0. \tag{3.100}$$

This inequality corresponds to the **second law** of black hole's mechanics which states that the event horizon area of the black hole is always increased by Penrose process. In fact, the second law can be obtained by any process, and can be shown by using Raychaudhuri's equation and focusing theorem.

Since the second law associates with the change of area of the event horizon, the expansion scalar is a useful quantity to determined can be such the change in terms of geometric description. The evolution of expansion scalar can be determined by the Raychaudhuri's equation found in eq.(2.78),

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma^{\mu\nu}\sigma_{\mu\nu} - R_{\mu\nu}k^\mu k^\nu \leq 0. \tag{3.101}$$

Note that we have considered the null geodesics being hypersurface orthogonal, therefore the rotation matrix vanishes $\omega_{\mu\nu} = 0$. Moreover, the null energy condition $R_{\mu\nu}k^\mu k^\nu \geq 0$, have been applied. From above equation, eq.(3.101), one found that the expansion scalar is always decreased and called "focusing theorem". This implies that the geodesic congruence will focus to a point. As a result, we obtain

$$\begin{aligned} \frac{d\theta}{d\lambda} &\leq -\frac{1}{2}\theta^2, \\ \theta^{-1}(\lambda) &\geq \theta^{-1}(\lambda = 0) + \frac{\lambda}{2}. \end{aligned} \tag{3.102}$$

This implies that if the congruence is initially converging $\theta(\lambda = 0) < 0$, then $\theta(\lambda) \to -\infty$ with affine parameter $\lambda \leq \frac{2}{|\theta(\lambda=0)|}$, Raychaudhuri's equation, eq.2.78, is not valid. In order to avoid such situation the initial value of expansion scalar must be not negative, $\theta(\lambda = 0) \geq 0$. As a result, we obtain

$$\theta(\lambda) \geq 0. \tag{3.103}$$

From eq.(2.80), the expansion scalar corresponds to the change of the area, therefore eq.(3.103) implies

$$\delta A \geq 0.$$

It is called "**area theorem**" which corresponds to the second law of black hole's mechanics validating for any process [31].

3.2.4 The Third Law

This law relate to obtaining extremal black hole. For extremal charge black hole, it obeys the condition $m = q^2$ while for the rotating one obeys $mG^2 = a^2$. It is useful to consider extremal state via surface gravity only for Kerr black hole in eq.(3.70) as follows

$$\kappa = \frac{\sqrt{(mG)^2 - a^2}}{r_H^2 + a^2} = \frac{\sqrt{(mG)^2 - a^2}}{2mr_H}. \tag{3.104}$$

Note that extremal state is achived when surface gravity vanishes. By substituting $a = \frac{J}{m}$ and $r_H = mG + \sqrt{(mG)^2 - a^2}$ into eq.(3.104), and setting $G = 1$ we obtain

$$\kappa = \frac{\sqrt{m^2 - (\frac{J}{m})^2}}{2m\left[m + \sqrt{m^2 - (\frac{J}{m})^2}\right]} = \frac{1}{2m}\left[\frac{\sqrt{1 - (\frac{J}{m^2})^2}}{1 + \sqrt{1 - (\frac{J}{m^2})^2}}\right]. \tag{3.105}$$

To obtain zero surface gravity, we must increase angluar momentum by throwing in the particle. Every this process, the mass of black hole is also increased. Thus, if we keep doing the process, $\frac{J}{m^2} \sim \frac{l}{m}$ approaches to zero, where l is angular frequency of black hole. So, surface gravity depends on the factor $\frac{1}{2m}$. We must do the process till mass of black hole reach infinite. In other words, it requires infinite times of process to obtain extremal state. Then the **third law** of black hole's mechanics states that "it is impossible to reduce surface gravity of black hole to zero with finite sequence operations"[42].

CHAPTER IV

BLACK HOLE'S ENTROPIES

4.1 BLACK HOLE THERMODYNAMICS

If we compare the laws of black hole's mechanics and thermodynamics, they are quite similar to each other as shown in table 1.

Table 1 Black hole's mechanics and thermodynamics law.

Laws	Black hole's mechanics	Thermodynamics
Zeroth	Surface gravity is constant on event horizon entirely.	At equilibrium, temperature of whole systems are constant.
First	$\delta m = \frac{\kappa}{8\pi G}\delta A_H$	$dU = TdS$
Second	$\delta A_H \geq 0$	$dS \geq 0$
Third	It is impossible to reduce surface gravity to zero with finite number of operations.	It is impossible to reduce temperature to zero with finite number of operations.

Notice that, some quantities of black hole and thermodynamic system have the similar behavior. For example, surface gravity is equivalent to temperature, and horizon area is equivalent to entropy. Therefore, it might be worthwhile to imagine that black hole is thermal object.

4.2 BEKENSTEIN-HAWKING ENTROPY

4.2.1 Temperature of Black Hole

Stephen Hawking shown that there is radiation from black hole, so called "Hawking radiation" by using quantum field in curved spacetime. He considered Klein-Gordon field in a spherically gravitational collapse spacetime [1]. The objective of [1] is finding the number of particle which is observed at spatial infinity. The field operator can be expressed by creation and annihilation operator as follows

$$\hat{\phi} = \int d\omega (\hat{a}_\omega f_\omega + \hat{a}^\dagger_\omega \bar{f}_\omega) \quad \text{with} \quad \hat{a}_\omega |0\rangle = 0,$$

where $\{f_\omega, \bar{f}_\omega\}$ is plane wave basis which is solution of Klein-Gordon equation, $\omega^2 = \vec{p}^2$ and $|0\rangle$ is vacuum state. Hawking separated the field operator into two part, namely, ingoing and outgoing particles. The ingoing means going to the event horizon and outgoing means going to observer at spatial infinity as follows

$$\hat{\phi} = \int d\omega \left[(\hat{b}_\omega p_\omega + \hat{b}^\dagger_\omega \bar{p}_\omega) + (\hat{c}_\omega q_\omega + \hat{c}^\dagger_\omega \bar{q}_\omega) \right], \tag{4.1}$$

where $\hat{b}$ is annihilation of ingoing particles and $\hat{c}$ is annihilation of outgoing particles. The ground states are defined by

$$\hat{b}_\omega |0\rangle_H = 0, \quad \hat{c}_\omega |0\rangle_\infty = 0 \quad \text{with} \quad \hat{c}_\omega |0\rangle_H \neq 0,$$

where H and ∞ stand for horizon and spatial infinity, respectively. The average number of particles at horizon which observed at spatial infinity can be written by

$${}_H\langle 0 | \hat{c}^\dagger_\omega \hat{c}_\omega | 0 \rangle_H = \langle N^\infty_\omega \rangle_H. \tag{4.2}$$

By using many complicated mathematical methods [1], Hawking found that the average number of particle can be expressed as follows

$$\langle N^\infty_\omega \rangle_H = \frac{\Gamma_\omega}{e^{\frac{2\pi}{\kappa}\omega} - 1}, \tag{4.3}$$

where Γ_ω is absorption coefficient at mode ω and κ is surface gravity. It is similar to the Planck's law of black-body radiation as shown in Appendix A. By setting $k_B = 1$ and $h = 1$, the average number of radiated particle of the blackbody at temperature T is expressed by

$$\langle n\rangle_{bb} = \frac{1}{e^{\frac{1}{T}\nu} - 1}. \tag{4.4}$$

Notice that eqs.(4.3) and (4.4) look similar in the case of $\Gamma_\omega = 1$. Thus we can interpret temperature of black hole as

$$T_H = \frac{\kappa}{2\pi}, \tag{4.5}$$

is called "Hawking temperature" [1, 43, 44]. Since Hawking radiation matches up with eq.(A.5), and eq.(A.5) obeys Gibbs-Boltzmann statistics, BH entropy is Gibbs entropy type.

Moreover, black hole does not behave like black-body, it behaves like grey-body because it is factorized by Γ_ω which is called "Grey -body factor". This quantity can be interpreted as probability of outgoing wave to reach spatial infinity, or transmission coefficient in quantum mechanics [45].

Recalling the first law of black hole thermodynamics of Schwarzschild space-time in eq.(3.89), and using eq.(4.5), one obtains

$$\delta m = \frac{T_H}{4G}\delta A_H. \tag{4.6}$$

Since temperature can be defined on the black hole, we can treat black hole as thermodynamics object. It is supported by similarity between the mechanic laws and thermodynamic law. In this sense, mass of black hole, eq.(4.6), can play the role of internal energy, $dU = TdS$. Then the entropy of black hole can be written by

$$S_{BH} = \frac{A_H}{4G}, \tag{4.7}$$

which is known as "**Bekenstein-Hawking entropy**". Since Hawking radiation matches up with eq.(A.5), and eq.(A.5) obeys Gibbs-Boltzmann statistics, BH entropy is Gibbs entropy type.

4.2.2 Euler's Theorem and Smarr's Formula

By using Euler's theorem of homogeneous function, we would show that there is another approach for obtaining Smarr's formula of black hole. Euler's theorem states that if $f(x_1, \cdots, x_n)$ is homogeneous function of k^{th} order then $f(x)$ satisfies

$$kf(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i \frac{\partial f(x_1, \cdots, x_n)}{\partial x_i}, \tag{4.8}$$

where k satisfies the relation

$$f(\lambda x_1, \ldots, \lambda x_n) = \lambda^k f(x_1, \ldots, x_n). \tag{4.9}$$

Applying Hawking temperature (4.5) and Bekenstein-Hawking entropy (4.7), the Smarr's formula of Schwarzschild black hole found in eq.(3.86) can be reexpressed as

$$m = 2\left(\frac{\kappa}{2\pi}\right)\frac{A}{4G} = 2T_H S_{BH}. \tag{4.10}$$

We will find this formula by using Euler's theorem as following. From $1 - \frac{2mG}{r_H} = 0$, the mass of the black hole can be written in terms of Bekenstein-Hawking entropy as function of entropy,

$$m = \frac{r_H}{2G} = \frac{1}{4G}\sqrt{\frac{A_H}{\pi}} = \frac{1}{2\sqrt{\pi G}}\sqrt{\frac{A_H}{4G}} = \frac{1}{2}\sqrt{\frac{S_{BH}}{\pi G}}, \tag{4.11}$$

where $A_H = 4\pi r_H^2$. This mass is the homogeneous function of S_{BH} degree 1/2 which can be expressed as follows

$$m(\lambda S_{BH}) = \frac{1}{2}\sqrt{\frac{\lambda S_{BH}}{\pi G}} = \lambda^{1/2} m(S_{BH}). \tag{4.12}$$

Applying Euler's theorem (4.11), we obtain

$$\frac{1}{2}m(S_{BH}) = S_{BH}\frac{\partial m(S_{BH})}{\partial S_{BH}}. \tag{4.13}$$

The conjugate variable of entropy can be expressed as $\partial m/\partial S_{BH}$,

$$\frac{\partial}{\partial S_{BH}}\left(\frac{1}{2}\sqrt{\frac{S_{BH}}{\pi G}}\right) = \frac{1}{4\sqrt{S_{BH}\pi G}} = \frac{1}{2\sqrt{A_H\pi}} = \frac{1}{4\pi r_H} = \frac{1}{4}\frac{1}{2mG\pi} = \frac{\kappa_{sch}}{2\pi}.$$

This is exactly the same expression of Hawking temperature. Therefore, the Hawking temperature can be written in terms of conjugate variable of the entropy

$$\frac{\partial m}{\partial S_{BH}} = T_H. \tag{4.14}$$

As a result, substituting eq.(4.14) into eq.(4.13), we obtain

$$\frac{1}{2}m = T_H S_{BH}.$$

This is the Smarr's formula derived from Euler's theorem which is exactly the same with eq.(4.10) [26]. Furthermore, the first law of black hole's mechanics can be written in terms black hole's thermodynamic quantities as follows

$$\delta m = \frac{\partial m}{\partial S_{BH}}\delta S_{BH} = T_H\delta S_{BH}. \tag{4.15}$$

4.3 WALD ENTROPY

In 1913, Bardeen proposed the four laws of black hole's mechanics [4]. These laws of black hole is quite similar to the thermodynamic laws. 4 years later, the black hole's entropy was proposed by the Bekenstein known as "Bekenstein-Hawking entropy" [46]. These results are obtained by treating gravitational theory as General Relativity. Meanwhile, the modification of General Relativity has been investigated in order to understand nature of Universe. It gives rise many kinds of theory of gravity. It is possible to find the black hole's solution for such a theory. Consequently, the entropy of such a black hole may be modified. In 1993, Wald proposed the black hole's entropy which is directly derived from Noether's conserved charge. Therefore, it can be used in wide range in the gravitational theory, even in beyond General Relativity such as the Symmetric Teleparallel Equivalent of General Relativity (STEGR) [5, 47]. This kind of entropy is known as "Wald entropy".

4.3.1 Higher Derivative Theory and Boundary Term

The derivation of the Wald entropy is based on the Wald's original paper [30]. According to this method, the differential form is executed whole consideration, the differential n-form is written by

$$\underline{\underline{F}} = F_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}. \tag{4.16}$$

In this convention, an underlining motivation refers component of n-form. For the n-form (4.16), we have

$$\underline{F} \equiv F_{\mu_1 \cdots \mu_n}, \tag{4.17}$$

where $F_{\mu_1 \cdots \mu_n} = F_{[\mu_1 \cdots \mu_n]}$. Then the Lagrangian n-form can be written in terms of Lagrangian density $\mathcal{L}$ as follows

$$\underline{L} = \underline{\varepsilon}\mathcal{L}, \tag{4.18}$$

where $\underline{\varepsilon} = \sqrt{-g}\epsilon_{\mu_1\mu_2\cdots\mu_n}$ is the Levi-Civita tensor. By requiring a theory must be diffeomorphism invariance, such Lagrangian density is given by

$$\mathcal{L} = \mathcal{L}\left(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_{\mu_1}R_{\mu\nu\rho\sigma}, \cdots, \nabla_{(\mu_1}\cdots\nabla_{\mu_k)}R_{\mu\nu\rho\sigma}, \Psi, \cdots, \nabla_{(\mu_1\cdots\mu_k)}\Psi\right), \tag{4.19}$$

where Ψ is a extra dynamical field [30]. The variation of Lagrangian (4.19) is

$$\begin{aligned}\delta\underline{L} = \underline{\varepsilon}\Bigg[&\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}}\delta g_{\mu\nu} + \frac{\partial\mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\delta R_{\mu\nu\rho\sigma} + \frac{\partial\mathcal{L}}{\partial(\nabla_{\mu_1}R_{\mu\nu\rho\sigma})}\delta(\nabla_{\mu_1}R_{\mu\nu\rho\sigma}) + \cdots\\ &+\frac{\partial\mathcal{L}}{\partial(\nabla_{(\mu_1}\cdots\nabla_{\mu_k)}R_{\mu\nu\rho\sigma})}\delta(\nabla_{(\mu_1}\cdots\nabla_{\mu_k)}R_{\mu\nu\rho\sigma}) + \frac{\partial\mathcal{L}}{\partial\Psi}\delta\Psi + \frac{\partial\mathcal{L}}{\partial(\nabla_{\mu_1}\Psi)}\delta(\nabla_{\mu_1}\Psi)\\ &+\cdots+\frac{\partial\mathcal{L}}{\partial(\nabla_{(\mu_1}\cdots\nabla_{\mu_k)}\Psi)}\delta(\nabla_{(\mu_1}\cdots\nabla_{\mu_k)}\Psi)\Bigg] + \mathcal{L}\delta\underline{\varepsilon},\end{aligned} \tag{4.20}$$

where $\delta\underline{\varepsilon} = \frac{1}{2}\underline{\varepsilon}g^{\mu\nu}\delta g_{\mu\nu}$. Let s consider the term of variation of ith covariant derivative of $R_{\mu\nu\rho\sigma}$, $\underline{\varepsilon}\frac{\partial\mathcal{L}}{\partial(\nabla_{(\mu_1}\cdots\nabla_{\mu_i)}R_{\mu\nu\rho\sigma})}\delta\nabla_{(\mu_1}\cdots\nabla_{\mu_i)}R_{\mu\nu\rho\sigma}$. A totally symmetric tensor $A_{\mu\nu} = A_{(\mu\nu)}$ and a tensor $B_{\mu\nu} = B_{(\mu\nu)} + B_{[\mu\nu]}$, the contraction of them is written by

$$A_{(\mu\nu)}B^{\mu\nu} = A_{(\mu\nu)}B^{(\mu\nu)}. \tag{4.21}$$

By using this equation, one obtains

$$\begin{aligned}&\underline{\varepsilon}\frac{\partial\mathcal{L}}{\partial(\nabla_{(\mu_1}\cdots\nabla_{\mu_i)}R_{\mu\nu\rho\sigma})}\delta\nabla_{(\mu_1}\cdots\nabla_{\mu_i)}R_{\mu\nu\rho\sigma}\\ &\qquad= \underline{\varepsilon}\frac{\partial\mathcal{L}}{\partial(\nabla_{(\mu_1}\cdots\nabla_{\mu_i)}R_{\mu\nu\rho\sigma})}\delta\nabla_{\mu_1}\cdots\nabla_{\mu_i}R_{\mu\nu\rho\sigma}.\end{aligned} \tag{4.22}$$

For convenience, one defines

$$\nabla_{\mu_2}\cdots\nabla_{\mu_i}R_{\mu\nu\rho\sigma} \equiv \mathscr{R}_{\mu_2\cdots\mu_i\mu\nu\rho\sigma} = \underset{\sim}{\mathscr{R}}_{2,\cdots,i}, \tag{4.23}$$

$$\check{\Gamma}^{\lambda}_{\mu_1\mu_2}\underset{\sim}{\mathscr{R}}_{\lambda,3,\cdots,i} \equiv \Gamma^{\lambda}_{\mu_1\mu_2}\mathscr{R}_{\lambda\mu_3\cdots\mu_i\mu\nu\rho\sigma} + \cdots + \Gamma^{\lambda}_{\mu_1\sigma}\mathscr{R}_{\mu_2\cdots\mu_i\mu\nu\rho\lambda}, \tag{4.24}$$

where the undertilde absorbs the indices $\mu\nu\rho\sigma$. Then, we obtain

$$\begin{aligned}\delta\left(\nabla_{\mu_1}\utilde{\mathscr{R}}_{2,\cdots,i}\right) &= \delta\left(\partial_{\mu_1}\utilde{\mathscr{R}}_{2,\cdots,i} - \check{\Gamma}^{\lambda}_{\mu_1\mu_2}\utilde{\mathscr{R}}_{\lambda,3,\cdots,i}\right)\\ &= \partial_{\mu_1}\left(\delta\utilde{\mathscr{R}}_{2,\cdots,i}\right) - \left(\delta\check{\Gamma}^{\lambda}_{\mu_1\mu_2}\right)\utilde{\mathscr{R}}_{\lambda,3,\cdots,i} - \check{\Gamma}^{\lambda}_{\mu_1\mu_2}\left(\delta\utilde{\mathscr{R}}_{\lambda,3,\cdots,i}\right)\\ &= \nabla_{\mu_1}\left(\delta\utilde{\mathscr{R}}_{2,\cdots,i}\right) - \left(\delta\check{\Gamma}^{\lambda}_{\mu_1\mu_2}\right)\utilde{\mathscr{R}}_{\lambda,3,\cdots,i}, \qquad (4.25)\end{aligned}$$

where one defined

$$\begin{aligned}\left(\delta\check{\Gamma}^{\lambda}_{\mu_1\mu_2}\right)\utilde{\mathscr{R}}_{\lambda,3,\cdots,i} &\equiv \left(\delta\Gamma^{\lambda}_{\mu_1\mu_2}\right)\mathscr{R}_{\lambda\mu_3\cdots\mu_i\mu\nu\rho\sigma} + \cdots + \left(\delta\Gamma^{\lambda}_{\mu_1\sigma}\right)\mathscr{R}_{\mu_2\cdots\mu_i\mu\nu\rho\lambda},\\ \check{\Gamma}^{\lambda}_{\mu_1\mu_2}\left(\delta\utilde{\mathscr{R}}_{\lambda,3,\cdots,i}\right) &\equiv \Gamma^{\lambda}_{\mu_1\mu_2}\left(\delta\mathscr{R}_{\lambda\mu_3\cdots\mu_i\mu\nu\rho\sigma}\right) + \cdots + \Gamma^{\lambda}_{\mu_1\sigma}\left(\delta\mathscr{R}_{\mu_2\cdots\mu_i\mu\nu\rho\lambda}\right).\end{aligned}$$

According to eq.(2.215), the term $\delta\check{\Gamma}^{\lambda}_{\mu\nu} \propto \delta\Gamma^{\lambda}_{\mu\nu}$ is proportional to $\nabla\delta g_{\mu\nu}$, eq.(4.25) can be rewritten by

$$\delta\left(\nabla_{\mu_1}\cdots\nabla_{\mu_i}R_{\mu\nu\rho\sigma}\right) = \nabla_{\mu_1}\left(\delta\utilde{\mathscr{R}}_{2,\cdots,i}\right) + \textcircled{\scriptsize $\nabla\delta g$};, \qquad (4.26)$$

where $\textcircled{\scriptsize $\nabla\delta g$}$ represents the terms that proportional to covariant derivative of $\delta g_{\mu\nu}$. Next step is rewritting the expression of the term $\varepsilon\frac{\partial\mathcal{L}}{\partial(\nabla_{(\mu_1}\cdots\nabla_{\mu_i)}R_{\mu\nu\rho\sigma})}\delta\nabla_{\mu_1}\cdots\nabla_{\mu_i}R_{\mu\nu\rho\sigma}$. Using the notation (4.23) and eq.(4.26), one has

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial(\nabla_{\mu_1}\utilde{\mathscr{R}}_{(2,\cdots,i)})}\delta\left(\nabla_{\mu_1}\utilde{\mathscr{R}}_{2,\cdots,i}\right) &= \frac{\partial\mathcal{L}}{\partial\left(\nabla_{\mu_1}\utilde{\mathscr{R}}_{(2,\cdots,i)}\right)}\nabla_{\mu_1}\left(\delta\utilde{\mathscr{R}}_{2,\cdots,i}\right) + \textcircled{\scriptsize $\nabla\delta g$}\\ &= \nabla_{\mu_1}\left[\frac{\partial\mathcal{L}}{\partial\left(\nabla_{\mu_1}\utilde{\mathscr{R}}_{(2,\cdots,i)}\right)}\delta\utilde{\mathscr{R}}_{2,\cdots,i}\right]\\ &\quad - \nabla_{\mu_1}\left[\frac{\partial\mathcal{L}}{\partial\left(\nabla_{\mu_1}\utilde{\mathscr{R}}_{(2,\cdots,i)}\right)}\right]\delta\utilde{\mathscr{R}}_{2,\cdots,i} + \nabla_{\mu_1}\textcircled{\scriptsize δg} - \textcircled{\scriptsize δg}\\ &= \nabla_{\mu_1}\left[\frac{\partial\mathcal{L}}{\partial\left(\nabla_{\mu_1}\utilde{\mathscr{R}}_{(2,\cdots,i)}\right)}\delta\utilde{\mathscr{R}}_{2,\cdots,i} + \textcircled{\scriptsize δg}\right]\\ &\quad - \nabla_{\mu_1}\left[\frac{\partial\mathcal{L}}{\partial\left(\nabla_{\mu_1}\utilde{\mathscr{R}}_{(2,\cdots,i)}\right)}\right]\delta\utilde{\mathscr{R}}_{2,\cdots,i} + \textcircled{\scriptsize δg}, \qquad (4.27)\end{aligned}$$

where $\textcircled{\scriptsize δg}$ is the term which is proportional to $\delta g_{\mu\nu}$. The first term corresponds to the boundary term. The third term can contribute to the $\delta g_{\mu\nu}$'s equation of motion. The second term can be rewritten as follows

$$
\begin{aligned}
&\nabla_{\mu_1}\left[\frac{\partial\mathcal{L}}{\partial\left(\nabla_{\mu_1}\underset{\sim}{\mathscr{R}}_{(2,\cdots,i)}\right)}\right]\delta\underset{\sim}{\mathscr{R}}_{2,\cdots,i}\\
&\quad=\nabla_{\mu_1}\left[\frac{\partial\mathcal{L}}{\partial\left(\nabla_{(\mu_1}\nabla_{\mu_2}\cdots\nabla_{\mu_i)}R_{\mu\nu\rho\sigma}\right)}\right]\delta\nabla_{\mu_2}\cdots\nabla_{\mu_i}R_{\mu\nu\rho\sigma}\\
&\quad=\nabla_{\mu_i}\left[\frac{\partial\mathcal{L}}{\partial\left(\nabla_{(\mu_i}\nabla_{\mu_1}\cdots\nabla_{\mu_{i-1})}R_{\mu\nu\rho\sigma}\right)}\right]\delta\nabla_{\mu_1}\cdots\nabla_{\mu_{i-1}}R_{\mu\nu\rho\sigma},
\end{aligned}
\tag{4.28}
$$

where I relabel indices, $1\to i$, $2\to 1$, $\cdots$, $i-2\to i-1$. Then one keeps repeat the procedure in eq.(4.27) until reaching the $\delta R_{\mu\nu\rho\sigma}$. Thus the equation of motion with respect to $\delta R_{\mu\nu\rho\sigma}$ which derived from the term of $\underset{\sim}{\varepsilon}\frac{\partial\mathcal{L}}{\partial(\nabla_{(\mu_1}\cdots\nabla_{\mu_i)}R_{\mu\nu\rho\sigma})}\delta\nabla_{(\mu_1}\cdots\nabla_{\mu_i)}R_{\mu\nu\rho\sigma}$, is written by

$$
(-1)^i\nabla_{(\mu_1}\cdots\nabla_{\mu_i)}\left[\frac{\partial\mathcal{L}}{\partial\left(\nabla_{(\mu_1}\cdots\nabla_{\mu_i)}R_{\mu\nu\rho\sigma}\right)}\right]\delta R_{\mu\nu\rho\sigma}.
\tag{4.29}
$$

