อภินันทนาการ

สัญญาเลขที่ สัญญาเลขที่ R255 7C078

รายงานวิจัยฉบับสมบูรณ์

โครงการ : ทฤษฎีบทจุดตรึงสำหรับการส่งแบบแอลฟากึ่งไม่ขยาย

FIXED POINT THEOREMS FOR α-QUASI-NONEXPANSIVE MAPPINGS

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สนับสนุนโดยกองทุนวิจัยมหาวิทยาลัยนเรศวร

กิตติกรรมประกาศ (Acknowledgement)

รายงานการวิจัยฉบับนี้สำเร็จลุล่วงได้ ข้าพเจ้าและคณะผู้ทำวิจัยขอขอบพระคุณทางมหาวิทยาลัยนเรศวร ที่ให้ทุนอุดหนุนการวิจัยจากงบประมาณรายได้ กองทุนวิจัยมหาวิทยาลัยนเรศวร ประจำปีงบประมาณ พ.ศ. 2553 เป็นจำนวนเงินทั้งสิ้น 180,000 บาท

ผู้วิจัยขอกราบขอบพระคุณ ศ.ดร. สุเทพ สวนใต้ และครูอาจารย์ที่ประสิทธิประสาทวิชาความรู้เป็นอย่างสูงที่มีส่วนช่วยในการให้คำชี้แนะและเป็นผู้ที่สร้างแรงบันดาลใจให้ทำงานวิจัยชิ้นนี้ขึ้นมา และครอบครัวที่คอยเป็นกำลังกำลังใจให้ข้าพเจ้าตลอดมา

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Fixed point theorems for α-quasi nonexpansive mappings

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ได้รับทุนอุดหนุนการวิจัยจาก กองทุนวิจัยมหาวิทยาลัยนเรศวร

จำนวนเงิน 180,000 บาท

ระยะเวลาการทำวิจัย 1 ปี

บทคัดย่อ

ในงานวิจัยนี้ เราได้ศึกษาทฤษฎีบทจุดตรึงสำหรับการส่งแบบแอลฟากึ่งไม่ขยายและได้ขยายแนวคิดแนะนำปริภูมิดิสโลเคทเตดกึ่งบีเมตริกซ์และได้พิสูจน์การมีจุดตรึงสำหรับการส่งแบบหดตัววัฏจักรในปริภูมิดังกล่าว

ABSTRACT

In this research, we study fixed point theorems for α-quasi nonexpansive mappings and introduced dislocated quasi-b-metric spaces and prove the existence of fixed point theorems for such spaces

Contents

Chapter 1
Introduction

Let X be a nonempty set and $T : X \to X$ a self map. We say that $p \in X$ is a *fixed point* of T if $p = Tp$ and denote by $F(T)$ the set of all fixed points of T. Having in view that many of the most important nonlinear problems of applied mathematics reduce to solving a given equation which in turn may be reduced to finding the fixed points of a certain operator, on the other hand, the metrical fixed point theory has developed significantly in the second part of the 20th century.

The fixed point theory is concerned with finding conditions on the structure that the set X must be endowed as well as on the properties of the operator $T : X \to X$, in order to obtain results on;
1. the existence (and uniqueness) of fixed points;
2. the structure of the fixed point sets;
3. the approximation of fixed points.

The ambient spaces X involved in fixed point theorems cover a variety of spaces: metric space, normed linear space, generalized metric space, uniform space, linear topological space etc., while the conditions imposed on the operator T are generally metrical or compactness type conditions. A plethora of metrical fixed point theorems have been obtained, more or less important from a theoretical point of view, which establish usually the existence, or the existence and uniqueness of fixed points for a certain nonlinear mappings. Among these fixed point theorems, only a small number are important from a practical point of view, that, they offer a constructive method for finding the fixed points.

However, from a practical point of view it is important not only to know the fixed point exists (and, possible, is unique), but also to be able to construct that fixed points.

As the constructive methods used in metrical fixed point theory are prevailingly iterative procedures, that is, approximate methods, it is also of crucial importance to have a priori or/and a posteriori error estimates (or rate of convergence) for such method. For example, the Banach fixed point theorem concerns certain mappings (contractions) of a complete metric space into itself.

