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อันดับ n ที่มีสัมประสิทธิ์เป็นตัวแปร

ชื่อผู้วิจัย นางสาว ฐาปะนีย์ไกรวีระเดชาชัย

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On the distributional solutions of some n-th order differential equations with variable coefficients

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ON THE DISTRIBUTIONAL SOLUTIONS OF SOME N-TH ORDER DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

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ABSTRACT

In this research, we study the distributional solutions of n-th order differential equation of the form $xy^n + (m-x)y^{(n-1)}(x) - py^{(n-2)}(x) - my^{(n-3)}(x) = 0$ where $p, m \in R$, $n \geq 3$ and x is a real variable. These solutions are obtained in the form of infinite series of the dirac delta functions and its derivatives. We employ these solutions to observe their interesting features.

ACKNOWLEDGEMENT

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CHAPTER I

INTRODUCTION

Recently, there has been considerable interest in problems concerning the existence of solutions to linear ordinary differential equations and functional differential equations in various spaces of generalized functions. Many important areas in theoretical and mathematical physics, theory of partial differential equations, quantum electrodynamics, operational calculus, and functional analysis use the methods of the distribution theory. Yet for ordinary differential equations research in this direction is insufficiently developed and remains restricted to isolated results for some second-order equations or special higher-order systems. It is well known that the normal linear homogeneous system of the ordinary differential equation with infinitely smooth coefficients have no distributional solutions other them the classical ones. However, distributional solutions may appear in the case of the equations whose coefficients have singularities. A simple example is the first-order ordinary differential equation

$$x^2 \frac{dy}{dx} + 2y = 0. \tag{1.1.1}$$

The point $x = 0$ is an essential singularity of this equation. It is readily verified that the infinite series

$$y = \sum_{n=0}^{\infty} \frac{2^{n+1} \delta^{(n)}(x)}{n!(n+1)!}, \tag{1.1.2}$$

formally satisfies (1.1.1). Although (1.1.2) does not define a distribution, Kim and Kwon [7] established that the series defines a hyperfunction concentrated at $\{0\}$. Here $\delta(x)$ is the Dirac delta function and the superscript k stands for kth order differentiation.

In 1982, Wiener [14] studied various differential equations with singular coefficients and obtained their distributional solutions. Wiener and Shah [16] surveyed the work in this field and have exhibited a unified approach in the study

of both distributional and entire solutions to some classes of linear ordinary differential equations.

In 1987, Littlejohn and Kanwal [10] studied the distributional solutions to the hypergeometric differential equation. These solutions were obtained in the form of infinite series of the Dirac delta functions and its derivatives. Another motivation for studying solutions of the form of infinite series of the Dirac delta functions and its derivatives to ordinary differential equations comes from the works of Morton and Krall [11], Krall [8], and Littlejohn [9], Wiener and Cooke [1, 15, 17], and Hernandez-Estrada [3]. These researchers have collectively shown that weight distributions for a certain class of orthogonal polynomials have the form of infinite series of the Dirac delta functions and its derivatives, and simultaneously satisfy a system of ordinary differential equations.

In [4], Kamke studied the solutions of the differential equation of the form

$$xy'''(x) + (a+b)y''(x) - xy'(x) - ay(x) = 0. \tag{1.1.3}$$

He found that such equation has a distributional solution, iff a is a positive integer and b is a even positive.

In this research, our aim is to present the solutions of the form

$$\sum_{n=0}^{\infty} a_n \delta^{(n)}(x) \tag{1.1.4}$$

for nth-order differential equation of the form

$$xy^{(n)}(x) + (m\text{-}x)y^{(n-1)}(x) - py^{(n-2)}(x) - my^{(n-3)}(x) = 0, \tag{1.1.5}$$

where , m $\in \mathbb{R}$, $n \geq 3$ and x i s a real variable. It is of course interesting to derive these solutions for their intrinsic value. But we also want to display their uses and exhibit their interplay with related results in the theory of ordinary differential equations. For instance, if we let $n = 3$, the equation (1.1.5) is reduced to the equation (1.1.3).