This process can be visualized by diagram 30. Repeating the same procedure to

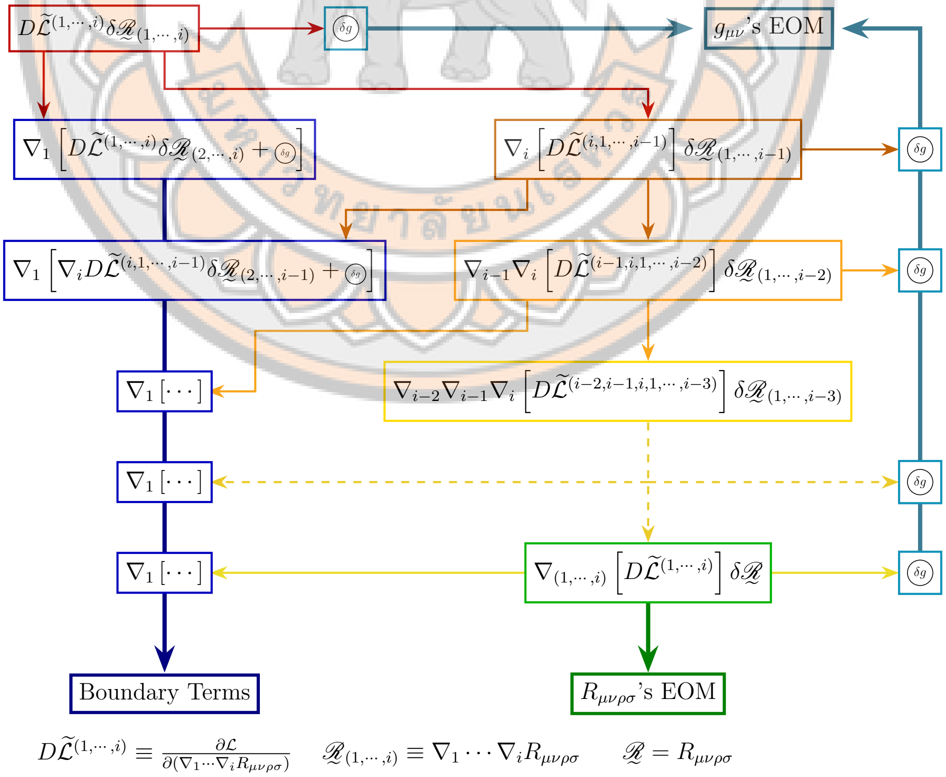


Figure 30 Procedure to obtain the equations of motion and boundary term.

every terms which multiplying with variation covariant derivative of $R_{\mu\nu\rho\sigma}$ and the extra field Ψ in eq.(4.20), we would obtain

$$\delta\underline{L} = \underline{\varepsilon}\left(E^{\mu\nu}_{(g)}\delta g_{\mu\nu} + E^{\mu\nu\rho\sigma}_{(R)}\delta R_{\mu\nu\rho\sigma} + E_{(\Psi)}\delta\Psi\right) + \underline{\varepsilon}\nabla_{\mu_1}\tilde{\Theta}^{\mu_1}, \tag{4.30}$$

where $E_{(\Psi)}$ is the equation of motion with respect to Ψ, $\underline{E}^{\mu\nu}_{(g)}$ is the equation of motion w.r.t. $g_{\mu\nu}$. Here, The equation of motion with respect to $R_{\mu\nu\rho\sigma}$ which written as follows

$$\begin{aligned} E^{\mu\nu\rho\sigma}_{(R)} = \Bigg[\frac{\partial\mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} - \nabla_{\mu_1}\frac{\partial\mathcal{L}}{\partial\left(\nabla_{\mu_1}R_{\mu\nu\rho\sigma}\right)} + \cdots \\ +(-1)^k\nabla_{(\mu_1}\cdots\nabla_{\mu_k)}\frac{\partial\mathcal{L}}{\partial\left(\nabla_{(\mu_1}\cdots\nabla_{\mu_k)}R_{\mu\nu\rho\sigma}\right)}\Bigg], \end{aligned} \tag{4.31}$$

The last term in eq.(4.30) is the boundary terms, i.e., the first term in eq.(4.27). The current associated to the boundary term is explicitly expressed as

$$\begin{aligned} \tilde{\Theta}^{\mu_1} = S^{\mu_1\mu\nu}\delta g_{\mu\nu} + \sum_{i=1}^{m-1}T^{\mu\nu\rho\sigma\mu_1\cdots\mu_i}_{(i)}\delta\nabla_{(\mu_2}\cdots\nabla_{\mu_i)}R_{\mu\nu\rho\sigma} \\ + \sum_{i=1}^{l-1}U^{\mu_1\cdots\mu_i}_{(i)}\delta\nabla_{(\mu_2}\cdots\nabla_{\mu_i)}\Psi, \end{aligned} \tag{4.32}$$

where S, $T_{(i)}$ and $U_{(i)}$ are function of dynamical fields, i.e., $g_{\mu\nu}$, $R_{\mu\nu\rho\sigma}$ and Ψ. In fact, $R_{\mu\nu\rho\sigma}$ is not an independent field, it can be expressed in term of $\delta g_{\mu\nu}$. According to the reference [30], we have

$$\delta R_{\mu\nu\rho\sigma} = 2\nabla_\mu\nabla_\sigma\delta g_{\nu\rho} + R_{\mu\nu\rho}{}^{\lambda}\delta g_{\sigma\lambda}. \tag{4.33}$$

Then the variation of Lagrangian (4.30) can be written by

$$\begin{aligned} \delta\underline{L} &= \underline{\varepsilon}\left[\left(E^{\mu\nu}_{(g)} + E^{\sigma\lambda\rho\mu}_{(R)}R_{\sigma\lambda\rho}{}^{\nu}\right)\delta g_{\mu\nu} + \left(2E^{\rho\mu\nu\sigma}_{(R)}\nabla_\rho\nabla_\sigma\delta g_{\mu\nu}\right) + E_{(\Psi)}\delta\Psi\right] + \underline{\varepsilon}\nabla_{\mu_1}\tilde{\Theta}^{\mu_1} \\ &= \underline{\tilde{E}}_{(\phi)}\delta\phi + \underline{\varepsilon}\nabla_{\mu_1}\left[\tilde{\Theta}^{\mu_1} + 2E^{\mu_1\mu\nu\sigma}_{(R)}\nabla_\sigma\delta g_{\mu\nu} - 2\left(\nabla_\rho E^{\rho\mu\nu\mu_1}_{(R)}\right)\delta g_{\mu\nu}\right] \\ &= \underline{\tilde{E}}_{(\phi)}\delta\phi + \underline{\varepsilon}\nabla_{\mu_1}\Theta^{\mu_1}, \end{aligned} \tag{4.34}$$

where we have used the integration by part twice times on the term $2E^{\rho\mu\nu\sigma}_{(R)}\nabla_\rho\nabla_\sigma\delta g_{\mu\nu}$ in the second line. The dynamical fields are understood as

$$\phi = (g_{\mu\nu}, \Psi),$$

which its equation of motion is

$$\underline{\tilde{E}}_{(\phi)}\delta\phi = \underline{\varepsilon}\left[\left(E^{\mu\nu}_{(g)} + E^{\sigma\lambda\rho\mu}_{(R)} R_{\sigma\lambda\rho}{}^{\nu} + 2\nabla_\rho\nabla_\sigma E^{\rho\mu\nu\sigma}_{(R)}\right)\delta g_{\mu\nu} + E_{(\Psi)}\delta\Psi\right], \tag{4.35}$$

and a new boundary current is

$$\Theta^{\mu_1} \equiv \bar{\Theta}^{\mu_1} + 2E^{\mu_1\mu\nu\sigma}_{(R)}\nabla_\sigma\delta g_{\mu\nu}, \tag{4.36}$$

where we have defined

$$\bar{\Theta}^{\mu_1} = \tilde{\Theta}^{\mu_1} - 2\left(\nabla_\rho E^{\rho\mu\nu\mu_1}_{(R)}\right)\delta g_{\mu\nu}. \tag{4.37}$$

In language of differential form, the **boundary current** Θ is $n-1$ form which is defined by

$$\underline{\Theta} = \varepsilon_{\mu_1\mu_2\cdots\mu_n}\Theta^{\mu_1}. \tag{4.38}$$

Then the variation of Lagrangian (4.34) can written as follows

$$\delta\underline{L} = \underline{\tilde{E}}_{\phi}\delta\phi + d\underline{\Theta}, \tag{4.39}$$

where d is exterior derivative.

4.3.2 Noether's Charge

According to a Noether's current (2.200), we can said that a Noether's current comprises from a boundary term and the dot product between a vector ξ and a Lagrangian. So, the Noether's current of such a theory in eq.(4.18) can be given by

$$J^\mu = \Theta^\mu\left(\phi, \delta_\xi\phi\right) - \xi^\mu\mathcal{L}. \tag{4.40}$$

It satisfies the condition $\nabla_\mu J^\mu = 0$,

$$\begin{aligned}\nabla_\mu J^\mu &= \nabla_\mu \Theta^\mu - \nabla_\mu \left(\xi^\mu \mathcal{L}\right) \\ &= \delta_\xi \mathcal{L} - \tilde{E}_\phi \delta_\xi \phi - \delta_\xi \mathcal{L} \\ &= -\tilde{E}_\phi \delta_\xi \phi, \end{aligned} \tag{4.41}$$

where one has used eq.(4.34). According to eq.(2.198), we can substitute $\delta_\xi \mathcal{L} = \nabla_\mu \left(\xi^\mu \mathcal{L}\right)$ into the second term in the first line. The Noether's current of above is conserved when the dynamical filed satisfies the equation of motion. The differential form's version of the Noether's current is given by $n-1$ form as following

$$\underline{J} = \varepsilon_{\mu_1 \mu_2 \cdots \mu_n} J^{\mu_1} \tag{4.42}$$

$$= \underline{\Theta}(\phi, \delta_\xi \phi) - \xi \cdot \underline{L}, \tag{4.43}$$

where a centered dot denotes contraction to the first index. We would substitute the expression of Noether's current (4.40) into eq.(4.42), we have that

$$\underline{J} = \underline{\varepsilon} \cdot \left[\left(\bar{\Theta}^\mu(\phi, \delta_\xi \phi) + 2E_R^{\mu\nu\rho\sigma} \nabla_\sigma \delta_\xi g_{\rho\nu} \right) - \xi^\mu \mathcal{L} \right]. \tag{4.44}$$

Since we were considering conserved quantity under transformation along a vector ξ, one would replace $\delta_\xi \phi$ by Lie derivative of dynamical fields $\hat{L}_\xi \phi$. According to the formula of Lie derivative (2.54), the ith derivative terms, $\delta_\xi \nabla_{\mu_1} \cdots \nabla_{\mu_i} \phi^\mu = \hat{L}_\xi \nabla_{\mu_1} \cdots \nabla_{\mu_i} \phi^\mu$, give raise the expression with linear on ξ^μ and $\nabla_\mu \xi^\nu$. Thereafter, the current $\bar{\Theta}^\mu$ can be expressed by

$$\bar{\Theta}^\mu(\phi, \hat{L}_\xi \phi) = V^{\mu\nu}\left(\phi\right) \xi_\nu + W^{\mu\rho\nu}\left(\phi\right) \nabla_\rho \xi_\nu, \tag{4.45}$$

where $V^{\mu\nu}$ and $W^{\mu\rho\nu}$ are the collection of S, T and U in eq.(4.32). According to the *Lemma 1* in [48], it proved that a closed n form $\underline{a}^{(i)}$ such that

$$a^{(i)}_{\mu_1 \cdots \mu_p} = A^{(i)}{}_{\mu_1 \cdots \mu_p}{}^{\nu_1 \cdots \nu_i}{}_\rho \nabla_{(\nu_1} \cdots \nabla_{\nu_i)} \phi^\rho, \tag{4.46}$$

where $A^{(i)}{}_{\mu_1 \cdots \mu_p}{}^{\nu_1 \cdots \nu_i}{}_\rho = A^{(i)}{}_{[\mu_1 \cdots \mu_p]}{}^{(\nu_1 \cdots \nu_i)}{}_\rho$ it can be expressed by

$$a^{(i)}_{\mu_1 \cdots \mu_p} = \nabla_{\mu_1} \left\{ \left(\frac{i}{n-p+i} \right) A^{(i)}{}_{\sigma \mu_2 \cdots \mu_p}{}^{\sigma \nu_2 \cdots \nu_i}{}_\rho \nabla_{\nu_2} \cdots \nabla_{\nu_i} \phi^\rho \right\}, \tag{4.47}$$

when $\underline{a}^{(i)}$ is closed. This implies that $\underline{a}^{(i)}$ can be written in term of reducing $i-1$ derivative, and a linear term with ϕ^ρ is zero. Thus, the term of Lagrangian $\xi^\mu \mathcal{L}$ and $V^{\mu\nu}\xi_\nu$ in eqs.(4.45) and (4.44), respectively, do not contribute to the conserved charge. The boundary term (4.45) can be written by

$$\begin{aligned}\underline{\varepsilon}\cdot\bar{\Theta}^\mu &= \varepsilon_{\mu\mu_2\cdots\mu_n} W^{\mu\rho\nu}\nabla_\rho\xi_\nu \\ &= \nabla_{\mu_2}\left\{\frac{1}{2}\varepsilon_{\lambda_1\lambda_2\mu_3\cdots\mu_n} W^{\lambda_1\lambda_2\nu}\xi_\nu\right\}.\end{aligned} \tag{4.48}$$

The second term in eq.(4.44) is expressed by

$$2E^{\mu\rho\nu\sigma}_{(R)}\nabla_\sigma \hat{L}_\xi g_{\rho\nu} = 2E^{\mu\rho\nu\sigma}_{(R)}\nabla_\sigma\left(\nabla_\rho\xi_\nu + \nabla_\nu\xi_\rho\right). \tag{4.49}$$

We would consider the second term, if we have anti-symmetric tensor $T^{\rho\sigma}$ then contact it with $\nabla_\rho\nabla_\sigma\xi^\mu$, we have

$$\begin{aligned}T^{\rho\sigma}\nabla_\rho\nabla_\sigma\xi^\mu &= T^{\rho\sigma}\left(\nabla_\sigma\nabla_\rho\xi^\mu + R^\mu{}_{\nu\sigma\rho}\xi^\nu\right) \\ &= T^{\sigma\rho}\left(\nabla_\rho\nabla_\sigma\xi^\mu + R^\mu{}_{\nu\rho\sigma}\xi^\nu\right) \\ &= -T^{\rho\sigma}\left(\nabla_\rho\nabla_\sigma\xi^\mu + R^\mu{}_{\nu\sigma\rho}\xi^\nu\right)\end{aligned}$$

where the relation $\left(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu\right)V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma$ is used. Adding $T^{\rho\sigma}\nabla_\rho\nabla_\sigma\xi^\mu$ to both sides, we obtain

$$T^{\rho\sigma}\nabla_\rho\nabla_\sigma\xi^\mu = \frac{1}{2}R^\mu{}_{\nu\sigma\rho}\xi^\nu. \tag{4.50}$$

The above equation implies that the second term in eq.(4.49) is linear on vector ξ^μ, then it does not contribute in the conserved charge. Only the first term is a remainder in eq.(4.49). The following expression

$$\begin{aligned}\nabla_\sigma\nabla_\rho\xi_\nu &= \nabla_{(\sigma}\nabla_{\rho)}\xi_\nu + \nabla_{[\sigma}\nabla_{\rho]}\xi_\nu \\ &= \nabla_{(\sigma}\nabla_{\rho)}\xi_\nu + \frac{1}{2}R_{\nu\lambda\sigma\rho}\xi^\lambda,\end{aligned} \tag{4.51}$$

will be used for manipulating eq.(4.49). Using the above equation, one would rewrite this term as following

$$
\begin{aligned}
\underline{\varepsilon} \cdot 2E_{(R)}^{\mu\rho\nu\sigma}\nabla_\sigma\nabla_\rho\xi_\nu &= 2\varepsilon_{\mu\mu_2\cdots\mu_n}E_{(R)}^{\nu\sigma\mu\rho}\nabla_\sigma\nabla_\rho\xi_\nu \\
&= 2\varepsilon_{\mu\mu_2\cdots\mu_n}E_{(R)}^{\nu\sigma\mu\rho}\left(\nabla_{(\sigma}\nabla_{\rho)}\xi_\nu + R_{\nu\lambda\sigma\rho}\xi^\lambda\right) \\
&= \varepsilon_{\mu\mu_2\cdots\mu_n}\left[\left(-{E_{(R)}}_\nu{}^{\sigma\rho\mu} - {E_{(R)}}_\nu{}^{\rho\sigma\mu}\right)\nabla_{(\sigma}\nabla_{\rho)}\xi^\nu + E_{(R)}^{\nu\sigma\mu\rho}R_{\nu\lambda\sigma\rho}\xi^\lambda\right] \\
&= -\nabla_{\mu_2}\left[\frac{2}{3}\varepsilon_{\mu\sigma\mu_3\cdots\mu_n}\left({E_{(R)}}_\nu{}^{\sigma\rho\mu} + {E_{(R)}}_\nu{}^{\rho\sigma\mu}\right)\nabla_\rho\xi^\nu\right], \qquad (4.52)
\end{aligned}
$$

where we applied eq.(4.47) in the third line with $i = 2(i = 0)$ and $p = n - 1$ for the first(second) term. The properties of $E_R^{\mu\nu\rho\sigma}$ is inherited from $R^{\mu\nu\rho\sigma}$, then following properties,

$$E_{(R)}^{\mu\nu\rho\sigma} = E_{(R)}^{\rho\sigma\mu\nu}, \qquad (4.53)$$

$$E_{(R)}^{\mu\nu\rho\sigma} = -E_{(R)}^{\mu\nu\sigma\rho}, \qquad (4.54)$$

are used in the first and second line, respectively. We would continue calculation of the right-hand side of eq.(4.52), and then denoting this term with $\underline{Q}_R$, we have

$$
\begin{aligned}
\underline{Q}_{(R)} &= \frac{2}{3}\varepsilon_{\lambda_1\lambda_2\mu_3\cdots\mu_n}\left(E_{(R)}^{\nu\lambda_2\rho\lambda_1} + E_{(R)}^{\nu\rho\lambda_2\lambda_1}\right)\nabla_\rho\xi_\nu \\
&= \frac{2}{3}\varepsilon_{\lambda_1\lambda_2\mu_3\cdots\mu_n}\left(-E_{(R)}^{\nu\lambda_2\lambda_1\rho} + E_{(R)}^{\nu\rho\lambda_2\lambda_1}\right)\nabla_\rho\xi_\nu \\
&= \frac{2}{3}\varepsilon_{\lambda_1\lambda_2\mu_3\cdots\mu_n}\left(\frac{-E_{(R)}^{\nu\lambda_2\lambda_1\rho} + E_{(R)}^{\nu\lambda_1\lambda_2\rho}}{2} + E_{(R)}^{\nu\rho\lambda_2\lambda_1}\right)\nabla_\rho\xi_\nu \\
&= \frac{2}{3}\varepsilon_{\lambda_1\lambda_2\mu_3\cdots\mu_n}\left(\frac{-E_{(R)}^{\nu\lambda_2\lambda_1\rho} - E_{(R)}^{\nu\rho\lambda_1\lambda_2} - E_{(R)}^{\nu\lambda_2\rho\lambda_1}}{2} + E_{(R)}^{\nu\rho\lambda_2\lambda_1}\right)\nabla_\rho\xi_\nu \\
&= \frac{2}{3}\varepsilon_{\lambda_1\lambda_2\mu_3\cdots\mu_n}\left(\frac{E_{(R)}^{\nu\lambda_2\rho\lambda_1} - E_{(R)}^{\nu\lambda_2\rho\lambda_1}}{2} + \frac{1}{2}E_{(R)}^{\nu\rho\lambda_2\lambda_1} + E_{(R)}^{\nu\rho\lambda_2\lambda_1}\right)\nabla_\rho\xi_\nu \\
&= -\varepsilon_{\lambda_1\lambda_2\mu_3\cdots\mu_n}E_{(R)}^{\nu\rho\lambda_1\lambda_2}\nabla_\rho\xi_\nu. \qquad (4.55)
\end{aligned}
$$

Bianchi's identity of $R^{\mu\nu\rho\sigma}$

$$E_{(R)}^{\mu\nu\rho\sigma} + E_{(R)}^{\mu\sigma\nu\rho} + E_{(R)}^{\mu\rho\sigma\mu} = 0, \qquad (4.56)$$

is used to obtain the fourth line. The antisymmetry of Levi-Civita tensor $\varepsilon_{\mu_1\cdots\mu_n}$ is used in the third line as follows

$$\begin{aligned}\varepsilon_{\lambda_1\lambda_2\mu_3\cdots\mu_n}E_R^{\nu\lambda_2\lambda_1\rho} &= \varepsilon_{\lambda_2\lambda_1\mu_3\cdots\mu_n}E_R^{\nu\lambda_1\lambda_2\rho}\\ &= -\varepsilon_{\lambda_1\lambda_2\mu_3\cdots\mu_n}E_R^{\nu\lambda_1\lambda_2\rho}. \end{aligned}\tag{4.57}$$

Finally, the Noether's current (4.44) is written by

$$\begin{aligned}\underline{J} &= \nabla_{\mu_2}\left(\frac{1}{2}\varepsilon_{\lambda_1\lambda_2\mu_3\cdots\mu_n}W^{\lambda_1\lambda_2\nu}\xi_\nu\right) + \nabla_{\mu_2}\left(\varepsilon_{\lambda_1\lambda_2\mu_3\cdots\mu_n}E_{(R)}^{\nu\rho\lambda_1\lambda_2}\nabla_\rho\xi_\nu\right)\\ &= \nabla_{\mu_2}\left(\underline{\tilde{W}}^{\nu}\xi_\nu + \underline{X}^{\nu\rho}\nabla_\rho\xi_\nu\right), \end{aligned}\tag{4.58}$$

where one has defined $\underline{\tilde{W}}^{\nu} \equiv \frac{1}{2}\varepsilon_{\lambda_1\lambda_2\mu_3\cdots\mu_n}W^{\lambda_1\lambda_2\nu}$, and

$$\underline{X}^{\mu\nu} \equiv \varepsilon_{\lambda_1\lambda_2\mu_3\cdots\mu_n}E_{(R)}^{\mu\nu\lambda_1\lambda_2}.\tag{4.59}$$

According to reference [48], the **Noether charge** is given by differential $n-2$ form as follows

$$\underline{J} = d\underline{Q}.\tag{4.60}$$

Therefore, the Noether charge which correspond to the current (4.58) can be written by

$$\underline{Q} = \underline{\tilde{W}}^{\mu}\xi_\mu + \underline{X}^{\mu\nu}\nabla_\nu\xi_\mu.\tag{4.61}$$

4.3.3 The First Law of Black Hole's Mechanics

Let's consider the variation the Noether's current $\underline{J}$ (4.43), we obtain

$$\begin{aligned}\delta\underline{J} &= \delta\underline{\Theta}\left(\phi,\delta_\xi\phi\right) - \xi\cdot\delta\underline{L}\\ &= \delta\underline{\Theta}\left(\phi,\delta_\xi\phi\right) - \xi\cdot d\underline{\Theta}\left(\phi,\delta\phi\right)\\ &= \delta\underline{\Theta}\left(\phi,\delta_\xi\phi\right) - \delta_\xi\underline{\Theta}\left(\phi,\delta\phi\right) + d\left[\xi\cdot\underline{\Theta}\left(\phi,\delta\phi\right)\right], \end{aligned}\tag{4.62}$$

where one has used eq.(4.39) and the identity as follows

$$\delta_\xi\underline{\alpha} = \hat{L}_\xi\underline{\alpha}$$

$$= \xi \cdot d\underline{\alpha} + d\left(\xi \cdot \underline{\alpha}\right), \tag{4.63}$$

for a differential n form $\underline{\alpha}$ [5]. Moreover, we used the condition $\underline{\bar{E}}_{\phi} = 0$ in the second line in eq.(4.62), thereby the arbitrary variation $\delta\phi$ satisfies the equation of motion. The first two terms in eq.(4.62) correspond to the quantity ω in eq.(2.186),

$$\underline{\omega}(\phi, \delta\phi, \delta_{\xi}\phi) = \delta\underline{\Theta}(\phi, \delta_{\xi}\phi) - \delta_{\xi}\underline{\Theta}(\phi, \delta\phi). \tag{4.64}$$

After that, eq.(4.62) can be written by

$$\underline{\omega}\left(\phi, \delta\phi, \delta_{\xi}\phi\right) = \delta\underline{J} - d\left(\xi \cdot \underline{\Theta}\right). \tag{4.65}$$

It is the variation of Hamiltonian-like density $\delta\mathcal{H}$ by the definition in the eq.(2.185). The value of Hamiltonian-like is obtained by using integral mapping of $(n-1)$-form $\underline{\underline{\omega}}$ to 0-form δH via integral mapping in eq.(2.42) as follows

$$\int_{\Sigma} : \underline{\underline{\omega}} \rightarrow \delta H, \tag{4.66}$$

where Σ is a hypersurface with fixing time. The we have

$$\begin{aligned} \delta H &= \int_{\Sigma} \underline{\underline{\omega}} \\ &= \delta \int_{\Sigma} d\underline{\underline{Q}} - \int_{\Sigma} d\left(\xi \cdot \underline{\underline{\Theta}}\right) \\ &= \int_{\partial\Sigma} \left(\delta\underline{\underline{Q}} - \xi \cdot \underline{\underline{\Theta}}\right), \end{aligned} \tag{4.67}$$

where the divergence theorem and the equation $\underline{J} = d\underline{Q}$ are used, and $\partial\Sigma$ is boundary surface of Σ. We assume that there exists a $n-1$ form $\underline{B}$ such as

$$\underline{\Theta} = \delta\underline{B}, \tag{4.68}$$

then we obtain

$$\int_{\partial\Sigma} \xi \cdot \underline{\underline{\Theta}} = \delta \int_{\partial\Sigma} \xi \cdot \underline{\underline{B}}. \tag{4.69}$$

Finally, the Hamiltonian can be expressed as follows

$$H = \int_{\partial\Sigma} \left(\underline{\underline{Q}} - \xi \cdot \underline{\underline{B}}\right). \tag{4.70}$$

If we choose a vector ξ^μ generates **asymptotic time translation**, namely, ξ^μ is chosen as the timelike Killing vector $K^\mu_{(t)}$, then the corresponding conserved quantity is the energy [31]. So that, the **canonical energy $\mathcal{E}$** can be defined by the Hamiltonian-value which is written as follows

$$\mathcal{E} = \int_{\partial\Sigma_\infty} \left(\underline{\underline{Q}} \left[K_{(t)} \right] - K_{(t)} \cdot \underline{\underline{B}} \right), \tag{4.71}$$

where the subscript ∞ means evaluation at spatial infinity, or asymptotically flat surface. Reference [30] prove that the energy E coincides the ADM mass in General Relativity.

