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FIXED POINT THEOREMS FOR CYCLIC CONTRACTIONS IN C*-ALGEBRA-VALUED b-METRIC SPACES

โดย

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ชื่อผู้วิจัย ผู้ช่วยศาสตราจารย์ ดร.จักรกฤษ กลิ่นเอี่ยม

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บทคัดย่อ

ในงานวิจัยนี้ เราแนะนำแนวคิดของปริภูมิบีเมตริกค่าพีชคณิตซีสตาร์ซึ่งได้ผสมแนวคิดของปริภูมิบีเมตริกและปริภูมิบีเมตริกค่าพีชคณิตซีสตาร์เข้าด้วยกัน และได้ศึกษาสมบัติพื้นฐานของปริภูมิดังกล่าวนอกจากนี้เรายังพิสูจน์ทฤษฎีบทจุดตรึงสำหรับการส่งแบบวัฏจักรชนิดหดตัว

ABSTRACT

In this research, we we introduce concept of C_-algebra-valued b-metric space, which is mixture of concept of b-metric space and idea of C*-algebra-valued metric space, study its fundamental properties and we give some fixed point theorems for cyclic mapping with contractive type.

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Chapter 1

Introduction

Firstly, we begin with the basic concept of C^*-algebras. A real or a complex linear space $\mathbb{A}$ is algebra if vector multiplication is defined for every pair of element of $\mathbb{A}$ satisfying two condition such that $\mathbb{A}$ is a ring with respect to vector addition and vector multiplication and for every scalar α and every pair of elements $x, y \in \mathbb{A}$, $\alpha(xy) = (\alpha x)y = x(\alpha y)$. A norm $\|\cdot\|$ on $\mathbb{A}$ is said to be submultiplicative if $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathbb{A}$. In this case $(\mathbb{A}, \|\cdot\|)$ is called normed algebra. A complete normed algebra is called Banach algebra. An involution on algebra $\mathbb{A}$ is conjugate linear map $a \mapsto a^*$ on $\mathbb{A}$ such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathbb{A}$. $(A, *)$ is called $*$-algebra. A Banach $*$-algebra $\mathbb{A}$ is $*$-algebra $\mathbb{A}$ with a complete submultiplicative norm such that $\|a^*\| = \|a\|$ for all $a \in \mathbb{A}$. C^*-algebra is Banach $*$-algebra such that $\|a^*a\| = \|a\|^2$. There are many example of C^*-algebra, such as set of complex numbers, the set of all bounded linear operators on a Hilbert space H, $L(H)$ and the set of $n \times n$-matrices ,$M_n(\mathbb{C})$. If a normed algebra $\mathbb{A}$ admits a unit I, $aI = Ia = a$ for all $a \in \mathbb{A}$ and $\|I\| = 1$, we say that $\mathbb{A}$ is a unital normed algebra. A complete unital normed algebra $\mathbb{A}$ is called Unital Banach algebra. For properties in C^*-algebras, we refer to [12, 10, 31] and the references therein.

It is well known that contractive mapping principle, appeared in the Ph.D. dissertation of S. Banach in 1920, let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a contraction if there exists $r \in [0, 1)$ such that

$$d(Tx, Ty) \leq rd(x, y) \quad \text{for all } x, y \in X$$

,which was published in 1922 [3]. The Banach's contraction principle has become one of the most important tool used for the existence of solutions of many nonlinear problems in many branches of science and has been extensively studied in many spaces which are more general than metric space by serveral mathematictians, see for example, Quasi-metric spaces [41, 11], Dislocated metric spaces [15], Dislocated quasi metric spaces [43], G-metric spaces [32, 33, 34], b-metric spaces [2, 8, 9], Metric-type spaces [24, 25], Metric-like spaces [13], b-metric-like spaces (or Dislocated b-metric spaces) [1, 17], Quasi b-metric spaces [38] and Dislocated quasi-b-metric spaces [26]. Note that the Banach contraction principle requires that the mapping T satisfies the contractive condition each point of $X \times X$ and ranges of T is positive real numbers.

Consider the operator equation

$$X - \sum_{n=1}^{\infty} L_n^* X L_n = Q$$

where $\{L_1, L_2, \ldots, L_n\}$ is subset of the set of linear bounded operators on Hilbert space H, $X \in L(H)$ and $Q \in L(H)_+$: positive linear bounded operators on Hilbert space H. Then we convert the operator equation to the mapping $F : L(H) \to L(H)$ is defined by

$$F(X) = \sum_{n=1}^{\infty} L_n^* X L_n + Q.$$

Observe that the range of the mapping F is not real numbers but it is linear bounded operators on Hilbert space H. Therefore the Banach contraction principle can not be applied with this problem. Afterward, the question is risen that does such mapping have a fixed point which is equivalent to a solution of operator equation. In 2014, Z. Ma, L. Jiang and H. Sun [29] introduced a new spaces, called C^*-algebra-valued metric spaces which is more general than metric space, replacing the set of real numbers by a C^*-algebras, and establish a fixed point theorem for self-maps with contractive or expansive conditions on such spaces, analogous to the Banach contraction principle. As applications , existence and uniqueness results for a type of integral equation and operator equation is given, was able to solve the above problem if $L_1, L_2, \ldots, L_n \in L(H)$ satisfy $\sum_{n=1}^{\infty} \|L_n\|^2 < 1$.

Later, many authors extend and improve the result of Ma *et al*. For example in [4] S. Batul and T. Kamran generalized the notation of C^*-valued contraction mappings by weakening the contractive condition introduced by Ma *et al*, the mapping is called C^*-valued contractive type mappings, and establish a fixed point theorem for such mapping and which is more generalize than the result of Ma *et al*, in [39] D. Shehwar and T. Kamran extend and improve the result of Ma *et al* [29] and Jachymski by proving a fixed point theorem for self-mappings on C^*-valued metric spaces satisfying the contractive condition for those pairs of elements from the metric space which form edges of a graph in the metric space. In 2015, Z. Ma and L. Jiang [30] introduced a concept of C^*-algebra-valued b-metric spaces which generalize an ordinary C^*-algebra-valued metric space and give some fixed point theorems for self-map with contractive condition on such spaces. As applications, existence and uniqueness results for a type of operator equation and an integral equation are given.

Generally, in order to use the Banach contraction principle, a self-mapping T must be Lipschitz continuous, with the Lipschitz constant $r \in [0, 1)$. In particular, T must be continuous at all element of its domain. That is one major drawback. Next, many authors could find contrac-

tive conditions which imply the existence of fixed point in complete metric space but not imply continuity. We refer to [19, 20] (Kannan-type mappings) and [7] (Chatterjea-type mapping).

Theorem 1.1. *[19] If (X, d) is a complete metric space and the mapping $T : X \to X$ satisfies*

$$d(Tx, Ty) \leq r[d(x, Tx) + d(y, Ty)],$$

where $0 \leq r < \frac{1}{2}$ and $x, y \in X$, then T has a unique fixed point.

Theorem 1.2. *[7] If (X, d) is a complete metric space and the mapping $T : X \to X$ satisfies*

$$d(Tx, Ty) \leq r[d(x, Ty) + d(y, Tx)],$$

where $0 \leq r < \frac{1}{2}$ and $x, y \in X$, then T has a unique fixed point.

In 2003, Kirk et al. [23] introduced the following notation of a cyclic representation and characterized the Banach contraction principle in context of a cyclic mapping as follow :

Theorem 1.3. *Let $A_1, A_2, \ldots, A_m$ be a nonempty closed subsets of a complete metric space (X, d). Assume that a mapping $T : \bigcup_{i=1}^{m} A_i \to \bigcup_{i=1}^{m} A_i$ satisfies the following conditions*

(i) $T(A_i) \subseteq A_{i+1}$ for all $1 \leq i \leq m$ and $A_{m+1} = A_1$;

(ii) there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x \in A_i$, $y \in A_{i+1}$ for $1 \leq i \leq m$.

Then T has a unique fixed point.

In 2011, E. Karapinar and I. M. Erhan introduced Kannan type cyclic contraction [21] and Chatterjea type cyclic contraction. Moreover, they derive some fixed point theorems for such cyclic contractions in complete metric spaces as follow;

Theorem 1.4. *(Fixed point theorem for Kannan type cyclic contraction) Let A and B be a nonempty subsets of a metric spaces (X, d) and a cyclic mapping $T : A \cup B \to A \cup B$ satisfies*

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)], \quad \text{for all} \quad x \in A \quad \text{and} \quad y \in B$$

where $0 \leq k < \frac{1}{2}$. Then T has a unique fixed point in $A \cap B$.

Theorem 1.5. *(Fixed point theorem for Chatterjea type cyclic contraction) Let A and B be a nonempty subsets of a metric spaces (X, d) and a cyclic mapping $T : A \cup B \to A \cup B$ satisfies*

$$d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)], \quad \textit{for all} \quad x \in A \quad \textit{and} \quad y \in B$$

where $0 \leq k < \frac{1}{2}$. Then T has a unique fixed point in $A \cap B$.

The purpose of this paper, we study fundamental properties of C^*-algebra-valued b-metric space which was introduced by Z. Ma and L. Jiang [30] and give some fixed point theorems for cyclic mapping with contractive and expansive condition on such space analogous to the results presented in [30].

Chapter 2

Preliminaries

In this section, we recollect some basic notation, defintion and results will be used in main result. Firstly, we begin with the concept of b-metric spaces.

Definition 2.1. [2, 8] Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}$ is called b-metric if there exists a real number $b \geq 1$ such that for every $x, y, z \in X$, we have

(i) $d(x, y) \geq 0$

(ii) $d(x, y) = 0$ if and only if $x = y$

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, z) \leq b[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a $b-$metric space.

The class of b-metric spaces is larger than the calass of metric spaces, since a b-metric space is a metric when $b = 1$ in the fourth condition in above definition. There exist many example in some work showing that the class of b-metric is efficiently larger than that metric spaces. (see also [2, 9, 42, 5])

Example 2.2. [2] The set $l_p(\mathbb{R})$ with $0 < p < 1$, where $l_p(\mathbb{R}) := \{\{x_n\} \subseteq \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$, together with the function $d : l_p(\mathbb{R}) \times l_p(\mathbb{R}) \to \mathbb{R}$,

$$d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}},$$

where $x = \{x_n\}, y = \{y_n\} \in l_p(\mathbb{R})$, is a b-metric space with coefficient $b = 2^{\frac{1}{p}} > 1$. Observe that the result hold for the general case $l_p(X)$ with $0 < p < 1$, where X is a Banach space.

Example 2.3. [2] The space $L_p(0 < p < 1)$ of all real functions $x(t)$, $t \in [0, 1]$, such that $\int_0^1 |x(t)|^p dt < \infty$,, together with the function

$$d(x, y) = \left(\int_0^1 |x(t) - y(t)|^p dt \right)^{\frac{1}{p}}, \quad \text{for all } x, y \in L_p[0, 1],$$

is a b-metric space with $b = 2^{\frac{1}{p}}$.

Example 2.4. [42] Let (X, d_1) be a metric space and $d_2(x, y) = (d_1(x, y))^p$, where $p > 1$ is natural numbers. Then d_2 is a b-metric with $b = 2^{p-1}$

The notation convergence, compactness, closedness and completeness in b-metric space are given in the same way as in metric space.

Next, we give concept of spectrum of element in C^*-algebra $\mathbb{A}$.

Definition 2.5. [31] We say that $a \in \mathbb{A}$ is invertible if there is an element $b \in \mathbb{A}$ such that $ab = ba = I$. In this case b is unique and written a^{-1}. The set

$$Inv(\mathbb{A}) = \{a \in \mathbb{A} | a \text{ is invertible } \}$$

is a qroup under multiplication. We define spectrum of an element a to be the set

$$\sigma(a) = \sigma_A(a) = \{\lambda \in \mathbb{C} | \lambda I - a \notin Inv(\mathbb{A})\}.$$

Theorem 2.6. *[31] Let $\mathbb{A}$ be a unital Banach algebra and a be an element of $\mathbb{A}$ such that $\|a\| < 1$. Then $I - a \in Inv(\mathbb{A})$ and*

$$(I - a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

Theorem 2.7. *[31] Let $\mathbb{A}$ be a unital C^*-algebra with a unit I, then*

(1) $I^* = I$,

(2) For any $a \in Inv(\mathbb{A})$, $(a^*)^{-1} = (a^{-1})^*$.

(3) For any $a \in \mathbb{A}$, $\sigma(a^*) = \sigma(a)^* = \{\overline{\lambda} \in \mathbb{C} : \lambda \in \sigma(a)\}$.

All over this paper, $\mathbb{A}$ mean a unital C^*-algebra with a unit I. $\mathbb{R}$ is set of real numbers and $\mathbb{R}_+$ is the set of nonnegetive real numbers. $M_n(\mathbb{R})$ is $n \times n$ matrix with entries $\mathbb{R}$.

Definition 2.8. [31] The set of hermitain elements of $\mathbb{A}$ is denoted by $\mathbb{A}_h$, that is $\mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$. An element x in $\mathbb{A}$ is positive element which is denoted by $\theta \preceq x$, where θ means the zero element in $\mathbb{A}$ if and only if $x \in \mathbb{A}_h$ and $\sigma(x)$ is a subset of nonnegative real numbers. We define a partial ordering $\mathbb{A}_h$ by using definition of positive element as $x \preceq y$ if and only if $y - x \succeq \theta$. The set of positive element in $\mathbb{A}$ is denoted by $\mathbb{A}_+ = \{x \in \mathbb{A} : x \succeq \theta\}$.

The following are definition and some properties of positive element of a C^*-algebra $\mathbb{A}$.

Lemma 2.9. *[31] The sum of two positive elements in a C^*-algebra is a positive element.*

Theorem 2.10. *[31] If a is an arbitrary element of a C^*-algebra $\mathbb{A}$, then a^*a is positive*

We summarise some elementary facts about $\mathbb{A}_+$ in the following results.

Theorem 2.11. *[31] Let $\mathbb{A}$ be a C^*-algebra.*

(1) The set $\mathbb{A}_+$ is closed cone in $\mathbb{A}$. [A cone C in a real or complex vector space is a subset closed under addition and under scalar multiplication by $\mathbb{R}_+$]

(2) The set $\mathbb{A}_+$ is equal to $\{a^*a : a \in \mathbb{A}\}$.

(3) If $\theta \preceq a \preceq b$, then $\|a\| \leq \|b\|$.

(4) If $\mathbb{A}$ is unital and a, b are positive invertible elements, then $a \preceq b \Rightarrow \theta \preceq b^{-1} \preceq a^{-1}$.

Theorem 2.12. *[31] Let $\mathbb{A}$ be a C^*-algebra. If $a, b \in \mathbb{A}_+$ and $a \preceq b$, then for any $x \in \mathbb{A}$ both x^*ax and x^*bx are positive elements and $x^*ax \preceq x^*bx$.*

Lemma 2.13. *[31] Suppose that $\mathbb{A}$ is a unital C^*-algebra with a unit I.*

(1) If $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$, then $I - a$ is invertible and $\|a(I-a)^{-1}\| < 1$.

(2) Suppose that $a, b \in \mathbb{A}$ with $a, b \succeq \theta$ and $ab = ba$, then $ab \succeq \theta$.

(3) Define $\mathbb{A}' = \{a \in \mathbb{A} : ab = ba, \forall b \in \mathbb{A}\}$. Let $a \in \mathbb{A}'$, if $b, c \in \mathbb{A}$ with $b \succeq c \succeq \theta$ and $I - a \in \mathbb{A}'_+$ is invertible operator, then

$$(I-a)^{-1}b \succeq (I-a)^{-1}c.$$

Definition 2.14. [12] Let T be an operator on the Hilbert space H. T is positive if and only if $\langle Tx, x\rangle \geq 0$ for all $x \in H$, T is self-adjoint or hermitain if and only if $T = T^*$.

In 2014, Z. Ma, L. Jiang and H. Sun [29] introduced concept of C^*-algebra-valued metric space by using the concept of positive elements in $\mathbb{A}$. The following is definition C^*-algebra-valued metric.

Definition 2.15. [29] Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{A}$ is called C^*-algebra-valued metric on X satisfies following conditions,

(1) $d(x, y) \succeq \theta$ for all $x, y \in X$;

(2) $d(x, y) = \theta$ if and only if $x = y$;

(3) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(4) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a C^*-algebra-valued metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra-valued metric space.

We know that range of mapping d in metric space is the set of real numbers which is C^*-algebra, then the space generalize metric space. In such paper, Ma *et al.* state the notation of convergence, Cauchy sequence, completeness in C^*-algebra-valued metric space. For detail, a sequence $\{x_n\}$ in a C^*-algebra-valued metric space $(X, \mathbb{A}, d)$ is said to converges to $x \in X$ with respect to $\mathbb{A}$ if for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\|d(x_n, x)\| < \varepsilon$ for all $n \geq N$. We write it as $\lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ is called a Cauchy sequence with respect to $\mathbb{A}$ if for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\|d(x_m, x_n)\| < \varepsilon$ for all $n, m \geq N$. The $(X, \mathbb{A}, d)$ is said to be a complete C^*-algebra-valued metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent. Moreover, they introduce definition of contractive and expansive mapping and give some related fixed point theorems for self-maps with C^*-algebra-valued contractive and expansive mapping, analogous to Banach contraction principle. The following is definition of contractive mapping and the related fixed point theorem.

Definition 2.16. [29] Suppose that $(X, \mathbb{A}, d)$ is a C^*-algebra-valued metric space. A mapping $T : X \to X$ is called C^*-algebra-valued contractive mapping on X, if there is an $\lambda \in \mathbb{A}$ with $\|\lambda\| < 1$ such that

$$d(Tx, Ty) \preceq \lambda^* d(x,y)\lambda, \text{ for all } x, y \in X.$$

Theorem 2.17. *[29] If $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued metric space and $T : X \to X$ satisfy Defintion 2.16, then T has a unique fixed point in X.*

In the same way, the concept of expansive mapping is defined in the following way

Definition 2.18. [29] Let X a nonempty set. A mapping T is a C^*-algebra-valued expansive mapping on X, if $T : X \to X$ satisfies :

(1) $T(X) = X$;

(2) $d(Tx, Ty) \succeq \lambda^* d(x,y)\lambda$, for all $x, y \in X$,

where $\lambda \in \mathbb{A}$ is an invertible element and $\|\lambda^{-1}\| < 1$.

The following is the related fixed point theorem for C^*-algebra-valued expansive mapping.

Theorem 2.19. *[29] Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra-valued metric space. If a $T : X \to X$ satisfies Defintion 2.18, then T has a unique fixed point in X.*

Chapter 3

Fundamental properties of C^*-algebra-valued b-metric spaces

In this section, we begin with the concept of C^*-algebra-valued b-metric space which was introduced by Z. Ma and L. Jiang [30] as follow;

Definition 3.1. [30] Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{A}$ is called C^*-algebra-valued b-metric on X if there exists $b \in \mathbb{A}'$ such that $b \succeq I$ satisfies following conditions,

(1) $d(x, y) \succeq \theta$ for all $x, y \in X$;

(2) $d(x, y) = \theta$ if and only if $x = y$;

(3) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(4) $d(x, y) \preceq b[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then $(X, d, \mathbb{A})$ is called a C^*-algebra-valued b-metric space.

Remark 3.2. If $b = I$, then a C^*-algebra-valued b-metric spaces is a C^*-algebra-valued metric spaces. In particular, If $\mathbb{A}$ is set of real numbers and $b = 1$, then the C^*-algebra-valued b-metric spaces is the metric spaces.

Definition 3.3. [30] Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space. A sequence $\{x_n\}$ in $(X, \mathbb{A}, d)$ is said to converges to x if and only if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\|d(x_n, x)\| \leq \varepsilon$. Then $\{x_n\}$ is said to be convergent with respect to $\mathbb{A}$ and x is called limit point of $\{x_n\}$. We denote it by $\lim_{n\to\infty} x_n = x$.

A sequence $\{x_n\}$ is called a Cauchy seqeunce with respect to $\mathbb{A}$ if and only if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $\|d(x_n, x_m)\| \leq \varepsilon$.

We say $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued b-metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent sequence with respect to $\mathbb{A}$.

The following is an example of complete C^*-algebra-valued b-metric space.