Fixed point Theory is studying, losing many of which can be seen from the works of many authors [4, 6, 12, 14, 16, 17]. Banach contraction principle was introduced in 1922 by Banach [3] as follows:
(i) Let (X, d) be a metric space and let $T : X \to X$. Then T is called a Banach contraction mapping if there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$.

The concept of Kannan mapping was introduced in 1969 by Kannan [8] as follows: (ii) T is called a Kannan mapping if there exists $r \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq rd(x, Tx) + rd(y, Ty)$$

for all $x, y \in X$.

Now, we recall definition of Cyclic map. Let Let A and B be nonempty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$. T is called a cyclic map iff $T(A) \subseteq B$ and $T(B) \subseteq A$.

In 2003, Kirk, Srinivasan and Veeramani [11] introduced cyclic contraction as follows:

(iii) A cyclic map $T : A \cup B \to A \cup B$ is said to be a cyclic contraction if there exists $a \in [0, 1)$ such that

$$d(Tx, Ty) \leq ad(x, y)$$

for all $x \in A$ and $y \in B$.

In 2010, Karapinar and Erhan [9] introduced Kannan type cyclic contraction as follows:

(iv) A cyclic map $T : A \cup B \to A \cup B$ is called a Kannan type cyclic contraction if there exists $b \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq bd(x, Tx) + bd(y, Ty)$$

for all $x \in A$ and $y \in B$.

If (X, d) is complete metric spaces, at least one of (i), (ii), (iii) and (iv) holds, then have a unique fixed point [3,8,11,9]. Next, we discuss the development of spaces. Conception of quasi-metric spaces was introduced by Wilson [19] in 1931 as a generalization of metric space, and in 2000 Hitzler and Seda [7] introduced dislocated metric space as a generalization of metric space, [28] generalized the result of Hitzler, Seda and Wilson and introduced the concept of dislocated quasi-metric space. Włodarczyk et al. [20-27] have created uniform spaces as this is the concept of metric spaces. In 1989, Bakhtin [2] introduced b-metric space as a generalization of metric space, Moreover, Czerwik [5] make the results of the Bakhtin is known more in 1998. Finally, in many other generalized b-metric space, such as, quasi b-metric space [15], b-metric-like space [1], quasi b-metric-like space [30].

In this research, we prove fixed point theorems for α-nonexpansive mappings and introduced dislocated quasi b-metric spaces which generalizes the quasi b-metric spaces and b-metric-like spaces and introduced the notion of Geraghty type dqb-cyclic-Banach Contraction, dqb-cyclic-Kannan mapping and derive the existence of fixed point theorems for such space. Our main theorems extends and unifies existing results in the recent literature.

Chapter 2
Preliminaries

We begin with the following definition as a recall from [7, 19].

Definition 2.0.1. *[3, 7, 19] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to [0, \infty)$ satisfies the following conditions:*

(d_1) $d(x, x) = 0$ *for all* $x \in X$;

(d_2) $d(x, y) = d(y, x) = 0$ *implies* $x = y$ *for all* $x, y \in X$;

(d_3) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(d_4) $d(x, y) \leq [d(x, z) + d(z, y)]$, *for all* $x, y, z \in X$.

If d satisfies the condition (d_1), (d_2) and (d_4) then d is called a quasi metric *on X. If d satisfies the condition (d_2), $(d_3)$and (d_4) then d is called a* dislocated metric *on X. If d satisfies the condition $(d_1) - (d_4)$ then d is called a* metric *on X.*

In 2005 the concept of dislocated quasi-metric space [28], which is a new generalization of quasi b-metric space and dislocated b-metric space, by definition 2.0.1 setting the condition (d_2) and (d_4) hold true then d is called a *dislocated quasi-metric* on X.

Remark 2.0.2. *It is obvious that metric spaces is quasi metric spaces and dislocated metric but conversely is not true.*

In 1989, Bakhthin[2] introduced the concept of b-metric spaces and investigated some fixed point theorems in such spaces.