For the angular momentum, the vector ξ^μ is chosen to be **asymptotic rotation** $K^\mu_{(\varphi)}$ which is tangent to the surface $\partial\Sigma_\infty$ with fixing the coordinates x^{μ_1} and x^{μ_2}. We would consider the term $\int_{\Sigma_\infty} K_{(\varphi)} \cdot \underline{\underline{B}}$. We would use eq.(2.42) and the expression of $n-2$ surface element in reference [31],

$$\begin{aligned} dS_{\mu_0\mu_1} &= \varepsilon_{\mu_1\mu_2\mu_3\cdots\mu_n} dx^{\mu_3} \wedge \cdots \wedge dx^{\mu_n} \\ &= -2n_{[\mu_1} r_{\mu_2]} \sqrt{\sigma} d^{n-2}\theta, \end{aligned} \tag{4.72}$$

where σ is the determinant of the induced metric on $\partial\Sigma$, and n^{μ_1} and r^{μ_2} are normal vectors corresponding to ones with fixing coordinates x^{μ_1} and x^{μ_2}, respectively. One obtains the result as follows

$$\begin{aligned} \int_{\partial\Sigma_\infty} K_{(\varphi)} \cdot \underline{\underline{\Theta}} &= \int_{\partial\Sigma_\infty} K^{\nu_2}_{(\varphi)} \Theta^{\nu_1} \varepsilon_{\nu_1\nu_2\mu_3\cdots\mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \\ &= -2 \int_{\partial\Sigma_\infty} K^{\nu_2}_{(\varphi)} \Theta^{\nu_1} n_{[\nu_1} r_{\nu_2]} \sqrt{\sigma} d^{n-2}\theta \\ &= 0. \end{aligned} \tag{4.73}$$

Note that $K_{(\varphi)} \cdot \underline{\underline{\Theta}} = K^{\nu_2}_{(\varphi)} \Theta^{\nu_1} \varepsilon_{\nu_1\nu_2\mu_3\cdots\mu_n} dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n}$ is $n-2$ form. The integration vanishes because the vector $K^\mu_{(\varphi)}$ is orthogonal to the vectors n^μ and r^μ. Consequently, the integration $\int_{\partial\Sigma} K_{(\varphi)} \cdot \underline{\underline{B}}$ is vanished. Therefore, the **canonical angular momentum $\mathcal{J}$** is given by the definition as following

$$\mathcal{J} = -\int_{\partial\Sigma_\infty} \underline{\underline{Q}} \left[K_{(\varphi)} \right]. \tag{4.74}$$

References [5, 30] show that above expression of angular momentum coincides to the Komar's angular momentum.

Now, the energy and angular momentum are obtained. The first law of black hole's mechanics can be achieved by choosing a vector ξ^μ as a **symmetry** of the dynamical fields [47]. Then, the variation of fields along the vector K^μ is

$$\delta_\xi \phi = \hat{L}_\xi \phi = 0. \tag{4.75}$$

Let's consider the quantity $\underline{\omega}$ (4.64). The boundary current $\underline{\Theta}$ (4.36) can be written in terms of linear equation such as

$$\underline{\Theta} = a(\phi)\delta\phi + \sum_{i=1} b^{(i)}(\phi)\delta\nabla_{(i)}\phi. \tag{4.76}$$

The variation of higher derivative of ϕ, $\delta\nabla_{(i)}\phi = \delta\nabla_{(\mu_1}\cdots\nabla_{\mu_i)}\phi$, arise from the current $\tilde{\Theta}^\mu$. According to eq.(4.25), this implies that the variation of higher derivative of ϕ can be written in terms of the variation ϕ as follows

$$\delta\nabla_{(\mu_1}\cdots\nabla_{\mu_i)}\phi \sim \delta\phi. \tag{4.77}$$

Using this assumption, the current $\underline{\Theta}$ can be expressed by

$$\underline{\Theta}(\phi, \delta\phi) = \underline{A}(\phi)\delta\phi. \tag{4.78}$$

Recalling the expression of $\underline{\omega}$ (4.64) and analogy to eq.(2.185), one obtains the result as following

$$\begin{aligned} \underline{\omega}(\phi, \delta\phi, \delta_\xi\phi) &= \delta\left[\underline{A}(\phi)\delta_\xi\phi\right] - \delta_\xi\left[\underline{A}(\phi)\delta\phi\right] \\ &= \left[\underline{A}'(\phi)\delta\phi\right]\delta_\xi\phi - \left[\underline{A}'(\phi)\delta_\xi\phi\right]\delta\phi \\ &= 0, \end{aligned} \tag{4.79}$$

where $\underline{A}'$ is the derivative of $\underline{A}$. Here, we used eq.(4.75) in the second line. Therefore, eq.(4.65) becomes

$$\delta d\underline{Q} - d\left(\xi\cdot\underline{\Theta}\right) = 0, \tag{4.80}$$

where we have used $\underline{J} = d\underline{Q}$. We would integrate eq.(4.80) over **instant time** hypersurface Σ_∞, and assign a vector ξ as the Killing vector

$$K^\mu = K^\mu_{(t)} + \Omega^{(\varphi)}_H K^\mu_{(\varphi)}, \tag{4.81}$$

where Ω_H is angular velocities at event horizon. One obtains the following equation

$$\int_{\Sigma_\infty} \left\{ d\delta \underline{\underline{Q}}[K] - d\left(K \cdot \underline{\underline{\Theta}}\right) \right\} = 0. \tag{4.82}$$

The formulation obtained above is applicable for the case of spacetime without inner boundary. However, for spacetime posses a black hole, there are inner boundaries which is the event horizons of the black hole. We interest the spacetime with black hole. In this case, one prefers the event horizon which is the bifurcation sphere $\Sigma_{\mathcal{B}}$. Therefore, the bifurcation sphere $\Sigma_{\mathcal{B}}$ is included as the inner boundary in the integration (4.82). One obtains the following equation,

$$\int_{\Sigma_{\mathcal{B}}}^{\Sigma_\infty} \left\{ \delta d \underline{\underline{Q}}[K] - d\left(K \cdot \underline{\underline{\Theta}}\right) \right\} = \left(\int_{\partial\Sigma_\infty} - \int_{\partial\Sigma_{\mathcal{B}}} \right) \left\{ \delta \underline{\underline{Q}}[K] - K \cdot \underline{\underline{\Theta}} \right\} = 0, \tag{4.83}$$

where the divergence theorem is used and $\partial\Sigma$ denotes the boundary of Σ. According to the definitions of mass and angular momentum in eqs.(4.71) and (4.74), the integration at spatial infinity Σ_∞ can be evaluated by

$$\int_{\partial\Sigma_\infty} \left\{ \delta \underline{\underline{Q}}[K] - K \cdot \underline{\underline{\Theta}} \right\} = \delta\mathcal{E} - \Omega^{(\varphi_i)}_H \delta\mathcal{J}_{(\varphi_i)}. \tag{4.84}$$

Since the Killing vector vanishes at the bifurcation sphere, the integration $\int_{\partial\Sigma_{\mathcal{B}}} K \cdot \underline{\underline{\Theta}} = 0$ vanishes here. Thereafter, eq.(4.83) can be written by

$$\delta \int_{\partial\Sigma_{\mathcal{B}}} \underline{\underline{Q}}\,[K] = \delta\mathcal{E} - \Omega^{(\varphi_i)}_H \delta\mathcal{J}_{(\varphi_i)}, \tag{4.85}$$

where eq.(4.84) is used. This equation is similar to the first law (3.88), then eq.(4.85) can be said to be the first law of black hole. Comparing to the first law with Bekenstein-Hawking entropy S_{BH} (4.15), the left-hand side can be read as

$$\delta \int_{\partial\Sigma_{\mathcal{B}}} \underline{\underline{Q}}\,[K] = \frac{\kappa}{2\pi} \delta S_W, \tag{4.86}$$

where κ is surface gravity, and S_W is known as the **Wald's entropy**. Note that the temperature in this presentation is still the Hawking temperature T_H, because the derivation of the temperature was done in the theory of quantum field in curved spacetime [1]. It can be applied for all general theory of gravity.

4.3.4 Wald's Entropy

In this section, one would find an exact expression of the Wald's entropy. Firstly, we would substitute the Noether charge $\underline{Q}$ (4.61), $\underline{Q} = \underline{\tilde{W}}^{\mu} K_{\mu} + \underline{X}^{\mu\nu} \nabla_{\nu} K_{\mu}$, into $\int_{\partial\Sigma_{\mathcal{B}}} \underline{\underline{Q}}\,[K]$. Then, we have

$$\int_{\partial\Sigma_{\mathcal{B}}} \underline{\underline{Q}}[K] = \int_{\partial\Sigma_{\mathcal{B}}} \underline{\underline{X}}^{\mu\nu} \nabla_{\nu} K_{\mu}. \tag{4.87}$$

The integration $\int_{\Sigma_{\mathcal{B}}} \underline{\underline{\tilde{W}}}^{\mu} K_{\mu}$ disappears, since the Killing vector vanishes at $\Sigma_{\mathcal{B}}$. Note that the quantity $\nabla_{\mu} K_{\nu}$ does not necessary to be zero at $\Sigma_{\mathcal{B}}$. To obtain the expression of $\nabla_{\mu} K_{\nu}$, it would be analogous to the normal vector of a surface. For the surface $\Phi = 0$, the normal vector n^{μ} is written by

$$\partial_{\mu} \left(\Phi = 0 \right) \propto n_{\mu}. \tag{4.88}$$

Since eq.(4.87) is calculated at the bifurcation sphere where the Killing vector vanishes there, $K^{\mu} = 0$, the quantity $\nabla_{\mu} \left(K_{\nu} = 0 \right)$ can infer to the normal vectors of the bifurcation sphere as follows

$$\nabla_{\mu} K_{\nu} \propto b_{\mu\nu}, \tag{4.89}$$

where $b_{\mu\nu}$ is the **binormal tensor**. Actually, such a tensor is the combination of 2 normal vectors of the bifurcation sphere. Obviously, it is antisymmetric tensor which is followed by antisymmetry of $\nabla_{\mu} K_{\nu}$ in eq.(2.55). The crucial property of $b_{\mu\nu}$ is expressed by

$$b^{\mu\nu} b_{\mu\nu} = -2. \tag{4.90}$$

The proportionality constant in the relation (4.89) can be assigned as the surface gravity. Thus, eq.(4.87) is expressed as follows

$$\int_{\mathcal{B}} \underline{\underline{Q}}[K] = \kappa \int_{\mathcal{B}} \underline{\underline{X}}^{\mu\nu} b_{\nu\mu}, \tag{4.91}$$

where one relabeled the notation of $\partial\Sigma_{\mathcal{B}}$ to $\mathcal{B}$. The surface gravity can be pulled out the integrand, since it is constant through the horizon which follows the first law of black hole's mechanics. According to reference [5], the quantity $\int_{\mathcal{B}} \underline{\underline{Q}}[K]$ is the same for all arbitrary horizon cross-section, this means that the choices of $b_{\mu\nu}$ are arbitrary. However, it must satisfy eq.(4.90). Finally, by comparing eq.(4.86) with eq.(4.91), the Wald's entropy is written by

$$S_W = -2\pi \int_H \underline{\underline{X}}^{\mu\nu} b_{\mu\nu}. \tag{4.92}$$

It is worthwhile to introduce a horizon-surface element, since this will be used to calculate the exact form of S_W. This surface element is written by

$$\begin{aligned} dS_{\mu\nu} &= \varepsilon_{\mu\nu\lambda_3\lambda_4\cdots\lambda_n} dx^{\lambda_3} \wedge dx^{\lambda_4} \wedge \cdots \wedge dx^{\lambda_n} \\ &= b_{\mu\nu}\sqrt{\sigma} d^{n-2}\theta, \end{aligned} \tag{4.93}$$

where σ is the determinant of induced metric of $n-2$ surface [49, 50].

For example, we would evaluate the Wald's entropy in the theory of General Relativity in 4 dimensions. This theory is given by the Einstein-Hilbert's Lagrangian which is written by

$$L_{EH} = \frac{R}{16\pi}, \tag{4.94}$$

where R is the trace of Ricci tensor. Then the quantity $E^{\mu\nu\rho\sigma}$ can be evaluated as following

$$
\begin{aligned}
E^{\mu\nu\rho\sigma}_{(GR)} &= \frac{1}{16\pi}\frac{\partial R}{\partial R_{\mu\nu\rho\sigma}} \\
&= \frac{1}{16\pi}\frac{\partial}{\partial R_{\mu\nu\rho\sigma}}\left(g^{\lambda_1\lambda_3}g^{\lambda_2\lambda_4}R_{\lambda_1\lambda_2\lambda_3\lambda_4}\right) \\
&= \frac{1}{16\pi}g^{\lambda_1\lambda_3}g^{\lambda_2\lambda_4}\delta^{\mu}_{\lambda_1}\delta^{\nu}_{\lambda_2}\delta^{\rho}_{\lambda_3}\delta^{\sigma}_{\lambda_4} \\
&= \frac{1}{16\pi}g^{\mu\rho}g^{\nu\sigma}. \qquad (4.95)
\end{aligned}
$$

Substituting this equation into eq.(4.59) and then eq.(4.92) and using the expression of null hypersurface $dS_{\mu\nu}$ in eq.(4.93), we obtain

$$
\begin{aligned}
S_{W,GR} &= -2\pi\int_H \underline{X}^{\mu\nu}b_{\mu\nu}dx^{\mu_2}\wedge dx^{\mu_3} \\
&= -2\pi\int_H X_{\mu_2\mu_3}{}^{\mu\nu}b_{\mu\nu}dx^{\mu_2}\wedge dx^{\mu_3} \\
&= -2\pi\int_H \varepsilon_{\lambda_1\lambda_2\mu_3\mu_4}E^{\lambda_1\lambda_2\mu\nu}_{(GR)}b_{\mu\nu}dx^{\mu_3}\wedge dx^{\mu_4} \\
&= -2\pi\int_H b_{\lambda_1\lambda_2}\frac{1}{16\pi}g^{\lambda_1\mu}g^{\lambda_2\nu}b_{\mu\nu}\sqrt{\sigma}d^2\theta \\
&= -\frac{1}{8}\int_H b^{\mu\nu}b_{\mu\nu}\sqrt{\sigma}d^2\theta \\
&= \frac{1}{4}\int_H \sqrt{\sigma}d^2\theta \\
&= \frac{A_H}{4}, \qquad (4.96)
\end{aligned}
$$

where eq.(4.90) is used. This corresponds to the Bekenstein-Hawking entropy, as we expected. Note that the quantity $\underline{\underline{X}}^{\mu\nu}N_{[\mu}K_{\nu]} = X_{\mu_3\cdots\mu_n}{}^{\mu\nu}N_{[\mu}K_{\nu]}dx^{\mu_3}\wedge\cdots\wedge dx^{n}$ is $n-2$ form.

4.4 BLACK HOLE'S NONEXTENSIVE ENTROPIES

4.4.1 Non-Extensivity of Bekenstein-Hawking Entropy

Thermodynamic variable is divided into 2 types, viz., intensive and extensive variable. The extensive variable depends on size of a system, and vice versa. For the extensive variable X, it satisfies the condition as follows

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}X_i<\infty. \tag{4.97}$$

Examples of extensive variable are entropy S and volume V. So, it is useful to investigate the extensivity of Bekenstein-Haking entropy. The entropy and the mass of the Schwarzschild black hole are related via

$$m=\frac{r_H}{2G}=\frac{1}{4}\sqrt{\frac{S_{BH}}{\pi G}}\quad\Rightarrow\quad S_{BH}\propto m^2. \tag{4.98}$$

Suppose that $m_{total}=m_1+m_2$, then the entropy of the system can be written in terms of subsystem as follows

$$\begin{aligned}S_{BH,total}\propto(m_1+m_2)^2=m_1^2+m_2^2+2m_1m_2,\\ S_{BH,total}=S_{BH,1}+S_{BH,2}+2\sqrt{S_{BH,1}S_{BH,2}}.\end{aligned} \tag{4.99}$$

Bekenstein-Hawking entropy is a **non-extensive entropy**, because it does not satisfies the relation (4.97). Moreover, the black hole's entropy is a **non-additive entropy**, where an additive quantity satisfies the condition

$$X_{12}=X_1+X_2. \tag{4.100}$$

Thus, for Gibbs-Bolztmann statistics seems not to be applicable for describing BH's entropy with non-extensive property.

4.4.1.1 Zeroth Law Incompatibility

The equilibrium state is related to zeroth law of thermodynamics which states that temperature of entire system are equal at equilibrium state. If we consider a composite system as shown in figure 14, the zeroth law can be written in terms of mathematics as follows

$$T_1 = T_2. \tag{4.101}$$

The zeroth law in eq.(4.101) requires additive property of internal energy and entropy. In order to show that, let us consider the composite system as shown in figure 14 and assume that internal energy and entropy are additive. Since they are isolated, total internal energy of this system is constant $U_{\text{total}} = U_1 + U_2$. Thus, infinitesimal change of total internal energy is zero $dU_{\text{total}} = dU_1 + dU_2 = 0$, we obtain

$$dU_1 = -dU_2. \tag{4.102}$$

For total entropy, it is the same with total internal energy as follows $S_{\text{total}} = S_1 + S_2$ and $dS_{\text{total}} = dS_1 + dS_2 = 0$. According to the **fundamental relation of thermodynamics** which is given by

$$dU = TdS - PdV + \mu_i dN^i.$$

From the fundamental relation of thermodynamics, we can have entropy function as follows

$$dS = \frac{1}{T}dU + \frac{P}{T}dV - \frac{\mu_i}{T}dN^i. \tag{4.103}$$

This can give us a relation of derivative of entropy with respect to internal energy as follows

$$\frac{\partial S}{\partial U} = \frac{1}{T}. \tag{4.104}$$

By fixing volume and number of particle, the infinitesimal change of entropy of such a system figure 14 can be written by

$$dS = \left(\frac{\partial S}{\partial U}\right) dU. \tag{4.105}$$

Substituting the relation (4.105) into the infinitesimal change of total entropy $dS_1 + dS_2 = 0$ and using the relation $dU_1 = -dU_2$ and eq.(4.104), we obtain

$$\begin{aligned} dS_1 &= -dS_2, \\ \left(\frac{dS_1}{dU_1}\right) dU_1 &= -\left(\frac{dS_2}{dU_2}\right) dU_2, \\ \frac{1}{T_1} dU_1 &= -\frac{1}{T_2}(-dU_1), \\ T_1 &= T_2. \end{aligned} \tag{4.106}$$

Therefore, the zeroth law of thermodynamics requires **additivity** of internal energy and entropy [22].

4.4.2 Black Hole's Rényi Entropy

In order to investigate the extensivity of Bekenstein-Hawking entropy, we need to redefine S_{BH} in other statistics. One proposes the **Tsallis entropy**,

$$S_T = \frac{1}{q-1}(1 - \sum_i p_i^q). \tag{4.107}$$

The composition rule of Tsallis entropy can be expressed as

$$S_T(p_i, p_j) = S_T(p_i) + S_T(p_j) + (q-1) S_T(p_i) S_T(p_j). \tag{4.108}$$

where p_i and p_j are independent variables. Tsallis entropy is generalized entropy from Gibbs-Boltzmann entropy where

$$\lim_{q \to 1} S_T = S_{GB}. \tag{4.109}$$

From The Khinchin axioms for entropy, the "q" parameter must satisfy the range as follows $(0, \infty)$.

To study about non-extensive property of black hole, we should define a entropy of the black hole as non-extensive entropy. We define the Tsallis entropy as the area of the black hole as follows

$$S_{GB} = \frac{A}{4G} \quad \rightarrow \quad \frac{A}{4G} \equiv S_T.$$

For consistence with zeroth law of thermodynamics, such a entropy must be additive as shown in section 4.4.1.1. The additive form of Tsallis entropy can be obtained by using logarithm map as follows

$$\mathrm{add}(S_T) \sim \ln\left[1 + (1-q)S_T\right].$$

For recovering Gibbs-Boltzmann entropy, we need to add the factor $\frac{1}{1-q}$. The additive form of Tsallis entropy is

$$\mathrm{add}(S_T) = \frac{1}{1-q}\ln\left[1 + (1-q)S_T\right], \tag{4.110}$$

where $\lim_{q\to 1}\mathrm{add}(S_T) = \lim_{q\to 1} S_T = S_{GB}$. This known as **Rényi entropy** given by

$$S_R = \frac{1}{1-q}\ln\left(\sum_i p_i^q\right). \tag{4.111}$$

The composition rule of the Rényi entropy is

$$S_R(p_i, p_j) = S_R(p_i) + S_R(p_j). \tag{4.112}$$

For convenient, we redefine the non-extensive parameter as $\lambda \equiv 1-q$. Thus, the entropy of black hole can be written by

$$S_{bh} = \frac{1}{\lambda}\ln\left(1 + \lambda\frac{A_H}{4G}\right), \tag{4.113}$$

so-called "**black hole's Reńyi entropy**". Note that subscript "bh" refer to black hole instead of Bekenstein-Hawking which is dented by "BH". Therefore, mass function can be written in terms of black hole's Rényi entropy as follows

$$m = \frac{r_H}{2G} = \frac{1}{2}\sqrt{\frac{e^{\lambda S_{bh}} - 1}{\lambda\pi G}}, \tag{4.114}$$

where $r_H = \frac{1}{2}\sqrt{\frac{A_H}{\pi}}$ and $A_H = \frac{4G}{\lambda}(e^{\lambda S_{bh}} - 1)$.