Example 3.4. [30] Let $X = \mathbb{R}$ and $\mathbb{A} = M_n(\mathbb{R})$. Define

$$d(x,y) = diag((x-y)^p, \alpha_1|x-y|^p, \alpha_2|x-y|^p, \ldots, \alpha_{n-1}|x-y|^p)$$
$$= \begin{bmatrix} |x-y|^p & 0 & 0 & \cdots & 0 \\ 0 & \alpha_1|x-y|^p & 0 & \cdots & 0 \\ 0 & 0 & \alpha_2|x-y|^p & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-1}|x-y|^p \end{bmatrix}$$

where $x, y \in \mathbb{R}$, $\alpha_i > 0$ for all $i = 1, 2, \ldots, n-1$ are constants and p is a natural number such that $p \geq 2$. A norm $\|\cdot\|$ on $\mathbb{A}$ is defined by

$$\|A\| = \max_{i,j} |a_{ij}|^{\frac{1}{p}}$$

where $A = (a_{ij})_{n\times n} \in \mathbb{A}$. The involution is given by $A^* = (\overline{A})^T$, conjugate transpose of matrix A.

$$A^* = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}^* = \begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} & \cdots & \overline{a_{n1}} \\ \overline{a_{12}} & \overline{a_{22}} & \cdots & \overline{a_{n2}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \overline{a_{2n}} & \cdots & \overline{a_{nn}} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

It is easy to verify d is a C^*-algebra-valued b-metric space and $(X, M_2(\mathbb{R}), d)$ is a complete C^*-algebra-valued b-metric space be completeness of $\mathbb{R}$.

Proof. An element $A \in \mathbb{A} = M_n(\mathbb{R})$ is positive element, denote it by

$$A \succeq \theta, \text{ if and only if } A \text{ is positive semidefinite.}$$

We define a partial ordering $\preceq$ on $\mathbb{A}$ as follows :

$$A \preceq B \text{ if and only if } \theta \preceq B - A,$$

where θ mean the zero matrix in $M_n(\mathbb{R})$. Firstly, it clears that $\preceq$ is partially order relation. Next, we show that d is a C^*-algebra-valued b-metric space. Let $x, y, z \in X$. It easy to see that d satifies condition (1) ,(2) and (3) of Definition 3.1. We will only show condition (4) that $d(x,y) \preceq b[d(x,z) + d(z,y)]$ with

$$b = \begin{bmatrix} 2^{p-1} & 0 & \cdots & 0 \\ 0 & 2^{p-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2^{p-1} \end{bmatrix}_{n\times n}.$$

Since function $f(x) = |x|^p$ is convex function for all $p \geq 2$ and $x \in \mathbb{R}$, this implies that

$$\left|\frac{a+c}{2}\right|^p = \left|\frac{1}{2}a + (1-\frac{1}{2})c\right|^p \leq \frac{1}{2}|a|^p + \left(1-\frac{1}{2}\right)|c|^p = \frac{1}{2}(|a|^p + |c|^p)$$

and hence $|a+c|^p \leq 2^{p-1}(|a|^p + |c|^p)$ for all $a, c \in \mathbb{R}$. We substitute $a = x - y$ and $c = y - z$, then

$$|x-z|^p = |x-y+y-z|^p \leq 2^{p-1}(|x-y|^p + |y-z|^p).$$

Hence, setting $M_0 = (|x-y|^p + |y-z|^p)$ and $M_1 = |x-z|^p$, we obtain that

$$\begin{bmatrix} 2^{p-1}M_0 - M_1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_1(2^{p-1}M_0 - M_1) & 0 & \cdots & 0 \\ 0 & 0 & \alpha_2(2^{p-1}M_0 - M_1) & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-1}(2^{p-1}M_0 - M_1) \end{bmatrix}$$

$$= \begin{bmatrix} 2^{p-1}M_0 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_1 2^{p-1}M_0 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_2 2^{p-1}M_0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-1}2^{p-1}M_0 \end{bmatrix} - \begin{bmatrix} M_1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_1 M_1 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_2 M_1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-1}M_1 \end{bmatrix}$$

$$= \begin{bmatrix} 2^{p-1} & 0 & \cdots & 0 \\ 0 & 2^{p-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2^{p-1} \end{bmatrix} \begin{bmatrix} M_0 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_1 M_0 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_2 M_0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-1}M_0 \end{bmatrix}$$

$$- \begin{bmatrix} M_1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_1 M_1 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_2 M_1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-1}M_1 \end{bmatrix}$$

$$= b(d(x,y) + d(y,z)) - d(x,z)$$

implies that each eigenvalue of $b[d(x,z) + d(z,y)] - d(x,y)$ is nonnegative. Since each eigenvalue of a positive semidefinite matrix is a nonnegative real number, we have $b[d(x,z) +$

$d(z,y)] - d(x,y)$ is positive semidefinite, i.e. $b[d(x,z)+d(z,y)] - d(x,y) \succeq 0$, that is $d(x,y) \preceq b[d(x,z)+d(z,y)]$, where $b = 2^{p-1}I \in \mathbb{A}'$ and $b \succeq I$ by $2^{p-1} > 1$. But $|x-y|^p \leq |x-z|^p + |z-y|^p$ is impossible for all $x, y, z \in \mathbb{R}$. Hence $(X, M_n(\mathbb{R}), d)$ is C^*-algebra-valued b-metric spaces but not C^*-algebra-valued metric spaces.

Finally, we show that $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued b-metric space. Suppose that $\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$. Then, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|d(x_m, x_n)\| \leq \varepsilon$ for all $m, n \geq N$, that is

$$\max\{(|x_m - x_n|^p)^{\frac{1}{p}}, (\alpha_1|x_m - x_n|^p)^{\frac{1}{p}}, (\alpha_2|x_m - x_n|^p)^{\frac{1}{p}}, \ldots, (\alpha_{n-1}|x_m - x_n|^p)^{\frac{1}{p}}\} \leq \varepsilon$$

for all $m, n \geq N$. Therefore

$$\begin{aligned} |x_m - x_n| &= (|x_m - x_n|^p)^{\frac{1}{p}} \\ &\leq \max\{(|x_m - x_n|^p)^{\frac{1}{p}}, (\alpha_1|x_m - x_n|^p)^{\frac{1}{p}}, (\alpha_2|x_m - x_n|^p)^{\frac{1}{p}}, \ldots, (\alpha_{n-1}|x_m - x_n|^p)^{\frac{1}{p}}\} \\ &\leq \varepsilon \end{aligned}$$

for all $m, n \geq N$. Hence $\{x_n\}$ is a Cauchy sequnce in $\mathbb{R}$. By completeness of $\mathbb{R}$, there exists $x \in \mathbb{R}$ such that $\lim_{n\to\infty} x_n = x$, that is $\lim_{n\to\infty} |x_n - x| = 0$. Then, we have

$$\|d(x_n, x)\| = \max\{(|x_n - x|^p)^{\frac{1}{p}}, (\alpha_1|x_n - x|^p)^{\frac{1}{p}}, (\alpha_2|x_n - x|^p)^{\frac{1}{p}}, \ldots, (\alpha_{n-1}|x_n - x|^p)^{\frac{1}{p}}\}$$

converges to 0 as $n \to \infty$. Therefore, $\{x_n\}$ is convergent with respect to $\mathbb{A}$ and $\{x_n\}$ converges to x, so $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued b-metric space. □

Next, we disscus some fundamental properties of C^*-algebra-valued b-metric spaces.

Theorem 3.5. *Let $(X, \mathbb{A}, d)$ be C^*-algebra-valued b-metric space. If $\{x_n\}$ is a convergent sequence with respect to $\mathbb{A}$, then $\{x_n\}$ is Cauchy sequence with respect to $\mathbb{A}$.*

Proof. Assume that $\{x_n\}$ is a convergent sequence with respect to $\mathbb{A}$, then there exists a $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Let $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\|d(x_n, x)\| \leq \frac{\varepsilon}{2\|b\|}.$$

Consider, for $m, n \in N$, we get that

$$d(x_m, x_n) \preceq b[d(x_m, x) + d(x, x_n)].$$

By Theorem 2.11, for $m, n \geq N$ we have

$$\begin{aligned}\|d(x_m, x_n)\| &\leq \|b[d(x_m, x) + d(x, x_n)]\|, \\ &\leq \|b\|\|d(x_m, x) + d(x, x_n)\|, \\ &\leq \|b\|\|d(x_m, x)\| + \|b\|\|d(x, x_n)\|, \\ &\leq \|b\|\frac{\varepsilon}{2\|b\|} + \|b\|\frac{\varepsilon}{2\|b\|} = \varepsilon.\end{aligned}$$

This implies that $\{x_n\}$ is Cauchy sequence with respect to $\mathbb{A}$. □

Definition 3.6. A subset S of a C^*-algebra-valued b-metric space $(X, \mathbb{A}, d)$ is bounded with respect to $\mathbb{A}$ if there exists $\overline{x} \in X$ and a nonnegetive real numbers M such that

$$\|d(x, \overline{x})\| \leq M, \quad \text{for all} \quad x \in X.$$

Theorem 3.7. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space and let $\{x_n\}$ be a sequence in X and $x \in X$ Then :*

1. *$x_n \to x$ if and only if $d(x_n, x) \to \theta$,*
2. *A convergent sequence in X is bounded with respect to $\mathbb{A}$ and its limit is unique,*
3. *A Cauchy sequence in X is bounded with respect to $\mathbb{A}$.*

Proof. (1) Assume that $x_n \to x$. For any $\varepsilon > 0$ is given. Then, there exists $N_0 \in \mathbb{N}$ such that

$$\|d(x_n, x) - \theta\| = \|d(x_n, x)\| \leq \varepsilon.$$

This implies that $d(x_n, x) \to \theta$. Conversely, assume that $d(x_n, x) \to \theta$. Then, for any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$\|d(x_n, x) - \theta\| \leq \varepsilon \Rightarrow \|d(x_n, x)\| \leq \varepsilon,$$

that is $x_n \to x$.

(2) Let $\{x_n\}$ be a convergent sequence with respect to $\mathbb{A}$. Suppose that $x_n \to x$. Then taking $\varepsilon = 1$, we can find $N \in \mathbb{N}$ such that

$$d(x_n, x) \leq 1, \forall n \geq N.$$

Let $K = \max\{\|d(x_1, x)\|, \|d(x_2, x)\|, \ldots, \|d(x_N, x)\|\}$. Setting $M = \max\{1, K\}$. This implies that

$$\|d(x_n, x)\| \leq M, \quad \text{for all} \quad n \in \mathbb{N}.$$

Next, suppose that $x_n \to x$ and $x_n \to y$. Consider, $d(x,y) \preceq b[d(x,x_n) + d(x_n,y)]$, by Theorem 2.11, we have

$$\|d(x,y)\| \leq \|b\|[\|d(x_n,x)\| + \|d(x_n,y)\|].$$

From (1), letting $n \to \infty$, we obtian that $\|d(x,y)\| = 0$, that is $x = y$.

(3) Assume that $\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$. In particular, $\varepsilon = 1$, there exists $N_1 \in \mathbb{N}$ such that

$$\|d(x_m,x_n)\| \leq 1 \quad \text{for all } m,n \geq N_1$$

Let $K = \max\{\|d(x_1,x_{N_1})\|, \|d(x_2,x_{N_1})\|, \ldots, \|d(x_{N_1-1},x_{N_1})\|\}$. Then,

$$\|d(x_n,x_{N_1})\| \leq K \quad \text{for all } n < N_1.$$

Setting $M = \max\{1,K\}$. Then, we get that

$$\|d(x_n,x_{N_1})\| \leq M \quad \text{for all } n \in \mathbb{N}.$$

□

Theorem 3.8. *Let $\{x_n\}$ be a convergent sequence in a C^*-algebra-valued b-metric space $(X,\mathbb{A},d)$ and $\lim_{n\to\infty} x_n = x$. Then, every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is convergent and has the same limit x.*

Proof. Let $\varepsilon > 0$ be given. Then, there exists $N \in \mathbb{N}$ such that

$$\|d(x_n,x)\| \leq \varepsilon, \quad \text{for all } n \geq N.$$

Since $n_1 < n_2 < \cdots < n_k < \cdots$ is an increasing sequence of natural numbers, it is easily proved (by Induction) that $n_k \geq k$. Hence, if $k \geq N$, we also have $n_k \geq k \geq N$ so that

$$\|d(x_{n_k},x)\| \leq \varepsilon, \quad \text{for all } n_k \geq N.$$

Therefore the subsequence $\{x_{n_k}\}$ also converges to x. □

Theorem 3.9. *Let $(X,\mathbb{A},d)$ be a C^*-algebra-valued b-metric space. Then every subsequence of a Cauchy sequence is Cauchy sequence.*

Proof. Let $\{x_{n_k}\}$ be a subsequence of Cauchy sequence $\{x_n\}$ in a C^*-algebra-valued b-metric space. Then for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $r,s \geq N$, we have $\|d(x_r,x_s)\| \leq \varepsilon$. Similar facts in proof of previous theorem, we have $n_r \geq r \geq N$ and $n_s \geq s \geq N$. Hence, we obtain that $\|d(x_{n_r},x_{n_s})\| \leq \varepsilon$. Therefore $\{x_{n_k}\}$ is Cauchy sequence. □

Theorem 3.10. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space and let $\{x_n\}$ be a Cauchy sequence with respect to $\mathbb{A}$. If $\{x_n\}$ contains its convergent subsequence, then $\{x_n\}$ is convergent sequence.*

Proof. Let $\varepsilon > 0$. Since $\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$, there exists a $N_0 \in \mathbb{N}$ such that

$$\|d(x_m, x_p)\| \leq \frac{1}{2\|b\|}\varepsilon, \quad \text{for all } m, p \geq N_0.$$

Let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}$ and $x_{n_k} \to x$ $(k \to \infty)$. Then, there exists $N_1 \in \mathbb{N}$ such that

$$\|d(x_{n_k}, x)\| \leq \frac{1}{2\|b\|}\varepsilon, \quad \text{for all } n_k \geq N_1.$$

Let $N = \max\{N_0, N_1\}$. For $n, k \geq N$, we have

$$d(x_n, x) \preceq b[d(x_n, x_{n_k}) + d(x_{n_k}, x)].$$

By Theorem 2.11, we also have

$$\begin{aligned}
\|d(x_n, x)\| &\leq \|b\|\|d(x_n, x_{n_k}) + d(x_{n_k}, x)\| \\
&\leq \|b\|\|d(x_n, x_{n_k})\| + \|b\|\|d(x_{n_k}, x)\| \\
&\leq \|b\|[\frac{\varepsilon}{2\|b\|} + \frac{\varepsilon}{2\|b\|}] \\
&\leq \varepsilon.
\end{aligned}$$

Therefore $x_n \to x$ as $n \to \infty$ □

Theorem 3.11. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space. Suppose that $\{x_n\}$ and $\{y_n\}$ are convergent with respect to $\mathbb{A}$ and converge to x and y, respectively. Then $d(x_n, y_n)$ converges to $b^2 d(x, y)$.*

Proof. Let $\varepsilon > 0$. Since $x_n \to x$ and $y_n \to y$, there exist $N_0, N_1 \in \mathbb{N}$ such that

$$\|d(x_n, x)\| \leq \frac{\varepsilon}{2\|b\|}, \ \forall n \geq N_0 \quad \text{and} \quad \|d(y_n, y)\| \leq \frac{\varepsilon}{2\|b\|^2}, \ \forall n \geq N_1.$$

Since $d(x_n, y_n) \preceq bd(x_n, x) + b^2 d(x, y) + b^2 d(y, y_n)$, By Theorem 2.11, we have

$$\|d(x_n, y_n) - b^2 d(x, y)\| \leq \|b\|\|d(x_n, x)\| + \|b\|^2\|d(y, y_n)\| \leq \varepsilon.$$

Therefore $d(x_n, y_n) \to b^2 d(x, y)$. □

Theorem 3.12. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space. Suppose that $\{x_n\}$ and $\{y_n\}$ are convergent with respect to $\mathbb{A}$ and converge to x and y, respectively. Then,*

$$\frac{1}{\|b\|^2}\|d(x,y)\| \leq \liminf_{n\to\infty}\|d(x_n,y_n)\| \leq \limsup_{n\to\infty}\|d(x_n,y_n)\| \leq \|b\|^2\|d(x,y)\|.$$

In particular, if $x = y$, then we have $\lim_{n\to\infty}\|d(x_n,y_n)\| = 0$. Moreover for any $z \in X$, we have

$$\frac{1}{\|b\|}\|d(x,z)\| \leq \liminf_{n\to\infty}\|d(x_n,z)\| \leq \limsup_{n\to\infty}\|d(x_n,z)\| \leq \|b\|\|d(x,z)\|$$

Proof. By defintion of C^*-algebra-valued b-metric space, it easy to see that

$$d(x,y) \preceq bd(x,x_n) + b^2d(x_n,y_n) + b^2d(y_n,y)$$

and

$$d(x_n,y_n) \preceq bd(x_n,x) + b^2d(x,y) + b^2d(y,y_n).$$

Using Theorem 2.11, we have

$$\|d(x,y)\| \leq \|b\|\|d(x,x_n)\| + \|b\|^2\|d(x_n,y_n)\| + \|b\|^2\|d(y_n,y)\|$$

and

$$\|d(x_n,y_n)\| \leq \|b\|\|d(x_n,x)\| + \|b\|^2\|d(x,y)\| + \|b\|^2\|d(y,y_n)\|.$$

Taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality, this complete the first result. In particular, If $x = y$, we have

$$\|d(x_n,y_n)\| \leq \|b\|\|d(x_n,x)\| + \|b\|^2\|d(y,y_n)\|.$$

Taking the limit as $n \to \infty$ in this inequality, we obtain that $\lim_{n\to\infty}\|d(x_n,y_n)\| = 0$.
Since

$$d(x,z) \preceq b[d(x,x_n) + d(x_n,z)] \quad \text{and} \quad d(x_n,z) \preceq b[d(x_n,x) + d(x,z)],$$

by Theorem 2.11, we have

$$\|d(x,z)\| \leq \|b\|\|d(x,x_n)\| + \|b\|\|d(x_n,z)\| \quad \text{and} \quad \|d(x_n,z)\| \leq \|b\|\|d(x_n,x)\| + \|b\|\|d(x,z)\|.$$

Again taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality, we obtain that the second desired result. □

Definition 3.13. Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space. A subset F of $(X, \mathbb{A}, d)$ is called a closed set if a sequence $\{x_n\}$ in X and $x_n \to x$ with respect to $\mathbb{A}$ imply $x \in F$.

Chapter 4

Fixed point theorems for cyclic contractions

Theorem 4.1. *Let A and B be nonempty closed subset of a complete C^*-algebra-valued b-metric space $(X, \mathbb{A}, d)$. Assume that $T : A \cup B \to A \cup B$ is cyclic mapping that satisfies*

$$d(Tx, Ty) \preceq \lambda^* d(x, y)\lambda, \quad \forall\ x \in A \ \text{ and } \ \forall y \in B$$

where $\lambda \in \mathbb{A}$ with $\|\lambda\| < \frac{1}{\|b\|}$. Then T has a unique fixed point in $A \cap B$.

Proof. Let x_0 be any point in A. Since T is cyclic mapping, we have $Tx_0 \in B$ and $T^2x_0 \in A$. Using the contractive condition of the mapping T, we get

$$d(Tx_0, T^2x_0) = d(Tx_0, T(Tx_0)) \preceq \lambda^* d(x_0, Tx_0)\lambda.$$

For all $n \in \mathbb{N}$, we have

$$d(T^n x_0, T^{n+1}x_0) \preceq (\lambda^*)^n d(x_0, Tx_0)\lambda^n = (\lambda^*)^n \beta \lambda^n$$

where $\beta = d(x_0, Tx_0)$. Consider, for any $m, n \in \mathbb{N}$ such that $m \leq n$, then

$$\begin{aligned}
d(T^m x_0, T^n x_0) &\preceq b[d(T^m x_0, T^{m+1}x_0) + d(T^{m+1}x_0, T^n x_0)] \\
&\preceq bd(T^m x_0, T^{m+1}x_0) + b^2[d(T^{m+1}x_0, T^{m+2}x_0) + d(T^{m+2}x_0, T^n x_0)] \\
&\preceq \cdots \\
&\preceq bd(T^m x_0, T^{m+1}x_0) + b^2 d(T^{m+1}x_0, T^{m+2}x_0) + \cdots + b^{n-m} d(T^{n-1}x_0, T^n x_0) \\
&\preceq b(\lambda^*)^m \beta \lambda^m + b^2(\lambda^*)^{m+1}\beta\lambda^{m+1} + \cdots + b^{n-m}(\lambda^*)^{n-1}\beta\lambda^{n-1} \\
&= \sum_{k=m}^{n-1} b^{k-m+1}(\lambda^*)^k \beta \lambda^k.
\end{aligned}$$

From Theorem 2.11, we have

$$\begin{aligned}
\|d(T^m x_0, T^n x_0)\| &\leq \|\sum_{k=m}^{n-1} b^{k-m+1}(\lambda^*)^k \beta \lambda^k\| \\
&\leq \sum_{k=m}^{n-1} \|b^{k-m+1}(\lambda^*)^k \beta \lambda^k\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=m}^{n-1} \|b^{k-m+1}\| \|(\lambda^*)^k\| \|\beta\| \|\lambda^k\| \\
&\leq \|\beta\| \sum_{k=m}^{n-1} \|b^{k-m+1}\| \|(\lambda)^k\|^2 \\
&\leq \|\beta\| \sum_{k=m}^{n-1} \|b\|^{k-m+1} \|\lambda\|^{2k} \\
&\leq \|\beta\| \sum_{k=m}^{n-1} \|b\|^{2k} \|\lambda\|^{2k} \\
&\leq \|\beta\| \sum_{k=m}^{\infty} (\|b\| \|\lambda\|)^{2k} \\
&= \|\beta\| \frac{(\|b\| \|\lambda\|)^{2m}}{1 - (\|b\| \|\lambda\|)}
\end{aligned}
$$

Since $0 \leq \|\lambda\| < \frac{1}{\|b\|}$, we have $\|\beta\| \frac{(\|b\| \|\lambda\|)^{2m}}{1-(\|b\| \|\lambda\|)} \to 0$ as $m \to \infty$. Therefore $\{T^n x_0\}$ is Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, d)$, there exists an element $x \in X$ such that $x = \lim_{n\to\infty} T^n x_0$.