CHAPTER II

BASIC CONCEPTS AND PRELIMINARIES

In this chapter, we review some basic knowledges of the test functions, distributions, ordinary differential operators, which will be used in our work.

2.1 Test Functions

Let $\mathbb{R}^n$ be the n-dimensional real space in which we have a Cartesian system of coordinates such that a point P is denoted by $x = (x_1, x_2, \ldots, x_n)$ and the distance r, of P from the origin, is $r = |x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. Let k be an n-tuple of nonnegative integer, $k = (k_1, k_2, \ldots, k_n)$, the so-called *multiindex* of order n, then we define

$$|k| = k_1 + k_2 + \cdots + k_n, \quad x^k = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$$

and

$$D^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}} = \frac{\partial^{k_1 + k_2 + \cdots + k_n}}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}} = D_1^{k_1} D_2^{k_2} \cdots D_n^{k_n},$$

where $D_j = \partial/\partial x_j, j = 1, 2, \ldots, n$. For the one-dimensional case, D^k reduces to d/dx. Furthermore, if any component of k is zero, the differentiation with respect to the corresponding variable is omitted.

Example 2.1.1 In $\mathbb{R}^3$, with $k = (3, 0, 4)$, we have

$$D^k = \partial^7/\partial x_1^3 \partial x_3^4 = D_1^3 D_3^4.$$

Definition 2.1.2 A function $f(x)$ is *locally integrable* in $\mathbb{R}^n$ if $\int_R |f(x)| dx$ exists for every bounded region R in $\mathbb{R}^n$. A function $f(x)$ is locally integrable on a hypersurface in $\mathbb{R}^n$ if $\int_S |f(x)| dS$ exists for every bounded region S in $\mathbb{R}^{n-1}$.

Definition 2.1.3 The *support* of a function $f(x)$ is the closure of the set of all points x such that $f(x) \neq 0$. We shall denote the support of f by supp f.

Example 2.1.4 For $f(x) = \sin x, x \in \mathbb{R}$, the support of $f(x)$ consists of the whole real line, even though $\sin x$ vanishes at $x = n\pi$.

Definition 2.1.5 If supp f is a bounded set, then f is said to have a *compact support*.

Example 2.1.6 The support of the function

$$f(x) = \begin{cases} 0, & \text{for } -\infty < x \leq -1, \\ x+1, & \text{for } -1 < x < 0, \\ 1-x, & \text{for } 0 \leq x < 1, \\ 0, & \text{for } 1 \leq x < \infty \end{cases}$$

is $[-1, 1]$, which is compact.

Definition 2.1.7 The space $\mathscr{D}$ is a linear space consist of **all real-valued** functions $\phi(x) = \phi(x_1, x_2, \ldots, x_n)$, such that the following conditions hold:

(1) $\phi(x)$ is an infinitely differentiable function defined at every point of $\mathbb{R}^n$. This means that $D^k\phi$ exists for all multiindices k. Such a function is also called a C^∞ *function*.

(2) There exists a number A such that $\phi(x)$ vanishes for $r > A$. This means that $\phi(x)$ has a compact support.

Then $\phi(x)$ is called a *test function*.

Example 2.1.8 The prototype of a test function belonging to $\mathscr{D}$ is

$$\phi(x, a) = \begin{cases} \exp\left(-\frac{a^2}{a^2 - r^2}\right), & \text{for } r < a, \\ 0, & \text{for } r \geq a, \end{cases} \tag{2.1.1}$$

for a is a constant and $r = |x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. Its support is clearly $r \leq a$.

In particular, if we consider in $\mathbb{R}$ and by taking $a = 1$, then (2.1.1) reduces to

$$\phi(x) = \begin{cases} e^{-\frac{1}{1-x^2}}, & \text{for } x \in (-1, 1), \\ 0, & \text{for } x \in (-\infty, -1] \cup [1, \infty), \end{cases} \tag{2.1.2}$$

and the support of $\phi(x)$ is [-1, 1].

The following properties of the test functions are evident.