4.4.2.1 Smarr's formula and First law

To obtain Smarr's formula, mass function (4.114) should be homogeneous function. One finds that mass must be function of entropy and λ^{-1} as follows

$$m(\alpha S_{bh}, \alpha\lambda^{-1}) = \frac{1}{2}\sqrt{\frac{(e^{\lambda S_{bh}} - 1)}{\pi G}\frac{\alpha}{\lambda}} = \alpha^{1/2} m(S_{bh}, \lambda^{-1}). \tag{4.115}$$

According to Euler's theorem found in eq.(4.8), we can write

$$\frac{1}{2}m = \left(\frac{\partial m}{\partial S_{bh}}\right) S_{bh} + \left(\frac{\partial m}{\partial \lambda^{-1}}\right)\lambda^{-1}. \tag{4.116}$$

Since $d\lambda^{-1} = -\lambda^{-2} d\lambda$ and we define "Rényi temperature" $T_R \equiv \frac{\partial m}{\partial S_{bh}}$ and conjugate variable of non-extensive parameter $\Psi_R \equiv \frac{\partial m}{\partial \lambda}$, Smarr's formula becomes

$$\frac{1}{2}m = \left(\frac{\partial m}{\partial S_{bh}}\right) S_{bh} - \left(\frac{\partial m}{\partial \lambda}\right)\lambda = T_R S_{bh} - \Psi_R \lambda. \tag{4.117}$$

Here, the Rényi temperature is expressed as follows

$$T_R = \frac{1}{4\pi r_H}\left(1 + \lambda\frac{\pi r_H^2}{G}\right). \tag{4.118}$$

We should consider physical meaning of Ψ_R. In reference [26], the authors interpret Ψ_R as thermodynamic volume since the leading order term of small λ expansion is proportional to volume as follows

$$\Psi_R \approx \frac{\pi r_H^3}{8G^2} - \frac{\pi^2 r_H^5}{24G^3}\lambda + \frac{\pi^3 r_H^7}{48G^4}\lambda^2 + ... \tag{4.119}$$

By defining

$$V_R = \frac{2}{3}G^2\Psi_R, \tag{4.120}$$

we can interpret Ψ_R as "Rényi volume". Consequently, non-extensive parameter λ can be interpreted as "Rényi pressure"

$$\lambda = \frac{3}{2}G^2 P_R. \tag{4.121}$$

Now, Smarr's formula is written by

$$\frac{1}{2}m = T_R S_{bh} - P_R V_R. \tag{4.122}$$

According to eq.(4.117), the derivative form of mass function is

$$dm = \left(\frac{\partial m}{\partial S_{bh}}\right) dS_{bh} + \left(\frac{\partial m}{\partial \lambda}\right) d\lambda.$$

By using the definition of Rényi temperature, Rényi volume and Rényi pressure, one obtains

$$dm = T_R dS_{bh} + V_R dP_R. \tag{4.123}$$

The mass of the black hole is found to be the **enthalpy** in thermodynamics rather than internal energy.

CHAPTER V

THERMODYNAMIC STABILITY OF BLACK HOLE IN MODIFIED GRAVITY

5.1 MODIFIED GRAVITY

5.1.1 Modified Gravity: Importance

In 1915, there was a scientific theory that revolutionizes the ordinary institution of gravity which is discovered by Newton in 1687. It is the "General Relativity", so-called GR, which was proposed by Einstein. One of the advantages of GR is solving the Newton's unsolvable problem, e.g., the anomalous perihelion precession of Mercury. There are evidences that support GR, namely, gravitational redshift, gravitational lensing, and the defection of light by the Sun [51]. Moreover, people daily use GR via the Global Positioning System, GPS. GR also predicts the mysterious astronomical objects, for example, the black hole and the gravitational wave. Nowadays, we have advanced technology to probe the physical phenomena. For example, the M87 black hole's image from the EHT [16, 17, 18, 19, 20, 21], and the gravitational wave from LIGO and VIRGO [52, 53, 54].

Unfortunately, there are still the unsolvable problems with the ordinary theory, such as the dynamics of the Universe at early and late times, the black hole's singularity, the graviton and etc. Fortunately, some problem can be solved, that is late-time accelerated expansion of the Universe. One of the solutions is to modify GR, by adding the cosmological constant Λ. Such a theory is so-called the Λ cold dark matter (ΛCDM) model. Actually, it gives rise to the coincidence problem. But at least the accelerated expansion had been solved. This implies that the ordinary theory is not enough and a solution may be obtained by **modifying** GR in other complicated ways. The research field which is associated with pursuing the unsolvable problem by modifying GR, is called the "Modified Gravity Theories"[11].

5.1.2 Modified Gravity: Implement

There are several ways to modify GR. In ΛCDM model, we just add the cosmological constant. One can also treat a function of R, as a Lagrangian instead of R which is Einstein-Hilbert Lagrangian, known as the $f(R)$ gravity, as well as to modify the matter sector $f(T)$ where T is the trace of energy-momentum tensor. The extension of dimensions is one of the modifications, namely, the "Lovelock's theory of gravity" which is the most general metric theory that preserves the second order equation of motion. Moreover, we can relax some assumption to construct the new theory, e.g., the "Symmetric Teleparallel Equivalent of General Relativity", STEGR, which relax the torsion free condition and metric compatibility in GR. Alike to the ΛCDM model, one can add scalar field or vector field to be a new degree of freedom besides the metric. These are a few examples of the **modified gravity**.

5.2 VECTOR-TENSOR HORNDESKI THEORY

In references [6, 7, 8, 9, 10], the authors proposed that the acceleration of Universe's expansion is affected by a vector field. This might play the role of the dark energy. One of the proposed theories is "vector-tensor Horndeski theory" [15]. This theory is the most general theory for $U(1)$ gauge-invariant vector-tensor theories that provides the second-order differential equation of motion [11, 12, 13, 14]. Since it is the theory with the second-order differential equation of motion, then it is free from the Ostrogradsky instability [55]. The action of the vector-tensor Horndeski theory is given by

$$\mathcal{S} = \int d^4x\sqrt{-g}\left(\frac{m_{Pl}^2}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \beta L^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\right), \tag{5.1}$$

where $m_{Pl} \equiv \frac{1}{\sqrt{8\pi G}}$ is the reduced Planck's mass, β is the coupling constant, and $L^{\mu\nu\alpha\beta}$ is the double dual Riemann tensor which is defined as follows

$$L^{\mu\nu\alpha\beta} = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}R_{\rho\sigma\gamma\delta}, \tag{5.2}$$

where the Levi-Civita tensor $\epsilon_{\mu\nu\alpha\beta}$ is defined in eq.(2.32). The last term in eq.(5.1) corresponds to the interaction between gravitational and gauge fields, so-called "vector-tensor Horndeski interaction".

Absolutely, the dynamical fields are the gauge field A^μ and metric $g^{\mu\nu}$. By varying the action with respect to A^μ, we obtain the following equations of motion

$$\nabla_\mu\left(F^{\mu\nu} - 4\beta F_{\alpha\beta}L^{\alpha\beta\mu\nu}\right) = 0. \tag{5.3}$$

This is the modification of the equation of motion for the gauge field eq.(2.109). The variation with respect to $g^{\mu\nu}$ gives rise to the modified Einstein's field equation as follows

$$m_{Pl}^2 G_{\mu\nu} - F_\mu{}^\beta F_{\nu\beta} + \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \beta H_{\mu\nu} = 0, \tag{5.4}$$

where

$$\begin{aligned} H_{\mu\nu} \equiv g_{\mu\nu}L^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} + 2\tilde{F}_{\mu\sigma}\tilde{F}^{\gamma\delta}R^\sigma{}_{\nu\gamma\delta} - 4\nabla^\sigma\tilde{F}_{\gamma\nu}\nabla^\gamma\tilde{F}_{\mu\sigma} \\ - 4\tilde{F}_{\gamma\nu}\left(R^{\rho\gamma}\tilde{F}_{\mu\rho} + R^\gamma{}_{\sigma\rho\mu}\tilde{F}^{\rho\sigma}\right), \end{aligned} \tag{5.5}$$

and the dual strength tensor is defined by

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}. \tag{5.6}$$

5.3 VECTOR-TENSOR HORNDESKI BLACK HOLES

5.3.1 Spherical Symmetric and Static Spacetime

The first black hole's solution of the Einstein's field equation is incorporated with spherically symmetric and static spacetime. Then the black hole is identified by the existence of event horizons. This type of black hole is the simplest black hole solution. Thus, the static and spherically symmetric solution is quite an easiest one in any theory of gravity.

For the spherically symmetric and static spacetime, the metric in eq.(2.89) can be reexpressed as

$$ds^2 = -a^2(r)f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2, \tag{5.7}$$

where $a(r)$ and $f(r)$ are arbitrary functions. This form of metric is suitable to find the temperature of the black hole. The components of the four-vector potential is read as $A_\mu = (\Phi_e, A_i)$ where Φ_e is the electric potential and A_i is the component of the magnetic vector potential. For static electromagnetic field, the associated four-vector potential is given by [15]

$$A_\mu = \left[\Phi_e(r), 0, 0, -Q_m \cos\theta\right], \tag{5.8}$$

where Q_m is a magnetic charge. The electric and magnetic fields are calculated in eq.(2.107). The equations of motion is obtained by varying the action with respect to $a(r)$, $f(r)$ and $\Phi_e(r)$ can be written respectively as

$$\left(m_{pl}^2 + 4\beta\frac{Q_m^2}{r^4}\right)a'(r) + 4\beta\left[\frac{3Q_m^2}{r^4} - \left(\frac{\Phi_e'(r)}{a(r)}\right)^2\right]\frac{a(r)}{r} = 0, \tag{5.9}$$

$$\begin{aligned} r\left(m_{Pl}^2 + 4\beta\frac{Q_m^2}{r^4}\right)f'(r) + \left(m_{Pl}^2 - 4\beta\frac{6Q_m^2}{r^4}\right)f(r) - m_{Pl}^2 + \frac{Q_m^2}{2r^2} \\ + \frac{1}{2}\left[\frac{8\beta\left(1 - f(r)\right) + r^2}{a^2(r)}\right]\Phi_e'^2(r) = 0, \end{aligned} \tag{5.10}$$

$$\left[\frac{8\beta\left(1 - f(r)\right) + r^2}{a(r)}\Phi_e'(r)\right]' = 0. \tag{5.11}$$

The quantity in the square bracket in the last equation is simply a constant, then one denotes this constant as "Q_e". The $\Phi_e(r)$'s equation of motion can be rewritten as

$$\Phi'_e(r) = \frac{a(r)Q_e}{8\beta\left(1 - f(r)\right) + r^2}. \tag{5.12}$$

In the limit $\beta \to 0$, the solution $a(r)$ of eq.(5.9) is a constant which we can choose to be 1, $a(r) = 1$. Then, the electric potential in eq.(5.12) becomes

$$\lim_{\beta \to 0} \Phi'_e(r) = \frac{Q_e}{r^2}. \tag{5.13}$$

Thus, the constant Q_e is interpreted as an electric charge. By substituting eq.(5.12) into the eqs.(5.9) and (5.10), one obtains

$$\left(m_{Pl}^2 + 4\beta\frac{Q_m^2}{r^4}\right)a'(r) + 4\beta\left\{\frac{3Q_m^2}{r^4} - \frac{Q_e^2}{\left[8\beta\left(1 - f(r)\right) + r^2\right]^2}\right\}\frac{a(r)}{r} = 0, \tag{5.14}$$

$$r\left(m_{Pl}^2 + 4\beta\frac{Q_m^2}{r^4}\right)f'(r) + \left(m_{Pl}^2 - 4\beta\frac{6Q_m^2}{r^4}\right)f(r) - m_{Pl}^2 + \frac{Q_m^2}{2r^2} + \frac{Q_e^2/2}{8\beta\left(1 - f(r)\right) + r^2} = 0. \tag{5.15}$$

The black hole's solutions have been investigated and analytical solution can be found in reference [56]. In this thesis, we will follow [56] by separating consideration as the purely magnetic and electric black holes.

5.3.2 Magnetic Black hole

The **magnetic**-Horndeski black hole (MHBH) can be obtained by solving the eqs.(5.14) and (5.15) in the limit $Q_e \to 0$. We impose the value of the Universal gravitational constant ($G = 1$) and $\pi = 1/4$. Then, the equations of motion for MHBH are written by

$$r\left(1 + \frac{8\beta Q_m^2}{r^4}\right)a'(r) + \frac{24Q_m^2\beta}{r^4}a(r) = 0, \tag{5.16}$$

$$r\left(1 + \frac{8Q_m^2\beta}{r^4}\right)f'(r) + \left(1 - \frac{48Q_m^2\beta}{r^4}\right)f(r) + \frac{Q_m^2}{r^2} - 1 = 0. \tag{5.17}$$

5.3.2.1 Homogeneous Linear ODE

The equation of motion for the function $a(r)$ is the first-order homogeneous linear ODE, and it can be rewritten in the form of $a'(r)+g(r)a(r)=0$ where $g(r)=24Q_m^2\beta/r^5\left(1+\frac{8\beta Q_m^2}{r^4}\right)$. This kind of equation is solved by following step

$$\begin{aligned} a'(r) &= -g(r)a(r), \\ \int \frac{1}{a(r)}da(r) &= -\int_r y(r')dr' + \mathbb{C}_1, \\ a(r) &= \mathbb{C}_1 \exp\left[-\int_r g(r')dr'\right] && (5.18) \\ &= \left(1+\frac{8Q_m^2\beta}{r^4}\right)^{3/4} && (5.19) \end{aligned}$$

where an arbitrary constant $\mathbb{C}_1$ is chosen to be 1.

5.3.2.2 Inhomogeneous Linear ODE

The equation of motion of $f(r)$ is quite complicated. We would define new variables as follows

$$z \equiv \frac{8Q_m^2\beta}{r^4}, \tag{5.20}$$

and

$$p \equiv \frac{Q_m}{\sqrt{8\beta}}. \tag{5.21}$$

Eq.(5.17) is written in terms of these variables as

$$4z(1+z)f'(z)-(1-6z)f(z)-p\sqrt{z}+1=0. \tag{5.22}$$

This is the first-order inhomogeneous linear ODE.

The first-order inhomogeneous linear ODE can be expressed in general form as

$$L_1(z)f'(z)+L_2(z)f(z)+L_3(z)=0, \tag{5.23}$$

where L_i is an arbitrary function. The ansatz solution can be given by

$$f(z)=f_0(z)u(z). \tag{5.24}$$

The function $y_0(z)$, so-called the "complementary solution", is the solution of the homogeneous equation as

$$L_1(z)f_0'(z)+L_2(z)y_0(z)=0. \tag{5.25}$$

This solution can be expressed in the same form as one in eq.(5.18) as

$$f_0(z)=\mathbb{C}_2\exp\left[-\int_z\frac{L_2(z')}{L_1(z')}dz'\right], \tag{5.26}$$

where $\mathbb{C}_2$ is the integration constant. Substituting eq.(5.24) into eq.(5.23) and using eq.(5.25), one obtains

$$L_1(z)f_0(z)u'(z)+L_3(z)=0. \tag{5.27}$$

The solution of this equation is simply calculated by

$$u(z)=-\int_z\frac{L_3(z')}{L_1(z')f_0(z')}dz'+\mathbb{C}_3, \tag{5.28}$$

where $\mathbb{C}_3$ is the integration constant. It is called the "particular solution".

According to eq.(5.22), L_1, L_2 and L_3 are replaced with

$$L_1=4z(1+z), \tag{5.29}$$

$$L_2=-(1-6z), \tag{5.30}$$

$$L_3=-p\sqrt{z}+1. \tag{5.31}$$

We need to solve the complementary solution of eq.(5.23). Using eq.(5.26), one reaches the complementary solution as follows

$$\begin{aligned} f_0(z)&=\mathbb{C}_2\exp\left[\int_z\frac{1-6z'}{4z'(1+z')}dz'\right]\\ &=\frac{z^{1/4}}{(1+z)^{7/4}}, \end{aligned} \tag{5.32}$$

where one has chosen $\mathbb{C}_2$ to be 1. Thereafter, the particular solution is calculated by

$$
\begin{aligned}
u(z) &= -\int_z \frac{(-p\sqrt{z'}+1)}{4z'(1+z')}\frac{(1+z')^{7/4}}{z'^{1/4}}dz' - \mu \\
&= \frac{1}{4}\int \left[p(1+z')^{3/4}z'^{-3/4} - (1+z')^{3/4}z'^{-5/4}\right]dz' - \mu,
\end{aligned}
\tag{5.33}
$$

where $-\mu$ is the integration constant. The integration can be evaluated by using the identity of Gauss's hypergeometric function ${}_2F_1$ as follows

$$
\int_z (1+z')^{\alpha} z'^{\beta} dz' = \frac{z^{1+\beta}}{1+\beta}{}_2F_1(-\alpha, \beta+1, \beta+2, -z). \tag{5.34}
$$

Then, the solution (5.33) can be written in terms of ${}_2F_1$ as

$$
u(z) = pz^{1/4}{}_2F_1(-\frac{3}{4}, \frac{1}{4}, \frac{5}{4}, -z) + z^{-1/4}{}_2F_1(-\frac{3}{4}, -\frac{1}{4}, \frac{3}{4}, -z) - \mu. \tag{5.35}
$$

Combining the complementary and particular solutions, the solution of eq.(5.22) is achieved by following expression

$$
f(z) = \frac{1}{(1+z)^{7/4}}\left[-\mu z^{1/4} + p\sqrt{z}{}_2F_1(-\frac{3}{4}, \frac{1}{4}, \frac{5}{4}, -z) + {}_2F_1(-\frac{3}{4}, -\frac{1}{4}, \frac{3}{4}, -z)\right]. \tag{5.36}
$$

The solution can be written in terms of r, Q_m and β as follows

$$
\begin{aligned}
f(r) = \left(1+\frac{8Q_m^2\beta}{r^4}\right)^{-7/4} & \left[-\frac{2m}{r} + \frac{Q_m^2}{r^2}{}_2F_1(-\frac{3}{4}, \frac{1}{4}, \frac{5}{4}, -\frac{8Q_m^2\beta}{r^4})\right. \\
& \left. + {}_2F_1(-\frac{3}{4}, -\frac{1}{4}, \frac{3}{4}, -\frac{8Q_m^2\beta}{r^4})\right],
\end{aligned}
\tag{5.37}
$$

where the ADM mass is determined as follows

$$
2m \equiv \mu\sqrt{Q_m}(8\beta)^{1/4}. \tag{5.38}
$$

5.3.3 Electric Black Hole

The **electric**-Horndeski black hole (EHBH) is achieved by solving a set of equations, namely, eqs.(5.12), (5.14) and (5.15) with the limit $Q_M \to 0$. They are expressed by

$$a'(r) - \frac{8Q_e^2\beta}{r\left\{8\beta\left[1-f(r)\right]+r^2\right\}^2}a(r) = 0, \tag{5.39}$$

$$rf'(r) + f(r) - 1 + \frac{Q_e^2}{8\beta\left[1-f(r)\right]+r^2} = 0, \tag{5.40}$$

$$\Phi_e'(r) - \frac{Q_e a(r)}{8\beta\left[1-f(r)\right]+r^2} = 0. \tag{5.41}$$

Since eq.(5.40) is a nonlinear ODE and coupled with other equations, it implies that these equations are difficult to solve for analytic solutions. The numerical method seems to be an appropriated way to solve these equations. We have chosen a computing programme, namely, the “Wolfram Mathematica 13.0”.

5.3.3.1 Mathematica’s NDSolve

To solve differential equations with numerical method, we use the command “NDSolve” in Mathematica. NDSolve requires differential equations, boundary conditions, and range of variable. This command will be completed when the number of boundary conditions corresponds to the order of the differential equations. For example, a second (first)-order DE requires 2 (1) boundary conditions, and so on. The range of variable is the interval of variable which we are interested in. Figure 31 shows implement of NDSolve. All of the inputs of the command NDSolve are numerics. Moreover, the parameters must be assigned with a number, then the input differential equations depend only the variable x. In order to obtain suitable boundary conditions, one needs to find a suitable asymptotic behavior of the solution. These give the relation between model’s parameter and boundary’s values of the variables.

Black holes’s solution possess event horizons in which the metric is undefined at such points of spacetime. One of the asymptotic behaviors of the solution is the

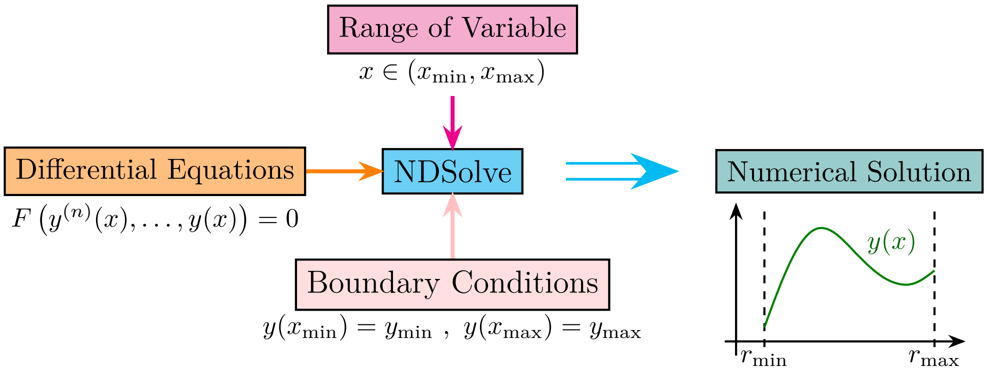


Figure 31 The flowchart of the command NDSolve.

condition of the event horizon denoted by

$$f(r = r_H) = 0. \tag{5.42}$$

Another asymptotic solution is determined by one at asymptotically flat spacetime which denoted by

$$\lim_{r\to\infty} f(r) = 1, \tag{5.43}$$

$$\lim_{r\to\infty} a(r) = 1. \tag{5.44}$$

Thus, the interval of outer horizon to spatial infinity is chosen to be the range of variable, $r \in (r_H, \infty)$. This range of r is useful for investigating black hole's thermodynamics. The relevant boundary conditions are given by ansatz solutions at event horizon and asymptotically flat spacetime.

5.3.3.2 Boundary Conditions

In order to find the suitable numeric boundary condition at the horizon, we can expand the function $f(r)$ around the horizon r_H as follows

$$f_H(r) = \sum_{i=1} H_i (r - r_H)^i, \tag{5.45}$$

where H_i is constant. The functions $a(r)$ and $\Phi_e(r)$ do not need to vanish at the horizon. Then, nearby the horizon, they can be expressed as follows

$$a_H(r) = \sum_{i=0} A_{H,i}(r - r_H)^i, \tag{5.46}$$

$$\Phi_{e,H}(r) = \sum_{i=0} V_{H,i}(r - r_H)^i, \tag{5.47}$$

where $A_{H,i}$ and $V_{H,i}$ are constant. The constants H_i, $A_{H,i}$, and $V_{H,i}$ are specified by applying the approximated solutions in the equation of motions. For example, if one keeps the solutions up to the first-order, we can set approximation $((r - r_H)^2) \approx 0$. Substituting eqs.(5.45), (5.46) and (5.47) into the equations of motion (5.39), (5.40) and (5.41) and using Taylor's expansion up to the first-order approximation, we attain

$$\left(H_1 r_H + \frac{Q_e^2}{r_H^2 + 8\beta} - 1\right) + 2\left[r_H H_2 + H_1 - \frac{Q_e^2(r_H - 4\beta H_1)}{(r_H^2 + 8\beta)^2}\right]\Delta = 0, \tag{5.48}$$

$$\begin{aligned}&\left(\frac{A_{H,1}}{2} - \frac{4Q_e^2\beta}{r_H(r_H^2 + 8\beta)^2}A_{H,0}\right) + 4\left\{\frac{A_{H,2}}{4}\right.\\ &\left.\quad - \frac{Q_e^2\beta\left[r_H(r_H^2 + 8\beta)A_{H,1} + (16\beta r_H H_1 - 5r_H^2 - 8\beta)A_{H,0}\right]}{r_H^2(r_H^2 + 8\beta)^3}\right\}\Delta = 0,\end{aligned} \tag{5.49}$$

$$\begin{aligned}&\left(V_{H,1} - \frac{Q_e A_{H,0}}{r_H^2 + 8\beta}\right) + \left[2V_{H,2} + \frac{2Q_e(r_H - 4\beta H_1)}{(r_H^2 + 8\beta)^2}A_{H,0}\right.\\ &\qquad\qquad\left. - \frac{Q_e}{r_H^2 + 8\beta}A_{H,1}\right]\Delta = 0,\end{aligned} \tag{5.50}$$

where $\Delta \equiv r - r_H$. The constants H_i, $a_{H,i}$ and $V_{H,i}$ are determined by solving the equations order by order of Δ^i as follows

$$\begin{aligned}H_1 &= \frac{r_H^2 + 8\beta - Q_e^2}{r_H(r_H^2 + 8\beta)}, & H_2 &= \frac{2Q_e^2(r_H^4 + 10r_H^2\beta + 16\beta^2) - (r_H^2 + 8\beta)^3 + 4Q_e^4\beta}{r_H^2(r_H^2 + 8\beta)^3},\\ A_{H,1} &= \frac{8Q_e^2\beta}{r_H(r_H^2 + 8\beta)^2}A_{H,0}, & A_{H,2} &= -\frac{4Q_e^2\beta\left[5r_H^4 + 32r_H^2\beta + 8(Q_e^2 - 8\beta)\beta\right]}{r_H^2(r_H^2 + 8\beta)^4}A_{H,0},\\ V_{H,1} &= \frac{Q_e}{r_H^2 + 8\beta}A_{H,0}, & V_{H,2} &= -\frac{Q_e(r_H^2 - 4\beta)}{r_H(r_H^2 + 8\beta)^2}A_{H,0}.\end{aligned} \tag{5.51}$$

Substituting them into eqs.(5.45), (5.46) and (5.47), the solutions $f(r)$, $a(r)$, and $V(r)$ up to the first-order approximation are written by

$$f_H(r) = \frac{r_H^2 + 8\beta - Q_e^2}{r_H(r_H^2 + 8\beta)^2}(r - r_H), \tag{5.52}$$

$$a_H(r) = A_{H,0} + \frac{8Q_e^2\beta A_{H,0}}{r_H(r_H^2 + 8\beta)}(r - r_H), \tag{5.53}$$

$$V_H(r) = V_{H,0} - \frac{Q_e A_{H,0}}{r_H^2 + 8\beta}(r - r_H). \tag{5.54}$$

Notice that there exists unspecified constants $A_{H,0}$ and $V_{H,0}$. If we extend the solution up to higher order, these constants still unidentify. They are “free parameters.”