Since $\{T^{2n} x_0\}$ is a sequence in A and $\{T^{2n-1} x_0\}$ is a sequence in B, we obtain that both sequence converges to the same limit x. Since A and B are closed set imply $x \in A \cap B$.

Next, we will complete the proof by showing that x is a unique fixed point of T. Since

$$
\begin{aligned}
\theta &\preceq d(Tx, x) \\
&\preceq b[d(Tx, T^{2n} x_0) + d(T^{2n} x_0, x)] \\
&\preceq b[\lambda^* d(x, T^{2n-1} x_0)\lambda + d(T^{2n} x_0, x)]
\end{aligned}
$$

by Theorem 2.11, we obtain that

$$
0 \leq \|d(Tx, x)\| \leq \|b\| \|\lambda\|^2 \|d(x, T^{2n-1} x_0)\| + \|b\| \|d(T^{2n} x_0, x)\| \to 0 \quad (n \to \infty).
$$

We have $Tx = x$, i.e. x is a fixed point of T.

Suppose that y is fixed point of T and $y \neq x$. Since

$$
\theta \preceq d(x, y) = d(Tx, Ty) \preceq \lambda^* d(x, y)\lambda,
$$

we have

$$
\|d(x, y)\| \leq \|\lambda^* d(x, y)\lambda\| \leq \|\lambda^*\| \|d(x, y)\| \|\lambda\| = \|\lambda\|^2 \|d(x, y)\| < \|d(x, y)\|.
$$

This is a contradiction. Therefore $x = y$ which implies that the fixed point is unique. □

Example 4.2. Let X be a set of real numbers and $\mathbb{A} = M_{2\times 2}(\mathbb{R})$ with $\|A\| = \max_{i,j}|a_{ij}|$ where a_{ij} are entries of the matrix $A \in M_{2\times 2}(\mathbb{R})$. Then $(X, \mathbb{A}, d)$ is a C^*-algebra-valued b-metric space with $b = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, where the involution is given by $A^* = (\overline{A})^T$,

$$d(x,y) = \begin{bmatrix} |x-y|^2 & 0 \\ 0 & |x-y|^2 \end{bmatrix}$$

and partial ordering on $\mathbb{A}$ is given as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \preceq \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \Leftrightarrow a_{ij} \leq b_{ij} \quad \text{for all} \quad i,j = 1,2,3,4.$$

Define a mapping $T : X \to X$ by

$$Tx = \begin{cases} -\frac{x+\frac{1}{3}}{3}|\sin(\frac{1}{x})| - \frac{1}{3} & ; \ x \in (\infty, -\frac{1}{3}] \\ -\frac{1}{3} & ; \ x \in (-\frac{1}{3}, 0] \\ -\frac{1}{2} & ; \ x \in (0, +\infty) \end{cases}$$

It clear that T is not continuous at all element of X. Therefore Theorem 2.16 can not imply the existence of fixed point of the mapping T.

Suppose that $A = [-\frac{1}{2}, -\frac{1}{3}]$ and $B = [-\frac{1}{3}, 0]$. Firstly, we will show that $T : A \cup B \to A \cup B$ is cyclic mapping. Let $x \in B$, that is $-\frac{1}{3} \leq x \leq 0$. Then $Tx = -\frac{1}{3} \in A$. Again, let $y \in A$, that is $-\frac{1}{2} \leq x \leq -\frac{1}{3}$. Indeed, we consider

$$\begin{aligned} -\frac{1}{2} \leq x \leq -\frac{1}{3} &\Rightarrow -\frac{1}{6} \leq x + \frac{1}{3} \leq 0 \\ &\Rightarrow -\frac{1}{18} \leq \frac{x+\frac{1}{3}}{3} \leq 0 \\ &\Rightarrow 0 \leq -\left(\frac{x+\frac{1}{3}}{3}\right) \leq \frac{1}{18} \\ &\Rightarrow 0 \leq -\left(\frac{x+\frac{1}{3}}{3}\right)|\sin(\frac{1}{x})| \leq \frac{1}{18}|\sin(\frac{1}{x})| \leq \frac{1}{18} \\ &\Rightarrow -\frac{1}{3} \leq -\left(\frac{x+\frac{1}{3}}{3}\right)|\sin(\frac{1}{x})| - \frac{1}{3} \leq \frac{1}{18} - \frac{1}{3} \leq 0, \end{aligned}$$

this implies that $Tx \in [-\frac{1}{3}, 0] = B$. For any $x \in A$ and $y \in B$, since $-\frac{1}{2} \leq x \leq -\frac{1}{3}$ and $-\frac{1}{3} \leq y$, we have $\frac{1}{9} \leq -\frac{x}{3} \leq \frac{1}{6}$ and $-\frac{1}{9} \leq \frac{y}{3}$. Hence, we obtain that

$$0 \leq -\frac{x}{3} - \frac{1}{9} \leq -\frac{x}{3} + \frac{y}{3}.$$

Next, we consider

$$\begin{aligned}
|Tx - Ty|^2 &= \left| -\left(\frac{x+\frac{1}{3}}{3}\right) |\sin(\frac{1}{x})| - \frac{1}{3} - (-\frac{1}{3}) \right|^2 \\
&= \left| -\left(\frac{x+\frac{1}{3}}{3}\right) |\sin(\frac{1}{x})| \right|^2 \\
&\preceq \left| -\left(\frac{x+\frac{1}{3}}{3}\right) \right|^2 \\
&= \left| -\frac{x}{3} - \frac{1}{9} \right|^2 \\
&\leq \left| -\frac{x}{3} + \frac{y}{3} \right|^2 \\
&\leq \frac{1}{9} |x-y|^2.
\end{aligned}$$

Then, we have

$$\begin{aligned}
d(Tx, Ty) &= \begin{bmatrix} |Tx-Ty|^2 & 0 \\ 0 & |Tx-Ty|^2 \end{bmatrix} \\
&\preceq \begin{bmatrix} \frac{1}{9}|x-y|^2 & 0 \\ 0 & \frac{1}{9}|x-y|^2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} |x-y|^2 & 0 \\ 0 & |x-y|^2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \\
&= \lambda^* d(x,y) \lambda,
\end{aligned}$$

where $\lambda = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$. Then $\|\lambda\| = \frac{1}{3} < \frac{1}{2} = \frac{1}{\|b\|}$. Thus T satisfies contraction of 5.10 imply that T has a unique fixed point in $A \cap B$, i.e $\{-\frac{1}{3}\} = F(T)$.

Corollary 4.3. *Suppose that $(X, \mathbb{A}, d)$ is a C^*-algebra-valued b-metric space. Assume that $T : X \to X$ is called a C^*-algebra-valued b-contractive mapping on X, that is T satisfies*

$$d(Tx, Ty) \preceq \lambda^* d(x,y) \lambda, \quad \forall\, x, y \in X$$

where $\lambda \in \mathbb{A}$ with $\|\lambda\| < \frac{1}{\|b\|}$. Then T has a unique fixed point in X.

Proof. Putting $A = B = X$, by Theorem 5.10, this implies that T has a unique fixed point in $A \cap B = X$. □

Theorem 4.4. *Suppose that $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued b-metric space. Assume that a mapping $T : X \to X$ satisfies*

(1) $T(X) = X$;

(2) $d(Tx, Ty) \succeq \lambda^* d(x, y)\lambda$ *for all* $x, y \in X$

where $\lambda \in \mathbb{A}$ *is an invertible element and* $\|\lambda^{-1}\| < \frac{1}{\|b\|}$ *such that* T *is a* C^**-algebra-valued* b*-expansive mapping on* X*. Then* T *has a unique fixed point in* X*.*

Proof. We will begin to prove this theorem by showing that T is injective. Let x, y be an element in X such that $x \neq y$ that is $d(x, y) \neq 0$. Assume that $Tx = Ty$. We have

$$\theta = d(Tx, Ty) \succeq \lambda^* d(x, y)\lambda = \lambda^* d(x, y)^{\frac{1}{2}} d(x, y)^{\frac{1}{2}} \lambda = (d(x, y)^{\frac{1}{2}}\lambda)^*(d(x, y)^{\frac{1}{2}}\lambda) \succeq \theta.$$

This implies that $\lambda^* d(x, y)\lambda = \theta$. Since λ is invertible, we have $d(x, y) = \theta$ which leads to contradiction. Thus T is injective. By the first condition of mapping T, we obtain that T is bijective which implies that T is invertibe and T^{-1} is bijective.

Next, we will show that T has a unique fixed point in X. In fact, since T is C^*-algebra-valued b-expansive and invertible mapping, we substitute x, y with $T^{-1}x, T^{-1}y$ in the second condition of T,respectively, which implies that

$$d(T(T^{-1}x), T(T^{-1}y)) \succeq \lambda^* d(T^{-1}x, T^{-1}y)\lambda, \quad \forall x, y \in X.$$

That is

$$d(x, y) \succeq \lambda^* d(T^{-1}x, T^{-1}y)\lambda, \quad \forall x, y \in X.$$

Since $d(x, y)$ and $\lambda^* d(T^{-1}x, T^{-1}y)\lambda$ are positive element in $\mathbb{A}$, $\lambda^* d(T^{-1}x, T^{-1}y) \preceq \lambda d(x, y)$ and $\lambda^{-1} \in \mathbb{A}$. By condition (2) of Theorem 2.7 and Theorem 2.12, we have

$$\begin{aligned} d(T^{-1}x, T^{-1}y) &= (\lambda\lambda^{-1})^* d(T^{-1}x, T^{-1}y)\lambda(\lambda^{-1}) \\ &= (\lambda^{-1})^* \lambda^* d(T^{-1}x, T^{-1}y)\lambda(\lambda^{-1}) \\ &\preceq (\lambda^{-1})^* d(x, y)\lambda^{-1}. \end{aligned}$$

Therefore T^{-1} is b-contractive mapping. Using Corollary 5.12, there exists a unique x such that $T^{-1}x = x$, which means there has a unique fixed point $x \in X$ such that $Tx = T(T^{-1}x) = (TT^{-1})x = Ix = x$. □

Theorem 4.5. *(Cyclic Kannan-Type) Let* A *and* B *be nonempty closed subset of a complete* C^**-algebra-valued* b*-metric space* $(X, \mathbb{A}, d)$*. Assume that* $T : A \cup B \to A \cup B$ *is cyclic mapping that satisfies*

$$d(Tx, Ty) \preceq \lambda[d(x, Tx) + d(y, Ty)], \quad \forall\, x \in A \ \text{ and } \ \forall y \in B$$

where $\lambda \in \mathbb{A}'_+$ *with* $\|\lambda\| < \frac{1}{2\|b\|}$*. Then* T *has a unique fixed point in* $A \cap B$*.*

Proof. Without loss of generality, we can assume that $\lambda \neq \theta$. Since $\lambda \in \mathbb{A}'_+$ and $\theta \preceq d(x,Tx) + d(y,Ty)$, by the second condition of Lemma 2.13, we have $\theta \preceq \lambda\{d(x,Tx) + d(y,Ty)\}$.

Let x_0 be any element in A. Since T is cyclic mapping, we have $Tx_0 \in B$ and $T^2x_0 \in A$. Consider,

$$\begin{aligned} d(Tx_0, T^2x_0) &= d(Tx_0, T(Tx_0)) \\ &\preceq \lambda[d(x_0, Tx_0) + d(Tx_0, T^2x_0)] \\ &= \lambda d(x_0, Tx_0) + \lambda d(Tx_0, T^2x_0), \end{aligned}$$

that is

$$(I-\lambda)d(Tx_0, T^2x_0) \preceq \lambda d(x_0, Tx_0)$$

Since $\lambda \in \mathbb{A}'_+$ and $\|\lambda\| < \frac{1}{2\|b\|} < \frac{1}{2}$, by the first condition of Lemma 2.13, we have $I-\lambda$ is invertible and $\|(I-\lambda)^{-1}\lambda\| < 1$. From the third condition of Lemma 2.13, we have

$$d(Tx_0, T^2x_0) \preceq (I-\lambda)^{-1}\lambda d(x_0, Tx_0).$$

Similarly, we get that

$$d(T^2x_0, T^3x_0) \preceq (I-\lambda)^{-1}\lambda d(Tx_0, T^2x_0).$$

Since $(I-\lambda)^{-1}\lambda \in \mathbb{A}'_+$ and $\theta \preceq (I-\lambda)^{-1}\lambda d(x_0, Tx_0) - d(Tx_0, T^2x_0)$, the second condition of Lemma 2.13, we have

$$\theta \preceq (I-\lambda)^{-1}\lambda\{(I-\lambda)^{-1}\lambda d(x_0, Tx_0) - d(Tx_0, T^2x_0)\}.$$

that is

$$(I-\lambda)^{-1}\lambda d(Tx_0, T^2x_0) \preceq [(I-\lambda)^{-1}\lambda]^2 d(x_0, Tx_0)$$

Hence

$$d(T^2x_0, T^3x_0) \preceq (I-\lambda)^{-1}\lambda d(Tx_0, T^2x_0) \preceq [(I-\lambda)^{-1}\lambda]^2 d(x_0, Tx_0).$$

Continue this proces, we have

$$d(T^nx_0, T^{n+1}x_0) \preceq [(I-\lambda)^{-1}\lambda]^n d(x_0, Tx_0) = \alpha^n\beta$$

where $\alpha = (I-\lambda)^{-1}\lambda$ and $\beta = d(x_0, Tx_0)$. Next, we will show that $\{T^nx_0\}$ is Cauchy sequence with respect to $\mathbb{A}$. Consider for any $m, n \in \mathbb{N}$ and $m \leq n$, we have

$$\begin{aligned} d(T^mx_0, T^nx_0) &\preceq bd(T^mx_0, T^{m+1}x_0) + b^2d(T^{m+1}x_0, T^{m+2}x_0) + \cdots + b^{n-m}d(T^{n-1}x_0, T^nx_0) \\ &\preceq b\alpha^m\beta + b^2\alpha^{m+1}\beta + \cdots + b^{n-m}\alpha^{n-1}\beta \\ &= \sum_{k=m}^{n-1} b^{k-m+1}\alpha^k\beta. \end{aligned}$$

From Theorem 2.11, we get that

$$\begin{aligned}
\|d(T^m x_0, T^n x_0)\| &\leq \|\sum_{k=m}^{n-1} b^{k-m+1}\alpha^k\beta\| \\
&\leq \sum_{k=m}^{n-1} \|b^{k-m+1}\alpha^k\beta\| \\
&\leq \sum_{k=m}^{n-1} \|b\|^{k-m+1}\|\alpha\|^k\|\beta\| \\
&\leq \sum_{k=m}^{n-1} \|b\|^k\|\alpha\|^k\|\beta\| \\
&= \|\beta\| \sum_{k=m}^{n-1} (\|b\|\|\alpha\|)^k \\
&\leq \|\beta\| \sum_{k=m}^{\infty} (\|b\|\|\alpha\|)^k \\
&= \|\beta\| \frac{(\|b\|\|\alpha\|)^m}{1-(\|b\|\|\alpha\|)}.
\end{aligned}$$

Consider,

$$\begin{aligned}
\|b\|\|\alpha\| &= \|b\|\|\lambda(I-\lambda)^{-1}\| \\
&\leq \|b\|\|\lambda\|\|(I-\lambda)^{-1}\| \\
&= \|b\|\|\lambda\|\|\sum_{i=0}^{\infty}(\lambda)^i\| \\
&\leq \|b\|\|\lambda\|\sum_{i=0}^{\infty}\|(\lambda)\|^i \\
&< \|b\|(\frac{1}{2\|b\|})\frac{1}{1-\|\lambda\|} \\
&< \frac{1}{2}\frac{1}{1-\frac{1}{2}} = 1.
\end{aligned}$$

Therefore $\|\beta\|\frac{(\|b\|\|\alpha\|)^{2m}}{1-(\|b\|\|\alpha\|)} \to 0$ as $m \to \infty$. Therefore $\{T^n x_0\}$ is Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, d)$, there exists an element $x \in X$ such that $x = \lim_{n\to\infty} T^n x_0$.

Since $\{T^{2n}x_0\}$ is a sequence in A and $\{T^{2n-1}x_0\}$ is a sequence in B, we obtain that both sequence converges to the same limit x. Since A and B are closed set imply $x \in A \cap B$. Next,

we will show that x is a unique fixed point of T. Consider,

$$\begin{aligned} d(Tx,x) &\preceq b[d(Tx,T^{2n}x_0)+d(T^{2n}x_0,x)] \\ &= bd(Tx,T(T^{2n-1}x_0))+bd(T^{2n}x_0,x) \\ &\preceq b\lambda[d(x,Tx)+d(T^{2n-1}x_0,T^{2n}x_0)]+bd(T^{2n}x_0,x) \\ &\preceq b\lambda d(x,Tx)+b^2\lambda d(T^{2n-1}x_0,x)+b^2\lambda d(x,T^{2n}x_0)+bd(T^{2n}x_0,x), \end{aligned}$$

by Theorem 2.11 and submultiplicative, we obtian that

$$\|d(Tx,x)\| \leq \|b\|\|\lambda\|\|d(x,Tx)\|+\|b\|^2\|\lambda\|\|d(T^{2n-1}x_0,x)\|+\|b\|^2\|\lambda\|\|d(x,T^{2n}x_0)\|+\|b\|\|d(T^{2n}x_0$$

Letting $n \to \infty$, we get that

$$\|d(Tx,x)\| \leq \|b\|\|\lambda\|\|d(x,Tx)\|,$$

and so

$$\|d(Tx,x)\| \leq \|b\|\frac{1}{2\|b\|}\|d(x,Tx)\| < \frac{1}{2}\|d(x,Tx)\|.$$

This implies that $\|d(Tx,x)\| = 0$, that is $d(Tx,x) = \theta$ and so $Tx = x$. i.e. x is fixed point of T. Now if y is another fixed point of T and $y \neq x$, then

$$\theta \preceq d(x,y) = d(Tx,Ty) \preceq \lambda(d(x,Tx)+,d(y,Ty)) = \lambda(d(x,x)+d(y,y)) = \theta,$$

which leads to contradiction. Therefore, $x = y$, we complete the proof. □

Example 4.6. Let $X = [-1,1]$ and $\mathbb{A} = M_{2\times 2}(\mathbb{R})$ with $\|A\| = \max_{i,j}|a_{ij}|$ where a_{ij} are entries of the matrix $A \in M_{2\times 2}(\mathbb{R})$. Then $(X,\mathbb{A},d)$ is a C^*-algebra-valued b-metric space with $b = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, where the involution is given by $A^* = (\overline{A})^T$,

$$d(x,y) = \begin{bmatrix} |x-y|^2 & 0 \\ 0 & |x-y|^2 \end{bmatrix}$$

and partial ordering on $\mathbb{A}$ is given as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \preceq \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \Leftrightarrow a_{ij} \leq b_{ij} \quad \text{for all} \quad i,j = 1,2,3,4.$$

Suppose that $A = [-1,0]$ and $B = [0,1]$. Define a mapping $T : A \cup B \to A \cup B$ by $Tx = -\frac{x}{4}$. Firstly, we will show that T is cyclic mapping. Let x be an element in A, that is

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$-1 \leq x \leq 0$. Then $0 \leq -\frac{x}{4} \leq 1$ imply $Tx \in B$. Similarly, let $y \in B$, so $0 \leq y \leq 1$. Then $-\frac{1}{4} \leq -\frac{y}{4} \leq 0$. Hence $Ty \in A$.