(1) If ϕ_1 and ϕ_2 are in $\mathscr{D}$, then so is $c_1\phi_1 + c_2\phi_2$, where c_1 and c_2 are real numbers. Thus $\mathscr{D}$ is a linear space.

(2) If $\phi \in \mathscr{D}$, then so is $D^k\phi$.

(3) For a C^∞ function $f(x)$ and $\phi \in \mathscr{D}, f\phi \in \mathscr{D}$.

(4) If $\phi(x_1, x_2, \ldots, x_m)$ is an m-dimensional test function and $\psi(x_{m+1}, x_{m+2}, \ldots, x_n)$ is an $(n-m)$-dimensional test function, then $\phi\psi$ is an n-dimensional test function in the variables $x_1, x_2, \ldots, x_n$.

Definition 2.1.9 A sequence $\{\phi_m\}, m = 1, 2, \ldots$ where $\phi_m \in \mathscr{D}$, *converges* to ϕ_0 if the following two conditions are satisfied:

(1) All ϕ_m as well as ϕ_0 vanish outside a common region.

(2) $D^k\phi_m \to D^k\phi_0$ uniformly over $\mathbb{R}^n$ as $m \to \infty$ for all multiindices k.

It is not difficult to show that $\phi_0 \in \mathscr{D}$ and hence that $\mathscr{D}$ is closed (or is complete) with respect to this definition of convergence. For the special case $\phi_0 = 0$, the sequence $\{\phi_m\}$ is called a *null sequence*.

2.2   Distributions

Definition 2.2.1 A *linear functional* f on the space $\mathscr{D}$ of test functions is an operation (or a rule) by which we assign to every test function $\phi(x)$ a real number denoted $\langle f, \phi \rangle$, such that

$$\langle f, c_1\phi_1 + c_2\phi_2 \rangle = c_1 \langle f, \phi_1 \rangle + c_2 \langle f, \phi_2 \rangle, \qquad (2.2.1)$$

for arbitrary test functions ϕ_1 and ϕ_2 and real numbers c_1 and c_2.

Definition 2.2.2 A linear functional f on $\mathscr{D}$ is *continuous* if and only if the sequence of numbers $\langle f, \phi_m \rangle$ converges to $\langle f, \phi \rangle$ when the sequence of test functions $\{\phi_m\}$ converges to the test function ϕ. Thus

$$\lim_{m \to \infty} \langle f, \phi_m \rangle = \left\langle f, \lim_{m \to \infty} \phi_m \right\rangle.$$

We now have all the tools for defining the concept of distributions.

Definition 2.2.3 A continuous linear functional on the space $\mathscr{D}$ of test functions is called a *distribution*. The space of all distributions on $\mathscr{D}$ is denoted by $\mathscr{D}'$.

The set of distributions that are most useful are those generated by locally integrable functions. Indeed, every locally integrable $f(x)$ generates a distribution through the formula

$$\langle f, \phi \rangle = \int_{\mathbb{R}^n} f(x)\phi(x)dx. \tag{2.2.2}$$

Linearity of this functional is obvious. To prove its continuity, observe that

$$|\langle f, \phi \rangle| \leq \max_{x \in \text{ supp } \phi} |\phi(x)| \int_{\text{supp } \phi} |f(x)|dx < \infty.$$

Thus, if the sequence $\{\phi_m\}$ converges to zero, then so does $\langle f, \phi_m \rangle$. Hence, it is continuous.

Definition 2.2.4 Distributions defined by (2.2.2) are called *regular*. All other distributions are called *singular*. However, we may use (2.2.2) symbolically for a singular distribution also.

Example 2.2.5 The Heaviside distribution in $\mathbb{R}^n$ is $\langle H_R, \phi \rangle = \int_R \phi(x)dx$, where

$$H_R(x) = \begin{cases} 1, & \text{for } x \in \mathbb{R}_n^+, \\ 0, & \text{for } x \notin \mathbb{R}_n^+. \end{cases} \tag{2.2.3}$$

For $\mathbb{R}$, (2.2.3) becomes

$$\langle H, \phi \rangle = \int_0^\infty \phi(x)dx. \tag{2.2.4}$$

Since $H(x)$ is a piecewise continuous function, this is a regular distribution.