According to the conditions (5.43) and (5.44), the functions $f(r)$ and $a(r)$, around the asymptotically flat spacetime, can be expanded as

$$f_f(r) = 1 + \sum_{i=1} F_i r^{-i}, \tag{5.55}$$

$$a_f(r) = 1 + \sum_{i=1} A_{f,i} r^{-i}, \tag{5.56}$$

where F_i and $A_{f,i}$ are constants. The function $\Phi_e(r)$ does not need to vanish at asymptotically flat spacetime. Its expression can be given by

$$\Phi_{e,f}(r) = \sum_{i=1} V_{f,i} r^{-i}, \tag{5.57}$$

where $V_{f,i}$ is a constant. We use the same method as done for the event horizon’s case in determining the constants F_i, $A_{f,i}$ and $V_{f,i}$. For example, one keeps the solutions up to the fifth order, $(r^{-6}) \approx 0$. Substituting eqs.(5.55), (5.56) and (5.57) into eqs.(5.39), (5.40) and (5.41) and using Taylor’s series up to the fifth order, one obtains

$$\frac{Q_e^2 - F_2}{r^2} - \frac{2F_3}{r^3} - \frac{3F_4}{r^4} + \frac{8Q_e^2\beta F_1 - 4F_5}{r^5} + \mathcal{O}(r^{-6}) = 0, \tag{5.58}$$

$$\frac{1}{2}\frac{A_{f,1}}{r^2} - \frac{A_{f,2}}{r^3} - \frac{3}{2}\frac{A_{f,3}}{r^4} - \frac{4Q_e^2\beta + 2A_{f,4}}{r^5} + \mathcal{O}(r^{-6}) = 0, \tag{5.59}$$

$$\begin{aligned} &-\frac{Q_e + V_{f,1}}{r^2} - \frac{Q_e A_{f,1} + 2V_{f,2}}{r^3} - \frac{Q_e A_{f,2} + 3V_{f,3}}{r^4} \\ &\qquad - \frac{Q_e(A_{f,2} + 8\beta F_1) + 4V_{f,4}}{r^5} + \mathcal{O}(r^{-6}) = 0. \end{aligned} \tag{5.60}$$

Solving the equations order by order of r^{-i}, one achieves

$$
\begin{aligned}
&F_2 = Q_e^2, &&F_3 = 0, &&F_4 = 0, &&F_5 = 2Q_e^2\beta F_1,\\
&A_{f,1} = 0, &&A_{f,2} = 0, &&A_{f,3} = 0, &&A_{f,4} = -2Q_e^2\beta,\\
&V_{f,1} = -Q_e, &&V_{f,2} = -\frac{Q_e}{2}A_{f,1}, &&V_{f,3} = \frac{-Q_e}{3}A_{f,2}, &&V_{f,4} = \frac{-Q_e}{4}(A_{f,3} + 8\beta F_1).
\end{aligned}
$$

Applying them to the eq.(5.55), (5.56) and (5.57), the solutions are expressed as

$$f_f(r) = 1 + \frac{F_1}{r} + \frac{Q_e^2}{r^2} + \frac{2Q_e^2\beta}{r^5}F_1, \tag{5.61}$$

$$a_f(r) = 1 - \frac{2Q_e^2\beta}{r^4}, \tag{5.62}$$

$$V_f(r) = -\frac{Q_e}{r} - \frac{2Q_e\beta}{r^4}F_1. \tag{5.63}$$

In this case, the free parameter is only F_1 which can be interpreted as the ADM mass.

5.3.3.3 Outer Horizon

According to section 3.1.3, a charged black hole possesses 2 event horizons determined by $g^{rr} = 0$. This behavior is shown explicitly in Figure 32. Therefore, we need to distinguish the inner and outer horizons numerically. From this figure, one can see that the slope of g^{rr} is negative (positive) at the inner (outer) horizon. As a result, one can specify the outer horizon by determining the positiveness of the slope of g^{rr}, or $f(r)$, at event horizon. Since we consider $f(r)$ at the horizon,

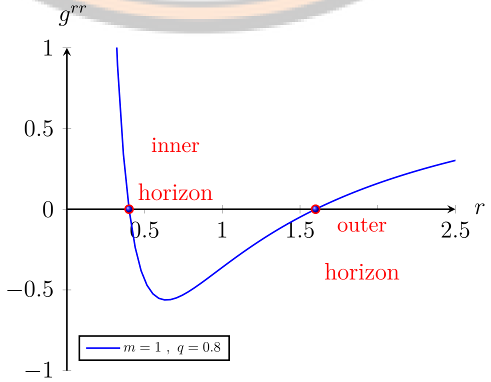


Figure 32 Plot of g^{rr} of Reissner-Nordström spacetime.

one can use the series's expansion of $f(r)$ in eq.(5.45) for calculating the slope of $f(r)$. As a result, the slope of $f(r)$ at the horizon can be determined by the first coefficient in the expansion as follows

$$f'_H(r = r_H) = H_1. \tag{5.64}$$

Thus the outer horizon is just given by the condition $H_1 > 0$, or

$$r_H^2 > Q_e^2 - 8\beta, \tag{5.65}$$

where we used the expression of H_1 in eq.(5.51).

5.3.3.4 Calculation

So far, we have 6 boundary conditions, that is, the event horizon solutions of $f_H(r)$, $a_H(r)$ and $\Phi_{e,H}(r)$, and the asymptotically flat solutions of $f_f(r)$, $a_f(r)$ and $\Phi_{e,f}(r)$. NDSolve requires the numbers of boundary conditions to be equal to the numbers of differential equations, but one has 3 differential equations. So, we have to add other 3 differential equations. Technically, one can construct differential equations from the **free parameters**, $F_1, A_{H,0}$ and $V_{H,0}$. There are 3 free parameters, the additional differential equations are written by

$$F'_1(r) = 0, \quad A'_{H,0}(r) = 0, \quad V'_{H,0}(r) = 0. \tag{5.66}$$

This technique is useful in numerical calculation when the number of the boundary conditions exceed that of the differential equations. This still satisfies the original differential equations, since the solution of above equations still be constant coefficients. In the command NDSolve, we have totally 2 free parameters, namely, Q_m and β to specify the numerical calculation. Note that, the parameter r_H can be determined from Q_e and β by using eq.(5.65). A process of our calculation is illustrated in Figure 33.

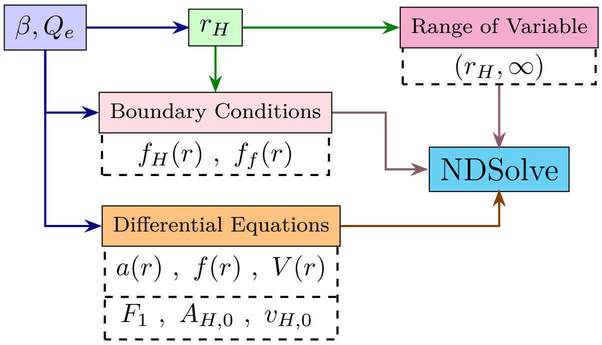


Figure 33 The flowchart of NDSolve with specific value (β, Q_e)**.**

5.4 BLACK HOLE'S THERMODYNAMIC STABILITY

In this section, we investigate thermodynamic stability of the vector-tensor Horndeski black holes through the local and global stabilities. The local stability, considered in this section, is determined through only heat transfer via the sign of heat capacity, namely, the heat capacity must be positive which is inferred from eq.(2.289). For a thermal system undergoes a process with fixing thermodynamic variables X and Y, the local stability can be written by

$$C_{X,Y} \propto \left(\frac{\delta S_{BH}}{\delta T_H}\right)_{X,Y} > 0. \tag{5.67}$$

For the global stability, it is obtained by the condition (2.282). This implies a preferred state of the system, between hot gas and black hole. If the free energy of hot gas' state is chosen to be zero, the stability's condition can be expressed as $\Delta G = G_{\text{hot gas}} - G_{\text{ black hole}} > 0$, or

$$G_{\text{ black hole}} < 0. \tag{5.68}$$

In general, the global condition can be hold for other type of **thermodynamic potentials**. For example, the Helmholtz free energy F and grand potential Φ also satisfy the above equation for the global stability.

5.4.1 Black Holes' Entropy

The vector-Horndeski theory possesses the interaction term between Riemann and field strength tensor. Thus, Wald entropy is more appropriated than Bekenstein-Hawking entropy for black holes in this theory. Let us determine Wald entropy by using eq.(4.92). Firstly, the $R_{\mu\nu\rho\sigma}$'s equation of motion must be evaluated. Using eq.(4.31) with the Lagrangian in eq.(5.1), one obtains

$$\begin{aligned}
E^{\mu\nu\rho\sigma}_{(H)} &= \frac{\partial}{\partial R_{\mu\nu\rho\sigma}} L_H, \\
&= \frac{\partial}{\partial R_{\mu\nu\rho\sigma}} \left(\frac{m_{Pl}^2}{2} R + \beta L^{\lambda_1\lambda_2\lambda_3\lambda_4} F_{\lambda_1\lambda_2} F_{\lambda_3\lambda_4} \right) \\
&= E^{\mu\nu\rho\sigma}_{(GR)} + \frac{\beta}{4} \epsilon^{\lambda_1\lambda_2\alpha_1\alpha_2} \epsilon^{\lambda_3\lambda_4\alpha_3\alpha_4} F_{\lambda_1\lambda_2} F_{\lambda_3\lambda_4} \frac{\partial R_{\alpha_1\alpha_2\alpha_3\alpha_4}}{\partial R_{\mu\nu\rho\sigma}}, \\
&= E^{\mu\nu\rho\sigma}_{(GR)} + \frac{\beta}{4} \epsilon^{\lambda_1\lambda_2\alpha_1\alpha_2} \epsilon^{\lambda_3\lambda_4\alpha_3\alpha_4} \delta^{\mu}_{\alpha_1} \delta^{\nu}_{\alpha_2} \delta^{\rho}_{\alpha_3} \delta^{\sigma}_{\alpha_4} F_{\lambda_1\lambda_2} F_{\lambda_3\lambda_4} \\
&= E^{\mu\nu\rho\sigma}_{(GR)} + E^{\mu\nu\rho\sigma}_{(\beta)},
\end{aligned} \tag{5.69}$$

where $E^{\mu\nu\rho\sigma}_{(GR)} = \frac{m_{Pl}^{-2}}{2} g^{\mu\rho} g^{\nu\sigma}$ represents Bekenstein-Hawking entropy, and

$$E^{\mu\nu\rho\sigma}_{(\beta)} = \frac{\beta}{4} \epsilon^{\lambda_1\lambda_2\mu\nu} \epsilon^{\lambda_3\lambda_4\rho\sigma} F_{\lambda_1\lambda_2} F_{\lambda_3\lambda_4} \tag{5.70}$$

represents a part of the Wald entropy resulting from the interaction term. Then, the tensor $\underline{X}^{\mu\nu}$ is written as

$$\underline{X}^{\mu\nu} = \epsilon_{\lambda_1\lambda_2\mu_3\mu_4} E^{\mu\nu\lambda_1\lambda_2}_{(H)}, \tag{5.71}$$

where one used eq.(4.59). Since there are many choice for binormal tensor $b_{\mu\nu}$ and the event horizon is the Killing horizon where Killing vector is the normal vector [31, 37], we can choose $b_{\mu\nu}$ as the combination of the Killing vector K^μ and auxiliary null vector N^μ as follows

$$b_{\mu\nu} = 2N_{[\mu} K_{\nu]}. \tag{5.72}$$

In this spacetime, it is static one and possesses the timelike Killing vector as follows

$$K^\mu = \delta^\mu_{(t)} = (1, 0, 0, 0). \tag{5.73}$$

The auxiliary null vector N^μ must satisfy the conditions $N_\mu N^\mu|_{r_H} = 0$ and $N^\mu K_\mu|_{r_H} = -1$. In order to obtain the exact form of N^μ satisfying such these conditions, one can choose its form as follows

$$N^\mu = (\frac{1}{a^2(r)f(r)}, -\frac{1}{a(r)}, 0, 0). \tag{5.74}$$

Substituting them into the Wald formula (4.92), one obtains

$$\begin{aligned}
S_{W,H} &= -2\pi \int_{r_H} \epsilon_{\alpha_1\alpha_2\mu_3\mu_4} E^{\alpha_1\alpha_2\rho\sigma}_{(H)} b_{\rho\sigma} dx^{\mu_3} \wedge dx^{\mu_4} \\
&= -2\pi \oint_{r_H} b_{\alpha_1\alpha_2} E^{\alpha_1\alpha_2\rho\sigma}_{(H)} b_{\rho\sigma} \sqrt{\sigma} d^2\theta \\
&= -2\pi \oint_{r_H} 2N_{[\alpha_1} K_{\alpha_2]} \left(E^{\alpha_1\alpha_2\rho\sigma}_{(GR)} + E^{\alpha_1\alpha_2\rho\sigma}_{(\beta)} \right) 2N_{[\rho} K_{\sigma]} \sqrt{\sigma} d^2\theta \\
&= S_{BH} + (-2\pi\beta) \oint_{r_H} N_{[\alpha_1} K_{\alpha_2]} \epsilon^{\lambda_1\lambda_2\alpha_1\alpha_2} \epsilon^{\lambda_3\lambda_4\rho\sigma} F_{\lambda_1\lambda_2} F_{\lambda_3\lambda_4} N_{[\rho} K_{\sigma]} \sqrt{\sigma} d^2\theta \\
&= \pi r_H^2 - \frac{32\pi^2 Q_m^2}{r_H^2}\beta,
\end{aligned} \tag{5.75}$$

where we used eq.(4.93) in the first line. According to the electric potential in eq.(5.13), this implies that we are working on the unit $\pi = \frac{1}{4}$ as mentioned earlier. The Wald entropy in eq.(5.75) in this convention is written by

$$S_{W,H} = \frac{r_H^2}{4} - \frac{2Q_m^2\beta}{r_H^2}. \tag{5.76}$$

Furthermore, the expression of black hole's entropy in above equation can be calculated by using the other choice of binormal tensor as follows

$$b_{\mu\nu} = 2n_{[\mu} r_{\nu]}$$

where n^μ and r^μ respectively are normal vectors of fixing time t and radius r which expressed as

$$n^\mu = \frac{1}{\sqrt{n^\nu n_\nu}}(-1, 0, 0, 0), \quad \text{and} \quad r^\mu = \frac{1}{\sqrt{r^\nu r_\nu}}(0, 1, 0, 0).$$

5.4.2 Magnetic Black Hole

5.4.2.1 Thermodynamic system

By using eq.(5.37), the mass function can be obtained by solving $f(r_H) = 0$ as follows

$$m = \frac{Q_m^2}{2r_H}{}_2F_1\left(-\frac{3}{4}, \frac{1}{4}, \frac{5}{4}, -\frac{8Q_m^2\beta}{r_H^4}\right) + \frac{r_H}{2}{}_2F_1\left(-\frac{3}{4}, -\frac{1}{4}, \frac{3}{4}, -\frac{8Q_m^2\beta}{r_H^4}\right). \tag{5.77}$$

In order to investigate the homogeneous function of mass, the mass of black hole has to be treated as a function of black hole's entropy S_{bh}, magnetic charge Q_m^2 and coupling constant β. The black hole's entropy is represented by Wald entropy expressed in eq.(5.76), $S_{bh} = S_{W,H}$. By using this equation, the horizon radius r_H can be written in terms of S_{bh}, Q_m^2 and β as

$$r_H = \sqrt{2S_{bh} + 2\sqrt{S_{bh}^2 + 2Q_m^2\beta}}. \tag{5.78}$$

Applying the above equation to the mass of black hole in eq.(5.77), we obtain

$$\begin{aligned} m(S_{bh}, Q_m^2, \beta) = & \frac{Q_m^2}{2r_H(S_{bh}, Q_m^2, \beta)}{}_2F_1\left(-\frac{3}{4}, \frac{1}{4}, \frac{5}{4}, -\frac{8Q_m^2\beta}{r_H^4(S_{bh}, Q_m^2, \beta)}\right) \\ & + \frac{r_H(S_{bh}, Q_m^2, \beta)}{2}{}_2F_1\left(-\frac{3}{4}, -\frac{1}{4}, \frac{3}{4}, -\frac{8Q_m^2\beta}{r_H^4(S_{bh}, Q_m^2, \beta)}\right), \end{aligned} \tag{5.79}$$

where the horizon function $r_H(S_{bh}, Q_m^2, \beta)$ is given by eq.(5.78). By rescaling the variables of mass S_{bh}, Q_m^2 and β with a non-zero parameter α, the mass function can be written as a homogeneous function degree 1/2 as follows

$$m(\alpha S_{bh}, \alpha Q_m^2, \alpha\beta) = \alpha^{1/2} m(S_{bh}, Q_m^2, \beta), \tag{5.80}$$

where we used the relation $r_H(\alpha S_{bh}, \alpha Q_m^2, \alpha\beta) = \alpha^{1/2} r_H(S_{bh}, Q_m^2, \beta)$. Applying the Euler's theorem (4.8), the Smarr formula can be expressed as

$$\begin{aligned}\frac{1}{2}m(S_{bh}, Q_m, \beta) &= \left(\frac{\partial m}{\partial S_{bh}}\right) S_{bh} + \left(\frac{\partial m}{\partial Q_M^2}\right) Q_m^2 + \left(\frac{\partial m}{\partial \beta}\right) \beta \\ &= \left(\frac{\partial m}{\partial S_{bh}}\right) S_{bh} + \frac{1}{2}\left(\frac{\partial m}{\partial Q_m}\right) Q_m + \left(\frac{\partial m}{\partial \beta}\right) \beta \\ &= T_H S_{bh} + \frac{1}{2}\Phi_m Q_m + B_\beta \beta. \qquad (5.81)\end{aligned}$$

Note that, the Hawking temperature T_H is assigned as the conjugate variable of S_{bh} which can be expressed as

$$T_H(r_H, Q_m, \beta) = \frac{\partial m(S_{bh}, Q_m, \beta)}{\partial S_{bh}} = \frac{r_H^2 - Q_m^2}{r_H^3\left(1 + \frac{8Q_m^2\beta}{r_H^4}\right)^{1/4}}. \qquad (5.82)$$

The magnetic potential Φ_m, a conjugate variable of Q_m, is written by

$$\begin{aligned}\Phi_m(r_H, Q_m, \beta) &= \frac{m(S_{bh}, Q_m, \beta)}{\partial Q_m} \\ &= \frac{Q_m}{16 r_H}\left[9\,{}_2F_1(\frac{1}{4}, \frac{1}{4}, \frac{5}{4}, -\frac{8Q_m^2\beta}{r_H^4}) - \frac{32\beta}{r_H^2}\,{}_2F_1(\frac{1}{4}, \frac{3}{4}, \frac{7}{4}, -\frac{8Q_m^2\beta}{r_H^4})\right. \\ &\left.\quad + \left(1 + \frac{8Q_m^2\beta}{r_H^4}\right)^{-1/4}\left(7 - \frac{8Q_m^2\beta}{r_H^4} - \frac{64\beta}{r_H^2}\right)\right]. \qquad (5.83)\end{aligned}$$

The conjugate variable of β denoted by B_β can be expressed as follows

$$\begin{aligned}B_\beta(r_H, Q_m, \beta) &= \frac{\partial m(S_{bh}, Q_m, \beta)}{\partial \beta} \qquad (5.84) \\ &= \frac{Q_m}{32 r_H \beta}\left[3\,{}_2F_1(\frac{1}{4}, \frac{3}{4}, \frac{7}{4}, -\frac{8Q_m^2\beta}{r_H^4}) - \frac{32 r_H\beta}{r_H^2}\,{}_2F_1(\frac{1}{4}, \frac{3}{4}, \frac{7}{4}, -\frac{8Q_m^2\beta}{r_H^4})\right. \\ &\left.\quad + \left(1 + \frac{8Q_m^2\beta}{r_H^4}\right)^{-1/4}\left(-3 + \frac{40Q_m^2\beta}{r_H^4} - \frac{64\beta}{r_H^2}\right)\right]. \qquad (5.85)\end{aligned}$$

By using Smarr formula (5.81), the first law of MHBH is expressed as

$$\begin{aligned}\delta m(S_{bh}, Q_m, \beta) &= \left(\frac{\partial m}{\partial S_{bh}}\right)\delta S_{bh} + \left(\frac{\partial m}{\partial Q_m}\right)\delta Q_m + \left(\frac{\partial m}{\partial \beta}\right)\delta\beta \\ &= T_H\delta S_{bh} + \Phi_m\delta Q_m + B_\beta\delta\beta. \qquad (5.86)\end{aligned}$$

There are 4 possible thermal processes of black hole which are associated with mass in eq.(5.86). They are heat transferring while 1. fixing (Q_m, β), 2.

fixing (Q_m, B_β), 3. fixing (Φ_m, β) and 4. fixing (Φ_m, B_β). For each process, one can determine the thermodynamic stability by considering the associated heat capacity for local stability and associated free energy for global stability of such a process.

However, we do not interest the phase transition of MHBH. We would rather consider the consistent of linear stability and thermodynamic one.

5.4.2.2 Calculation of Thermodynamic stability

The condition of local stability is positive heat capacity which is given by eq.(5.67). The heat capacities are evaluated by following procedure. We would define X_i are an exact variables, i.e., Q_m and β, and Y_i is a conjugate variable of X_i. Y_i is the function of X_i, i.e., Φ_m and B_β. Let us consider a function $f^{(i)}(r_H, X_1, X_2)$, one assigns $f^{(1)} = S_{bh}(r_H, Q_m, \beta)$ and $f^{(2)} = T_H(r_H, Q_m, \beta)$. The variation of the function $f^{(i)}(r_H, X_1, X_2)$ is written as

$$\begin{aligned} \delta f(r_H, X_1, X_2) &= \left(\frac{\partial f}{\partial r_H}\right)\delta r_H + \left(\frac{\partial f}{\partial X_1}\right)\delta X_1 + \left(\frac{\partial f}{\partial X_2}\right)\delta X_2 \\ &= \left[\left(\frac{\partial f}{\partial r_H}\right) + \left(\frac{\partial f}{\partial X_1}\right)\frac{\delta X_1}{\delta r_H} + \left(\frac{\partial f}{\partial X_2}\right)\frac{\delta X_2}{\delta r_H}\right]\delta r_H. \end{aligned} \tag{5.87}$$

There are 3 kinds of evaluable process, namely, 1. fixing variables (X_1, X_2) and 2. fixing variable and conjugate variable (X_i, Y_j) where $i \neq j$, and 3. fixing conjugate variables (Y_1, Y_2).

For the first case, fixing (X_1, X_2), the conditions are certainly obtained by

$$\delta X_1 = 0, \tag{5.88}$$

$$\delta X_2 = 0. \tag{5.89}$$

Thus, the variation of $f(r_H, X_1, X_2)$ with fixing (X_1, X_2) is expressed by

$$\delta f|_{X_1, X_2} = \left(\frac{\partial f}{\partial r_H}\right)\delta r_H. \tag{5.90}$$

In the case of fixing (X_i, Y_j), we would give an example's calculation by the process fixing (X_1, Y_2). The condition of fixing X_1 is already obtained by eq.(5.88).