For any $x \in A$ and $y \in B$, we consider

$$\begin{aligned}
|Tx - Ty|^2 &= |\frac{-x}{4} - \frac{-y}{4}|^2 \\
&= \frac{1}{16}|x - y|^2 \\
&\leq \frac{1}{16}(|x| + |y|)^2 \\
&\leq \frac{1}{16}(2|x|^2 + 2|y|^2) \\
&= \frac{2}{25}(\frac{25}{16}|x|^2 + \frac{25}{16}|y|^2) \\
&= \frac{2}{25}(|x + \frac{x}{4}|^2 + |y + \frac{y}{4}|^2) \\
&= \frac{2}{25}(|x - Tx|^2 + |y - Ty|^2).
\end{aligned}$$

Then, we have

$$\begin{aligned}
d(Tx, Ty) &= \begin{bmatrix} |Tx - Ty|^2 & 0 \\ 0 & |Tx - Ty|^2 \end{bmatrix} \\
&\preceq \begin{bmatrix} \frac{2}{25}(|x - Tx|^2 + |y - Ty|^2) & 0 \\ 0 & \frac{2}{25}(|x - Tx|^2 + |y - Ty|^2) \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{25} & 0 \\ 0 & \frac{2}{25} \end{bmatrix} \begin{bmatrix} (|x - Tx|^2 + |y - Ty|^2) & 0 \\ 0 & (|x - Tx|^2 + |y - Ty|^2) \end{bmatrix} \\
&= \lambda[d(x, Tx) + d(y, Ty)]
\end{aligned}$$

where $\lambda = \begin{bmatrix} \frac{2}{25} & 0 \\ 0 & \frac{2}{25} \end{bmatrix}$. Then $\|\lambda\| = \frac{2}{25} < \frac{1}{4} = \frac{1}{2\|b\|}$. Thus T satisfies contraction of 5.14 imply that T has a unique fixed point in $A \cap B$, i.e $\{0\} = F(T)$.

Theorem 4.7. *(Cyclic Chatterjea-Type) Let A and B be nonempty closed subset of a complete C^*-algebra-valued b-metric space $(X, \mathbb{A}, d)$. Assume that $T : A \cup B \to A \cup B$ is cyclic mapping that satisfies*

$$d(Tx, Ty) \preceq \lambda[d(y, Tx) + d(x, Ty)], \quad \forall\, x \in A \ \text{ and } \ \forall y \in B$$

where $\lambda \in \mathbb{A}'_+$ with $\|\lambda\| < \frac{1}{2\|b\|^2}$. Then T has a unique fixed point in $A \cap B$.

Proof. Without loss of generality, we can assume that $\lambda \neq \theta$. Since $\lambda \in \mathbb{A}'_+$ and $\theta \preceq d(y,Tx) + d(x,Ty)$, by the second condition of Lemma 2.13, we have $\theta \preceq \lambda\{d(y,Tx) + d(x,Ty)\}$.

Let x_0 be any element in A, Since T is cyclic mapping, we have $Tx_0 \in B$ and $T^2x_0 \in A$. Consider,

$$\begin{aligned} d(Tx_0, T^2x_0) &= d(Tx_0, T(Tx_0)) \\ &\preceq \lambda[d(Tx_0, Tx_0) + d(x_0, T^2x_0)] \\ &\preceq b\lambda[d(x_0, Tx_0) + d(Tx_0, T^2x_0)], \end{aligned}$$

that is

$$(I - b\lambda)d(Tx_0, T^2x_0) \preceq b\lambda d(x_0, Tx_0)$$

Since $\lambda \in \mathbb{A}'_+$ and $b \in \mathbb{A}'_+$, From the second condition of Lemma 2.13, we get that $b\lambda \in \mathbb{A}'_+$. Since $\|b\lambda\| < \|b\|\frac{1}{2\|b\|^2} \leq \frac{1}{2}$ and $b\lambda \in \mathbb{A}'_+$, by the first condition of Lemma 2.13, we have $(I - b\lambda)^{-1} \in \mathbb{A}'_+$ and $(b\lambda)(I - b\lambda)^{-1} \in \mathbb{A}'_+$ with $\|(b\lambda)(I - b\lambda)^{-1}\| < 1$. From the third condition of Lemma 2.13, we have

$$d(Tx_0, T^2x_0) \preceq (b\lambda)(I - b\lambda)^{-1}d(x_0, Tx_0).$$

Similarly, we get that

$$d(T^2x_0, T^3x_0) \preceq (b\lambda)(I - b\lambda)^{-1}d(Tx_0, T^2x_0).$$

Since $(b\lambda)(I - b\lambda)^{-1} \in \mathbb{A}'_+$ and $\theta \preceq (b\lambda)(I - b\lambda)^{-1}d(x_0, Tx_0) - d(Tx_0, T^2x_0)$, the second condition of Lemma 2.13, we have

$$\theta \preceq (b\lambda)(I - b\lambda)^{-1}\{(b\lambda)(I - b\lambda)^{-1}d(x_0, Tx_0) - d(Tx_0, T^2x_0)\}.$$

that is

$$(b\lambda)(I - b\lambda)^{-1}d(Tx_0, T^2x_0) \preceq [(b\lambda)(I - b\lambda)^{-1}]^2 d(x_0, Tx_0)$$

Hence

$$d(T^2x_0, T^3x_0) \preceq (b\lambda)(I - b\lambda)^{-1}d(Tx_0, T^2x_0) \preceq [(b\lambda)(I - b\lambda)^{-1}]^2 d(x_0, Tx_0).$$

Continue this proces, we have

$$d(T^nx_0, T^{n+1}x_0) \preceq [(b\lambda)(I - b\lambda)^{-1}]^n d(x_0, Tx_0) = \omega^n\beta$$

where $\omega = (b\lambda)(I - b\lambda)^{-1}$ and $\beta = d(x_0, Tx_0)$. Next, we will show that $\{T^n x_0\}$ is Cauchy sequence with respect to $\mathbb{A}$. Consider for any $m, n \in \mathbb{N}$ and $m \leq n$, we have

$$\begin{aligned} d(T^m x_0, T^n x_0) &\preceq bd(T^m x_0, T^{m+1} x_0) + b^2 d(T^{m+1} x_0, T^{m+2} x_0) + \cdots + b^{n-m} d(T^{n-1} x_0, T^n x_0) \\ &\preceq b\omega^m \beta + b^2 \omega^{m+1} \beta + \cdots + b^{n-m} \omega^{n-1} \beta \\ &= \sum_{k=m}^{n-1} b^{k-m+1} \omega^k \beta. \end{aligned}$$

From Theorem 2.11, we get that

$$\begin{aligned} \|d(T^m x_0, T^n x_0)\| &\leq \| \sum_{k=m}^{n-1} b^{k-m+1} \omega^k \beta \| \\ &\leq \sum_{k=m}^{n-1} \| b^{k-m+1} \omega^k \beta \| \\ &\leq \sum_{k=m}^{n-1} \|b\|^{k-m+1} \|\omega\|^k \|\beta\| \\ &\leq \sum_{k=m}^{n-1} \|b\|^k \|\omega\|^k \|\beta\| \\ &= \|\beta\| \sum_{k=m}^{n-1} (\|b\| \|\omega\|)^k \\ &= \|\beta\| \sum_{k=m}^{\infty} (\|b\| \|\omega\|)^k \\ &= \|\beta\| \frac{(\|b\| \|\omega\|)^m}{1 - (\|b\| \|\omega\|)}. \end{aligned}$$

Consider,

$$\begin{aligned} \|b\| \|\omega\| &= \|b\| \|b\lambda (I - b\lambda)^{-1}\| \\ &\leq \|b\| \|b\lambda\| \|(I - b\lambda)^{-1}\| \\ &= \|b\| \|b\lambda\| \| \sum_{i=0}^{\infty} (b\lambda)^i \| \\ &\leq \|b\| \|b\lambda\| \sum_{i=0}^{\infty} \|(b\lambda)\|^i \\ &< \|b\| (\frac{\|b\|}{2\|b\|^2}) \frac{1}{1 - \|b\lambda\|} \\ &< \frac{1}{2} \frac{1}{1 - \frac{1}{2}} = 1, \end{aligned}$$

Therefore $\|\beta\|\frac{(\|b\|\|\omega\|)^{2m}}{1-(\|b\|\|\omega\|)} \to 0$ as $m \to \infty$. Therefore $\{T^n x_0\}$ is Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, d)$, there exists an element $x \in X$ such that $x = \lim_{n\to\infty} T^n x_0$.

Since $\{T^{2n}x_0\}$ is a sequence in A and $\{T^{2n-1}x_0\}$ is a sequence in B, we obtain that both sequence converges to the same limit x. Since A and B are closed set imply $x \in A \cap B$.

Next, we will complete the proof by showing that x is a unique fixed point of T. Since

$$\begin{aligned} d(Tx, x) &\preceq b[d(Tx, T^{2n}x_0) + d(T^{2n}x_0, x)] \\ &= bd(Tx, T(T^{2n-1}x_0)) + bd(T^{2n}x_0, x) \\ &\preceq b\lambda[d(x, T^{2n}x_0) + d(T^{2n-1}x_0, Tx)] + bd(T^{2n}x_0, x) \\ &= b\lambda d(x, T^{2n}x_0) + b\lambda d(T^{2n-1}x_0, Tx) + bd(T^{2n}x_0, x) \\ &\preceq b\lambda d(x, T^{2n}x_0) + b^2\lambda d(T^{2n-1}x_0, x) + b^2\lambda d(x, Tx) + bd(T^{2n}x_0, x), \end{aligned}$$

by Theorem 2.11, we have

$$\|d(Tx, x)\| \leq \|b\|\|\lambda\|\|d(x, T^{2n}x_0)\| + \|b\|^2\|\lambda\|\|d(T^{2n-1}x_0, x)\| + \|b\|^2\|\lambda\|\|d(x, Tx)\| + \|b\|\|d(T^{2n}x_0$$

Letting $n \to \infty$, we get that

$$\|d(Tx, x)\| \leq \|b\|^2\|\lambda\|\|d(x, Tx)\|,$$

and so

$$\|d(Tx, x)\| \leq \|b\|^2 \frac{1}{2\|b\|^2}\|d(x, Tx)\| < \frac{1}{2}\|d(x, Tx)\|.$$

This implies that $\|d(Tx, x)\| = 0$, that is $d(Tx, x) = \theta$ and so $Tx = x$. i.e. x is fixed point of T. Now if y is another fixed point of T and $y \neq x$, then

$$\theta \preceq d(x, y) = d(Tx, Ty) \preceq \lambda(d(y, Tx) +, d(x, Ty)) = 2\lambda d(x, y),$$

From Theorem 2.11, we get that

$$\|d(x, y)\| \leq \|2\lambda d(x, y)\| \leq 2\|\lambda\|\|d(x, y)\| < 2(\frac{1}{2\|b^2\|})\|d(x, y)\| \leq \|d(x, y)\|,$$

which leads to a contradiction. Therefore $x = y$ which implies that the fixed point is unique.

□

Example 4.8. Let $X = [0, 1]$ and $\mathbb{A} = M_{2\times 2}(\mathbb{R})$ with $\|A\| = \max_{i,j} |a_{ij}|$ where a_{ij} are entries of the matrix $A \in M_{2\times 2}(\mathbb{R})$. Then $(X, \mathbb{A}, d)$ is a C^*-algebra-valued b-metric space

with $b = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, where the involution is given by $A^* = (\overline{A})^T$,

$$d(x,y) = \begin{bmatrix} |x-y|^2 & 0 \\ 0 & |x-y|^2 \end{bmatrix}$$

and partial ordering on $\mathbb{A}$ is given as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \preceq \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \Leftrightarrow a_{ij} \leq b_{ij} \quad \text{for all} \quad i,j = 1,2,3,4.$$

Suppose that $A = [0,1]$ and $B = [0, \frac{1}{2}]$. Define a mapping $T : A \cup B \to A \cup B$ by $Tx = \frac{x}{5}$. Firstly, we will show that T is cyclic mapping. Let $x \in A$, that is $0 \leq x \leq 1$. Then $0 \leq \frac{x}{5} \leq \frac{1}{5}$ imply $Tx \in B$. Similarly, let $y \in B$, so $0 \leq y \leq \frac{1}{2}$. Then $0 \leq \frac{y}{5} \leq \frac{1}{10}$. Hence $Ty \in A$.

Now, we will show that T satisfies the contraction of 5.16. Consider,

$$\frac{(x-y)}{5} = \frac{1}{6}\frac{(6(x-y))}{5} = \frac{1}{6}(x - \frac{y}{5} + \frac{x}{5} - y)$$

and so

$$\begin{aligned} \left(\frac{(x-y)}{5}\right)^2 &= \left(\frac{1}{6}(x - \frac{y}{5} + \frac{x}{5} - y)\right)^2 \\ &= \frac{1}{36}\left((x - \frac{y}{5}) + (\frac{x}{5} - y)\right)^2 \\ &\leq \frac{1}{36}\left(2(x - \frac{y}{5})^2 + 2(\frac{x}{5} - y)^2\right) \\ &= \frac{1}{18}\left(|x - Ty|^2 + |Tx - y|^2\right) \end{aligned}$$

Then, we have

$$\begin{aligned} d(Tx,Ty) &= \begin{bmatrix} |Tx-Ty|^2 & 0 \\ 0 & |Tx-Ty|^2 \end{bmatrix} \\ &\preceq \begin{bmatrix} \frac{1}{18}\left(|x - Ty|^2 + |Tx - y|^2\right) & 0 \\ 0 & \frac{1}{18}\left(|x - Ty|^2 + |Tx - y|^2\right) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{18} & 0 \\ 0 & \frac{1}{18} \end{bmatrix} \begin{bmatrix} \left(|x - Ty|^2 + |Tx - y|^2\right) & 0 \\ 0 & \left(|x - Ty|^2 + |Tx - y|^2\right) \end{bmatrix} \\ &= \lambda[d(x,Ty) + d(y,Tx)] \end{aligned}$$

where $\lambda = \begin{bmatrix} \frac{1}{18} & 0 \\ 0 & \frac{1}{18} \end{bmatrix}$. Then $\|\lambda\| = \frac{1}{18} < \frac{1}{8} = \frac{1}{2\|b\|^2}$. Thus T satisfies contraction of 5.16 imply that T has a unique fixed point in $A \cap B$.

Chapter 5

Conclusion

5.1 Fundamental properties of C^*-algebra-valued b-metric spaces

Theorem 5.1. *Let $(X, \mathbb{A}, d)$ be C^*-algebra-valued b-metric space. If $\{x_n\}$ is a convergent sequence with respect to $\mathbb{A}$, then $\{x_n\}$ is Cauchy sequence with respect to $\mathbb{A}$.*

Definition 5.2. A subset S of a C^*-algebra-valued b-metric space $(X, \mathbb{A}, d)$ is bounded with respect to $\mathbb{A}$ if there exists $\overline{x} \in X$ and a nonnegetive real numbers M such that

$$\|d(x, \overline{x})\| \leq M, \quad \text{for all} \quad x \in X.$$

Theorem 5.3. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space and let $\{x_n\}$ be a sequence in X and $x \in X$ Then :*

1. *$x_n \to x$ if and only if $d(x_n, x) \to \theta$,*
2. *A convergent sequence in X is bounded with respect to $\mathbb{A}$ and its limit is unique,*
3. *A Cauchy sequence in X is bounded with respect to $\mathbb{A}$.*

Theorem 5.4. *Let $\{x_n\}$ be a convergent sequence in a C^*-algebra-valued b-metric space $(X, \mathbb{A}, d)$ and $\lim_{n\to\infty} x_n = x$. Then, every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is convergent and has the same limit x.*

Theorem 5.5. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space. Then every subsequence of a Cauchy sequence is Cauchy sequence.*

Theorem 5.6. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space and let $\{x_n\}$ be a Cauchy sequence with respect to $\mathbb{A}$. If $\{x_n\}$ contains its convergent subsequence, then $\{x_n\}$ is convergent sequence.*

Theorem 5.7. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space. Suppose that $\{x_n\}$ and $\{y_n\}$ are convergent with respect to $\mathbb{A}$ and converge to x and y, respectively. Then $d(x_n, y_n)$ converges to $b^2 d(x, y)$.*

Theorem 5.8. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space. Suppose that $\{x_n\}$ and $\{y_n\}$ are convergent with respect to $\mathbb{A}$ and converge to x and y, respectively. Then,*

$$\frac{1}{\|b\|^2}\|d(x,y)\| \leq \liminf_{n\to\infty}\|d(x_n,y_n)\| \leq \limsup_{n\to\infty}\|d(x_n,y_n)\| \leq \|b\|^2\|d(x,y)\|.$$

In particular, if $x = y$, then we have $\lim_{n\to\infty}\|d(x_n,y_n)\| = 0$. Moreover for any $z \in X$, we have

$$\frac{1}{\|b\|}\|d(x,z)\| \leq \liminf_{n\to\infty}\|d(x_n,z)\| \leq \limsup_{n\to\infty}\|d(x_n,z)\| \leq \|b\|\|d(x,z)\|$$

Definition 5.9. Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space. A subset F of $(X, \mathbb{A}, d)$ is called a closed set if a sequence $\{x_n\}$ in X and $x_n \to x$ with respect to $\mathbb{A}$ imply $x \in F$.

5.2 Fixed point theorem for cyclic contractions

Theorem 5.10. *Let A and B be nonempty closed subset of a complete C^*-algebra-valued b-metric space $(X, \mathbb{A}, d)$. Assume that $T : A \cup B \to A \cup B$ is cyclic mapping that satisfies*

$$d(Tx,Ty) \preceq \lambda^* d(x,y)\lambda, \quad \forall\, x \in A \ \text{ and } \ \forall y \in B$$

where $\lambda \in \mathbb{A}$ with $\|\lambda\| < \frac{1}{\|b\|}$. Then T has a unique fixed point in $A \cap B$.

Example 5.11. Let X be a set of real numbers and $\mathbb{A} = M_{2\times 2}(\mathbb{R})$ with $\|A\| = \max_{i,j}|a_{ij}|$ where a_{ij} are entries of the matrix $A \in M_{2\times 2}(\mathbb{R})$. Then $(X, \mathbb{A}, d)$ is a C^*-algebra-valued b-metric space with $b = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, where the involution is given by $A^* = (\overline{A})^T$,

$$d(x,y) = \begin{bmatrix} |x-y|^2 & 0 \\ 0 & |x-y|^2 \end{bmatrix}$$

and partial ordering on $\mathbb{A}$ is given as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \preceq \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \Leftrightarrow a_{ij} \leq b_{ij} \quad \text{for all} \quad i,j = 1,2,3,4.$$

Define a mapping $T : X \to X$ by

$$Tx = \begin{cases} -\frac{x+\frac{1}{3}}{3}|\sin(\frac{1}{x})| - \frac{1}{3} & ; \ x \in (\infty, -\frac{1}{3}] \\ -\frac{1}{3} & ; \ x \in (-\frac{1}{3}, 0] \\ -\frac{1}{2} & ; \ x \in (0, +\infty) \end{cases}$$

It clear that T is not continuous at all element of X. Therefore Theorem 2.16 can not imply the existence of fixed point of the mapping T.

Suppose that $A = [-\frac{1}{2}, -\frac{1}{3}]$ and $B = [-\frac{1}{3}, 0]$. Then T has a unique fixed point in $A \cap B$, i.e $\{-\frac{1}{3}\} = F(T)$.

Corollary 5.12. *Suppose that $(X, \mathbb{A}, d)$ is a C^*-algebra-valued b-metric space. Assume that $T : X \to X$ is called a C^*-algebra-valued b-contractive mapping on X, that is T satisfies*

$$d(Tx, Ty) \preceq \lambda^* d(x, y)\lambda, \quad \forall\ x, y \in X$$

where $\lambda \in \mathbb{A}$ with $\|\lambda\| < \frac{1}{\|b\|}$. Then T has a unique fixed point in X.

Theorem 5.13. *Suppose that $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued b-metric space. Assume that a mapping $T : X \to X$ satisfies*

(1) $T(X) = X$;

(2) $d(Tx, Ty) \succeq \lambda^* d(x, y)\lambda$ *for all* $x, y \in X$

where $\lambda \in \mathbb{A}$ is an invertible element and $\|\lambda^{-1}\| < \frac{1}{\|b\|}$ such that T is a C^*-algebra-valued b-expansive mapping on X. Then T has a unique fixed point in X.

Theorem 5.14. *(Cyclic Kannan-Type) Let A and B be nonempty closed subset of a complete C^*-algebra-valued b-metric space $(X, \mathbb{A}, d)$. Assume that $T : A \cup B \to A \cup B$ is cyclic mapping that satisfies*

$$d(Tx, Ty) \preceq \lambda[d(x, Tx) + d(y, Ty)], \quad \forall\ x \in A \ \text{ and } \ \forall y \in B$$

where $\lambda \in \mathbb{A}'_+$ with $\|\lambda\| < \frac{1}{2\|b\|}$. Then T has a unique fixed point in $A \cap B$.