Definition 2.0.3. *[2] Let X be a nonempty set. Suppose that the mapping $b : X \times X \to [0, \infty)$ such that the constant $s \geq 1$ satisfies the following conditions:*

(b_1) $b(x, y) = b(y, x) = 0 \Leftrightarrow x = y$ *for all* $x, y \in X$;

(b_2) $b(x, y) = b(y, x)$ *for all* $x, y \in X$;

(b_3) $b(x, y) \leq s[b(x, z) + b(z, y)]$, *for all* $x, y, z \in X$.

The pair (X, b) is then called a b-metric space.

Remark 2.0.4. *It is obvious that metric spaces are b-metric spaces, but conversely is not true.*

In 2012, Shah and Hussain[15] introduced the concept of quasi b-metric spaces and verify some fixed point theorems in quasi b-metric spaces.

Definition 2.0.5. *[15] Let X be a nonempty set. Suppose that the mapping $q : X \times X \to [0, \infty)$ such that constant $s \geq 1$ satisfies the following conditions:*

(q_1) $q(x, y) = q(y, x) = 0 \Leftrightarrow x = y$ *for all* $x, y \in X$;

(q_2) $q(x, y) \leq s[q(x, z) + q(z, y)]$, *for all* $x, y, z \in X$.

The pair (X, b) is then called a quasi b-metric spaces.

Remark 2.0.6. *It is obvious that b-metric spaces is quasi b-metric spaces, but conversely is not true.*

Recently, the concept of b-metric-like spaces, which is a new generalization of metric-like spaces, was introduced by Alghamdi et al. [3].

Definition 2.0.7. *[15] Let X be a nonempty set. Suppose that the mapping $D : X \times X \to [0, \infty)$ such that constant $s \geq 1$ satisfies the following conditions:*

(D_1) $D(x, y) = 0 \;\Rightarrow\; x = y$ *for all* $x, y \in X$;

(D_2) $D(x, y) = D(y, x)$ *for all* $x, y \in X$;

(D_3) $D(x, y) \leq s[D(x, z) + D(z, y)]$, *for all* $x, y, z \in X$.

The pair (X, D) is then called a b-metric-like spaces(or dislocated b-metric spaces).

Remark 2.0.8. *It is obvious that b-metric spaces is b-metric-like spaces, but conversely is not true.*

In this research, we introduced dislocated quasi b-metric spaces which generalizes the quasi b-metric spaces and b-metric-like spaces and introduced the notion of Geraghty type dqb-cyclic-Banach Contraction, dqb-cyclic-Kannan mapping and derive the existence of fixed point theorems for such space. Our main theorems extends and unifies existing results in the recent literature.

Chapter 3
Main Results

3.1 α-quasi nonexpansive mapping

In this section, we introduce a new mapping which is called α-quasi nonexpansive mapping and prove some fixed point theorems for α-quasi nonexpansive mappings.

Definition 3.1.1. *For a given multiindex $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ satisfies $\alpha_i \geq 0, i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \alpha_i = 1$. Let C be a nonempty closed and convex subset of Banach space E.*
A mapping $T : C \to C$ is said to be α-quasi nonexpansive mapping, if

$$\sum_{i=1}^{n} \alpha_i \|T^i x - p\| \leq \|x - p\|, \forall p \in F(T), \forall x \in C.$$

The last observation is that the mapping implies that for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ the mapping

$$T_\alpha x = \sum_{i=1}^{n} \alpha_i T^i x, \forall x \in C,$$

is quasi nonexpansive. Indeed

$$\begin{aligned}
\|T_\alpha x - p\| &= \| \sum_{i=1}^{n} \alpha_i T^i x - p\| \\
&= \| \sum_{i=1}^{n} \alpha_i T^i x - \sum_{i=1}^{n} \alpha_i p\| \\
&= \sum_{i=1}^{n} \alpha_i \|T^i x - p\| \\
&\leq \|x - p\|.
\end{aligned}$$