Definition 2.2.6 The product of a distribution t and an infinitely differentiable function f is defined by

$$\langle ft, \phi \rangle = \langle t, f\phi \rangle, \tag{2.2.5}$$

where ϕ and $f\phi$ are element of $\mathscr{D}$.

Example 2.2.7 For an infinitely differentiable function f and $\phi \in \mathscr{D}$. Hence

$$\langle f\delta, \phi \rangle = \langle \delta, f\phi \rangle = f(0)\phi(0) = f(0)\langle \delta, \phi \rangle = \langle f(0)\delta, \phi \rangle$$

or

$$f(x)\delta(x) = f(0)\delta(x). \tag{2.2.6}$$

It follows that in this special case it is sufficient for the function $f(x)$ to be continuous at the origin. More generally,

$$f(x)\delta(x - \xi) = f(0)\delta(x - \xi). \tag{2.2.7}$$

2.3   The Dirac Delta Function

We consider the Dirac delta function in the applied engineering, physics, and in other fields of sciences. For example, when the tennis ball is hit, when using hammer hit the objects, when playing drum, or when the large an electric current through the circuit in a short period. Therefore, these problems occur when in a short period of time.

$$f_k(t) = \begin{cases} 1/k & \text{for} \quad a \le t \le a + k, \\ 0 & \text{for} \quad otherwise, \end{cases} \tag{2.3.1}$$

for $a \ge 0$. If $k \to 0$, then the height of the rectangle is increased, the width is decreased but the area of the square equals a unit always remains.

$$\int_0^\infty f_k(t)dt = \int_a^{a+k} \frac{1}{k}dt = 1. \tag{2.3.2}$$

The function $f_k(t)$ can be written in the form of *Heaviside function* $H(t-a)$, that is,

$$f_k(t) = \frac{1}{k}[H(t - a) - H(t - (a + k))], \tag{2.3.3}$$

where

$$H(t - a) = \begin{cases} 0 & \text{for} \quad t < a, \\ 1 & \text{for} \quad t > a. \end{cases} \tag{2.3.4}$$

Definition 2.3.1 Let $f(t)$ be a function of t, for $t \geq 0$. The Laplace transform of $f(t)$, denoted $\mathscr{L}\{f(t)\}$ or $\mathscr{F}(s)$, is defined by

$$\mathscr{L}\{f(t)\} = \mathscr{F}(s) = \int_0^\infty e^{-st} f(t) dt \qquad (2.3.5)$$

By using the formula $\mathscr{L}\{H(t-a)\} = \frac{e^{-as}}{s}$ for $s > 0$ and $a \geq 0$, we have

$$\mathscr{L}\{f_k(t)\} = \frac{1}{ks}\left[e^{-as} - e^{-(a+k)s}\right] = e^{-as}\left[\frac{1 - e^{-ks}}{ks}\right]. \qquad (2.3.6)$$

Limits of $f_k(t)$ when $k \to 0$ denoted by $\delta(t{-}a)$ and called *Dirac delta function* or *unit impulse function*. By L'Hospital's rule, we obtain

$$\lim_{k \to 0} e^{-as}\left[\frac{1 - e^{-ks}}{ks}\right] = e^{-as}. \qquad (2.3.7)$$

Thus, equation (2.3.6) becomes,

$$\mathscr{L}\{\delta(t-a)\} = e^{-as}, \qquad (2.3.8)$$

and if $a \to 0$, then

$$\mathscr{L}\{\delta(t)\} = 1. \qquad (2.3.9)$$

Note that $\delta(t{-}a)$ is not ordinary function as a general introductory calculus. Since, if $k \to 0$, then we have

$$\delta(t - a) = \begin{cases} \infty & \text{for} \quad t = a, \\ 0 & \text{for} \quad t \neq a. \end{cases} \qquad (2.3.10)$$

and

$$\int_0^\infty \delta(t - a) dt = 1. \qquad (2.3.11)$$

But ordinary function which is zero everywhere except at one point just to the integral of zero. However, in impulse problems are described by the Dirac delta function $\delta(t - a)$, which has several significant properties:

1. $\int_0^\infty \delta(t - a) f(t) dt = f(a),$

2. $\int_0^\infty \delta^{(n)}(t - a) f(t) dt = (-1)^n f^{(n)}(a).$

Example 2.3.2 Let $f(t)$ be a function, t and $a \geq 0$

$$\mathscr{L}\{f(t)\delta(t-a)\} = \int_0^\infty e^{-st}f(t)\delta(t-a)dt$$

$$= e^{-sa}f(a).$$

$$\mathscr{L}\{\delta^{(n)}(t-a)\} = \int_0^\infty e^{-st}\delta^{(n)}(t-a)dt$$

$$= (-1)^n\frac{d^n}{dt^n}(e^{-st})\,|_{t=a}$$

$$= s^n e^{-sa}.$$

2.4 Ordinary Differential Operators

Definition 2.4.1 Consider the differential operator L defined by,

$$Lt = \left[a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1(x)\frac{d}{dx} + a_0(x)\right]t$$

$$= \sum_{m=0}^n a_m(x)\frac{d^m t}{dx^m}, \tag{2.4.1}$$

where the coefficients $a_m(x)$ are infinitely differentiable function.

The solution of the ordinary differential equation

$$Lt = \sum_{m=0}^n a_m(x)\frac{d^m t}{dx^m} = \tau, \tag{2.4.2}$$

where τ is an arbitrary known distribution. A distribution t is a solution of (2.4.2) if for every test function ϕ, we have

$$\langle Lt, \phi \rangle = \langle \tau, \phi \rangle. \tag{2.4.3}$$

It is well known that the fundamental solution is the solution for $\tau = \delta(x)$. In searching for a solution t of differential equation (2.4.2) we may have the following situations:

(1) The solution t is a sufficiently smooth function, so that the operation in (2.4.2) can be performed in the classical sense and resulting equation is an identity. Then t is the *classical solution*.

(2) The solution t is not sufficiently smooth, so that the operation in (2.4.2) cannot be performed, but it satisfies (2.4.3) as a distribution. It is then a *weak solution*.

(3) The solution t is a singular distribution and satisfies (2.4.3). It is then a *distributional solution*.

All these solution are called *generalized solutions*.

Example 2.4.2 To fined the general solution of the equation

$$\frac{x^m dt}{dx} = 0, \quad m \geq 1, \tag{2.4.4}$$

we appeal to the relation $dH/dx = \delta(x)$ and use the derivatives of $\delta(x)$. Indeed, we assume that

$$t(x) = c_1 + c_2 H(x) + c_3 \delta(x) + c_4 \delta'(x) + \cdots + c_{m+1} \delta^{(m-2)}(x), \tag{2.4.5}$$

so that

$$t'(x) = c_2 \delta(x) + c_3 \delta'(x) + \cdots + c_{m+1} \delta^{(m-1)}(x).$$

Example 2.4.3

$$\langle x^m t'(x), \phi \rangle = \langle c_2 x^m \delta(x), \phi(x) \rangle + \langle c_3 x^m \delta'(x), \phi(x) \rangle$$
$$+ \cdots + \langle c_{m+1} x^m \delta^{(m-1)}(x), \phi(x) \rangle = 0,$$

and we have $x^m t' = 0$ as required. Hence, (2.4.5) is the general solution of (2.4.4).

For $m = 1$ the solution reduces to

$$t(x) = c_1 + c_2 H(x). \tag{2.4.6}$$

The Heaviside function $H(x)$, although an ordinary function, is not differentiable. Therefore (2.4.6) is a weak solution.

For $m \geq 2$, (2.4.5) is the distributional solution.