Example 5.15. Let $X = [-1, 1]$ and $\mathbb{A} = M_{2\times 2}(\mathbb{R})$ with $\|A\| = \max_{i,j} |a_{ij}|$ where a_{ij} are entries of the matrix $A \in M_{2\times 2}(\mathbb{R})$. Then $(X, \mathbb{A}, d)$ is a C^*-algebra-valued b-metric space with $b = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, where the involution is given by $A^* = (\overline{A})^T$,

$$d(x, y) = \begin{bmatrix} |x - y|^2 & 0 \\ 0 & |x - y|^2 \end{bmatrix}$$

and partial ordering on $\mathbb{A}$ is given as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \preceq \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \Leftrightarrow a_{ij} \leq b_{ij} \quad \text{for all} \quad i, j = 1, 2, 3, 4.$$

Suppose that $A = [-1, 0]$ and $B = [0, 1]$. Define a mapping $T : A \cup B \to A \cup B$ by $Tx = -\frac{x}{4}$. Then T has a unique fixed point in $A \cap B$, i.e $\{0\} = F(T)$.

Theorem 5.16. *(Cyclic Chatterjea-Type) Let A and B be nonempty closed subset of a complete C^*-algebra-valued b-metric space $(X, \mathbb{A}, d)$. Assume that $T : A \cup B \to A \cup B$ is cyclic mapping that satisfies*

$$d(Tx, Ty) \preceq \lambda[d(y, Tx) + d(x, Ty)], \quad \forall\, x \in A \quad \text{and} \quad \forall y \in B$$

where $\lambda \in \mathbb{A}'_+$ with $\|\lambda\| < \frac{1}{2\|b\|^2}$. Then T has a unique fixed point in $A \cap B$.

Example 5.17. Let $X = [0, 1]$ and $\mathbb{A} = M_{2\times 2}(\mathbb{R})$ with $\|A\| = \max_{i,j} |a_{ij}|$ where a_{ij} are entries of the matrix $A \in M_{2\times 2}(\mathbb{R})$. Then $(X, \mathbb{A}, d)$ is a C^*-algebra-valued b-metric space with $b = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, where the involution is given by $A^* = (\overline{A})^T$,

$$d(x, y) = \begin{bmatrix} |x - y|^2 & 0 \\ 0 & |x - y|^2 \end{bmatrix}$$

and partial ordering on $\mathbb{A}$ is given as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \preceq \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \Leftrightarrow a_{ij} \leq b_{ij} \quad \text{for all} \quad i, j = 1, 2, 3, 4.$$

Suppose that $A = [0, 1]$ and $B = [0, \frac{1}{2}]$. Define a mapping $T : A \cup B \to A \cup B$ by $Tx = \frac{x}{5}$. Then T has a unique fixed point in $A \cap B$.

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Research Article

Fixed Point Theorems for Cyclic Contractions in C^*-Algebra-Valued b-Metric Spaces

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We study fundamental properties of C^*-algebra-valued b-metric space which was introduced by Ma and Jiang (2015) and give some fixed point theorems for cyclic mapping with contractive and expansive condition on such space analogous to the results presented in Ma and Jiang, 2015.

1. Introduction

Firstly, we begin with the basic concept of C^*-algebras. A real or a complex linear space $\mathbb{A}$ is algebra if vector multiplication is defined for every pair of elements of $\mathbb{A}$ satisfying two conditions such that $\mathbb{A}$ is a ring with respect to vector addition and vector multiplication and for every scalar α and every pair of elements $x, y \in \mathbb{A}$, $\alpha(xy) = (\alpha x)y = x(\alpha y)$. A norm $\|\cdot\|$ on $\mathbb{A}$ is said to be submultiplicative if $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathbb{A}$. In this case $(\mathbb{A}, \|\cdot\|)$ is called normed algebra. A complete normed algebra is called Banach algebra. An involution on algebra $\mathbb{A}$ is conjugate linear map $a \mapsto a^*$ on $\mathbb{A}$ such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathbb{A}$. $(A, *)$ is called $*$-algebra. A Banach $*$-algebra $\mathbb{A}$ is $*$-algebra $\mathbb{A}$ with a complete submultiplicative norm such that $\|a^*\| = \|a\|$ for all $a \in \mathbb{A}$. C^*-algebra is Banach $*$-algebra such that $\|a^*a\| = \|a\|^2$. There are many examples of C^*-algebra, such as the set of complex numbers, the set of all bounded linear operators on a Hilbert space H, $L(H)$, and the set of $n \times n$-matrices, $M_n(\mathbb{C})$. If a normed algebra $\mathbb{A}$ admits a unit I, $aI = Ia = a$ for all $a \in \mathbb{A}$, and $\|I\| = 1$, we say that $\mathbb{A}$ is a unital normed algebra. A complete unital normed algebra $\mathbb{A}$ is called unital Banach algebra. For properties in C^*-algebras, we refer to [1–3] and the references therein.

Let (X, d) be a complete metric space. The well-known Banach's contraction principle, which appeared in the Ph.D. dissertation of S. Banach in 1920, runs as follows: a mapping $T : X \to X$ is said to be a contraction if there exists $r \in [0, 1)$ such that

$$d(Tx, Ty) \leq rd(x, y) \quad \forall x, y \in X. \tag{1}$$

Then, T has a unique fixed point in X which was published in 1922 [4]. Banach's contraction principle has become one of the most important tools used for the existence of solutions of many nonlinear problems in many branches of science and has been extensively studied in many spaces which are more general than metric space by serveral mathematictians; see, for example, quasimetric spaces [5, 6], dislocated metric spaces [7], dislocated quasimetric spaces [8], G-metric spaces [9–11], b-metric spaces [12–14], metric-type spaces [15, 16], metric-like spaces [17], b-metric-like spaces (or dislocated b-metric spaces) [18, 19], quasi b-metric spaces [20], and dislocated quasi-b-metric spaces [21]. Note that the Banach contraction principle requires that mapping T satisfies the contractive condition that each point of $X \times X$ and ranges of T are positive real numbers. Consider the operator equation

$$X - \sum_{n=1}^{\infty} L_n^* X L_n = Q, \tag{2}$$

where $\{L_1, L_2, \ldots, L_n\}$ is subset of the set of linear bounded operators on Hilbert space H, $X \in L(H)$, and $Q \in L(H)_+$ is positive linear bounded operators on Hilbert space H. Then,

we convert the operator equation to the mapping $F : L(H) \to L(H)$ which is defined by

$$F(X) = \sum_{n=1}^{\infty} L_n^* X L_n + Q. \tag{3}$$

Observe that the range of mapping F is not real numbers but it is linear bounded operators on Hilbert space H. Therefore, the Banach contraction principle can not be applied with this problem. Afterward, does such mapping have a fixed point which is equivalent to the solution of operator equation? In 2014, Ma et al. [22] introduced new spaces, called C^*-algebra-valued metric spaces, which are more general than metric space, replacing the set of real numbers by C^*-algebras, and establish a fixed point theorem for self-maps with contractive or expansive conditions on such spaces, analogous to the Banach contraction principle. As applications, existence and uniqueness results for a type of integral equation and operator equation are given and were able to solve the above problem if $L_1, L_2, \ldots, L_n \in L(H)$ satisfy $\sum_{n=1}^{\infty} \|L_n\|^2 < 1$.

Later, many authors extend and improve the result of Ma et al. For example, in [23], Batul and Kamran generalized the notation of C^*-valued contraction mappings by weakening the contractive condition introduced by Ma et al. (the mapping is called C^*-valued contractive type mappings) and establish a fixed point theorem for such mapping which is more generalized than the result of Ma et al.; in [24], Shehwar and Kamran extend and improve the result of Ma et al. [22] and Jachymski [25] by proving a fixed point theorem for self-mappings on C^*-valued metric spaces satisfying the contractive condition for those pairs of elements from the metric space which form edges of a graph in the metric space. In 2015, Ma and Jiang [26] introduced a concept of C^*-algebra-valued b-metric spaces which generalize an ordinary C^*-algebra-valued metric space and give some fixed point theorems for self-map with contractive condition on such spaces. As applications, existence and uniqueness results for a type of operator equation and an integral equation are given.

Generally, in order to use the Banach contraction principle, a self-mapping T must be Lipschitz continuous, with the Lipschitz constant $r \in [0, 1)$. In particular, T must be continuous at all elements of its domain. That is one major drawback. Next, many authors could find contractive conditions which imply the existence of fixed point in complete metric space but not imply continuity. We refer to [27, 28] (Kannan-type mappings) and [29] (Chatterjea-type mapping).

Theorem 1 (see [27]). *If (X, d) is a complete metric space and mapping $T : X \to X$ satisfies*

$$d(Tx, Ty) \le r\left[d(x, Tx) + d(y, Ty)\right], \tag{4}$$

where $0 \le r < 1/2$ and $x, y \in X$, then T has a unique fixed point.

Theorem 2 (see [29]). *If (X, d) is a complete metric space and mapping $T : X \to X$ satisfies*

$$d(Tx, Ty) \le r\left[d(x, Ty) + d(y, Tx)\right], \tag{5}$$

where $0 \le r < 1/2$ and $x, y \in X$, then T has a unique fixed point.

In 2003, Kirk et al. [30] introduced the following notation of a cyclic representation and characterized the Banach contraction principle in context of a cyclic mapping as follows.

Theorem 3. *Let $A_1, A_2, \ldots, A_m$ be nonempty closed subsets of a complete metric space (X, d). Assume that a mapping $T : \bigcup_{i=1}^{m} A_i \to \bigcup_{i=1}^{m} A_i$ satisfies the following conditions:*

(i) *$T(A_i) \subseteq A_{i+1}$ for all $1 \le i \le m$ and $A_{m+1} = A_1$.*

(ii) *There exists $k \in [0, 1)$ such that $d(Tx, Ty) \le kd(x, y)$ for all $x \in A_i$ and $y \in A_{i+1}$ for $1 \le i \le m$.*

Then, T has a unique fixed point.

In 2011, Karapinar and Erhan [31] introduced Kannan-type cyclic contraction and Chatterjea-type cyclic contraction. Moreover, they derive some fixed point theorems for such cyclic contractions in complete metric spaces as follows.

Theorem 4 (fixed point theorem for Kannan-type cyclic contraction). *Let A and B be nonempty subsets of metric spaces (X, d) and a cyclic mapping $T : A \cup B \to A \cup B$ satisfies*

$$d(Tx, Ty) \le k\left[d(x, Tx) + d(y, Ty)\right], \quad \forall x \in A,\ y \in B, \tag{6}$$

where $0 \le k < 1/2$. Then, T has a unique fixed point in $A \cap B$.

Theorem 5 (fixed point theorem for Chatterjea-type cyclic contraction). *Let A and B be nonempty subsets of a metric spaces (X, d) and a cyclic mapping $T : A \cup B \to A \cup B$ satisfies*

$$d(Tx, Ty) \le k\left[d(x, Ty) + d(y, Tx)\right], \quad \forall x \in A,\ y \in B, \tag{7}$$

where $0 \le k < 1/2$. Then, T has a unique fixed point in $A \cap B$.

The purpose of this paper is to study fundamental properties of C^*-algebra-valued b-metric space which was introduced by Ma and Jiang [26] and give some fixed point theorems for cyclic mapping with contractive and expansive condition on such space analogous to the results presented in [26].

2. Preliminaries

In this section, we recollect some basic notations, defintions, and results that will be used in main result. Firstly, we begin with the concept of b-metric spaces.

Definition 6 (see [12, 13]). Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}$ is called b-metric if there exists a real number $b \ge 1$ such that, for every $x, y, z \in X$, we have

(i) $d(x, y) \ge 0$,

(ii) $d(x, y) = 0$ if and only if $x = y$,

(iii) $d(x, y) = d(y, x)$,

(iv) $d(x, z) \leq b[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space.

The class of b-metric spaces is larger than the calass of metric spaces, since a b-metric space is a metric when $b = 1$ in the fourth condition in the above definition. There exist many examples in some work showing that the class of b-metric is efficiently larger than those metric spaces (see also [12, 14, 32, 33]).

Example 7 (see [12]). The set $l_p(\mathbb{R})$ with $0 < p < 1$, where $l_p(\mathbb{R}) := \{\{x_n\} \subseteq \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$, together with the function $d : l_p(\mathbb{R}) \times l_p(\mathbb{R}) \to \mathbb{R}$,

$$d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{1/p}, \tag{8}$$

where $x = \{x_n\}, y = \{y_n\} \in l_p(\mathbb{R})$, is a b-metric space with coefficient $b = 2^{1/p} > 1$. Observe that the result holds for the general case $l_p(X)$ with $0 < p < 1$, where X is a Banach space.

Example 8 (see [12]). The space L_p $(0 < p < 1)$ of all real functions $x(t), t \in [0, 1]$, such that $\int_0^1 |x(t)|^p dt < \infty$, together with the function

$$d(x, y) = \left(\int_0^1 |x(t) - y(t)|^p \, dt\right)^{1/p}, \quad \forall x, y \in L_p[0, 1], \tag{9}$$

is a b-metric space with $b = 2^{1/p}$.

Example 9 (see [33]). Let (X, d_1) be a metric space and $d_2(x, y) = (d_1(x, y))^p$, where $p > 1$ is natural numbers. Then, d_2 is a b-metric with $b = 2^{p-1}$.

The notation convergence, compactness, closedness, and completeness in b-metric space are given in the same way as in metric space.

Next, we give concept of spectrum of element in C^*-algebra $\mathbb{A}$.

Definition 10 (see [3]). We say that $a \in \mathbb{A}$ is invertible if there is an element $b \in \mathbb{A}$ such that $ab = ba = I$. In this case, b is unique and written a^{-1}. The set

$$\mathrm{Inv}(\mathbb{A}) = \{a \in \mathbb{A} \mid a \text{ is invertible}\} \tag{10}$$

is a group under multiplication. We define spectrum of an element a to be the set

$$\sigma(a) = \sigma_A(a) = \{\lambda \in \mathbb{C} \mid \lambda I - a \notin \mathrm{Inv}(\mathbb{A})\}. \tag{11}$$

Theorem 11 (see [3]). *Let $\mathbb{A}$ be a unital Banach algebra and let a be an element of $\mathbb{A}$ such that $\|a\| < 1$. Then, $I - a \in \mathrm{Inv}(\mathbb{A})$ and*

$$(I - a)^{-1} = \sum_{n=0}^{\infty} a^n. \tag{12}$$

Theorem 12 (see [3]). *Let $\mathbb{A}$ be a unital C^*-algebra with a unit I, then*

(1) $I^* = I$,

(2) *For any $a \in \mathrm{Inv}(\mathbb{A})$, $(a^*)^{-1} = (a^{-1})^*$.*

(3) *For any $a \in \mathbb{A}$, $\sigma(a^*) = \sigma(a)^* = \{\overline{\lambda} \in \mathbb{C} : \lambda \in \sigma(a)\}$.*

All over this paper, $\mathbb{A}$ means a unital C^*-algebra with a unit I. $\mathbb{R}$ is set of real numbers and $\mathbb{R}_+$ is the set of nonnegetive real numbers. $M_n(\mathbb{R})$ is $n \times n$ matrix with entries $\mathbb{R}$.

Definition 13 (see [3]). The set of hermitain elements of $\mathbb{A}$ is denoted by $\mathbb{A}_h$; that is, $\mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$. An element x in $\mathbb{A}$ is positive element which is denoted by $\theta \preceq x$, where θ means the zero element in $\mathbb{A}$ if and only if $x \in \mathbb{A}_h$ and $\sigma(x)$ is a subset of nonnegative real numbers. We define a partial ordering $\mathbb{A}_h$ by using definition of positive element as $x \preceq y$ if and only if $y - x \succeq \theta$. The set of positive elements in $\mathbb{A}$ is denoted by $\mathbb{A}_+ = \{x \in \mathbb{A} : x \succeq \theta\}$.

The following are definitions and some properties of positive element of a C^*-algebra $\mathbb{A}$.

Lemma 14 (see [3]). *The sum of two positive elements in a C^*-algebra is a positive element.*

Theorem 15 (see [3]). *If a is an arbitrary element of a C^*-algebra $\mathbb{A}$, then a^*a is positive.*

We summarise some elementary facts about $\mathbb{A}_+$ in the following results.

Theorem 16 (see [3]). *Let $\mathbb{A}$ be a C^*-algebra:*

(1) *The set $\mathbb{A}_+$ is closed cone in $\mathbb{A}$ [a cone C in a real or complex vector space is a subset closed under addition and under scalar multiplication by $\mathbb{R}_+$].*

(2) *The set $\mathbb{A}_+$ is equal to $\{a^*a : a \in \mathbb{A}\}$.*

(3) *If $\theta \preceq a \preceq b$, then $\|a\| \leq \|b\|$.*

(4) *If $\mathbb{A}$ is unital and a and b are positive invertible elements, then $a \preceq b \Rightarrow \theta \preceq b^{-1} \preceq a^{-1}$.*

Theorem 17 (see [3]). *Let $\mathbb{A}$ be a C^*-algebra. If $a, b \in \mathbb{A}_+$ and $a \preceq b$, then for any $x \in \mathbb{A}$ both x^*ax and x^*bx are positive elements and $x^*ax \preceq x^*bx$.*

Lemma 18 (see [3]). *Suppose that $\mathbb{A}$ is a unital C^*-algebra with a unit I:*

(1) *If $a \in \mathbb{A}_+$ with $\|a\| < 1/2$, then $I - a$ is invertible and $\|a(I - a)^{-1}\| < 1$.*

(2) *Suppose that* $a, b \in \mathbb{A}$ *with* $a, b \succeq \theta$ *and* $ab = ba$*; then,* $ab \succeq \theta$.

(3) *Define* $\mathbb{A}' = \{a \in \mathbb{A} : ab = ba,\ \forall b \in \mathbb{A}\}$. *Let* $a \in \mathbb{A}'$*; if* $b, c \in \mathbb{A}$ *with* $b \succeq c \succeq \theta$ *and* $I - a \in \mathbb{A}'_+$ *is invertible operator, then*

$$(I - a)^{-1} b \succeq (I - a)^{-1} c. \tag{13}$$

Definition 19 (see [34]). A matrix $A \in M_n(\mathbb{C})$ is Hermitian if $A = A^*$, where A^* is a conjugate transpose matrix of A. A Hermitian matrix $A \in M_n(\mathbb{C})$ is positive definite if $x^* Ax > 0$ for all nonzero $x \in \mathbb{C}^n$, and it is positive semidefinite if $x^* Ax \geq 0$ for all nonzero $x \in \mathbb{C}^n$.

In 2014, Ma et al. [22] introduced the concept of C^*-algebra-valued metric space by using the concept of positive elements in $\mathbb{A}$. The following is definition of C^*-algebra-valued metric.

Definition 20 (see [22]). Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{A}$ is called C^*-algebra-valued metric on X if it satisfies the following conditions:

(1) $d(x, y) \succeq \theta$ for all $x, y \in X$.

(2) $d(x, y) = \theta$ if and only if $x = y$.

(3) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(4) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then, d is called a C^*-algebra-valued metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra-valued metric space.

We know that range of mapping d in metric space is the set of real numbers which is C^*-algebra; then, C^*-algebra-valued metric space generalizes the concept of metric spaces, replacing the set of real numbers by $\mathbb{A}_+$. In such paper, Ma et al. state the notation of convergence, Cauchy sequence, and completeness in C^*-algebra-valued metric space. For detail, a sequence $\{x_n\}$ in a C^*-algebra-valued metric space $(X, \mathbb{A}, d)$ is said to converge to $x \in X$ with respect to $\mathbb{A}$ if for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\|d(x_n, x)\| < \varepsilon$ for all $n \geq N$. We write it as $\lim_{n \to \infty} x_n = x$. A sequence $\{x_n\}$ is called a Cauchy sequence with respect to $\mathbb{A}$ if for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\|d(x_m, x_n)\| < \varepsilon$ for all $n, m \geq N$. The $(X, \mathbb{A}, d)$ is said to be a complete C^*-algebra-valued metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent. Moreover, they introduce definition of contractive and expansive mapping and give some related fixed point theorems for self-maps with C^*-algebra-valued contractive and expansive mapping, analogous to Banach contraction principle. The following is the definition of contractive mapping and the related fixed point theorem.

Definition 21 (see [22]). Suppose that $(X, \mathbb{A}, d)$ is a C^*-algebra-valued metric space. A mapping $T : X \to X$ is called C^*-algebra-valued contractive mapping on X, if there is an $\lambda \in \mathbb{A}$ with $\|\lambda\| < 1$ such that

$$d(Tx, Ty) \preceq \lambda^* d(x, y) \lambda, \quad \forall x, y \in X. \tag{14}$$

Theorem 22 (see [22]). *If* $(X, \mathbb{A}, d)$ *is a complete* C^**-algebra-valued metric space and* $T : X \to X$ *satisfies Defintion 21, then* T *has a unique fixed point in* X.

In the same way, the concept of expansive mapping is defined in the following way.