Theorem 3.1.2. *Let C be a nonempty closed convex subset of Banach space E and for all $n \in \mathbb{N}$, let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ such that $\alpha_i \geq 0$, $i = 1, 2, \ldots, n$, $\alpha_1 > 0$. $\sum_{i=1}^{n} \alpha_i = 1$. Let T be an α-quasi nonexpansive mapping from C into itself. We define mapping $T_\alpha x = \sum_{i=1}^{n} \alpha_i T^i x$, for all $x \in C$. Suppose that $\alpha_1 > \frac{1}{\sqrt[n]{2}}$ and T satisfies CP-condition. Then $F(T) = F(T_\alpha)$.*

Proof. It is clear that $F(T) \subseteq F(T_\alpha)$. Next, we show that $F(T_\alpha) \subseteq F(T)$. Let $x \in F(T_\alpha)$, we have

$$\begin{aligned}
\|x - Tx\| &= \|T_\alpha x - Tx\| \\
&= \|\sum_{i=1}^{n} \alpha_i T^i x - \sum_{i=1}^{n} \alpha_i Tx\| \\
&= \|(\alpha_1 Tx + \alpha_2 T^2 x + \cdots + \alpha_n T^n x) - (\alpha_1 + \alpha_2 + \cdots + \alpha_n)Tx\| \\
&= \|\alpha_1(Tx - Tx) + \alpha_2(T^2 x - Tx) + \alpha_3(T^3 x - Tx) + \cdots \\
&\quad + \alpha_n(T^n x - Tx)\| \\
&\leq \alpha_2(\|T^2 x - x\| + \|x - Tx\|) + \alpha_3(\|T^3 x - x\| + \|x - Tx\|) + \cdots \\
&\quad + \alpha_n(\|T^n x - x\| + \|x - Tx\|) \\
&= \alpha_2\|T^2 x - x\| + \alpha_2\|x - Tx\| + \alpha_3\|T^3 x - x\| + \alpha_3\|x - Tx\| + \cdots \\
&\quad + \alpha_n\|T^n x - x\| + \alpha_n\|x - Tx\| \\
&= (\alpha_2 + \alpha_3 + \cdots + \alpha_n)\|x - Tx\| + (\alpha_2\|T^2 x - x\| + \alpha_3\|T^3 x - x\| + \cdots \\
&\quad + \alpha_n\|T^n x - x\|) \\
&\leq (\alpha_2 + \alpha_3 + \cdots + \alpha_n)\|x - Tx\| + (\frac{\alpha_2}{\alpha_1}\|T^2 x - x\| + \frac{\alpha_3}{\alpha_1^2}\|T^3 x - x\| + \cdots \\
&\quad + \frac{\alpha_n}{\alpha_1^{n-1}}\|T^n x - x\|) \\
&\leq (1 - \alpha_1)\|x - Tx\| + (\frac{\alpha_2}{\alpha_1}\|Tx - x\| + \frac{\alpha_3}{\alpha_1^2}\|Tx - x\| + \cdots + \frac{\alpha_n}{\alpha_1^{n-1}}\|Tx - x\|) \\
&= (1 - \alpha_1 + \frac{\alpha_2}{\alpha_1} + \frac{\alpha_3}{\alpha_1^2} + \cdots + \frac{\alpha_n}{\alpha_1^{n-1}})\|Tx - x\| \\
&\leq \alpha_1(\frac{1-\alpha_1}{\alpha_1} + \frac{1-\alpha_1}{\alpha_1^2} + \frac{1-\alpha_1}{\alpha_1^3} + \cdots + \frac{1-\alpha_1}{\alpha_1^n})\|Tx - x\| \\
&= \alpha_1(\frac{1-\alpha_1^n}{\alpha_1^n})\|Tx - x\|.
\end{aligned}$$

since $\alpha_1 > \frac{1}{\sqrt[n]{2}}$, we have $\frac{1-\alpha_1^n}{\alpha_1^n} < 1$.This implies that $Tx = x$ and $x \in F(T)$. Hence $F(T) = F(T_\alpha)$, as required. □

Lemma 3.1.3. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be as in Theorem 3.1.2. and let C be a nonempty closed convex subset of Banach space E. Let T be an α-quasi nonexpansive mapping from C into itself. Suppose that $\alpha_1 > \frac{1}{\sqrt[n]{2}}$ and T satisfies CP-condition. Let $\{x_n\}$ be a sequence in C. Then $\|x_m - Tx_m\| \to 0$ if and only if $\|x_m - T_\alpha x_m\| \to 0$ as $m \to 0$.*