Example 2.4.4 The distribution $y(x) = \delta(x)$ is the solution of the following equations: The confluent hypergeometric equation

$$xy'' + (2 - x)y' - y = 0. \tag{2.4.7}$$

The hypergeometric equation

$$x(1 - x)y'' + (2 - 5x)y' - 3y = 0. \tag{2.4.8}$$

The Bessel equation

$$x^2 y'' + xy' + (x^2 - 1)y = 0. \tag{2.4.9}$$

Example 2.4.5 In fact, by applying the formula

$$x^m \delta^{(n)}(x) = \begin{cases} (-1)^m \dfrac{n!}{(n - m)!} \delta^{(n-m)}(x) & \text{for } n \geq m, \\ 0 & \text{for } n < m, \end{cases} \tag{2.4.10}$$

it is easy to verify that $y(x) = \delta(x)$ satisfy the equations mentioned above.

CHAPTER III

ON THE DISTRIBUTIONAL SOLUTIONS OF SOME nTH-ORDER DIFFERENTIAL EQUATIONS

3.1 Main Results

Theorem 3.1.1 *Suppose that $w(x) = \sum_{k=0}^{\infty} a_k \delta^{(k)}(x)$ is a formal **distributional** solution to nth-order differential equation of the form*

$$xy^{(n)}(x) + (\text{m-x})y^{(n-1)}(x) - py^{(n-2)}(x) - my^{(n-3)}(x) = 0, \qquad (3.1.1)$$

where $p, \text{m} \in \mathbb{R}, n \geq 3$ and x is a real variable. Then $w(x)$ satisfies the following properties:

(i) if $\text{m} \notin \{n-2, n-1\}$, then $w(x) \equiv 0$;

(ii) if $\text{m} = n-2$, then

$$w(x) = a_0 \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{k}(2j+\text{x})}{2^k k!} \delta^{(2k)}(x); \qquad (3.1.2)$$

(iii) if $\text{m} = n-1$, then

$$w(x) = a_1 \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{k}(2j+\text{x})}{2^k k!} \delta^{(2k+1)}(x). \qquad (3.1.3)$$

Proof. First we have to appeal to the basic concept of distribution theory, namely, use the test function $\phi(x) \in \mathscr{D}(\mathbb{R})$. Accordingly, we have to examine the quantity

$$\langle xy^{(n)} + (\text{m -x})y^{(n-1)} \text{ -p}y^{(n-2)} - my^{(n-3)}, \phi \rangle = \langle xy^{(n)}, \phi \rangle + \langle (\text{m-x})y^{(n-1)}, \phi \rangle$$
$$+ \langle -py^{(n-2)}, \phi \rangle + \langle -my^{(n-3)}, \phi \rangle .$$

$$(3.1.4)$$

Now

$$\langle xy^{(n)}, \phi \rangle = \langle y^{(n)}, x\phi \rangle = (-1)^n \langle y, (x\phi)^{(n)} \rangle = (-1)^n \langle y, x\phi^{(n)} + n\phi^{(n-1)} \rangle,$$
(3.1.5)

$$\langle (\text{m-x})y^{(n-1)}, \phi \rangle = \langle y^{(n-1)}, (\text{m-x})\phi \rangle = (-1)^{n-1} \langle y, (\text{m-x})\phi^{(n-1)} \rangle, \qquad (3.1.6)$$

$$\langle -py^{(n-2)}, \phi \rangle = \langle y^{(n-2)}, -p\phi \rangle = (-1)^{n-2} \langle y, (-p\phi)^{(n-2)} \rangle$$
$$= (-1)^{n-1} \langle y, p\phi^{(n-2)} + (n-2)\phi^{(n-3)} \rangle \qquad (3.1.7)$$

and

$$\langle -my^{(n-3)}, \phi \rangle = \langle y^{(n-3)}, -m\phi \rangle = (-1)^{n-3} \langle y, -m\phi^{(n-3)} \rangle. \qquad (3.1.8)$$