Definition 23 (see [22]). Let X be a nonempty set. A mapping T is a C^*-algebra-valued expansive mapping on X, if $T : X \to X$ satisfies

(1) $T(X) = X$,

(2) $d(Tx, Ty) \succeq \lambda^* d(x, y) \lambda$, for all $x, y \in X$,

where $\lambda \in \mathbb{A}$ is an invertible element and $\|\lambda^{-1}\| < 1$.

The following is the related fixed point theorem for C^*-algebra-valued expansive mapping.

Theorem 24 (see [22]). *Let* $(X, \mathbb{A}, d)$ *be a complete* C^**-algebra-valued metric space. If a* $T : X \to X$ *satisfies Defintion 23, then* T *has a unique fixed point in* X.

3. Fundamental Properties of C^*-Algebra-Valued b-Metric Spaces

In this section, we begin with the concept of C^*-algebra-valued b-metric space which was introduced by Ma and Jiang [26] as follows.

Definition 25 (see [26]). Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{A}$ is called C^*-algebra-valued b-metric on X if there exists $b \in \mathbb{A}'$ such that $b \succeq I$ satisfies following conditions:

(1) $d(x, y) \succeq \theta$ for all $x, y \in X$.

(2) $d(x, y) = \theta$ if and only if $x = y$.

(3) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(4) $d(x, y) \preceq b[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then, $(X, d, \mathbb{A})$ is called a C^*-algebra-valued b-metric space.

Remark 26. If $b = I$, then a C^*-algebra-valued b-metric spaces are C^*-algebra-valued metric spaces. In particular, if $\mathbb{A}$ is set of real numbers and $b = 1$, then the C^*-algebra-valued b-metric spaces is the metric spaces.

Definition 27 (see [26]). Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space. A sequence $\{x_n\}$ in $(X, \mathbb{A}, d)$ is said to converge to x if and only if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $\|d(x_n, x)\| \leq \varepsilon$. Then, $\{x_n\}$ is said to be convergent with respect to $\mathbb{A}$ and x is called limit point of $\{x_n\}$. We denote it by $\lim_{n \to \infty} x_n = x$.

A sequence $\{x_n\}$ is called a Cauchy seqeunce with respect to $\mathbb{A}$ if and only if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that, for all $n, m \geq N$, $\|d(x_n, x_m)\| \leq \varepsilon$.

We say $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued b-metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent sequence with respect to $\mathbb{A}$.

The following is an example of complete C^*-algebra-valued b-metric space.

Example 28 (see [26]). Let $X = \mathbb{R}$ and let $\mathbb{A} = M_n(\mathbb{R})$. Define

$$\begin{aligned} &d(x,y) \\ &= \operatorname{diag}\left((x-y)^p, \alpha_1|x-y|^p, \alpha_2|x-y|^p, \ldots, \alpha_{n-1}|x-y|^p\right) \\ &= \begin{bmatrix} |x-y|^p & 0 & 0 & \cdots & 0 \\ 0 & \alpha_1|x-y|^p & 0 & \cdots & 0 \\ 0 & 0 & \alpha_2|x-y|^p & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-1}|x-y|^p \end{bmatrix}, \end{aligned} \tag{15}$$

where $x, y \in \mathbb{R}$ and $\alpha_i > 0$ for all $i = 1, 2, \ldots, n-1$ are constants and p is a natural number such that $p \geq 2$. A norm $\|\cdot\|$ on $\mathbb{A}$ is defined by

$$\|A\| = \max_{i,j} \left|a_{ij}\right|^{1/p}, \tag{16}$$

where $A = (a_{ij})_{n\times n} \in \mathbb{A}$. The involution is given by $A^* = (\overline{A})^T$, conjugate transpose of matrix A:

$$\begin{aligned} A^* &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}^* = \begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} & \cdots & \overline{a_{n1}} \\ \overline{a_{12}} & \overline{a_{22}} & \cdots & \overline{a_{n2}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \overline{a_{2n}} & \cdots & \overline{a_{nn}} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}. \end{aligned} \tag{17}$$

It is easy to verify d is a C^*-algebra-valued b-metric space and $(X, M_2(\mathbb{R}), d)$ is a complete C^*-algebra-valued b-metric space be completeness of $\mathbb{R}$.

Proof. An element $A \in \mathbb{A} = M_n(\mathbb{R})$ is positive element; denote it by

$$A \succeq \theta, \quad \text{iff } A \text{ is positive semidefinite.} \tag{18}$$

We define a partial ordering $\preceq$ on $\mathbb{A}$ as follows:

$$A \preceq B \quad \text{iff } \theta \preceq B - A, \tag{19}$$

where θ mean the zero matrix in $M_n(\mathbb{R})$. Firstly, it clears that $\preceq$ is partially order relation. Next, we show that d is a C^*-algebra-valued b-metric space. Let $x, y, z \in X$. It is easy to see that d satifies conditions (1), (2), and (3) of Definition 25. We will only show condition (4) where $d(x, y) \preceq b[d(x, z) + d(z, y)]$ with

$$b = \begin{bmatrix} 2^{p-1} & 0 & \cdots & 0 \\ 0 & 2^{p-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2^{p-1} \end{bmatrix}_{n\times n}. \tag{20}$$

Since function $f(x) = |x|^p$ is convex function for all $p \geq 2$ and $x \in \mathbb{R}$, this implies that

$$\begin{aligned} \left|\frac{a+c}{2}\right|^p &= \left|\frac{1}{2}a + \left(1-\frac{1}{2}\right)c\right|^p \leq \frac{1}{2}|a|^p + \left(1-\frac{1}{2}\right)|c|^p \\ &= \frac{1}{2}\left(|a|^p + |c|^p\right) \end{aligned} \tag{21}$$

and hence $|a + c|^p \leq 2^{p-1}(|a|^p + |c|^p)$ for all $a, c \in \mathbb{R}$. We substitute $a = x - y$ and $c = y - z$; then,

$$\begin{aligned} |x-z|^p &= |x-y+y-z|^p \\ &\leq 2^{p-1}\left(|x-y|^p + |y-z|^p\right). \end{aligned} \tag{22}$$

Hence, setting $M_0 = (|x - y|^p + |y - z|^p)$ and $M_1 = |x - z|^p$, we obtain that

$$\begin{bmatrix} 2^{p-1}M_0 - M_1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_1\left(2^{p-1}M_0 - M_1\right) & 0 & \cdots & 0 \\ 0 & 0 & \alpha_2\left(2^{p-1}M_0 - M_1\right) & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-1}\left(2^{p-1}M_0 - M_1\right) \end{bmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} 2^{p-1}M_0 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_1 2^{p-1}M_0 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_2 2^{p-1}M_0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-1}2^{p-1}M_0 \end{bmatrix} - \begin{bmatrix} M_1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_1 M_1 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_2 M_1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-1}M_1 \end{bmatrix} \\
&= \begin{bmatrix} 2^{p-1} & 0 & \cdots & 0 \\ 0 & 2^{p-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2^{p-1} \end{bmatrix} \begin{bmatrix} M_0 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_1 M_0 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_2 M_0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-1}M_0 \end{bmatrix} - \begin{bmatrix} M_1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_1 M_1 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_2 M_1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-1}M_1 \end{bmatrix} \\
&= b\left(d\left(x, y\right) + d\left(y, z\right)\right) - d\left(x, z\right)
\end{aligned} \tag{23}
$$

implies that each eigenvalue of $b[d(x,z)+d(z,y)]-d(x,y)$ is nonnegative. Since each eigenvalue of a positive semidefinite matrix is a nonnegative real number, we have that $b[d(x,z)+d(z,y)]-d(x,y)$ is positive semidefinite; that is, $b[d(x,z)+d(z,y)]-d(x,y) \succeq \theta$, that is, $d(x,y) \preceq b[d(x,z)+d(z,y)]$, where $b = 2^{p-1}I \in \mathbb{A}'$ and $b \succeq I$ by $2^{p-1} > 1$. But $|x-y|^p \leq |x-z|^p + |z-y|^p$ is impossible for all $x, y, z \in \mathbb{R}$. Hence, $(X, M_n(\mathbb{R}), d)$ is C^*-algebra-valued b-metric spaces but not C^*-algebra-valued metric spaces.

Finally, we show that $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued b-metric space. Suppose that $\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$. Then, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|d(x_m, x_n)\| \leq \varepsilon$ for all $m, n \geq N$; that is,

$$
\begin{aligned}
&\max\left\{\left(|x_m - x_n|^p\right)^{1/p}, \left(\alpha_1 |x_m - x_n|^p\right)^{1/p}, \right. \\
&\left.\left(\alpha_2 |x_m - x_n|^p\right)^{1/p}, \ldots, \left(\alpha_{n-1} |x_m - x_n|^p\right)^{1/p}\right\} \leq \varepsilon
\end{aligned} \tag{24}
$$

for all $m, n \geq N$. Therefore,

$$
\begin{aligned}
|x_m - x_n| = \left(|x_m - x_n|^p\right)^{1/p} \leq \max\left\{\left(|x_m - x_n|^p\right)^{1/p}, \right. \\
\left(\alpha_1 |x_m - x_n|^p\right)^{1/p}, \left(\alpha_2 |x_m - x_n|^p\right)^{1/p}, \ldots, \\
\left.\left(\alpha_{n-1} |x_m - x_n|^p\right)^{1/p}\right\} \leq \varepsilon
\end{aligned} \tag{25}
$$

for all $m, n \geq N$. Hence, $\{x_n\}$ is a Cauchy sequnce in $\mathbb{R}$. By completeness of $\mathbb{R}$, there exists $x \in \mathbb{R}$ such that $\lim_{n\to\infty} x_n = x$; that is, $\lim_{n\to\infty} |x_n - x| = 0$. Then, we have that

$$
\begin{aligned}
\|d\left(x_n, x\right)\| = \max\left\{\left(|x_n - x|^p\right)^{1/p}, \left(\alpha_1 |x_n - x|^p\right)^{1/p}, \right. \\
\left.\left(\alpha_2 |x_n - x|^p\right)^{1/p}, \ldots, \left(\alpha_{n-1} |x_n - x|^p\right)^{1/p}\right\}
\end{aligned} \tag{26}
$$

converges to 0 as $n \to \infty$. Therefore, $\{x_n\}$ is convergent with respect to $\mathbb{A}$ and $\{x_n\}$ converging to x, so $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued b-metric space. □

Next, we disscus some fundamental properties of C^*-algebra-valued b-metric spaces.

Theorem 29. *Let $(X, \mathbb{A}, d)$ be C^*-algebra-valued b-metric space. If $\{x_n\}$ is a convergent sequence with respect to $\mathbb{A}$, then $\{x_n\}$ is Cauchy sequence with respect to $\mathbb{A}$.*

Proof. Assume that $\{x_n\}$ is a convergent sequence with respect to $\mathbb{A}$; then, there exists a $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Let $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$
\|d\left(x_n, x\right)\| \leq \frac{\varepsilon}{2\|b\|}. \tag{27}
$$

For $m, n \in N$, we get that

$$
d\left(x_m, x_n\right) \preceq b\left[d\left(x_m, x\right) + d\left(x, x_n\right)\right]. \tag{28}
$$

By Theorem 16, for $m, n \geq N$, we have

$$
\begin{aligned}
\|d\left(x_m, x_n\right)\| &\leq \|b\left[d\left(x_m, x\right) + d\left(x, x_n\right)\right]\| \\
&\leq \|b\| \|d\left(x_m, x\right) + d\left(x, x_n\right)\| \\
&\leq \|b\| \|d\left(x_m, x\right)\| + \|b\| \|d\left(x, x_n\right)\| \\
&\leq \|b\| \frac{\varepsilon}{2\|b\|} + \|b\| \frac{\varepsilon}{2\|b\|} = \varepsilon.
\end{aligned} \tag{29}
$$

This implies that $\{x_n\}$ is Cauchy sequence with respect to $\mathbb{A}$. □

Definition 30. A subset S of a C^*-algebra-valued b-metric space $(X, \mathbb{A}, d)$ is bounded with respect to $\mathbb{A}$ if there exists $\overline{x} \in X$ and a nonnegetive real number M such that

$$
\|d\left(x, \overline{x}\right)\| \leq M, \quad \forall x \in X. \tag{30}
$$

Theorem 31. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space and let $\{x_n\}$ be a sequence in X and $x \in X$. Then,*

(1) *$x_n \to x$ if and only if $d(x_n, x) \to \theta$,*

(2) *a convergent sequence in X is bounded with respect to $\mathbb{A}$ and its limit is unique,*

(3) *a Cauchy sequence in X is bounded with respect to $\mathbb{A}$.*

Proof. (1) Assume that $x_n \to x$. Let $\varepsilon > 0$ is given. Then, there exists $N_0 \in \mathbb{N}$ such that

$$\left\|d(x_n, x) - \theta\right\| = \left\|d(x_n, x)\right\| \leq \varepsilon. \tag{31}$$

This implies that $d(x_n, x) \to \theta$. Conversely, assume that $d(x_n, x) \to \theta$. Then, for any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$\left\|d(x_n, x) - \theta\right\| \leq \varepsilon \Longrightarrow \left\|d(x_n, x)\right\| \leq \varepsilon; \tag{32}$$

that is, $x_n \to x$.

(2) Let $\{x_n\}$ be a convergent sequence with respect to $\mathbb{A}$. Suppose that $x_n \to x$. Then, taking $\varepsilon = 1$, we can find $N \in \mathbb{N}$ such that

$$d(x_n, x) \leq 1, \quad \forall n \geq N. \tag{33}$$

Let $K = \max\{\|d(x_1, x)\|, \|d(x_2, x)\|, \ldots, \|d(x_N, x)\|\}$. Setting $M = \max\{1, K\}$. This implies that

$$\left\|d(x_n, x)\right\| \leq M, \quad \forall n \in \mathbb{N}. \tag{34}$$

Next, suppose that $x_n \to x$ and $x_n \to y$. Consider, $d(x, y) \leq b[d(x, x_n) + d(x_n, y)]$; by Theorem 16, we have

$$\left\|d(x, y)\right\| \leq \|b\| \left[\left\|d(x_n, x)\right\| + \left\|d(x_n, y)\right\|\right]. \tag{35}$$

From (1), letting $n \to \infty$, we obtian that $\|d(x, y)\| = 0$; that is $x = y$.

(3) Assume that $\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$. In particular, $\varepsilon = 1$; there exists $N_1 \in \mathbb{N}$ such that

$$\left\|d(x_m, x_n)\right\| \leq 1 \quad \forall m, n \geq N_1. \tag{36}$$

Let $K = \max\{\|d(x_1, x_{N_1})\|, \|d(x_2, x_{N_1})\|, \ldots, \|d(x_{N_1-1}, x_{N_1})\|\}$. Then,

$$\left\|d\left(x_n, x_{N_1}\right)\right\| \leq K \quad \forall n < N_1. \tag{37}$$

Set $M = \max\{1, K\}$. Then, we get that

$$\left\|d\left(x_n, x_{N_1}\right)\right\| \leq M \quad \forall n \in \mathbb{N}. \tag{38}$$

□

Theorem 32. *Let $\{x_n\}$ be a convergent sequence in a C^*-algebra-valued b-metric space $(X, \mathbb{A}, d)$ and $\lim_{n\to\infty} x_n = x$. Then, every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is convergent and has the same limit x.*

Proof. Let $\varepsilon > 0$ be given. Then, there exists $N \in \mathbb{N}$ such that

$$\left\|d(x_n, x)\right\| \leq \varepsilon, \quad \forall n \geq N. \tag{39}$$

Since $n_1 < n_2 < \cdots < n_k < \cdots$ is an increasing sequence of natural numbers, it is easily proved (by induction) that $n_k \geq k$. Hence, if $k \geq N$, we also have $n_k \geq k \geq N$ so that

$$\left\|d\left(x_{n_k}, x\right)\right\| \leq \varepsilon, \quad \forall n_k \geq N. \tag{40}$$

Therefore, subsequence $\{x_{n_k}\}$ also converges to x. □

Theorem 33. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space. Then, every subsequence of a Cauchy sequence is Cauchy sequence.*

Proof. Let $\{x_{n_k}\}$ be a subsequence of Cauchy sequence $\{x_n\}$ in a C^*-algebra-valued b-metric space. Then, for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that, for all $r, s \geq N$, we have $\|d(x_r, x_s)\| \leq \varepsilon$. Similar to the facts in proof of previous theorem, we have $n_r \geq r \geq N$ and $n_s \geq s \geq N$. Hence, we obtain that $\|d(x_{n_r}, x_{n_s})\| \leq \varepsilon$. Therefore, $\{x_{n_k}\}$ is Cauchy sequence. □

Theorem 34. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space and let $\{x_n\}$ be a Cauchy sequence with respect to $\mathbb{A}$. If $\{x_n\}$ contains its convergent subsequence, then $\{x_n\}$ is convergent sequence.*

Proof. Let $\varepsilon > 0$. Since $\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$, there exists an $N_0 \in \mathbb{N}$ such that

$$\left\|d\left(x_m, x_p\right)\right\| \leq \frac{1}{2\|b\|}\varepsilon, \quad \forall m, p \geq N_0. \tag{41}$$

Let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}$ and $x_{n_k} \to x$ $(k \to \infty)$. Then, there exists $N_1 \in \mathbb{N}$ such that

$$\left\|d\left(x_{n_k}, x\right)\right\| \leq \frac{1}{2\|b\|}\varepsilon, \quad \forall n_k \geq N_1. \tag{42}$$

Let $N = \max\{N_0, N_1\}$. For $n, k \geq N$, we have

$$d(x_n, x) \leq b\left[d\left(x_n, x_{n_k}\right) + d\left(x_{n_k}, x\right)\right]. \tag{43}$$

By Theorem 16, we also have

$$\begin{aligned}\left\|d(x_n, x)\right\| &\leq \|b\| \left\|d\left(x_n, x_{n_k}\right) + d\left(x_{n_k}, x\right)\right\| \\ &\leq \|b\| \left\|d\left(x_n, x_{n_k}\right)\right\| + \|b\| \left\|d\left(x_{n_k}, x\right)\right\| \\ &\leq \|b\| \left[\frac{\varepsilon}{2\|b\|} + \frac{\varepsilon}{2\|b\|}\right] \leq \varepsilon.\end{aligned} \tag{44}$$

Therefore, $x_n \to x$ as $n \to \infty$. □

Theorem 35. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space. Suppose that $\{x_n\}$ and $\{y_n\}$ are convergent with respect to $\mathbb{A}$ and converge to x and y, respectively. Then, $d(x_n, y_n)$ converges to $b^2 d(x, y)$.*

Proof. Let $\varepsilon > 0$. Since $x_n \to x$ and $y_n \to y$, there exist $N_0, N_1 \in \mathbb{N}$ such that

$$\|d(x_n, x)\| \leq \frac{\varepsilon}{2\|b\|}, \quad \forall n \geq N_0,$$
$$\|d(y_n, y)\| \leq \frac{\varepsilon}{2\|b\|^2}, \quad \forall n \geq N_1. \tag{45}$$

Since $d(x_n, y_n) \leq bd(x_n, x) + b^2 d(x, y) + b^2 d(y, y_n)$, by Theorem 16, we have

$$\|d(x_n, y_n) - b^2 d(x, y)\| \leq \|b\| \|d(x_n, x)\| + \|b\|^2 \|d(y, y_n)\| \leq \varepsilon. \tag{46}$$

Therefore, $d(x_n, y_n) \to b^2 d(x, y)$. □

Theorem 36. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space. Suppose that $\{x_n\}$ and $\{y_n\}$ are convergent with respect to $\mathbb{A}$ and converge to x and y, respectively. Then,*

$$\frac{1}{\|b\|^2} \|d(x, y)\| \leq \liminf_{n\to\infty} \|d(x_n, y_n)\| \leq \limsup_{n\to\infty} \|d(x_n, y_n)\| \leq \|b\|^2 \|d(x, y)\|. \tag{47}$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} \|d(x_n, y_n)\| = 0$. Moreover, for any $z \in X$, we have

$$\frac{1}{\|b\|} \|d(x, z)\| \leq \liminf_{n\to\infty} \|d(x_n, z)\| \leq \limsup_{n\to\infty} \|d(x_n, z)\| \leq \|b\| \|d(x, z)\|. \tag{48}$$

Proof. By defintion of C^*-algebra-valued b-metric space, it easy to see that

$$d(x, y) \leq bd(x, x_n) + b^2 d(x_n, y_n) + b^2 d(y_n, y),$$
$$d(x_n, y_n) \leq bd(x_n, x) + b^2 d(x, y) + b^2 d(y, y_n). \tag{49}$$

Using Theorem 16, we have

$$\|d(x, y)\| \leq \|b\| \|d(x, x_n)\| + \|b\|^2 \|d(x_n, y_n)\| + \|b\|^2 \|d(y_n, y)\|,$$
$$\|d(x_n, y_n)\| \leq \|b\| \|d(x_n, x)\| + \|b\|^2 \|d(x, y)\| + \|b\|^2 \|d(y, y_n)\|. \tag{50}$$

Taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality, this completes the first result. In particular, if $x = y$, we have

$$\|d(x_n, y_n)\| \leq \|b\| \|d(x_n, x)\| + \|b\|^2 \|d(y, y_n)\|. \tag{51}$$

Taking the limit as $n \to \infty$ in this inequality, we obtain that $\lim_{n\to\infty} \|d(x_n, y_n)\| = 0$. Since

$$d(x, z) \leq b[d(x, x_n) + d(x_n, z)],$$
$$d(x_n, z) \leq b[d(x_n, x) + d(x, z)], \tag{52}$$

by Theorem 16, we have

$$\|d(x, z)\| \leq \|b\| \|d(x, x_n)\| + \|b\| \|d(x_n, z)\|,$$
$$\|d(x_n, z)\| \leq \|b\| \|d(x_n, x)\| + \|b\| \|d(x, z)\|. \tag{53}$$

Again taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality, we obtain the second desired result. □

Definition 37. Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued b-metric space. A subset F of $(X, \mathbb{A}, d)$ is called a closed set if a sequence $\{x_n\}$ in X and $x_n \to x$ with respect to $\mathbb{A}$ imply $x \in F$.