For the condition of fixing $Y_2(X_1, X_2)$, it is calculated as follows

$$\begin{aligned} \delta Y_2(r_H, X_1, X_2) &= \left(\frac{\partial Y_2}{\partial r_H}\right)\delta r_H + \left(\frac{\partial Y_2}{\partial X_1}\right)\delta X_1 + \left(\frac{\partial Y_2}{\partial X_2}\right)\delta X_2, \\ 0 &= \left(\frac{\partial Y_2}{\partial r_H}\right)\delta r_H + \left(\frac{\partial Y_2}{\partial X_2}\right)\delta X_2, \end{aligned} \tag{5.91}$$

where $\delta X_1 = 0$ is applied in the second line. As a result, the quantity $\delta X_2/\delta r_H$ is expressed as

$$\frac{\delta X_2}{\delta r_H} = -\left(\frac{\partial Y_2}{\partial r_H}\right)\Big/\left(\frac{\partial Y_2}{\partial X_2}\right). \tag{5.92}$$

Thus the variation of the function $f(r_H, X_1, X_2)$ with fixing (X_1, Y_2) is expressed as

$$\delta f|_{X_1,Y_2} = \left[\left(\frac{\partial f}{\partial r_H}\right) + \left(\frac{\partial f}{\partial X_2}\right)\frac{\delta X_2}{\delta r_H}\right]\delta r_H, \tag{5.93}$$

where $\delta X_2/\delta r_H$ will be replaced by eq.(5.91). The process of fixing (X_2, Y_1) is calculated by the same way.

For the process with fixing (Y_1, Y_2), namely (Φ_m, B_β), the conditions are obtained by following. Recalling the first line of eq.(5.91), the relations $\delta Y_1(X_1, X_2) = 0$ and $\delta Y_2(X_1, X_2) = 0$ lead us the conditions as follows

$$\frac{\delta X_1}{\delta r_H} = -\left[\left(\frac{\partial Y_1}{\partial r_H}\right) + \left(\frac{\partial Y_1}{\partial X_2}\right)\frac{\delta X_2}{\delta r_H}\right]\Big/\left(\frac{\partial Y_1}{\partial X_1}\right), \tag{5.94}$$

$$\frac{\delta X_1}{\delta r_H} = -\left[\left(\frac{\partial Y_2}{\partial r_H}\right) + \left(\frac{\partial Y_2}{\partial X_2}\right)\frac{\delta X_2}{\delta r_H}\right]\Big/\left(\frac{\partial Y_2}{\partial X_1}\right), \tag{5.95}$$

where the first and second equations are solved by

$$\begin{aligned} \left(\frac{\partial Y_1}{\partial r_H}\right)\delta r_H + \left(\frac{\partial Y_1}{\partial X_1}\right)\delta X_1 + \left(\frac{\partial Y_1}{\partial X_2}\right)\delta X_2 = 0, \\ \left(\frac{\partial Y_2}{\partial r_H}\right)\delta r_H + \left(\frac{\partial Y_2}{\partial X_1}\right)\delta X_1 + \left(\frac{\partial Y_2}{\partial X_2}\right)\delta X_2 = 0, \end{aligned}$$

respectively. Equating eqs.(5.94) and (5.95), we would obtain the quantity $\delta X_2/\delta r_H$ as follows

$$\left[\left(\frac{\partial Y_1}{\partial r_H}\right) + \left(\frac{\partial Y_1}{\partial X_2}\right)\frac{\delta X_2}{\delta r_H}\right]\left(\frac{\partial Y_2}{\partial X_1}\right) = \left[\left(\frac{\partial Y_2}{\delta r_H}\right) + \left(\frac{\partial Y_2}{\partial X_2}\right)\frac{\delta X_2}{\delta r_H}\right]\left(\frac{\partial Y_1}{\partial X_1}\right),$$

$$\left[\left(\frac{\partial Y_1}{\partial X_2}\right)\left(\frac{\partial Y_2}{\partial X_1}\right) - \left(\frac{\partial Y_2}{\partial X_2}\right)\left(\frac{\partial Y_1}{\partial X_1}\right)\right]\frac{\delta X_2}{\delta r_H} = \left(\frac{\partial Y_2}{\partial r_H}\right)\left(\frac{\partial Y_1}{\partial X_1}\right) - \left(\frac{\partial Y_1}{\partial r_H}\right)\left(\frac{\partial Y_2}{\partial X_1}\right),$$

$$\frac{\delta X_2}{\delta r_H} = \frac{\left(\frac{\partial Y_2}{\partial r_H}\right)\left(\frac{\partial Y_1}{\partial X_1}\right) - \left(\frac{\partial Y_1}{\partial r_H}\right)\left(\frac{\partial Y_2}{\partial X_1}\right)}{\left(\frac{\partial Y_1}{\partial X_2}\right)\left(\frac{\partial Y_2}{\partial X_1}\right) - \left(\frac{\partial Y_2}{\partial X_2}\right)\left(\frac{\partial Y_1}{\partial X_1}\right)}. \tag{5.96}$$

Thereafter, the quantity $\delta X_1/\delta r_H$ can be evaluated by substituting $\delta X_2/\delta r_H$ in eq.(5.96) into the expression of $\delta X_1/\delta r_H$ in either eq.(5.94) or (5.95).

The condition of global stability is negative free energy which is given by eq.(5.68). There exists the appropriated thermodynamic potential for the process. This thermodynamic potential has to be a function of temperature T_H and fixing variables. For example the process with fixing (X_1, Y_2), the proper thermodynamic potential as the function of (T_H, X_1, Y_2) is calculated by the Legendre transformation of the mass function as follows

$$F(T_H, X_1, Y_2) = m(S_{bh}, X_1, X_2) - T_H S_{bh} - Y_2 X_2. \tag{5.97}$$

We summarize the conditions and proper thermodynamic potentials of various processes in table 2.

Table 2 Conditions and proper thermodynamic potentials of processes.

Process with fixing	**Condition**		**Thermodynamic potentials**
Q_m, β	$\delta Q_m = 0$	$\delta\beta = 0$	$F(T_H, Q_m, \beta) = m - T_H S_{bh}$
Q_m, B_β	$\delta Q_m = 0$	$\frac{\delta\beta}{\delta r_H} = -\left(\frac{\partial B_\beta}{\partial r_H}\right)\Big/\left(\frac{\partial B_\beta}{\partial \beta}\right)$	$F(T_H, Q_m, B_\beta) = m - T_H S_{bh} - B_\beta \beta$
Φ_m, β	$\frac{\delta Q_m}{\delta r_H} = -\left(\frac{\partial \Phi_m}{\partial r_H}\right)\Big/\left(\frac{\partial \Phi_m}{\partial Q_m}\right)$	$\delta\beta = 0$	$F(T_H, \Phi_m, \beta) = m - T_H S_{bh} - \Phi_m Q_m$
Φ_m, B_β	$\frac{\delta Q_m}{\delta r_H}$ in (5.94) or (5.95)	$\frac{\delta\beta}{\delta r_H}$ in (5.96)	$F(T_H, \Phi_m, \beta) = m - T_H S_{bh} - \Phi_m Q_m - B_\beta \beta$

5.4.2.3 Thermodynamic stability

We would find the region of magnetic charge Q_m and coupling constant β which corresponds to thermodynamic stability in eqs.(5.67) and (5.68). For example, Figure 34 shows the region of local and global stabilities along the fixing (Q_m, β) process. The brown (pink) area represents the local (global) stability. The thermodynamic stability is the region that local and global stability areas are intersected one. In this case, it locates at the top-left of the plot.

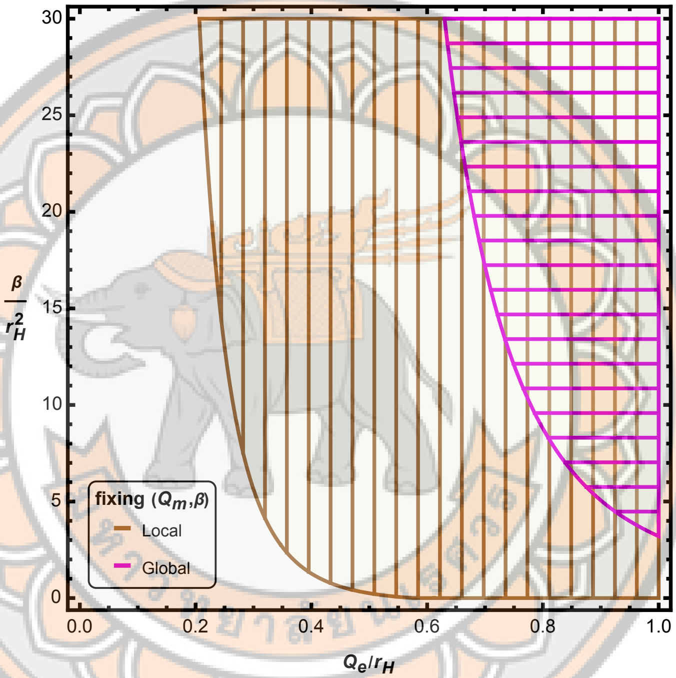


Figure 34 Stable MHBH in the fixing (Q_m, β) process.

In summary, the region of (Q_m, β) which agrees with thermodynamic stability of all calculable processes is presented in Figure 35. The red, green and blue areas represent the thermodynamic (both local and global) stability, of the process with fixing (Q_m, β), (Q_m, B_β), (Φ_m, β) and (Φ_m, B_β), respectively.

There exists the region in which areas of thermodynamic stability are intersected. Nevertheless, there is no the intersected region of all processes. There are the regions which 3 processes are coincided, viz., the intersected area of

$$\mathbb{TS}_1 \equiv \{\text{Area}(Q_m, \beta) \cap \text{Area}(Q_m, B_\beta) \cap \text{Area}(\Phi_m, \beta)\}\,, \tag{5.98}$$

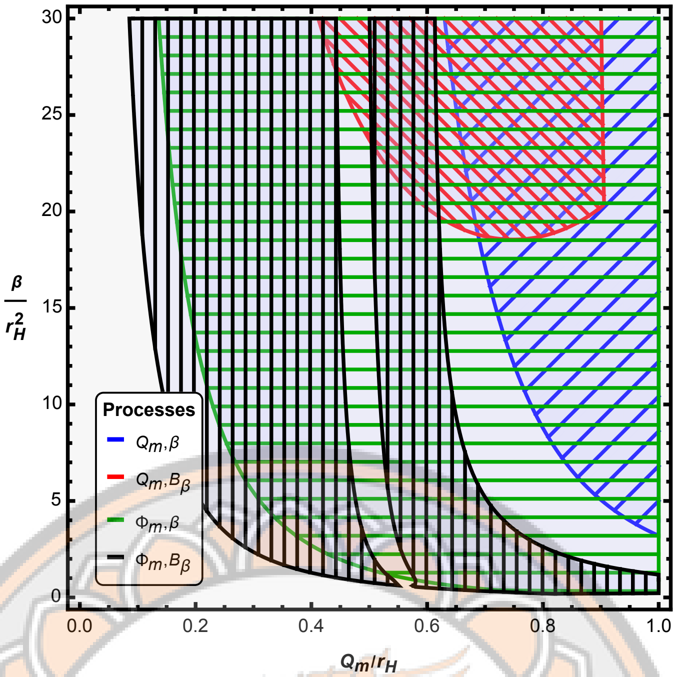


Figure 35 **The thermodynamic stability of MHBH for all processes.**

$$\mathbb{TS}_2 \equiv \{\text{Area}(\Phi_m, B_\beta) \cap \text{Area}(Q_m, B_\beta) \cap \text{Area}(\Phi_m, \beta)\} \,. \tag{5.99}$$

where Area(X_i, Y_j) stands for the region of (Q_m, β) which is thermodynamically stable undergoing the process with fixing (X_i, Y_j). The region $\mathbb{TS}_1$ is greater than $\mathbb{TS}_2$, where $\mathbb{TS}_1$ covers value of Q_m around $0.6 - 0.85$ and β for $18 - 30$. For the region $\mathbb{TS}_2$, it covers $0.5 - 0.6$ and $20 - 30$ for Q_m and β, respectively. Even if there is no the set of (Q_m, β) in which MHBH is stable undergoing all processes, but there exists the joint result, that is MHBH is **thermodynamically stable** with high-positive value of β for each a process. In other words, the coupling constant β has to dominate the magnetic charge Q_m.

5.4.2.4 Black hole's stability

A black hole is said to be stable if it can withstand any perturbations without being fundamentally altered. For example, the thermodynamic stability is studied through the thermal perturbation. Another type of disturbance is the **metric perturbation**. It is corresponding to **linear stability**.

The linear stability of MHBH was investigated in reference [15]. The conditions of such stability include with 1. absence of ghost instability and 2. positive radial and angular speed of gravitational wave. In this thesis, we would not consider the condition of positive angular speed. The mathematical expressions of the linear stablility condition are given by following

$$\begin{aligned}
m_{Pl}^2 r^4 + 4\beta Q_m^2 > 0, \\
m_{Pl}^2 r^6 + 4\beta m_{Pl}^2 \left[f(r) - 1\right] r^4 + 6\beta Q_m^2 r^2 - 384\beta^2 Q_m^2 f(r) > 0, \\
m_{Pl}^4 \left[l(l+1) - 2\right] r^8 + 2m_{Pl}^2 Q_m^2 r^6 + 4m_{Pl}^2 \beta Q_m^2 \left[l(l+1) + 2f(r) - 4\right] r^4 \\
+ 12\beta Q_m^4 r^2 - 768\beta^2 Q_m^4 f(r) > 0,
\end{aligned}$$

where $f(r)$ is the metric's component g_{rr} and l is a non-negative integer. Moreover, l is a parameter defining the specific spherical harmonic function $Y_l^m(\theta, \phi)$. Evaluating at horizon radius r_H where $f(r_H) = 0$, and replacing $m_{Pl}^2 = \frac{1}{2}$, we obtain the following conditions

$$\mathcal{F}_1 \equiv r_H^4 + 8\beta Q_m^2 > 0, \tag{5.100}$$

$$\mathcal{F}_2 \equiv r_H^6 - 8\beta m_{Pl}^2 r_H^4 + 12\beta Q_m^2 r_H^2 > 0, \tag{5.101}$$

$$\mathcal{F}_3 \equiv \left[l(l+1) - 2\right] r_H^8 + 4Q_m^2 r_H^6 + 8\beta Q_m^2 \left[l(l+1) - 4\right] r_H^4 + 48\beta Q_m^4 r_H^2 > 0. \tag{5.102}$$

We will consider the stability of MHBH where areas of the linear and thermodynamic stabilities are intersected.

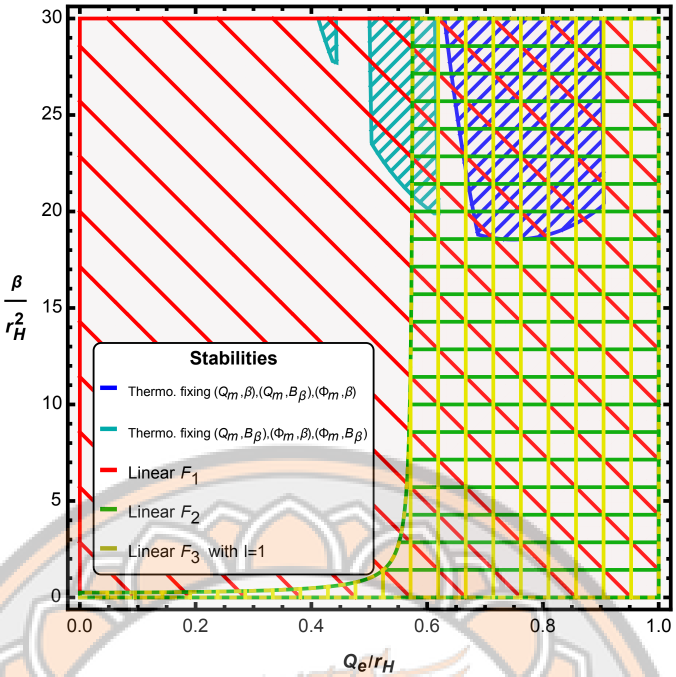


Figure 36 Thermodynamic stability and linear stability of MHBH.

Figure36 shows regions of stabilities including the linear and thermodynamic stabilities. The blue and cyan areas are regions of (Q_m, β) which MHBH is thermodynamic stable corresponding to $\mathbb{TS}_1$ and $\mathbb{TS}_2$, respectively. The red, green and yellow areas satisfy the conditions of the linear stability in eqs.(5.100), (5.101) and (5.102) with $l = 1$. We would denote the intersected region of eqs.(5.100), (5.101) and (5.102) such as

$$\mathbb{LS} \equiv \{\text{Area}(F_1) \cap \text{Area}(F_2) \cap \text{Area}(F_3)\}\,, \tag{5.103}$$

which covers the half-right part of the region-plot in figure 36. The region $\mathbb{LS}$ intersects the region $\mathbb{TS}_2$ for the interval of $0.55 \leq Q_m \leq 0.6$ and $20 \leq \beta \leq 30$. For the case of the regions $\mathbb{TS}_1$, $\mathbb{LS}$ totally covers the region $\mathbb{TS}_1$. Accordingly, MHBH is **stable** for all thermodynamic processes.

5.4.3 Electric Black Hole

5.4.3.1 Padmanabhan's method

It is impossible to obtain the first law of EHBH. Since the solution of $f(r)$ is the numerical one. There is a method for achieving the first law of black hole through the Einstein's equation. This is known as "Padmanabhan's method". The idea of such method is the Einstein's equation is a thermodynamic identity itself [57, 58]. For example, the first law of Schwarzchild black hole can be obtained by following steps. Recalling the Einstein's equation which is the second line of eq.(2.102) and replacing $e^{2\alpha(r)} = f(r)$, one obtains

$$f(r) + rf'(r) = 1. \tag{5.104}$$

One considers the above equation at horizon radius r_H, the term $f(r)$ is zero. Multiplying eq.(5.104) by $\frac{dr_H}{2}$, we attain the following expression,

$$\begin{aligned} \frac{dr_H}{2} &= r_H f'(r_H)\frac{dr_H}{2}, \\ d\left(\frac{r_H}{2}\right) &= \frac{f'(r_H)}{4\pi}d(\pi r^2), \\ dE &= \left(\frac{\kappa}{2\pi}\right)d\left(\frac{A_H}{4}\right) = T_H dS_{BH}, \end{aligned} \tag{5.105}$$

where we used the expression of surface gravity of the Schwarzschild black hole in eq.(3.57), and defined the internal energy as follows

$$E \equiv \frac{r_H}{2}. \tag{5.106}$$

In this case, the internal energy is explicitly the mass of black hole. Then, eq.(5.105) is consistent with the first law of black hole thermodynamics (4.15).

For the electric Reissner-Nordström black hole, the first law can be obtained by the same procedure. Recalling the Einstein's equation in the third line of eq.(2.138), replacing $e^{2\alpha(r)} = f(r)$ and then considering at horizon radius r_H, one acquires the following expression

$$r_H f'(r_H) - 1 = -\frac{q^2}{r_H^2}. \tag{5.107}$$

Multiplying the above equation by $\frac{dr_H}{2}$, we have

$$\begin{aligned} \frac{dr_H}{2} &= f'(r_H)\frac{r_H}{2}dr_H + \frac{q^2}{2r_H^2}dr_H \\ &= \frac{f'(r_H)}{4\pi}d\left(\pi r_H^2\right) - \frac{1}{2}qd\left(\frac{q}{r_H}\right), \\ dE &= T_H dS_{BH} - \frac{1}{2}qd\Phi_q, \end{aligned} \tag{5.108}$$

where the electric potential is expressed by $\Phi_q = \frac{q}{r}$ and the definition of internal energy in eq.(5.106) is used. This implies that the internal energy is the function of entropy S_{BH} and electric potential Φ_q, $E = E(S_{BH}, \Phi_q)$. According to eq.(35) of reference [26], the first law of the electric Reissner-Nordström black hole is expressed as

$$dm = T_H dS_{BH} + \Phi_q dq. \tag{5.109}$$

Thus, the mass does not coincide with the internal energy. Nevertheless, the mass function can be obtained from Legendre transformation of the internal energy as follows

$$m = E + \frac{1}{2}\Phi_q q. \tag{5.110}$$

The derivative of mass is calculated as follows

$$\begin{aligned} dm &= dE + d\left(\frac{1}{2}\Phi_q q\right) \\ &= T_H dS_{BH} - \frac{1}{2}qd_{r_H}\Phi_q + \frac{1}{2}d_q(\Phi_q q) + \frac{1}{2}d_{r_H}(\Phi_q q) \\ &= T_H dS_{BH} + \Phi_q dq, \end{aligned} \tag{5.111}$$

where d_{r_H} and d_q are the derivative with respect to r_H and q, respectively. Here, we used the relation,

$$d_q(\Phi_q q) = d_q\left(\frac{q^2}{r_H}\right) = \frac{2q}{r_H}dq = 2\Phi_q dq,$$

in the second line of eq.(5.111). Therefore, this kind of Legendre transformation can be used for defining the mass function of electric black holes. The transformation of electric charge to potential one has to include the factor 1/2.

5.4.3.2 Thermodynamic system

We will use the Padmanabhan's method and the Legendre transformation in eq.(5.110) for thermodynamic analysis of EHBH. Firstly, we would determine the temperature and entropy of EHBH. The entropy is obtained by taking the limit $Q_m \to 0$ of $S_{W,H}$ in eq.(5.75). Thereafter, the EHBH's entropy is simply the Bekenstein-Hawking entropy $S_{BH} = \pi r_H^2$. According to the definition of Hawking temperature $T_H = \frac{\kappa}{2\pi}$ and surface gravity in eq.(3.57), the temperature of EHBH can be expressed by

$$\begin{aligned} T_H &= \frac{1}{4\pi}\left[\sqrt{-g_{tt}g_{rr}}g^{rr}(-g_{tt})^{-1}\right]_{r_H} |\partial_r g_{tt}|_{r=r_H} \\ &= \frac{1}{4\pi a(r_H)}\left[2a(r)a'(r)f(r) + a^2(r)f'(r)\right]_{r=r_H} \\ &= \frac{a(r_H)f'(r_H)}{4\pi}. \end{aligned} \tag{5.112}$$

The Padmanabhan's method begins with the equation of motion for $f(r)$. Recalling eq.(5.40), evaluating at outer horizon r_H and multiplying it by $a(r_H)\frac{dr_H}{2}$, we achieve the following expression

$$\begin{aligned} \frac{a(r_H)}{2}dr_H &= \frac{a(r_H)f'(r_H)}{2}r_H dr_H + \frac{Q_e^2 a(r_H)}{2\left(8\beta + r_H^2\right)}dr_H \\ &= \frac{a(r_H)f'(r_H)}{4\pi}d(\pi r_H^2) + \frac{1}{2}Q_e\Phi_e'(r_H)dr_H, \\ dE &= T_H dS_{BH} + \frac{1}{2}Q_e d\Phi_e, \end{aligned} \tag{5.113}$$

where we used eq.(5.41) evaluating at r_H in the first line. Here, the internal energy is defined as follows

$$E \equiv \int^{r_H} \frac{a(r_H')}{2}dr_H'. \tag{5.114}$$

The mass function can be achieved in the same way as mass of RNBH in eq.(5.110). Then, the mass of EHBH is written by

$$m = E - \frac{1}{2}\Phi_e Q_e. \tag{5.115}$$

The first law of EHBH is the derivative form of the mass function which is calculated as

$$\begin{aligned} dm &= dE - d\left(\frac{1}{2}\Phi_q Q_e\right) \\ &= T_H dS_{BH} + \frac{1}{2}Q_e d_{r_H}\Phi_e - \frac{1}{2}d_{Q_e}(\Phi_e Q_e) - \frac{1}{2}d_{r_H}(\Phi_e Q_e) \\ &= T_H dS_{BH} - \Phi_e dQ_e. \end{aligned} \tag{5.116}$$

In the second line, we used the relation,

$$d_{Q_e}(\Phi_e Q_e) = d_{Q_e}[Q_e^2 v(r_H)] = 2Q_e v(r_H) dQ_e = 2\Phi_e dQ_e,$$

where $\Phi_e = Q_e v(r_H) = Q_e a(r_H)/(8\beta + r_H^2)$.

5.4.3.3 Functions of r_H and Mathematica's Interpolation

The thermodynamic quantities of black hole, i.e., temperature, entropy, internal energy and etc., all of them are the functions of the horizon radius r_H. Since the solution of EHBH is numerical, these kinds of functions cannot be obtained directly.