Next, we substitute the series $y(x) = \sum_{k=0}^{\infty} a_k \delta^{(k)}(x)$ in the right-hand side of (3.1.5) to (3.1.8), we have

$$\langle xy^{(n)}, \phi \rangle = \left\langle -\sum_{k=0}^{\infty} (n+k)a_k \delta^{(k+n-1)}(x), \phi \right\rangle$$

$$\langle (\text{m-x})y^{(n-1)}, \phi \rangle = \left\langle \sum_{k=0}^{\infty} (\text{m-x})a_k \delta^{(k+n-1)}(x), \phi \right\rangle$$

$$\langle -py^{(n-2)}, \phi \rangle = \left\langle \sum_{k=0}^{\infty} (n+k-2)a_k \delta^{(k+n-3)}(x), \phi \right\rangle$$

and

$$\langle -my^{(n-3)}, \phi \rangle = \left\langle -\sum_{k=0}^{\infty} ma_k \delta^{(k+n-3)}(x), \phi \right\rangle.$$

Substituting these value in (3.1.4), we obtain

$$
x\overset{(n)}{y}+\overset{(n-1)}{(\text{m-x})y}-\overset{(n-2)}{py}-\overset{(n-3)}{py},\phi\Big\rangle
$$

$$
=\Big\langle -\sum_{k=0}^{\infty}(n+k)a_k\delta^{(k+n-1)}(x)+\sum_{k=0}^{\infty}(\text{ m-x})a_k\delta^{(k+n-1)}(x)+\sum_{k=0}^{\infty}(n+k-2)a_k\delta^{(k+n-3)}(x)
$$

$$
-\sum_{k=0}^{\infty}ma_k\delta^{(k+n-3)}(x),\phi\Big\rangle
$$

$$
=\Big\langle \sum_{k=0}^{\infty}(\text{m-x}-n-k)a_k\delta^{(k+n-1)}(x)+\sum_{k=0}^{\infty}(n+k-\text{m}-2)a_k\delta^{(k+n-p-3)}(x),\phi\Big\rangle
$$

$$
=\Big\langle \sum_{r=2}^{\infty}(\text{m -x}-n-r+2)a_{r-2}\delta^{(r+n-3)}(x)+\sum_{r=0}^{\infty}(n+r-\text{m}-2)a_r\delta^{(r+n-3)}(x),\phi\Big\rangle
$$

$$
=\Big\langle \sum_{r=2}^{\infty}[(\text{m-x}-n-r+2)a_{r-2}+(n+r-\text{m}-2)a_r]\delta^{(r+n-3)}(x)
$$

$$
+(n-\text{m}-2)a_0\delta^{(n-3)}(x)+(n-\text{m}-1)a_1\delta^{(n-2)}(x),\phi\Big\rangle
\tag{3.1.9}
$$

From (3.1.9), it follows that, if $y(x)$ is a solution of (3.1.1), then $(n-\text{m}-2)a_0=0$ and $(n-\text{m}-1)a_1=0$ and for $r=2,3,4,\ldots$, we have recurrence relation

$$
(\text{m-x}-n-r+2)a_{r-2}-(\text{m}-n-r+2)a_r=0.
\tag{3.1.10}
$$

In order to find the coefficients a_r, we consider the following cases:

If $\text{m}\notin\{n-2,n-1\}$, then $\text{m}-n-r+2\neq 0$, and thus

$$
a_r=a_{r-2}\frac{(\text{m}+q-n+2)}{(\text{m}-n-r+2)}.
\tag{3.1.11}
$$

But since $a_0=0$ and $a_1=0$, we find that $a_r=0$ for all $r\geq 2$. This therefore yields $w(x)=0$.

If $p=n-2$, then $a_0\neq 0$. Using the recurrence relation (3.1.11), we have

$$
a_2=\frac{a_0(2+\text{x})}{2},
$$

$$
a_4=\frac{a_2(4+\text{x})}{4}=\frac{a_0(2+\text{x})(4+\text{x})}{2\cdot 4},
$$

$$
a_6=\frac{a_4(6+\text{x})}{6}=\frac{a_0(2+\text{x})(4+\text{x})(6+\text{x})}{2\cdot 4\cdot 6},
$$

$$
\vdots
$$

$$
a_{2k}=a_0\frac{(2+\text{x})(4+\text{x})(6+\text{x})\cdots(2k+\text{x})}{2^k k!}.
$$

for all $k \geq 1$. Therefore, the solution is

$$w(x) = a_0 \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{k}(2j - \mathrm{x})}{2^k k!} \delta^{(2k)}(x). \qquad (3.1.12)$$