4. Fixed Point Theorems for Cyclic Contractions

Theorem 38. *Let A and B be nonempty closed subset of a complete C^*-algebra-valued b-metric space $(X, \mathbb{A}, d)$. Assume that $T : A \cup B \to A \cup B$ is cyclic mapping that satisfies*

$$d(Tx, Ty) \leq \lambda^* d(x, y) \lambda, \quad \forall x \in A, \ \forall y \in B, \tag{54}$$

where $\lambda \in \mathbb{A}$ with $\|\lambda\| < 1/\|b\|$. Then, T has a unique fixed point in $A \cap B$.

Proof. Let x_0 be any point in A. Since T is cyclic mapping, we have $Tx_0 \in B$ and $T^2 x_0 \in A$. Using the contractive condition of mapping T, we get

$$d(Tx_0, T^2 x_0) = d(Tx_0, T(Tx_0)) \leq \lambda^* d(x_0, Tx_0) \lambda. \tag{55}$$

For all $n \in \mathbb{N}$, we have

$$d(T^n x_0, T^{n+1} x_0) \leq (\lambda^*)^n d(x_0, Tx_0) \lambda^n = (\lambda^*)^n \beta \lambda^n, \tag{56}$$

where $\beta = d(x_0, Tx_0)$. Consider, for any $m, n \in \mathbb{N}$ such that $m \leq n$; then,

$$\begin{aligned} & d\left(T^m x_0, T^n x_0\right) \\ & \preceq b\left[d\left(T^m x_0, T^{m+1} x_0\right) + d\left(T^{m+1} x_0, T^n x_0\right)\right] \\ & \preceq b d\left(T^m x_0, T^{m+1} x_0\right) \\ & \quad + b^2\left[d\left(T^{m+1} x_0, T^{m+2} x_0\right) + d\left(T^{m+2} x_0, T^n x_0\right)\right] \\ & \preceq \cdots \\ & \preceq b d\left(T^m x_0, T^{m+1} x_0\right) + b^2 d\left(T^{m+1} x_0, T^{m+2} x_0\right) \\ & \quad + \cdots + b^{n-m} d\left(T^{n-1} x_0, T^n x_0\right) \\ & \preceq b\left(\lambda^*\right)^m \beta \lambda^m + b^2\left(\lambda^*\right)^{m+1} \beta \lambda^{m+1} + \cdots \\ & \quad + b^{n-m}\left(\lambda^*\right)^{n-1} \beta \lambda^{n-1} = \sum_{k=m}^{n-1} b^{k-m+1}\left(\lambda^*\right)^k \beta \lambda^k. \end{aligned} \tag{57}$$

From Theorem 16, we have

$$\begin{aligned} \left\|d\left(T^m x_0, T^n x_0\right)\right\| &\leq \left\|\sum_{k=m}^{n-1} b^{k-m+1}\left(\lambda^*\right)^k \beta \lambda^k\right\| \\ &\leq \sum_{k=m}^{n-1}\left\|b^{k-m+1}\left(\lambda^*\right)^k \beta \lambda^k\right\| \\ &\leq \sum_{k=m}^{n-1}\left\|b^{k-m+1}\right\|\left\|\left(\lambda^*\right)^k\right\|\|\beta\|\left\|\lambda^k\right\| \\ &\leq \|\beta\| \sum_{k=m}^{n-1}\left\|b^{k-m+1}\right\|\left\|(\lambda)^k\right\|^2 \\ &\leq \|\beta\| \sum_{k=m}^{n-1}\|b\|^{k-m+1}\|\lambda\|^{2k} \\ &\leq \|\beta\| \sum_{k=m}^{n-1}\|b\|^{2k}\|\lambda\|^{2k} \\ &\leq \|\beta\| \sum_{k=m}^{\infty}(\|b\|\|\lambda\|)^{2k} \\ &= \|\beta\| \frac{(\|b\|\|\lambda\|)^{2m}}{1-(\|b\|\|\lambda\|)}. \end{aligned} \tag{58}$$

Since $0 \leq \|\lambda\| < 1/\|b\|$, we have $\|\beta\|(\|b\|\|\lambda\|)^{2m}/(1 - (\|b\|\|\lambda\|)) \to 0$ as $m \to \infty$. Therefore, $\{T^n x_0\}$ is Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, d)$, there exists an element $x \in X$ such that $x = \lim_{n\to\infty} T^n x_0$.

Since $\{T^{2n} x_0\}$ is a sequence in A and $\{T^{2n-1} x_0\}$ is a sequence in B, we obtain that both sequences converge to the same limit x. Since A and B are closed set, this implies that $x \in A \cap B$.

Next, we will complete the proof by showing that x is a unique fixed point of T. Since

$$\begin{aligned} \theta &\preceq d(Tx, x) \preceq b\left[d\left(Tx, T^{2n} x_0\right) + d\left(T^{2n} x_0, x\right)\right] \\ &\preceq b\left[\lambda^* d\left(x, T^{2n-1} x_0\right)\lambda + d\left(T^{2n} x_0, x\right)\right] \end{aligned} \tag{59}$$

by Theorem 16, we obtain that

$$\begin{aligned} & 0 \leq \|d(Tx, x)\| \\ & \leq \|b\|\|\lambda\|^2\left\|d\left(x, T^{2n-1} x_0\right)\right\| + \|b\|\left\|d\left(T^{2n} x_0, x\right)\right\| \\ & \longrightarrow 0 \quad (n \longrightarrow \infty). \end{aligned} \tag{60}$$

We have $Tx = x$; that is, x is a fixed point of T.

Suppose that y is fixed point of T and $y \neq x$. Since

$$\theta \preceq d(x, y) = d(Tx, Ty) \preceq \lambda^* d(x, y)\lambda, \tag{61}$$

we have

$$\begin{aligned} \|d(x, y)\| &\leq \left\|\lambda^* d(x, y)\lambda\right\| \leq \left\|\lambda^*\right\|\|d(x, y)\|\|\lambda\| \\ &= \|\lambda\|^2\|d(x, y)\| < \|d(x, y)\|. \end{aligned} \tag{62}$$

This is a contradiction. Therefore, $x = y$ which implies that the fixed point is unique. □

Example 39. Let X be a set of real numbers and $\mathbb{A} = M_{2\times 2}(\mathbb{R})$ with $\|A\| = \max_{i,j}|a_{ij}|$, where a_{ij} are entries of the matrix $A \in M_{2\times 2}(\mathbb{R})$. Then, $(X, \mathbb{A}, d)$ is a C^*-algebra-valued b-metric space with $b = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right]$, where the involution is given by $A^* = (\overline{A})^T$,

$$d(x, y) = \begin{bmatrix} |x-y|^2 & 0 \\ 0 & |x-y|^2 \end{bmatrix} \tag{63}$$

and partial ordering on $\mathbb{A}$ is given as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \preceq \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \Longleftrightarrow \quad a_{ij} \leq b_{ij} \quad \forall i, j = 1, 2, 3, 4. \tag{64}$$

Define a mapping $T : X \to X$ by

$$Tx = \begin{cases} -\dfrac{x + 1/3}{3}\left|\sin\left(\dfrac{1}{x}\right)\right| - \dfrac{1}{3}; & x \in \left(\infty, -\dfrac{1}{3}\right] \\ -\dfrac{1}{3}; & x \in \left(-\dfrac{1}{3}, 0\right] \\ -\dfrac{1}{2}; & x \in (0, +\infty). \end{cases} \tag{65}$$

It is clear that T is not continuous at all elements of X. Therefore, Theorem 22 cannot imply the existence of fixed point of mapping T.

Suppose that $A = [-1/2, -1/3]$ and $B = [-1/3, 0]$. Firstly, we will show that $T : A \cup B \to A \cup B$ is cyclic mapping. Let $x \in B$; that is, $-1/3 \le x \le 0$. Then, $Tx = -1/3 \in A$. Again, let $y \in A$; that is, $-1/2 \le x \le -1/3$. Indeed, we consider

$$\begin{aligned}
&-\frac{1}{2} \le x \le -\frac{1}{3} \Longrightarrow \\
&-\frac{1}{6} \le x + \frac{1}{3} \le 0 \Longrightarrow \\
&-\frac{1}{18} \le \frac{x+1/3}{3} \le 0 \Longrightarrow \\
&0 \le -\left(\frac{x+1/3}{3}\right) \le \frac{1}{18} \Longrightarrow \\
&0 \le -\left(\frac{x+1/3}{3}\right)\left|\sin\left(\frac{1}{x}\right)\right| \le \frac{1}{18}\left|\sin\left(\frac{1}{x}\right)\right| \le \frac{1}{18} \Longrightarrow \\
&-\frac{1}{3} \le -\left(\frac{x+1/3}{3}\right)\left|\sin\left(\frac{1}{x}\right)\right| - \frac{1}{3} \le \frac{1}{18} - \frac{1}{3} \le 0;
\end{aligned} \tag{66}$$

this implies that $Tx \in [-1/3, 0] = B$. For any $x \in A$ and $y \in B$, since $-1/2 \le x \le -1/3$ and $-1/3 \le y$, we have $1/9 \le -x/3 \le 1/6$ and $-1/9 \le y/3$. Hence, we obtain that

$$0 \le -\frac{x}{3} - \frac{1}{9} \le -\frac{x}{3} + \frac{y}{3}. \tag{67}$$

Next, we consider

$$\begin{aligned}
|Tx - Ty|^2 &= \left|-\left(\frac{x+1/3}{3}\right)\left|\sin\left(\frac{1}{x}\right)\right| - \frac{1}{3} - \left(-\frac{1}{3}\right)\right|^2 \\
&= \left|-\left(\frac{x+1/3}{3}\right)\left|\sin\left(\frac{1}{x}\right)\right|\right|^2 \\
&\le \left|-\left(\frac{x+1/3}{3}\right)\right|^2 = \left|-\frac{x}{3} - \frac{1}{9}\right|^2 \\
&\le \left|-\frac{x}{3} + \frac{y}{3}\right|^2 \le \frac{1}{9}|x-y|^2.
\end{aligned} \tag{68}$$

Then, we have

$$\begin{aligned}
d(Tx, Ty) &= \begin{bmatrix} |Tx-Ty|^2 & 0 \\ 0 & |Tx-Ty|^2 \end{bmatrix} \\
&\le \begin{bmatrix} \frac{1}{9}|x-y|^2 & 0 \\ 0 & \frac{1}{9}|x-y|^2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} |x-y|^2 & 0 \\ 0 & |x-y|^2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \\
&= \lambda^* d(x, y)\lambda,
\end{aligned} \tag{69}$$

where $\lambda = \left[\begin{smallmatrix} 1/3 & 0 \\ 0 & 1/3 \end{smallmatrix}\right]$. Then, $\|\lambda\| = 1/3 < 1/2 = 1/\|b\|$. Thus, T satisfies contraction of Theorem 38 implying that T has a unique fixed point in $A \cap B$; that is, $\{-1/3\} = F(T)$.

Corollary 40. *Suppose that $(X, \mathbb{A}, d)$ is a C^*-algebra-valued b-metric space. Assume that $T : X \to X$ is called a C^*-algebra-valued b-contractive mapping on X; that is, T satisfies*

$$d(Tx, Ty) \preceq \lambda^* d(x, y)\lambda, \quad \forall x, y \in X, \tag{70}$$

where $\lambda \in \mathbb{A}$ with $\|\lambda\| < 1/\|b\|$. Then, T has a unique fixed point in X.

Proof. Putting $A = B = X$, by Theorem 38, this implies that T has a unique fixed point in $A \cap B = X$. □

Theorem 41. *Suppose that $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued b-metric space. Assume that a mapping $T : X \to X$ satisfies*

(1) $T(X) = X$;

(2) $d(Tx, Ty) \succeq \lambda^* d(x, y)\lambda$ *for all* $x, y \in X$,

where $\lambda \in \mathbb{A}$ is an invertible element and $\|\lambda^{-1}\| < 1/\|b\|$ such that T is a C^*-algebra-valued b-expansive mapping on X. Then, T has a unique fixed point in X.

Proof. We will begin to prove this theorem by showing that T is injective. Let x, y be an element in X such that $x \ne y$; that is, $d(x, y) \ne 0$. Assume that $Tx = Ty$. We have

$$\begin{aligned}
\theta = d(Tx, Ty) &\succeq \lambda^* d(x, y)\lambda \\
&= \lambda^* d(x, y)^{1/2} d(x, y)^{1/2} \lambda \\
&= \left(d(x, y)^{1/2}\lambda\right)^* \left(d(x, y)^{1/2}\lambda\right) \succeq \theta.
\end{aligned} \tag{71}$$

This implies that $\lambda^* d(x, y)\lambda = \theta$. Since λ is invertible, we have $d(x, y) = \theta$ which leads to contradiction. Thus, T is injective. By the first condition of mapping T, we obtain that T is bijective which implies that T is invertibe and T^{-1} is bijective.

Next, we will show that T has a unique fixed point in X. In fact, since T is C^*-algebra-valued b-expansive and invertible mapping, we substitute x, y with $T^{-1}x, T^{-1}y$ in the second condition of T, respectively, which implies that

$$d\left(T\left(T^{-1}x\right), T\left(T^{-1}y\right)\right) \succeq \lambda^* d\left(T^{-1}x, T^{-1}y\right)\lambda, \quad \forall x, y \in X. \tag{72}$$

That is

$$d(x, y) \succeq \lambda^* d\left(T^{-1}x, T^{-1}y\right)\lambda, \quad \forall x, y \in X. \tag{73}$$

Since $d(x, y)$ and $\lambda^* d(T^{-1}x, T^{-1}y)\lambda$ are positive elements in $\mathbb{A}$, $\lambda^* d(T^{-1}x, T^{-1}y) \preceq \lambda d(x, y)$ and $\lambda^{-1} \in \mathbb{A}$. By condition (2) of Theorem 12 and Theorem 17, we have

$$\begin{aligned}
d\left(T^{-1}x, T^{-1}y\right) &= \left(\lambda\lambda^{-1}\right)^* d\left(T^{-1}x, T^{-1}y\right)\lambda\left(\lambda^{-1}\right) \\
&= \left(\lambda^{-1}\right)^* \lambda^* d\left(T^{-1}x, T^{-1}y\right)\lambda\left(\lambda^{-1}\right) \\
&\preceq \left(\lambda^{-1}\right)^* d(x, y)\lambda^{-1}.
\end{aligned} \tag{74}$$

Therefore, T^{-1} is b-contractive mapping. Using Corollary 40, there exists a unique x such that $T^{-1}x = x$, which means it has a unique fixed point $x \in X$ such that $Tx = T(T^{-1}x) = (TT^{-1})x = Ix = x$. □

Theorem 42 (cyclic Kannan-type). *Let A and B be nonempty closed subset of a complete C^*-algebra-valued b-metric space $(X, \mathbb{A}, d)$. Assume that $T : A \cup B \to A \cup B$ is cyclic mapping that satisfies*

$$d(Tx, Ty) \preceq \lambda [d(x, Tx) + d(y, Ty)], \quad \forall x \in A,\ \forall y \in B, \tag{75}$$

where $\lambda \in \mathbb{A}'_+$ with $\|\lambda\| < 1/2\|b\|$. Then, T has a unique fixed point in $A \cap B$.

Proof. Without loss of generality, we can assume that $\lambda \neq \theta$. Since $\lambda \in \mathbb{A}'_+$ and $\theta \preceq d(x, Tx) + d(y, Ty)$, by the second condition of Lemma 18, we have $\theta \preceq \lambda\{d(x, Tx) + d(y, Ty)\}$.

Let x_0 be any element in A. Since T is cyclic mapping, we have $Tx_0 \in B$ and $T^2x_0 \in A$. Consider

$$\begin{aligned} d\left(Tx_0, T^2x_0\right) &= d\left(Tx_0, T\left(Tx_0\right)\right) \\ &\preceq \lambda\left[d\left(x_0, Tx_0\right) + d\left(Tx_0, T^2x_0\right)\right] \\ &= \lambda d\left(x_0, Tx_0\right) + \lambda d\left(Tx_0, T^2x_0\right); \end{aligned} \tag{76}$$

that is,

$$(I - \lambda)\, d\left(Tx_0, T^2x_0\right) \preceq \lambda d\left(x_0, Tx_0\right). \tag{77}$$

Since $\lambda \in \mathbb{A}'_+$ and $\|\lambda\| < 1/2\|b\| < 1/2$, by the first condition of Lemma 18, we have that $I - \lambda$ is invertible and $\|(I-\lambda)^{-1}\lambda\| < 1$. From the third condition of Lemma 18, we have

$$d\left(Tx_0, T^2x_0\right) \preceq (I - \lambda)^{-1} \lambda d\left(x_0, Tx_0\right). \tag{78}$$

Similarly, we get that

$$d\left(T^2x_0, T^3x_0\right) \preceq (I - \lambda)^{-1} \lambda d\left(Tx_0, T^2x_0\right). \tag{79}$$

Since $(I - \lambda)^{-1}\lambda \in \mathbb{A}'_+$ and $\theta \preceq (I - \lambda)^{-1}\lambda d(x_0, Tx_0) - d(Tx_0, T^2x_0)$, the second condition of Lemma 18, we have

$$\begin{aligned} \theta &\preceq (I - \lambda)^{-1} \\ &\cdot \lambda\left\{(I - \lambda)^{-1} \lambda d\left(x_0, Tx_0\right) - d\left(Tx_0, T^2x_0\right)\right\}; \end{aligned} \tag{80}$$

that is,

$$\begin{aligned} &(I - \lambda)^{-1} \lambda d\left(Tx_0, T^2x_0\right) \\ &\preceq \left[(I - \lambda)^{-1} \lambda\right]^2 d\left(x_0, Tx_0\right). \end{aligned} \tag{81}$$

Hence,

$$\begin{aligned} d\left(T^2x_0, T^3x_0\right) &\preceq (I - \lambda)^{-1} \lambda d\left(Tx_0, T^2x_0\right) \\ &\preceq \left[(I - \lambda)^{-1} \lambda\right]^2 d\left(x_0, Tx_0\right). \end{aligned} \tag{82}$$

Continuing this process, we have

$$d\left(T^nx_0, T^{n+1}x_0\right) \preceq \left[(I - \lambda)^{-1} \lambda\right]^n d\left(x_0, Tx_0\right) = \alpha^n\beta, \tag{83}$$

where $\alpha = (I - \lambda)^{-1}\lambda$ and $\beta = d(x_0, Tx_0)$. Next, we will show that $\{T^nx_0\}$ is Cauchy sequence with respect to $\mathbb{A}$. Consider for any $m, n \in \mathbb{N}$ and $m \leq n$ that we have

$$\begin{aligned} d\left(T^mx_0, T^nx_0\right) &\preceq bd\left(T^mx_0, T^{m+1}x_0\right) \\ &\quad + b^2d\left(T^{m+1}x_0, T^{m+2}x_0\right) + \cdots \\ &\quad + b^{n-m}d\left(T^{n-1}x_0, T^nx_0\right) \\ &\preceq b\alpha^m\beta + b^2\alpha^{m+1}\beta + \cdots \\ &\quad + b^{n-m}\alpha^{n-1}\beta = \sum_{k=m}^{n-1} b^{k-m+1}\alpha^k\beta. \end{aligned} \tag{84}$$

From Theorem 16, we get that

$$\begin{aligned} \left\|d\left(T^mx_0, T^nx_0\right)\right\| &\leq \left\|\sum_{k=m}^{n-1} b^{k-m+1}\alpha^k\beta\right\| \\ &\leq \sum_{k=m}^{n-1} \left\|b^{k-m+1}\alpha^k\beta\right\| \\ &\leq \sum_{k=m}^{n-1} \|b\|^{k-m+1} \|\alpha\|^k \|\beta\| \\ &\leq \sum_{k=m}^{n-1} \|b\|^k \|\alpha\|^k \|\beta\| \\ &= \|\beta\| \sum_{k=m}^{n-1} (\|b\| \|\alpha\|)^k \\ &\leq \|\beta\| \sum_{k=m}^{\infty} (\|b\| \|\alpha\|)^k \\ &= \|\beta\| \frac{(\|b\| \|\alpha\|)^m}{1 - (\|b\| \|\alpha\|)}. \end{aligned} \tag{85}$$

Consider

$$\begin{aligned} \|b\| \|\alpha\| &= \|b\| \left\|\lambda (I - \lambda)^{-1}\right\| \leq \|b\| \|\lambda\| \left\|(I - \lambda)^{-1}\right\| \\ &= \|b\| \|\lambda\| \left\|\sum_{i=0}^{\infty} (\lambda)^i\right\| \leq \|b\| \|\lambda\| \sum_{i=0}^{\infty} \|(\lambda)\|^i \\ &< \|b\| \left(\frac{1}{2\|b\|}\right) \frac{1}{1 - \|\lambda\|} < \frac{1}{2} \frac{1}{1 - 1/2} = 1. \end{aligned} \tag{86}$$

Therefore, $\|\beta\|(\|b\|\|\alpha\|)^{2m}/(1 - (\|b\|\|\alpha\|)) \to 0$ as $m \to \infty$. Therefore, $\{T^nx_0\}$ is Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, d)$, there exists an element $x \in X$ such that $x = \lim_{n\to\infty} T^nx_0$.