According to the condition of outer horizon (5.65), there are many values of r_H which satisfy this condition. So we would solve the solutions of $a(r)$ and $\Phi_q(r)$, of a given (Q_e, β) with various r_H under the condition (5.65). There are 2 cases, namely, $Q_e^2 > 8\beta$ and $Q_e^2 < 8\beta$. For the case $Q_e^2 > 8\beta$ The minimum value of r_H is evaluated by $r_{H,\min} = \sqrt{Q_e^2 - 8\beta} + s_{r_H}$ where s_{r_H} is the step of r_H. The minimum value of r_H in the case of $Q^2 < 8\beta$ is chosen to be $r_{H,\min} = s_{r_H}$. The maximum values of both cases are the same, that is $r_{H,\max} = r_{H,\min} + n_{r_H} s_{r_H}$ where n_{r_H} is the number of r_H's value. After the solutions of every r_H are solved, we collect the value of solutions at each r_{H_i}. Now, we have n_{r_H} values of the solution at the horizon. The horizon's function is obtained by sewing the solutions at horizon together. This kind of tool is the **Interpolation** of Mathematica. The process is visualized with diagram 37.

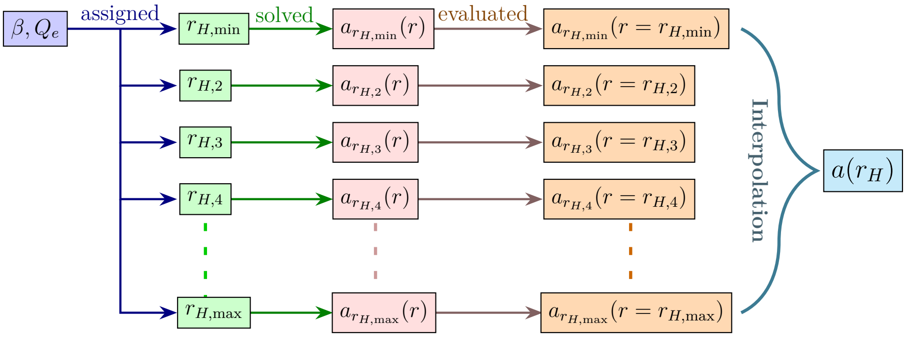


Figure 37 The flowchart of numerical calculating $a(r_H)$.

The command Interpolation of Mathematica generates an interpolating function which corresponds to the set of known data points. The input of this command is a set of points $\{(x_1, y_2), (x_2, y_2), (x_3, y_3), \ldots\}$, and then the output is the function $f(x)$ that satisfies $f(x = x_i) = y_i$. For example, the horizon's function $a(r_H)$, its input is the set expressed as follows

$$\big\{(r_{H,\min}, a_{r_{H,\min}}(r = r_{H,\min})), (r_{H,2}, a_{r_{H,2}}(r = r_{H,2})), (r_{H,3}, a_{r_{H,3}}(r = r_{H,3})), \ldots, \\ (r_{H,\max}, a_{r_{H,\max}}(r = r_{H,\max}))\big\} .$$

5.4.3.4 Thermodynamic stability

In order to investigate thermodynamic stability in various processes as the magnetic case, functions of (r_H, Q_e, β) can be found. Since only a function of r_H can be obtained, a possible calculable process is the one with fixing (Q_e, β). Consequently, all of functions are the function of r_H solely.

The local stability is studied through the heat capacity expressed as follows

$$C_{Q_e,\beta} \propto \left(\frac{\delta S_{BH}}{\delta T_H}\right)_{Q_e,\beta} \sim \frac{1}{\delta T_H} > 0.$$

Since the variation of S_{BH} is always positive, the sign of above expression depends only on the variation of temperature δT_H. Magnificently, the temperature can be written as

$$T_H(r_H) = \frac{a(r_H)}{4\pi r_H}\left(\frac{r_H^2 + 8\beta - Q_e^2}{r_H^2 + 8\beta}\right), \tag{5.117}$$

where eq.(5.40) is used with the expression of T_H in eq.(5.112). The global stability is examined by negative value of the proper thermodynamic potential as follows

$$F(T_H, Q_e, \beta) = m - T_H S_{BH} < 0, \tag{5.118}$$

where m is defined in eq.(5.115). This free energy is expressed as

$$F(r_H) = E(r_H) - \frac{1}{2}\Phi_e(r_H)Q_e - T_H(r_H)S_{BH}(r_H). \tag{5.119}$$

By choosing $Q_e = 0.9$ and $\beta = -27.5$, EHBH is **thermodynamically stable** as shown in Figure 38. The blue, cyan and red lines are the plot of temperature $T_H(r_H)$, variation of temperature $\delta T_H(r_H)$ and free energy $F(r_H)$, respectively. There exists an interval of r_H in which EHBH possesses positive temperature and negative free energy. Nevertheless, the variation δT_H is positive until the radius $r_H = 15.38$, the sign of δT_H becomes negative. The black hole is divided into 2 parts with the radius that black hole becomes unstable, $r_H = 15.38$. The black hole which is smaller (bigger) than $r_H = 15.38$ is the small (large) black hole. Therefore, this EHBH is said to be thermodynamically stable in the small size. The evaluation of values Q_e and β in which EHBH is thermodynamically stable is shown in appendix C.

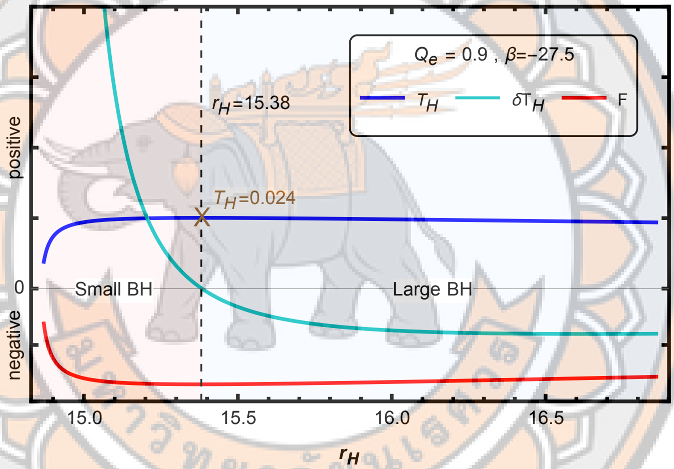


Figure 38 EHBH with $\beta = -27.5$ and $Q_e = 0.9$.

CHAPTER VI

THERMODYNAMIC STABILITY OF SCHWARZSCHILD-DE SITTER BLACK HOLE

6.1 SCHWARZSCHILD-DE SITTER BLACK HOLE

6.1.1 Expanding of Universe

The most successful theory for explaining the expanding of Universe with acceleration is the "Λ Cold Dark Matter" model, abbreviate with ΛCDM. This theory corresponds to many observations, e.g., the cosmic microwave back ground, elementary chemical element in the Universe, and accelerated expansion of the Universe in late time. The ΛCDM model is given by the action as follows

$$\mathcal{S} = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi G}R - 2\Lambda + \mathcal{L}_{\text{matter}}\right), \tag{6.1}$$

where Λ is the cosmological constant. The cosmological constant can be either negative and positive value where negative one corresponds to "anti de Sitter", otherwise, it corresponds to "de Sitter". The equation of motion is obtained by varying with respect to $g^{\mu\nu}$ as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{6.2}$$

Note that, the expanding Universe is described by positive cosmological constant.

6.1.2 Schwarzschild-AdS/dS Black Hole

In order to obtain Schwarzschild-like black hole, we solve the vacuum Einstein's field equation (6.2) with static and spherical symmetry. We use the following metric in this situation condition

$$ds^2 = -f(r)dt^2 + h^{-1}(r)dr^2 + r^2 d\Omega^2, \tag{6.3}$$

where $f(r)$ and $h(r)$ are arbitrary functions. Using the same method in section 2.2.3, we obtain the equations of motion as follows

$$E^{(f)} = rh'(r) + \left[h(r) + \Lambda r^2 - 1\right] = 0, \tag{6.4}$$

$$E^{(h)} = rh(r)f'(r) + \left[h(r) + \Lambda r^2 - 1\right] f(r) = 0, \tag{6.5}$$

where one has used the Euler-Lagrange equation (2.238). We subtract $E^{(h)}$ with $f(r)E^{(f)}$, one obtains

$$\frac{h'(r)}{h(r)} = \frac{f'(r)}{f(r)}, \tag{6.6}$$

then we can infer that

$$f(r) = h(r). \tag{6.7}$$

Eq.(2.238) satisfies with the solution as follows

$$h(r) = 1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2, \tag{6.8}$$

where one has set $G = 1$. This solution can recover the Schwarzschild solution via taking the limit $\Lambda \to 0$. The horizon structure can be investigated by solving $h(r_H) = 0$, we will not put the details in here. The black hole is categorized by the value of the cosmological constant. For the negative (positive) sign of the cosmological constant, the black hole is called "Schwarzschild-Anti de Sitter (de Sitter) black hole". In this thesis, we interest in the Schwarzschild-de Sitter black hole.

6.2 THERMODYNAMIC SYSTEM

In this situation, we treat the black hole's entropy as the black hole's Rényi entropy as follows

$$S_{bh} = \frac{1}{\lambda}\ln\left(1+\lambda S_{BH}\right) = \frac{1}{\lambda}\ln\left(1+\lambda\pi r_H^2\right). \tag{6.9}$$

The mass of the black hole is obtained by solving $h(r_H) = 0$ and written in terms of the black hole's entropy and cosmological constant as follows

$$m = \frac{\sqrt{S_{BH}}}{6\pi^{3/2}}\left(3\pi - \Lambda S_{BH}\right) = \frac{1}{6\pi^{3/2}\lambda}\sqrt{\frac{e^{\lambda S_{bh}}-1}{\lambda}}\left(3\pi\lambda + \Lambda - e^{\lambda S_{bh}}\Lambda\right), \tag{6.10}$$

we have used $S_{BH} = \frac{e^{\lambda S_{bh}}-1}{\lambda}$. The Smarr's formula can be obtained by using property of a homogeneous function together with applying Euler's theorem. Treating the mass function in eq. (6.10) as the homogeneous function of S_{bh}, Λ^{-1} and λ^{-1}, $m = m(S_{bh}, \Lambda^{-1}, \lambda^{-1})$, we obtain

$$m\left(\alpha S_{bh}, \alpha\Lambda^{-1}, \alpha\lambda^{-1}\right) = \frac{\alpha^{1/2}}{6\pi^{3/2}\lambda}\sqrt{\frac{e^{\lambda S_{bh}}-1}{\lambda}}\left(3\pi\lambda + \Lambda - e^{\lambda S_{bh}}\Lambda\right). \tag{6.11}$$

Thus, the mass function is the homogeneous function of degree 1/2. By applying the relation in eq.(4.8), one would get

$$\begin{aligned}\frac{1}{2}m &= \left(\frac{\partial m}{\partial S_{bh}}\right)S_{bh} + \left(\frac{\partial m}{\partial \Lambda^{-1}}\right)\Lambda^{-1} + \left(\frac{\partial m}{\partial \lambda^{-1}}\right)\lambda^{-1},\\ &= \left(\frac{\partial m}{\partial S_{bh}}\right)S_{bh} - \left(\frac{\partial m}{\partial \Lambda}\right)\Lambda - \left(\frac{\partial m}{\partial \lambda}\right)\lambda. \end{aligned} \tag{6.12}$$

According to eq.(4.117), the conjugate variable of the black hole's entropy is "Rényi temperature" which is expressed as follows

$$T_R = \frac{\partial m}{\partial S_{bh}} = -\frac{1}{4\pi r_H}\left(1+\lambda\pi r_H^2\right)\left(\Lambda r_H^2 - 1\right). \tag{6.13}$$

From eq.(6.12), one can see that the cosmological constant and the non-extensive parameter play the roles of thermodynamic variables.

For the cosmological constant, its conjugate variable can be expressed as follows

$$\frac{\partial m}{\partial \Lambda} = \frac{1}{6\pi^{3/2}\lambda}\sqrt{\frac{e^{\lambda S_{bh}}-1}{\lambda}}\left(1-e^{\lambda S_{bh}}\right). \tag{6.14}$$

By substituting $\frac{e^{\lambda S_{bh}}-1}{\lambda} = \pi r^2$ in eq.(6.14), one obtains $\frac{\partial m}{\partial \Lambda} = -\frac{1}{6}r^3$, which is proportional to 3-dimensional volume. Therefore, the conjugate variable of Λ can be interpreted as thermodynamic volume by following expression,

$$V_\Lambda = \frac{4}{3}\pi r_H^3 = -8\pi\left(\frac{\partial m}{\partial \Lambda}\right). \tag{6.15}$$

Thus, the cosmological constant is assigned as thermodynamic pressure: $P_\Lambda \equiv -\frac{\Lambda}{8\pi}$. In order to investigate the physical interpretation of the non-extensive parameter and its conjugate variable, λ and $\Psi_\lambda \equiv \frac{\partial m}{\partial \lambda}$, we do the Legendre transformation of the mass function as follows

$$E = m - P_\Lambda V_\Lambda - \Psi_\lambda \lambda, \tag{6.16}$$

where E is a thermodynamic potential. The derivative form of E is written as

$$dE = T_R dS_{bh} - P_\Lambda dV_\Lambda - \lambda d\Psi_\lambda. \tag{6.17}$$

Since the energy E is a function of $S_{bh}, V_\Lambda, \Psi_\lambda$, $E = E(S, V, \Psi_\lambda)$. It is reasonable to interpret E as internal energy. Therefore, one can interpret Ψ_λ as a number of particles and λ will play the role of chemical potential. In fact, by using Taylor's series expansion for small λ, Ψ_λ can be approximated as

$$\Psi_\lambda \approx \frac{1}{8}\pi r_H^3\left(1+8\pi r_H^2 P_\Lambda\right) - \frac{1}{24}\pi^2 r_H^5\left(1+8\pi r_H^2 P_\Lambda\right)\lambda + O\left(\lambda^2\right). \tag{6.18}$$

It is seen that the leading order is proportional to r_H^3. Since the leading order of Ψ_λ is scaled by the size, or volume, of the system, Ψ_λ is an extensive variable. This is reasonable to interpret Ψ_λ as the number of particles, $\Psi_\lambda = N_\lambda$. In this sense, the non-extensive parameter, λ, measures how the internal energy changes while there is a transfer of particles in the system. According to eq.(6.17), the chemical potential can be defined as

$$\mu_\lambda \equiv \left(-\frac{\partial E}{\partial N_\lambda}\right)_{S_{bh},V_\Lambda} = \lambda. \tag{6.19}$$

As a result, the first law of thermodynamics can be written by

$$dm = T_R dS_{bh} + V_\Lambda dP_\Lambda + N_\lambda d\mu_\lambda. \tag{6.20}$$

6.3 THERMODYNAMIC STABILITY

There are 2 types of thermodynamic stabilities that we need to concern, namely, local and global stability. In this work, we investigate the thermodynamic stability of the Schwarzschild-de Sitter black hole under the isobaric process of a closed system. According to section 2.3.4, the local stability is evaluation of the heat capacity along this process, that is isobaric heat capacity,

$$C_{P_\Lambda,N_\lambda} = \left(\frac{\delta Q}{\delta T}\right)_{P_\Lambda,N_\lambda} > 0. \tag{6.21}$$

For the global stability, one considers preferred states of black hole between hot gas and black hole states. The hot gas state is the black hole's system without horizons. We can set the Gibbs free energy of hot gas state is zero. As the minimum energy principle in 2.3.4, the global stability of black hole is given by

$$G_{\text{black hole}} < 0. \tag{6.22}$$

In order to examine the heat capacity and Gibbs free energy, we express every quantity as a function of horizon radius, cosmological constant($P_\Lambda = \Lambda$), and non-extensive parameter, $f = f(r_H, \Lambda, \lambda)$. The derivative form of such a function is written by

$$\delta f(r_H, \Lambda, \lambda) = \left(\frac{\partial f}{\partial r_H}\right)\delta r_H + \left(\frac{\partial f}{\partial \Lambda}\right)\delta\Lambda + \left(\frac{\partial f}{\partial \lambda}\right)\delta\lambda$$

$$= \left(\frac{\partial f}{\partial r_H} + \frac{\partial f}{\partial \Lambda}\frac{\delta \Lambda}{\delta r_H} + \frac{\partial f}{\partial \lambda}\frac{\delta \lambda}{\delta r_H} \right) \delta r_H. \tag{6.23}$$

The factors $\frac{\delta r_H}{\delta \lambda}$ and $\frac{\delta \Lambda}{\delta \lambda}$ are obtained from conditions of considered process. In this thesis, we consider the process of fixing pressure and the number of particles. For fixing pressure, since the thermodynamic quantities can be expressed in terms of pressure directly via $P_\Lambda = -\Lambda/(8\pi)$, it is easy to consider such quantities under the process by fixing Λ. For fixing the number of particles, one can find such a condition as follows

$$\begin{aligned} \delta N_\lambda &= \left(\frac{\partial N_\lambda}{\partial r_H} + \frac{\partial N_\lambda}{\partial \lambda}\frac{\delta \lambda}{\delta r_H} \right) \delta r_H = 0, \\ \frac{\delta \lambda}{\delta r_H} &= -\left(\frac{\partial N_\lambda}{\partial r_H} \right) \Big/ \left(\frac{\partial N_\lambda}{\partial \lambda} \right). \end{aligned} \tag{6.24}$$

By considering the Schwarzschild-de Sitter black hole, the heat capacity and the Gibbs free energy can be respectively written as

$$C_{P_\Lambda, N_\lambda} = T_R \left(\frac{\delta S_{bh}}{\delta T_R} \right)\Bigg|_{P_\Lambda, N_\lambda} \quad \text{and} \quad G_{bh} = m - T_R S_{bh} - \mu_\lambda N_\lambda. \tag{6.25}$$

Substituting the expression for the black hole thermodynamic variables into the above equations and using condition (6.24), the heat capacity and the Gibbs free energy can be computed explicitly in terms of r_H, Λ, λ. Note that from condition (6.24), it may not be possible to express λ in terms of N_λ, since the expression contains the complicated term of logarithmic function. In order to obtain the thermodynamic quantities with fixing N_λ, we use numerical methods.

6.3.1 Fixing N_λ and Λ

In order to calculations of δS_{bh} and δT_R under the fixing variable N_λ, we need to substitute the relation (6.24) into eq.(6.23). This is not enough because we need to ensure that the value of N_λ are the same for all value of $\delta f(r_H, \Lambda, \lambda)$. To do that, we need to assign the values of N_λ and Λ in the expression of N_λ as follows

$$N_\lambda = \frac{\partial m}{\partial \lambda} = \frac{(1 - \Lambda r_H^2)\left[-\lambda \pi r_H^2 + (1 + \lambda \pi r_H^2)\ln(1 + \lambda \pi r_H^2)\right]}{4\pi r_H \lambda^2}, \tag{6.26}$$

and then we will solve the value of λ of each r_H. For a given horizon radius r_H, there is the value of λ which corresponds to such horizon radius r_H through eq.(6.26). Assigning N_λ and Λ, the values of horizon radius $r_{H,1}$, $r_{H,2}$, $r_{H,3}$ and $r_{H,n}$ correspond to the values of λ as follows λ_1, λ_2, λ_3 and λ_n, respectively. This procedure can be illustrated in Figure 39. Thereafter, we would have the list of $(\Lambda, r_{H,i}, \lambda_i)$ as follows

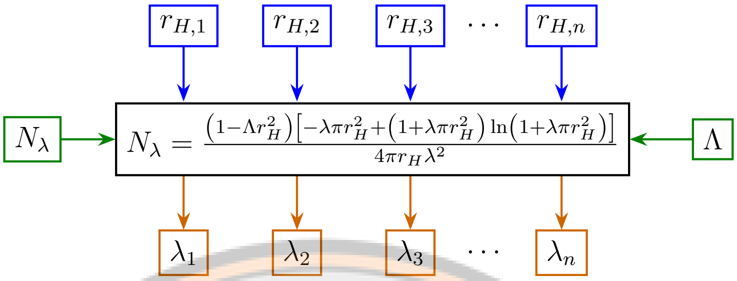


Figure 39 The flowchart for fixing N_λ and Λ.

$$\{(\Lambda, r_{H,1}, \lambda_1), (\Lambda, r_{H,2}, \lambda_2), (\Lambda, r_{H,3}, \lambda_3), \cdots, (\Lambda, r_{H,n}, \lambda_n)\}\,, \tag{6.27}$$

where the value of N_λ are the same for all member of this list. Thus, the values of $\delta f(r_H, \Lambda, \lambda)$ is evaluated by applying the value of $(r_{H_i}, \Lambda, \lambda_i)$ in the list (6.27). This procedure ensures that every values of $\delta f(r_H, \Lambda, \lambda)$ have the same value of N_λ.

6.3.2 Thermodynamic stability

By choosing $\Lambda = 0.2$ and $N_\lambda = 0.3$, we found that there exists a range of black hole's horizon in which the system is locally and globally stable as illustrated in Figure 40. From this figure, heat capacity and Gibbs free energy are presented in dark-blue and dark-brown lines. It shows that the heat capacity diverges at $r = 1.486$ represented as the dashed grey vertical line. The temperature is plotted via the dotted-blue line. A small range of horizon radius, in which the black hole is thermodynamically stable, is represented as the pink highlight. According to the fact that the horizon radius is known, we can track back to the mass of the black hole. For example, when a given cosmological constant is $\Lambda = 0.2\frac{m^2G^2}{c^4}$ where m

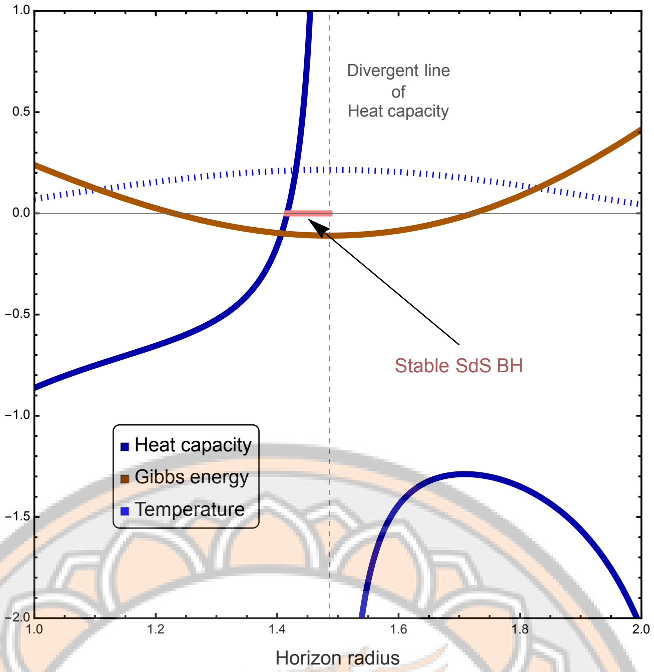


Figure 40 **Gibbs free energy and heat capacity of Sch-dS black hole.**

is mass of black hole, the Schwarzschild-de Sitter black hole is thermodynamically stable with a mass around 140 times the mass of the Earth.

In Figure 41, we show the Gibbs free energy versus temperature under an isobaric process with fixing the number of particles at given temperature represented by the green line. The value of the pressure $\Lambda = 0.2$ and the number of particles $N_\lambda = 0.3$ are chosen as the same as ones in Figure 40. The dashed cyan line represents Gibbs free energy of hot gas state. Therefore, there exist the phase transitions from the hot gas state to the black hole state at the temperature $T_R = 0.166$ and $T_R = 0.215$. For the hot gas with low temperature $T_R = 0.166$, it is the first-order phase transition, and then the system evolves to high temperature until $T_R = 0.215$. At this point, the heat capacity changes its sign corresponding to the second phase transition. Since the heat capacity of the black hole in this range $T_R = 0.166$ and $T_R = 0.215$, is negative, the black hole is locally unstable. The black hole will get higher temperature and smaller radius for this range. Until $T_R = 0.215$, a cusp of the graph, the heat capacity changes its sign corresponding

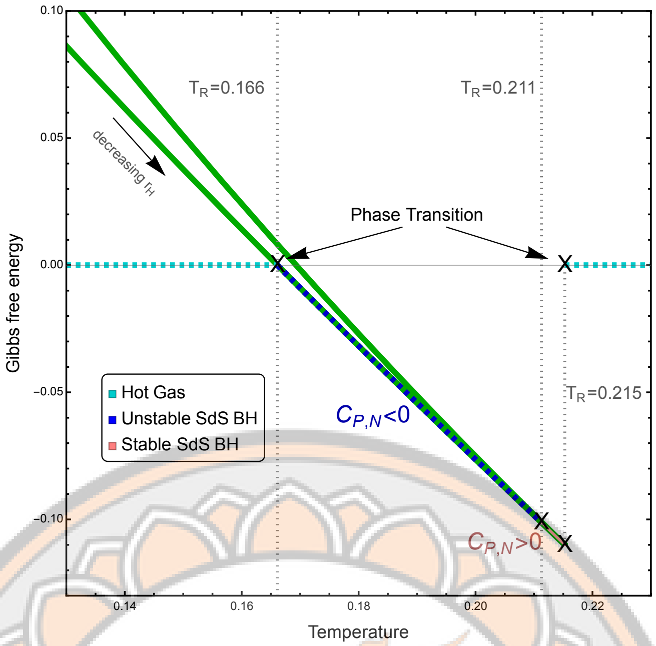


Figure 41 The plot of Gibbs free energy versus temperature of Sch-dS black hole.

to second-order phase transition. After that the system will evolve to a certain temperature suppose to be in the range of $r = 1.416$ to $r = 1.486$ denoted by the pink line as shown in Figure 40. In this phase, the black hole can exist in thermal equilibrium with the environment. In other words, the black hole can always evolve to the state with the same temperature as that of environment. Note that the pink line in Figure 41 represents the same range with one in Figure 40, corresponding to the range of temperature $T_R = 0.211$ to $T_R = 0.215$. The nth order phase transition is Gibbs free energy diverges at nth order differentiation with respect to the temperature.