Proof. Suppose that $\|x_m - Tx_m\| \to 0$. By α-quasi nonexpansive mapping, we have

$$
\begin{aligned}
\|T_\alpha x_m - x_m\| &= \|\sum_{i=1}^{n} \alpha_i T^i x_m - x_m\| \\
&= \|\sum_{i=1}^{n} \alpha_i T^i x_m - \sum_{i=1}^{n} \alpha_i x_m\| \\
&= \|(\alpha_1 T x_m + \alpha_2 T^2 x_m + \cdots + \alpha_n T^n x_m) - (\alpha_1 + \alpha_2 + \cdots + \alpha_n) x_m\| \\
&= \|\alpha_1 (T x_m - x_m) + \alpha_1 (T^2 x_m - x_m) + \cdots + \alpha_n (T^n x_m - x_m)\| \\
&\leq \alpha_1 \|T x_m - x_m\| + \alpha_2 \|T^2 x_m - x_m\| + \cdots + \alpha_n \|T^n x_m - x_m\| \\
&\leq \alpha_1 \|T x_m - x_m\| + \frac{\alpha_2}{\alpha_1} \|T x_m - x_m\| + \frac{\alpha_3}{\alpha_1^2} \|T x_m - x_m\| + \cdots \\
&\quad + \frac{\alpha_n}{\alpha_1^{n-1}} \|T x_m - x_m\| \\
&= (\alpha_1 + \frac{\alpha_2}{\alpha_1} + \frac{\alpha_3}{\alpha_1^2} + \cdots + \frac{\alpha_n}{\alpha_1^{n-1}}) \|T x_m - x_m\|.
\end{aligned}
$$

Which implies that $\|T_\alpha x_m - x_m\| \to 0$ as $m \to 0$. Next, suppose that $\|x_m - T_\alpha x_m\| \to 0$

$$
\begin{aligned}
\|x_m - T x_m\| &= \|x - T_\alpha x_m + T_\alpha x_m - T x_m\| \\
&\leq \|x_m - T_\alpha x_m\| + \|T_\alpha x_m - T x_m\| \\
&= \|x_m - T_\alpha x_m\| + \|\sum_{i=1}^{n} \alpha_i T^i x_m - \sum_{i=1}^{n} \alpha_i T x_m\| \\
&= \|x_m - T_\alpha x_m\| + \|(\alpha_1 T x_m + \alpha_2 T^2 x_m + \cdots + \alpha_n T^n x_m) \\
&\quad - (\alpha_1 + \alpha_2 + \cdots + \alpha_n) T x_m\| \\
&= \|x_m - T_\alpha x_m\| + \|\alpha_1 (T x_m - T x_m) + \alpha_2 (T^2 x_m - T x_m) + \cdots + \\
&\quad + \alpha_n (T^n x_m - T x_m)\| \\
&\leq \|x_m - T_\alpha x_m\| + \alpha_2 \|T^2 x_m - T x_m\| + \alpha_3 \|T^3 x_m - T x_m\| + \cdots \\
&\quad + \alpha_n \|T^n x_m - T x_m\| \\
&= \|x_m - T_\alpha x_m\| + \alpha_2 \|T^2 x_m - x_m + x_m - T x_m\| + \\
&\quad + \alpha_3 \|T^3 x_m - x_m + x_m - T x_m\| + \cdots + \alpha_n \|T^n x_m - x_m + x_m - T x_m\| \\
&\leq \|x_m - T_\alpha x_m\| + \alpha_2 \|T^2 x_m - x_m\| + \alpha_2 \|x_m - T x_m\| + \alpha_3 \|T^3 x_m - x_m\| + \\
&\quad + \alpha_3 \|x_m - T x_m\| + \cdots + \alpha_n \|T^n x_m - x_m\| + \alpha_n \|x_m - T x_m\| \\
&= \|x_m - T_\alpha x_m\| + (\alpha_2 \|T^2 x_m - x_m\| + \cdots + \alpha_n \|T^n x_m - x_m\|) + \\
&\quad \cdots + (\alpha_2 \|x_m - T x_m\| + \alpha_3 \|x_m - T x_m\| + \cdots + \alpha_n \|x_m - T x_m\|) \\
&\leq \|x_m - T_\alpha x_m\| + (\alpha_1 \|T x_m - x_m\| + \cdots + \alpha_n \|T^n x_m - x_m\|) + \\
&\quad \cdots + (\alpha_2 + \alpha_3 + \cdots + \alpha_n) \|x_m - T x_m\|) \\
&= \|x_m - T_\alpha x_m\| + \sum_{i=1}^{n} \alpha_i \|T^i x_m - x_m\| + (1 - \alpha) \|x_m - T x_m\| \\
&= \|x_m - T_\alpha x_m\| + \|T_\alpha x_m - x_m\| + (1 - \alpha) \|x_m - T x_m\|.
\end{aligned}
$$