Since $\{T^{2n}x_0\}$ is a sequence in A and $\{T^{2n-1}x_0\}$ is a sequence in B, we obtain that both sequences converge to

the same limit x. Since A and B are closed set, this implies $x \in A \cap B$. Next, we will show that x is a unique fixed point of T. Consider

$$\begin{aligned} d(Tx, x) &\preceq b\left[d\left(Tx, T^{2n}x_0\right) + d\left(T^{2n}x_0, x\right)\right] \\ &= bd\left(Tx, T\left(T^{2n-1}x_0\right)\right) + bd\left(T^{2n}x_0, x\right) \\ &\preceq b\lambda\left[d(x, Tx) + d\left(T^{2n-1}x_0, T^{2n}x_0\right)\right] \\ &\quad + bd\left(T^{2n}x_0, x\right) \\ &\preceq b\lambda d(x, Tx) + b^2\lambda d\left(T^{2n-1}x_0, x\right) \\ &\quad + b^2\lambda d\left(x, T^{2n}x_0\right) + bd\left(T^{2n}x_0, x\right); \end{aligned} \tag{87}$$

by Theorem 16 and submultiplicative, we obtian that

$$\begin{aligned} \|d(Tx, x)\| &\le \|b\| \|\lambda\| \|d(x, Tx)\| \\ &\quad + \|b\|^2 \|\lambda\| \left\|d\left(T^{2n-1}x_0, x\right)\right\| \\ &\quad + \|b\|^2 \|\lambda\| \left\|d\left(x, T^{2n}x_0\right)\right\| \\ &\quad + \|b\| \left\|d\left(T^{2n}x_0, x\right)\right\|. \end{aligned} \tag{88}$$

Letting $n \to \infty$, we get that

$$\|d(Tx, x)\| \le \|b\| \|\lambda\| \|d(x, Tx)\|, \tag{89}$$

and so

$$\|d(Tx, x)\| \le \|b\| \frac{1}{2\|b\|} \|d(x, Tx)\| < \frac{1}{2} \|d(x, Tx)\|. \tag{90}$$

This implies that $\|d(Tx, x)\| = 0$; that is, $d(Tx, x) = \theta$ and so $Tx = x$. That is, x is fixed point of T. Now if y is another fixed point of T and $y \neq x$, then

$$\begin{aligned} \theta \preceq d(x, y) &= d(Tx, Ty) \preceq \lambda\left(d(x, Tx) +, d(y, Ty)\right) \\ &= \lambda\left(d(x, x) + d(y, y)\right) = \theta, \end{aligned} \tag{91}$$

which leads to contradiction. Therefore, $x = y$; we complete the proof. □

Example 43. Let $X = [-1, 1]$ and $\mathbb{A} = M_{2\times 2}(\mathbb{R})$ with $\|A\| = \max_{i,j}|a_{ij}|$ where a_{ij} are entries of the matrix $A \in M_{2\times 2}(\mathbb{R})$. Then, $(X, \mathbb{A}, d)$ is a C^*-algebra-valued b-metric space with $b = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right]$, where the involution is given by $A^* = (\bar{A})^T$:

$$d(x, y) = \begin{bmatrix} |x - y|^2 & 0 \\ 0 & |x - y|^2 \end{bmatrix}, \tag{92}$$

and partial ordering on $\mathbb{A}$ is given as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \preceq \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \Longleftrightarrow \quad a_{ij} \le b_{ij} \quad \forall i, j = 1, 2, 3, 4. \tag{93}$$

Suppose that $A = [-1, 0]$ and $B = [0, 1]$. Define a mapping $T : A \cup B \to A \cup B$ by $Tx = -x/4$. Firstly, we will show that T is cyclic mapping. Let x be an element in A; that is, $-1 \le x \le 0$. Then, $0 \le -x/4 \le 1$ implies $Tx \in B$. Similarly, let $y \in B$, so $0 \le y \le 1$. Then, $-1/4 \le -y/4 \le 0$. Hence, $Ty \in A$.

For any $x \in A$ and $y \in B$, we consider

$$\begin{aligned} |Tx - Ty|^2 &= \left|\frac{-x}{4} - \frac{-y}{4}\right|^2 = \frac{1}{16}|x - y|^2 \\ &\le \frac{1}{16}(|x| + |y|)^2 \le \frac{1}{16}\left(2|x|^2 + 2|y|^2\right) \\ &= \frac{2}{25}\left(\frac{25}{16}|x|^2 + \frac{25}{16}|y|^2\right) \\ &= \frac{2}{25}\left(\left|x + \frac{x}{4}\right|^2 + \left|y + \frac{y}{4}\right|^2\right) \\ &= \frac{2}{25}\left(|x - Tx|^2 + |y - Ty|^2\right). \end{aligned} \tag{94}$$

Then, we have

$$\begin{aligned} d(Tx, Ty) &= \begin{bmatrix} |Tx - Ty|^2 & 0 \\ 0 & |Tx - Ty|^2 \end{bmatrix} \preceq \begin{bmatrix} \frac{2}{25}\left(|x - Tx|^2 + |y - Ty|^2\right) & 0 \\ 0 & \frac{2}{25}\left(|x - Tx|^2 + |y - Ty|^2\right) \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{25} & 0 \\ 0 & \frac{2}{25} \end{bmatrix} \begin{bmatrix} \left(|x - Tx|^2 + |y - Ty|^2\right) & 0 \\ 0 & \left(|x - Tx|^2 + |y - Ty|^2\right) \end{bmatrix} = \lambda\left[d(x, Tx) + d(y, Ty)\right], \end{aligned} \tag{95}$$

where $\lambda = \begin{bmatrix} 2/25 & 0 \\ 0 & 2/25 \end{bmatrix}$. Then, $\|\lambda\| = 2/25 < 1/4 = 1/2\|b\|$. Thus, T satisfies contraction of Theorem 42 implying that T has a unique fixed point in $A \cap B$; that is, $\{0\} = F(T)$.

Theorem 44 (cyclic Chatterjea-type). *Let A and B be nonempty closed subset of a complete C^*-algebra-valued b-metric space $(X, \mathbb{A}, d)$. Assume that $T : A \cup B \to A \cup B$ is cyclic mapping that satisfies*

$$d(Tx, Ty) \preceq \lambda [d(y, Tx) + d(x, Ty)], \quad \forall x \in A, \ \forall y \in B, \tag{96}$$

where $\lambda \in \mathbb{A}'_+$ with $\|\lambda\| < 1/2\|b\|^2$. Then, T has a unique fixed point in $A \cap B$.

Proof. Without loss of generality, we can assume that $\lambda \neq \theta$. Since $\lambda \in \mathbb{A}'_+$ and $\theta \preceq d(y, Tx) + d(x, Ty)$, by the second condition of Lemma 18, we have $\theta \preceq \lambda\{d(y, Tx) + d(x, Ty)\}$.

Let x_0 be any element in A, Since T is cyclic mapping, we have $Tx_0 \in B$ and $T^2x_0 \in A$. Consider

$$\begin{aligned} d(Tx_0, T^2x_0) &= d(Tx_0, T(Tx_0)) \\ &\preceq \lambda [d(Tx_0, Tx_0) + d(x_0, T^2x_0)] \\ &\preceq b\lambda [d(x_0, Tx_0) + d(Tx_0, T^2x_0)]; \end{aligned} \tag{97}$$

that is,

$$(I - b\lambda) d(Tx_0, T^2x_0) \preceq b\lambda d(x_0, Tx_0). \tag{98}$$

Since $\lambda \in \mathbb{A}'_+$ and $b \in \mathbb{A}'_+$, from the second condition of Lemma 18, we get that $b\lambda \in \mathbb{A}'_+$. Since $\|b\lambda\| < \|b\|(1/2\|b\|^2) \leq 1/2$ and $b\lambda \in \mathbb{A}'_+$, by the first condition of Lemma 18, we have $(I - b\lambda)^{-1} \in \mathbb{A}'_+$ and $(b\lambda)(I - b\lambda)^{-1} \in \mathbb{A}'_+$ with $\|(b\lambda)(I - b\lambda)^{-1}\| < 1$. From the third condition of Lemma 18, we have

$$d(Tx_0, T^2x_0) \preceq (b\lambda)(I - b\lambda)^{-1} d(x_0, Tx_0). \tag{99}$$

Similarly, we get that

$$d(T^2x_0, T^3x_0) \preceq (b\lambda)(I - b\lambda)^{-1} d(Tx_0, T^2x_0). \tag{100}$$

Since $(b\lambda)(I-b\lambda)^{-1} \in \mathbb{A}'_+$ and $\theta \preceq (b\lambda)(I-b\lambda)^{-1}d(x_0, Tx_0) - d(Tx_0, T^2x_0)$, the second condition of Lemma 18, we have

$$\begin{aligned} \theta &\preceq (b\lambda)(I - b\lambda)^{-1} \\ &\cdot \{(b\lambda)(I - b\lambda)^{-1} d(x_0, Tx_0) - d(Tx_0, T^2x_0)\}; \end{aligned} \tag{101}$$

that is,

$$\begin{aligned} &(b\lambda)(I - b\lambda)^{-1} d(Tx_0, T^2x_0) \\ &\preceq [(b\lambda)(I - b\lambda)^{-1}]^2 d(x_0, Tx_0). \end{aligned} \tag{102}$$

Hence,

$$\begin{aligned} d(T^2x_0, T^3x_0) &\preceq (b\lambda)(I - b\lambda)^{-1} d(Tx_0, T^2x_0) \\ &\preceq [(b\lambda)(I - b\lambda)^{-1}]^2 d(x_0, Tx_0). \end{aligned} \tag{103}$$

Continuing this process, we have

$$\begin{aligned} d(T^nx_0, T^{n+1}x_0) &\preceq [(b\lambda)(I - b\lambda)^{-1}]^n d(x_0, Tx_0) \\ &= \omega^n \beta, \end{aligned} \tag{104}$$

where $\omega = (b\lambda)(I - b\lambda)^{-1}$ and $\beta = d(x_0, Tx_0)$. Next, we will show that $\{T^nx_0\}$ is Cauchy sequence with respect to $\mathbb{A}$. Consider for any $m, n \in \mathbb{N}$ and $m \leq n$; we have

$$\begin{aligned} d(T^mx_0, T^nx_0) &\preceq bd(T^mx_0, T^{m+1}x_0) \\ &\quad + b^2 d(T^{m+1}x_0, T^{m+2}x_0) + \cdots \\ &\quad + b^{n-m} d(T^{n-1}x_0, T^nx_0) \\ &\preceq b\omega^m\beta + b^2\omega^{m+1}\beta + \cdots \\ &\quad + b^{n-m}\omega^{n-1}\beta = \sum_{k=m}^{n-1} b^{k-m+1}\omega^k\beta. \end{aligned} \tag{105}$$

From Theorem 16, we get that

$$\begin{aligned} \|d(T^mx_0, T^nx_0)\| &\leq \left\| \sum_{k=m}^{n-1} b^{k-m+1}\omega^k\beta \right\| \\ &\leq \sum_{k=m}^{n-1} \left\| b^{k-m+1}\omega^k\beta \right\| \\ &\leq \sum_{k=m}^{n-1} \|b\|^{k-m+1} \|\omega\|^k \|\beta\| \\ &\leq \sum_{k=m}^{n-1} \|b\|^k \|\omega\|^k \|\beta\| \\ &= \|\beta\| \sum_{k=m}^{n-1} (\|b\| \|\omega\|)^k \\ &= \|\beta\| \sum_{k=m}^{\infty} (\|b\| \|\omega\|)^k \\ &= \|\beta\| \frac{(\|b\| \|\omega\|)^m}{1 - (\|b\| \|\omega\|)}. \end{aligned} \tag{106}$$

Consider

$$\begin{aligned}\|b\|\|\omega\| &= \|b\|\left\|b\lambda\left(I-b\lambda\right)^{-1}\right\| \\ &\le \|b\|\|b\lambda\|\left\|\left(I-b\lambda\right)^{-1}\right\| \\ &= \|b\|\|b\lambda\|\left\|\sum_{i=0}^{\infty}(b\lambda)^{i}\right\| \le \|b\|\|b\lambda\|\sum_{i=0}^{\infty}\|(b\lambda)\|^{i} \\ &< \|b\|\left(\frac{\|b\|}{2\|b\|^{2}}\right)\frac{1}{1-\|b\lambda\|} < \frac{1}{2}\frac{1}{1-1/2} = 1.\end{aligned} \tag{107}$$

Therefore, $\|\beta\|(\|b\|\|\omega\|)^{2^m}/(1-(\|b\|\|\omega\|)) \to 0$ as $m \to \infty$. Therefore, $\{T^n x_0\}$ is Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, d)$, there exists an element $x \in X$ such that $x = \lim_{n\to\infty} T^n x_0$.

Since $\{T^{2n}x_0\}$ is a sequence in A and $\{T^{2n-1}x_0\}$ is a sequence in B, we obtain that both sequences converge to the same limit x. Since A and B are closed set, this implies $x \in A \cap B$.

Next, we will complete the proof by showing that x is a unique fixed point of T. Since

$$\begin{aligned}d(Tx,x) &\le b\left[d\left(Tx,T^{2n}x_0\right)+d\left(T^{2n}x_0,x\right)\right] \\ &= bd\left(Tx,T\left(T^{2n-1}x_0\right)\right)+bd\left(T^{2n}x_0,x\right) \\ &\le b\lambda\left[d\left(x,T^{2n}x_0\right)+d\left(T^{2n-1}x_0,Tx\right)\right] \\ &\quad + bd\left(T^{2n}x_0,x\right) \\ &= b\lambda d\left(x,T^{2n}x_0\right)+b\lambda d\left(T^{2n-1}x_0,Tx\right) \\ &\quad + bd\left(T^{2n}x_0,x\right) \\ &\le b\lambda d\left(x,T^{2n}x_0\right)+b^2\lambda d\left(T^{2n-1}x_0,x\right) \\ &\quad + b^2\lambda d\left(x,Tx\right)+bd\left(T^{2n}x_0,x\right),\end{aligned} \tag{108}$$

by Theorem 16, we have

$$\begin{aligned}\|d(Tx,x)\| &\le \|b\|\|\lambda\|\left\|d\left(x,T^{2n}x_0\right)\right\| \\ &\quad + \|b\|^2\|\lambda\|\left\|d\left(T^{2n-1}x_0,x\right)\right\| \\ &\quad + \|b\|^2\|\lambda\|\|d(x,Tx)\| \\ &\quad + \|b\|\left\|d\left(T^{2n}x_0,x\right)\right\|.\end{aligned} \tag{109}$$

Letting $n \to \infty$, we get that

$$\|d(Tx,x)\| \le \|b\|^2\|\lambda\|\|d(x,Tx)\|, \tag{110}$$

and so

$$\begin{aligned}\|d(Tx,x)\| &\le \|b\|^2\frac{1}{2\|b\|^2}\|d(x,Tx)\| \\ &< \frac{1}{2}\|d(x,Tx)\|.\end{aligned} \tag{111}$$

This implies that $\|d(Tx,x)\| = 0$; that is, $d(Tx,x) = \theta$ and so $Tx = x$. That is, x is fixed point of T. Now if y is another fixed point of T and $y \ne x$, then

$$\begin{aligned}\theta \le d(x,y) &= d(Tx,Ty) \\ &\le \lambda\left(d(y,Tx)+d(x,Ty)\right) = 2\lambda d(x,y).\end{aligned} \tag{112}$$

From Theorem 16, we get that

$$\begin{aligned}\|d(x,y)\| &\le \|2\lambda d(x,y)\| \le 2\|\lambda\|\|d(x,y)\| \\ &< 2\left(\frac{1}{2\|b^2\|}\right)\|d(x,y)\| \le \|d(x,y)\|,\end{aligned} \tag{113}$$

which leads to a contradiction. Therefore, $x = y$ which implies that the fixed point is unique. □

Example 45. Let $X = [0,1]$ and $\mathbb{A} = M_{2\times2}(\mathbb{R})$ with $\|A\| = \max_{i,j}|a_{ij}|$, where a_{ij} are entries of the matrix $A \in M_{2\times2}(\mathbb{R})$. Then, $(X, \mathbb{A}, d)$ is a C^*-algebra-valued b-metric space with $b = \left[\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right]$, where the involution is given by $A^* = (\overline{A})^T$:

$$d(x,y) = \begin{bmatrix}|x-y|^2 & 0\\ 0 & |x-y|^2\end{bmatrix}, \tag{114}$$

and partial ordering on $\mathbb{A}$ is given as

$$\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} \preceq \begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix} \iff a_{ij} \le b_{ij} \quad \forall i,j = 1,2,3,4. \tag{115}$$

Suppose that $A = [0,1]$ and $B = [0,1/2]$. Define a mapping $T : A \cup B \to A \cup B$ by $Tx = x/5$. Firstly, we will show that T is cyclic mapping. Let $x \in A$; that is, $0 \le x \le 1$. Then, $0 \le x/5 \le 1/5$ implies $Tx \in B$. Similarly, let $y \in B$, so $0 \le y \le 1/2$. Then, $0 \le y/5 \le 1/10$. Hence, $Ty \in A$.

Now, we will show that T satisfies the contraction of Theorem 44. Consider

$$\frac{(x-y)}{5} = \frac{1}{6}\frac{(6(x-y))}{5} = \frac{1}{6}\left(x-\frac{y}{5}+\frac{x}{5}-y\right) \tag{116}$$

and so

$$\begin{aligned}\left(\frac{(x-y)}{5}\right)^2 &= \left(\frac{1}{6}\left(x-\frac{y}{5}+\frac{x}{5}-y\right)\right)^2 \\ &= \frac{1}{36}\left(\left(x-\frac{y}{5}\right)+\left(\frac{x}{5}-y\right)\right)^2 \\ &\le \frac{1}{36}\left(2\left(x-\frac{y}{5}\right)^2+2\left(\frac{x}{5}-y\right)^2\right) \\ &= \frac{1}{18}\left(|x-Ty|^2+|Tx-y|^2\right).\end{aligned} \tag{117}$$

Then, we have

$$d(Tx,Ty)=\begin{bmatrix}|Tx-Ty|^2 & 0\\ 0 & |Tx-Ty|^2\end{bmatrix}\preceq\begin{bmatrix}\frac{1}{18}\left(|x-Ty|^2+|Tx-y|^2\right) & 0\\ 0 & \frac{1}{18}\left(|x-Ty|^2+|Tx-y|^2\right)\end{bmatrix}$$
$$=\begin{bmatrix}\frac{1}{18} & 0\\ 0 & \frac{1}{18}\end{bmatrix}\begin{bmatrix}\left(|x-Ty|^2+|Tx-y|^2\right) & 0\\ 0 & \left(|x-Ty|^2+|Tx-y|^2\right)\end{bmatrix}=\lambda\left[d(x,Ty)+d(y,Tx)\right], \tag{118}$$

where $\lambda=\begin{bmatrix}1/18 & 0\\ 0 & 1/18\end{bmatrix}$. Then, $\|\lambda\|=1/18<1/8=1/2\|b\|^2$. Thus, T satisfies contraction of Theorem 44 implying that T has a unique fixed point in $A\cap B$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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