For Gibbs-Boltzmann statistics, the heat capacity and the Gibbs free energy can be obtained by the taking limit $\lambda \to 0$ to eq.(6.25). One finds that the temperature is taken in the form of Hawking temperature,

$$T_H = \lim_{\lambda \to 0} \left(\frac{\partial m}{\partial S_R} \right) = \frac{\pi - \Lambda S_{BH}}{4\pi^{3/2}\sqrt{S_{BH}}}. \tag{6.28}$$

Since the existence of the black hole horizon is in the range of $r_h < \sqrt{\Lambda}$, or equivalently $S_{BH} < \frac{\pi}{\sqrt{\Lambda}}$, the temperature is always positive. By considering an isobaric process, the heat capacity is written as

$$C_{\Lambda} = T_H \frac{\delta S_{BH}}{\delta T_H}\bigg|_{\Lambda} = \frac{T_H}{(\partial T_H/\partial S_{BH})}\bigg|_{\Lambda} = -T_H \left(\frac{8\pi^{3/2} S_{BH}^{3/2}}{\pi + \Lambda S_{BH}} \right) < 0. \tag{6.29}$$

Since C_{Λ} is always negative, the black hole is locally unstable. For global stability, the Gibbs free energy in Gibbs-Boltzmann statistics is obtained by taking the limit $\lambda \to 0$ to the Gibbs free energy in eq.(6.25) as follows

$$G_{GB} = \lim_{\lambda \to 0} G_{bh} = m - T_H S_{BH} = \frac{\sqrt{S_{BH}}}{12\pi^{3/2}} (3\pi + \Lambda S_{BH}) > 0. \tag{6.30}$$

Since G_{GB} is always positive, the black hole is globally unstable. Therefore, the Schwarzschild-de Sitter black hole is thermodynamically unstable based on Gibbs-Boltzmann statistics.

CHAPTER VII

CONCLUSION AND DISCUSSION

So far, we considered 3 types of black holes, viz. magnetic-Horndeski black hole (MHBH), electric-Horndeski black hole (EHBH) and Schwarzschild-de Sitter black hole. The black hole's entropy of these black holes are different, as well as thermodynamic stability. Since they are in the modified gravity, the entropies of EHBH and MHBH are presented by the Wald entropy which is expressed as

$$S_W = S_{BH} - \frac{32\pi Q_m^2 \beta}{S_{BH}}.$$

The entropy of MHBH is exactly the Wald entropy, but EHBH's one is equivalent to the Bekenstein-Hawking entropy. For the Schwarzschild-de Sitter black hole, we prefer to use the Tsallis statistics to serve the zeroth law compatibility. So, the Rényi entropy would play the role of the black hole's entropy here, which is expressed as

$$S_{bh} = \frac{1}{\lambda} \ln[1 + \lambda S_{BH}].$$

The mass of MHBH can be written in terms of function of (S_{bh}, Q_m, β) where $S_{bh} = S_{W,H}$. So the first law of black hole is obtained by Euler's theorem of a homogeneous function. It is expressed by

$$dm = T_{bh} dS_{bh} + \Phi_m dQ_m + B_\beta d\beta$$

where B_β and Φ_m are conjugate variables of β and magnetic potential Φ_m. This allows us to investigate thermodynamic stability of black hole undergoing the processes of fixing (Q_m, β), (Q_m, B_β), (Φ_m, β) and Φ_m, B_β. The result is analyzed by the region plot of (Q_m, β) scaled with r_H and r_H^2, respectively. The MHBH is thermodynamically stable for all processes. There are 2 intersected regions that satisfy thermodynamic stability conditions of 3 processes, namely,

$$\mathbb{TS}_1 = \{\text{Area}(Q_m, \beta) \cap \text{Area}(Q_m, B_\beta) \cap \text{Area}(\Phi_m, \beta)\}\,,$$
$$\mathbb{TS}_2 = \{\text{Area}(\Phi_m, B_\beta) \cap \text{Area}(Q_m, B_\beta) \cap \text{Area}(\Phi_m, \beta)\}\,.$$

It is illustrated in Figure 35. Moreover, we included the linear stability. There are 3 linear stability conditions which are expressed in eqs.(5.100), (5.100) and (5.100). According to Figure 36, the region of linear stability,

$$\mathbb{LS} = \{\text{Area}(F_1) \cap \text{Area}(F_2) \cap \text{Area}(F_3)\}\,,$$

intersects both of thermodynamic stability's regions $\mathbb{TS}_1$ and $\mathbb{TS}_2$. Therefore, **MHBH is stable**, including to linear and thermodynamic stabilities. Moreover, the area of $\mathbb{TS}_1$ is totally covered by linear stability region. For the case of $\mathbb{TS}_2$, there is a small regime which coincides to $\mathbb{LS}$. This can be interpreted that if MHBH is observed in nature, it is likely to be the black hole which is characterized by the values of Q_m and β in the region $\mathbb{TS}_1$.

The solution of EHBH is the numerical solution. The first law can be obtained in the same way as MHBH. We used the Padmanabhan's method. Consequently, the Legendre transformation of electric term $\Phi_e Q_e$ and internal energy was reintroduced. The mass and internal energy is related by

$$m(S_{bh}, Q_e, \beta) = E(S_{bh}, \Phi_e, \beta) - \frac{1}{2}\Phi_e Q_e, \tag{7.1}$$

where Q_e and Φ_e are electric charge and potential, respectively. The calculable thermodynamic process is only fixing electric charge Q_e and coupling constant β. The result is displayed by the output of Mathematica's command. The EHBH is globally stable for all values of Q_e, negative β and entirely range of r_H. For the local stability, the EHBH is stable for some value of Q_e, with any negative β. The value of horizon radius r_H of each (Q_e, β) that satisfies local stability condition is around 25% of total number of the point in space of (Q_e, β, r_H). Figure 38 shows an example of thermodynamically stable black hole with ($Q_e = 0.9, \beta = -27.5$). It implies that **EHBH does stable** in the phase of small black hole. Thus, the EHBH is thermodynamically stable for small size of black hole.

The both of MHBH and EHBH are thermodynamically stable. The MHBH is stable on the positive value of β, meanwhile the EHBH does stable on negative value of β. However, the same thing is that the absolute value of β is greater than charges Q_m and Q_e. In other words, the vector-tensor Horndeski black hole is thermodynamic stable when the coupling constant β dominates over the electric charge Q_e and magnetic charge Q_m.

Moreover, the stability of EHBH was investigated (including to the linear stability). The conclusion of stability of MHBH can lead us to the conclusion that EHBH is stable for large β in order to dominate over the electric charge Q_e. Since it requires the complicated numerical method, the investigation for EHBH is out of scope of this thesis.

The first law of the Schwarzschild-de Sitter black hole with Rényi entropy can be obtained by using Euler's theorem of homogeneous function. Furthermore, we extend phase space by treating cosmological constant Λ and non-extensive parameter λ as thermodynamic pressure P_Λ and number of particle N_λ, respectively. The first law is written as

$$dm = T_{bh} dS_{bh} + V_\Lambda dP_\Lambda + N_\lambda d\mu_\lambda,$$

where $S_{bh} = S_R$. Here, the conjugate variable of λ and Λ are interpreted as the chemical potential μ_λ and thermodynamic volume V_Λ, respectively. There are 2 acceptable processes, that are fixing (P_Λ, N_λ) and (P_Λ, μ_λ) ines. Since undergoing the thermal process, the change of entropy leads to the change of horizon's radius, the process of fixing volume V_Λ is incompatible with the thermal perturbation. The black hole is thermodynamically unstable undergoing the fixing (P_Λ, μ_λ) process as shown in reference [29]. For the process of fixing (P_Λ, N_λ), there exists the

stable black hole in this process. Figure 40 presents the local and global stability conditions which are intersected on the short range of r_H. The phase transition of system from the hot gas phase to the black hole phase is shown in Figure 41 .

The Schwarzschild-de Sitter black hole in nature is possibly unstable. If we can detect the stable black hole, the entropy of the black hole should be described by Rényi entropy rather than Bekenstein-Hawking one. The closed thermodynamic system associated with the Schwarzschild-de Sitter black hole will be found to be stable undergoing an isobaric process.

Furthermore, the heat capacity is a “response function” that describes how one thermodynamic variable responds to a change of another one under controlled conditions. Since the first law of Schwarzschild-de Sitter black hole includes the work term as in found eq.(6.20), the Schwarzschild-de Sitter black hole able to respond with a pressure reservoir. Thus, we should consider another response function that corresponds to mechanical expansion which is known as “compressibility”. We leave this investigation for further work.

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APPENDIX

APPENDIX A PLANCK'S LAW OF BLACKBODY

The Planck's law of blackbody radiation set up the lowest radiated energy of particle of the system with $E_\nu = h\nu$ where ν is frequency of radiated particle and h is the Planck constant. An energy is the multiple of E_ν, $E_n = nE_\nu$ where $n = 1, 2, 3, \ldots$. According to Boltzmann distribution (2.300), the average energy is written by

$$\bar{E}_\nu = \sum_{n=0}^{\infty} E_n p(n) = \frac{\sum\limits_{n=0}^{\infty} (nh\nu) e^{-nh\nu/k_B T}}{\sum\limits_{m=0}^{\infty} e^{-mh\nu/k_B T}}, \tag{A.1}$$

where one substituted $E_n = nh\nu$. Defining $\frac{h\nu}{k_B T} \equiv x$, one does the following calculation

$$\begin{aligned} \bar{E}_\nu = h\nu \frac{\sum\limits_{n=0}^{\infty} n e^{-nx}}{\sum\limits_{m=0}^{\infty} e^{-nx}} = h\nu \frac{\frac{d}{dx} \sum\limits_{n=0}^{\infty} e^{-nx}}{\sum\limits_{n=0}^{\infty} e^{-nx}} &= h\nu \frac{d}{dx} \ln \left[\sum_{n=0}^{\infty} \left(e^{-x} \right)^n \right], \\ &= h\nu \frac{d}{dx} \ln \left(\frac{1}{1 - e^{-x}} \right), \\ &= h\nu \frac{e^{-x}}{1 - e^{-x}}, \\ &= \frac{h\nu}{e^x - 1}, \end{aligned} \tag{A.2}$$

where we used geometric series $\sum\limits_{i=0}^{n} a_i r^i = a_0(1 - r^n)/(1 - r)$ with $|r| < 1$. Substituting $x = \frac{h\nu}{k_B T}$ back to above equation, the average energy is written by

$$\bar{E}_\nu = \frac{h\nu}{e^{\frac{h\nu}{k_B T}} - 1}. \tag{A.3}$$

Moreover, the average energy can be written in terms of average number as

$$\bar{E} = \langle n \rangle h\nu. \tag{A.4}$$

Equating eq.(A.3) to eq.(A.4), we obtain the average number of radiated particle as follows

$$\langle n \rangle_{bb} = \frac{1}{e^{\frac{h\nu}{k_B T}} - 1}. \tag{A.5}$$

APPENDIX B EULER'S THEOREM OF HOMOGENEOUS FUNCTION

Euler's theorem is related to a function $f(x_1, x_2, \ldots, x_n) = f(x_i)$ which posses the property

$$f(\lambda x_1, \lambda x_2, \cdots, \lambda x_n) = \lambda^k f(x_1, x_2, ..., x_n), \tag{B.1}$$

where λ is non-zero parameter. This kind of function is **homogeneous function**. One considers the derivative of $f(x_i)$ with respect to λ,

$$\begin{aligned}\frac{\partial}{\partial\lambda} f(\lambda x_1, \cdots, \lambda x_n) &= \sum_{i=1}^{n} \frac{\partial f(\lambda x_1, \cdots, \lambda x_n)}{\partial(\lambda x_i)} \frac{\partial(\lambda x_i)}{\partial\lambda}, \\ &= \sum_{i=1}^{n} \frac{\partial[\lambda^k f(x_1, \cdots, x_n)]}{\lambda \partial x_i} \frac{x_i \partial\lambda}{\partial\lambda}, \\ &= \sum_{i=1}^{n} \frac{\lambda^k}{\lambda} \frac{\partial f(x_1, \cdots, x_n)}{\partial x_i} x_i. \end{aligned} \tag{B.2}$$

On the other hand, we have

$$\frac{\partial}{\partial\lambda} f(\lambda x_1, \cdots, \lambda x_n) = k\lambda^{k-1} f(x_1, \cdots, x_n). \tag{B.3}$$

By comparing eq.(B.3) to eq.(B.2), the Euler's theorem of homogeneous function is achieved by the following expression

$$k f(x_1, \ldots, x_i) = \sum_{i=1}^{n} \frac{\partial f(x_i)}{\partial x_i} x_i. \tag{B.4}$$

For the derivation of the function $f(x_1, \cdots, x_n)$, it can be written by

$$df(x_1, \cdots, x_n) = \sum_{i=1}^{n} \frac{\partial f(x_1, \cdots, x_n)}{\partial x_i} dx_i. \tag{B.5}$$

APPENDIX C COMMAND SELECT AND THERMODYNAMIC STABILITY OF EHBH

To find the interval of r_H which leads the value of $\delta T_H(r_H)$ in eq.(5.117) and $F(r_H)$ in eq.(5.119) as positive and negative values, respectively, one use the command **Select** in Mathematica. The input of Select is a list and condition, and then the output is the list in which all of the elements agree with such condition. We would construct the input list as the list of function $f(r_H)$ evaluating at a specific r_H as follows

$$\{f(r_H = r_{H,1}), f(r_H = r_{H,2}), f(r_H = r_{H,2}), \ldots, f(r_H = r_{H,200})\}\,.$$

In this research, one uses the value of β from -30 to 30 in which each value steps up for $s_\beta = 2.5$. For the magnetic charge Q_e, it runs from 0 to 1 and step for $s_{Q_e} = 0.1$. We impose the interval of r_H with $s_{r_H} = 0.01$ and calculate for $n_{r_H} = 200$ steps. We will obtain the set of e_i consisting of β, Q_e and number of points that satisfies the imposed conditions,

$$e_i = \{\beta, Q_e, \text{number of } F(r_{H,i}) < 0 \text{ (or } \delta T_H(r_{H,i}) > 0)\}.$$

The result of investigation is presented in Figure 42a. The black hole is global stable for negative value of β and all of Q_e and r_H. That can be seen by the number of 200 in the third member of each element. Note that the case of $Q_e = 0$ is equivalent to Schwarzschild black hole for all value of β. For the local stability, it is shown in Figure 42b. We found the black hole is still locally stable in the negative β's regime. Meanwhile, Q_e is not available for all values. The average number of $\delta T_H(r_{H,i})$ is around 31.

```
{{-30., 0., 200}, {-30., 0.1, 200}, {-30., 0.2, 200}, {-30., 0.3, 200}, {-30., 0.4, 200}, {-30., 0.5, 200},
 {-30., 0.6, 200}, {-30., 0.7, 200}, {-30., 0.8, 200}, {-30., 0.9, 200}, {-30., 1., 200}, {-27.5, 0., 200},
 {-27.5, 0.1, 200}, {-27.5, 0.2, 200}, {-27.5, 0.3, 200}, {-27.5, 0.4, 200}, {-27.5, 0.5, 200},
 {-27.5, 0.6, 200}, {-27.5, 0.7, 200}, {-27.5, 0.8, 200}, {-27.5, 0.9, 200}, {-27.5, 1., 200},
 {-25., 0., 200}, {-25., 0.1, 200}, {-25., 0.2, 200}, {-25., 0.3, 200}, {-25., 0.4, 200}, {-25., 0.5, 200},
 {-25., 0.6, 200}, {-25., 0.7, 200}, {-25., 0.8, 200}, {-25., 0.9, 200}, {-25., 1., 200}, {-22.5, 0., 200},
 {-22.5, 0.1, 200}, {-22.5, 0.2, 200}, {-22.5, 0.3, 200}, {-22.5, 0.4, 200}, {-22.5, 0.5, 200},
 {-22.5, 0.6, 200}, {-22.5, 0.7, 200}, {-22.5, 0.8, 200}, {-22.5, 0.9, 200}, {-22.5, 1., 200},
 {-20., 0., 200}, {-20., 0.1, 200}, {-20., 0.2, 200}, {-20., 0.3, 200}, {-20., 0.4, 200}, {-20., 0.5, 200},
 {-20., 0.6, 200}, {-20., 0.7, 200}, {-20., 0.8, 200}, {-20., 0.9, 200}, {-20., 1., 200}, {-17.5, 0., 200},
 {-17.5, 0.1, 200}, {-17.5, 0.2, 200}, {-17.5, 0.3, 200}, {-17.5, 0.4, 200}, {-17.5, 0.5, 200},
 {-17.5, 0.6, 200}, {-17.5, 0.7, 200}, {-17.5, 0.8, 200}, {-17.5, 0.9, 200}, {-17.5, 1., 200},
 {-15., 0., 200}, {-15., 0.1, 200}, {-15., 0.2, 200}, {-15., 0.3, 200}, {-15., 0.4, 200}, {-15., 0.5, 200},
 {-15., 0.6, 200}, {-15., 0.7, 200}, {-15., 0.8, 200}, {-15., 0.9, 200}, {-15., 1., 200}, {-12.5, 0., 200},
 {-12.5, 0.1, 200}, {-12.5, 0.2, 200}, {-12.5, 0.3, 200}, {-12.5, 0.4, 200}, {-12.5, 0.5, 200},
 {-12.5, 0.6, 200}, {-12.5, 0.7, 200}, {-12.5, 0.8, 200}, {-12.5, 0.9, 200}, {-12.5, 1., 200}, {-10., 0., 200},
 {-10., 0.1, 200}, {-10., 0.2, 200}, {-10., 0.3, 200}, {-10., 0.4, 200}, {-10., 0.5, 200}, {-10., 0.6, 200},
 {-10., 0.7, 200}, {-10., 0.8, 200}, {-10., 0.9, 200}, {-10., 1., 200}, {-7.5, 0., 200}, {-7.5, 0.1, 200},
 {-7.5, 0.2, 200}, {-7.5, 0.3, 200}, {-7.5, 0.4, 200}, {-7.5, 0.5, 200}, {-7.5, 0.6, 200}, {-7.5, 0.7, 200},
 {-7.5, 0.8, 200}, {-7.5, 0.9, 200}, {-7.5, 1., 200}, {-5., 0., 200}, {-5., 0.1, 200}, {-5., 0.2, 200},
 {-5., 0.3, 200}, {-5., 0.4, 200}, {-5., 0.5, 200}, {-5., 0.6, 200}, {-5., 0.7, 200}, {-5., 0.8, 200},
 {-5., 0.9, 200}, {-5., 1., 200}, {-2.5, 0., 200}, {-2.5, 0.1, 200}, {-2.5, 0.2, 200}, {-2.5, 0.3, 200},
 {-2.5, 0.4, 200}, {-2.5, 0.5, 200}, {-2.5, 0.6, 200}, {-2.5, 0.7, 200}, {-2.5, 0.8, 200}, {-2.5, 0.9, 199},
 {-2.5, 1., 199}, {0., 0., 3}, {2.5, 0., 3}, {5., 0., 3}, {7.5, 0., 3}, {10., 0., 3}, {12.5, 0., 3},
 {15., 0., 3}, {17.5, 0., 3}, {20., 0., 3}, {22.5, 0., 3}, {25., 0., 3}, {27.5, 0., 3}, {30., 0., 3}}
```

(a) The list of negative $F(r_H)$ of EHBH

```
{{-30., 0.1, 5}, {-30., 0.2, 11}, {-30., 0.3, 16}, {-30., 0.4, 22}, {-30., 0.5, 28}, {-30., 0.6, 34},
 {-30., 0.7, 40}, {-30., 0.8, 45}, {-30., 0.9, 51}, {-30., 1., 57}, {-27.5, 0.1, 5}, {-27.5, 0.2, 11},
 {-27.5, 0.3, 16}, {-27.5, 0.4, 22}, {-27.5, 0.5, 28}, {-27.5, 0.6, 34}, {-27.5, 0.7, 40}, {-27.5, 0.8, 45},
 {-27.5, 0.9, 51}, {-27.5, 1., 57}, {-25., 0.1, 5}, {-25., 0.2, 11}, {-25., 0.3, 16}, {-25., 0.4, 22},
 {-25., 0.5, 28}, {-25., 0.6, 34}, {-25., 0.7, 40}, {-25., 0.8, 46}, {-25., 0.9, 51}, {-25., 1., 57},
 {-22.5, 0.1, 5}, {-22.5, 0.2, 11}, {-22.5, 0.3, 16}, {-22.5, 0.4, 22}, {-22.5, 0.5, 28}, {-22.5, 0.6, 34},
 {-22.5, 0.7, 40}, {-22.5, 0.8, 46}, {-22.5, 0.9, 51}, {-22.5, 1., 57}, {-20., 0.1, 5}, {-20., 0.2, 11},
 {-20., 0.3, 16}, {-20., 0.4, 22}, {-20., 0.5, 28}, {-20., 0.6, 34}, {-20., 0.7, 40}, {-20., 0.8, 46},
 {-20., 0.9, 51}, {-20., 1., 57}, {-17.5, 0.1, 5}, {-17.5, 0.2, 11}, {-17.5, 0.3, 16}, {-17.5, 0.4, 22},
 {-17.5, 0.5, 28}, {-17.5, 0.6, 34}, {-17.5, 0.7, 40}, {-17.5, 0.8, 46}, {-17.5, 0.9, 51}, {-17.5, 1., 57},
 {-15., 0.1, 5}, {-15., 0.2, 11}, {-15., 0.3, 16}, {-15., 0.4, 22}, {-15., 0.5, 28}, {-15., 0.6, 34},
 {-15., 0.7, 40}, {-15., 0.8, 46}, {-15., 0.9, 52}, {-15., 1., 57}, {-12.5, 0.1, 5}, {-12.5, 0.2, 11},
 {-12.5, 0.3, 16}, {-12.5, 0.4, 22}, {-12.5, 0.5, 28}, {-12.5, 0.6, 34}, {-12.5, 0.7, 40}, {-12.5, 0.8, 46},
 {-12.5, 0.9, 52}, {-12.5, 1., 58}, {-10., 0.1, 5}, {-10., 0.2, 11}, {-10., 0.3, 16}, {-10., 0.4, 22},
 {-10., 0.5, 28}, {-10., 0.6, 34}, {-10., 0.7, 40}, {-10., 0.8, 46}, {-10., 0.9, 52}, {-10., 1., 58},
 {-7.5, 0.1, 5}, {-7.5, 0.2, 11}, {-7.5, 0.3, 16}, {-7.5, 0.4, 22}, {-7.5, 0.5, 28}, {-7.5, 0.6, 34},
 {-7.5, 0.7, 40}, {-7.5, 0.8, 46}, {-7.5, 0.9, 52}, {-7.5, 1., 58}, {-5., 0.1, 5}, {-5., 0.2, 11},
 {-5., 0.3, 16}, {-5., 0.4, 22}, {-5., 0.5, 28}, {-5., 0.6, 34}, {-5., 0.7, 40}, {-5., 0.8, 46},
 {-5., 0.9, 52}, {-5., 1., 58}, {-2.5, 0.1, 5}, {-2.5, 0.2, 11}, {-2.5, 0.3, 16}, {-2.5, 0.4, 22},
 {-2.5, 0.5, 28}, {-2.5, 0.6, 34}, {-2.5, 0.7, 40}, {-2.5, 0.8, 47}, {-2.5, 0.9, 53}, {-2.5, 1., 59}}
```

(b) The list of negative $\delta T_H(r_H)$ of EHBH

Figure 42 **The list of negative $F(r_H)$ and $\delta T_H(r_H)$ of EHBH.**