It follow that

$$\|x_m - Tx_m\| \leq \frac{2}{\cdots} \|T_\alpha x_m - x_m\|$$

This implies that $\|x_m - Tx_m\| \to 0$ as $m \to 0$. □

Theorem 3.1.4. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be as in Theorem 3.1.2 and let C be a nonempty compact convex subset of strictly convex Banach space. Suppose that T is α-quasi nonexpansive self mapping of C such that $\alpha_1 > \frac{1}{\sqrt[n]{2}}$, T satisfies CP-condition and S is continuous self mapping of C with $TS = ST$. Then $F(T) \cap F(S) \neq \emptyset$.*

Proof. By Theorem 3.1.2, We have $F(T) = F(T_\alpha)$ and Corollary ??, $F(T_\alpha)$ is a nonempty closed convex set. This implies that $F(T)$ is nonempty closed convex subset of the compact set C, that is $F(T)$ is compact. Since S is continuous self mapping of C which implies image of S is compact. Let $x \in S(F(T))$, so there exists $y \in F(T)$ such that $Sy = x$. Since $y \in F(T)$ then $Ty = y$ .It follow that

$$x = Sy = STy = TSy = Tx$$

It follow that $x \in F(T)$ and so $S : F(T) \to F(T)$. By Tychonoff fixed point theorem. We have $F(T) \cap F(S) \neq \emptyset$. □

Theorem 3.1.5. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be as in Theorem 3.1.2 and let C be a nonempty closed bounded convex subset of uniformly convex Banach space. Suppose that T is a α-quasi nonexpansive self-mapping of C such that $\alpha_1 > \frac{1}{\sqrt[n]{2}}$, T satisfies CP-condition and S is a self-mapping of C which is either nonexpansive or weakly continuous with $TS = ST$. Then $F(T) \cap F(S) \neq \emptyset$*

Proof. Since uniform convexity implies strictly convexity, by Theorem 3.1.2, We have $F(T) = F(T_\alpha)$ and Corollary ??, $F(T_\alpha)$ is a nonempty closed convex set. This implies that $F(T)$ is nonempty closed convex subset of the bounded set C. Since subset of bounded set is bounded set , so $F(T)$ is bounded set. Since $TS = ST$ we have $S(F(T)) \subset F(T)$. If S is nonexpansive then by the Browder fixed point theorem, S has a fixed point in $F(T)$, that is $F(T) \cap F(S) \neq \emptyset$. Next, Suppose that S is weakly continuous. we mean continuous with respect to the weak topology. Since uniformly convex Banach space E is reflexive and we know that E is reflexive if and only if every bounded closed convex subset of E is weakly compact. This implies that $F(T)$ is weakly compact which mean compact with respect to the weak topology. the weak topology is locally convex topological vector space. By the Tychonoff fixed point theorem gives us that S has a fixed point in $F(T)$,that is $F(T) \cap F(S) \neq \emptyset$. □

Theorem 3.1.6. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be as in Theorem 3.1.2 and let C be a nonempty weakly compact convex subset of a uniformly convex Banach space E. Suppose that $\{T_i\}$ is a family of α-quasi nonexpansive self-mappings of C such that $\alpha_1 > \frac{1}{\sqrt[n]{2}}$, T_i satisfies CP-condition and $T_\alpha T_i = T_i T_\alpha$,where $T_\alpha = \sum_{j=1}^{n} \alpha_j T_i^j$ for all $i \in I$. Then $\bigcap_{i \in I} F(T_i) \neq \emptyset$.*