And if $\mathrm{m} = n - 1$, then $a_1 \neq 0$. Using there currence relation(3.1.11)we have

$$a_3 = \frac{a_1(2 + \mathrm{x})}{2},$$

$$a_5 = \frac{a_3(4 + \mathrm{x})}{4} = \frac{a_1(2 + \mathrm{x})(4 + \mathrm{x})}{2 \cdot 4},$$

$$a_7 = \frac{a_5(6 + \mathrm{x})}{6} = \frac{a_1(2 + \mathrm{x})(4 + \mathrm{x})(6 + \mathrm{x})}{2 \cdot 4 \cdot 6},$$

$$\vdots$$

$$a_{2k+1} = a_1 \frac{(2 + \mathrm{x})(4 + \mathrm{x})(6 + \mathrm{x}) \cdots (2k + \mathrm{x})}{2^k k!},$$

for all $k \geq 1$. Therefore, the solution is

$$w(x) = a_1 \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{k}(2j + \mathrm{x})}{2^k k!} \delta^{(2k+1)}(x). \qquad (3.1.13)$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Corollary 3.1.2 *Consider the n-th order differential equation of the form*

$$xy^{(n)}(x) + (2\mathrm{p} + n - 2)y^{(n-1)}(x) - xy^{(n-2)}(x) - (n - 2)y^{(n-3)}(x) = 0, (3.1.14)$$

where p and n are positive integers with $n \geq 3$ for x is a real variable. Then the distributional solution of (3.1.14) as

$$y(x) = C \sum_{k=0}^{m-1} (-1)^{-k-1} \binom{\mathrm{p} - 1}{k} \delta^{(2k)}(x), \qquad (3.1.15)$$

where C is any constant.

Corollary 3.1.3 *Consider the n-th order differential equation of the form*

$$xy^{(n)}(x) + (2\mathrm{p} + n - 1)y^{(n-1)}(x) - xy^{(n-2)}(x) - (n - 1)y^{(n-3)}(x) = 0, (3.1.16)$$

where p and n are positive integers with $n \geq 3$ for x is a real variable. Then the distributional solution of (3.1.16) as

$$y(x) = C \sum_{k=0}^{m-1} (-1)^{-k-1} \binom{\mathrm{p} - 1}{k} \delta^{(2k+1)}(x), \qquad (3.1.17)$$

where C is any constant.

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ตามที่ข้าพเจ้า ดร.ฐานะนีย์ ไกรวีระเดชาชัย (ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์) ได้ส่งผลงานทางวิชาการการรายงานการวิจัย (เรื่อง) รายงานการวิจัยผลเฉลยเชิงดิสทริบิวชันของสมการเชิงอนุพันธ์อันดับ n ที่มีสัมประสิทธิ์เป็นตัวแปร

ปีที่พิมพ์   2557

ข้าพเจ้าขอรับรองว่า ผลงานทางวิชาการเป็นลิขสิทธิ์ของข้าพเจ้า ดร.ฐานะนีย์ ไกรวีระเดชาชัย เป็นเจ้าของลิขสิทธิ์ และเพื่อให้ผลงานทางวิชาการของข้าพเจ้าเป็นประโยชน์ต่อการศึกษาและสาธารณชน จึงอนุญาตให้เผยแพร่ผลงาน ดังนี้

    ✓   อนุญาตให้เผยแพร่

    ◯   ไม่อนุญาตให้เผยแพร่ เนื่องจาก...

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ลงชื่อ ....

(..........ฐานะนีย์ ไกรวีระเดชาชัย..........)

วันที่ ......12 กันยายน 2559......

หมายเหตุ   ลิขสิทธิ์ใดๆ ที่ปรากฏอยู่ในผลงานนี้เป็นความรับผิดชอบของเจ้าของผลงาน ไม่ใช่ของสำนัก**หอสมุด**