Proof. By Theorem 3.1.2, each $F(T_i)$ is nonempty closed and convex. Since $F(T_i)$ is closed in weakly compact convex C, so $F(T_i)$ weakly closed. Since every closed subset of compact space is compact, it follow that $F(T_i)$ is compact, it will be sufficient to show that the collection $F(T_i)$ has the finite intersection property. With the inductive hypothesis that any k of these sets have nonempty intersection, consider any $k+1$ of sets $F(T_1)$, $F(T_2)$, $F(T_3)$,..., $F(T_k)$, $F(T_{k+1})$. Let $D = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$. Since intersection of any number of closed is closed and $F(T_i)$ is weakly closed for all intersection of convex set is convex then D is convex. Since $T_\alpha T_i = T_i T_\alpha$ where $T_\alpha = \Sigma_{j=1}^{n} \alpha_j T_i^j$ for all $i \in \mathbb{N}$, consider $i = k+1$, that is $T_\alpha = \sum_{j=1}^{n} \alpha_j T_{k+1}^j$, since $T_\alpha T_i = T_i T_\alpha$ we have

$$T_\alpha(F(T_i)) \subset F(T_i) \Rightarrow \bigcap_{i=1}^{k} T_\alpha(F(T_i)) \subset \bigcap_{i=1}^{k} F(T_i)$$
$$\Rightarrow T_\alpha(\bigcap_{i=1}^{k} F(T_i)) \subset \bigcap_{i=1}^{k} F(T_i)$$

That is $T_\alpha(D) \subset D$. We choose point p, since D is nonempty closed and convex subset of a strictly convex Banach space E, there exists a unique point $q \in D$ such that $\|q-p\| = \inf\{\|p-z\| : z \in D\}$. Since T_α is quasi-nonexpansive mapping, we have $\|T_\alpha q - p\| \leq \|q-p\|$ and $q \in D$ implies $T_\alpha q \in D$. Thus $T_\alpha q = q$ that $q \in F(T\alpha)$, by Theorem 3.1.2, we have $F(T_{k+1}) = F(T_\alpha)$. We obtain that $q \in F(T_{k+1})$, and so $\bigcap_{i=1}^{k+1} F(T_i) \supset \{q\}$. □

3.2 Dislocated quasi-b-metric space

In this section, we begin with introducing the notion of a dislocated quasi-b-metric space.

Definition 3.2.1. *Let X be a nonempty set. Suppose that the mapping $d : X \times X \to [0,\infty)$ such that constant $s \geq 1$ satisfies the following conditions:*

(d1) $d(x,y) = d(y,x) = 0$ implies $x = y$ for all $x, y \in X$;

(d2) $d(x,y) \leq s[d(x,z) + d(z,y)]$, for all $x, y, z \in X$.

The pair (X,d) is then called a dislocated quasi b-metric space (or simply dqb-metric). *The number s is called to be the coefficient of (X,d).*

Remark 3.2.2. *It is obvious that b-metric spaces, quasi b-metric space and b-metric-like spaces is dislocated quasi b-metric space but conversely is not true.*

Example 3.2.3. *Let $X = \mathbb{R}$ and let*

$$d(x,y) = |x-y|^2 + \frac{|x|}{n} + \frac{|y|}{m},$$

where $n, m \in \mathbb{N} \setminus \{1\}$ with $n \neq m$.

Then (X,d) is a dislocated quasi b-metric space with the coefficient $s = 2$, but since $d(1,1) \neq 0$, we have (X,b) is not a quasi b-metric space and since $d(1,2) \neq d(2,1)$, we have (X,b) is not a b-metric-like space. And, (X,b) is not a dislocated quasi-metric space. Indeed,

Let $x, y, z \in X$. Suppose that $d(x,y) = 0$.

Then

$$|x-y|^2 + \frac{|x|}{n} + \frac{|y|}{m} = 0.$$

It implies that $|x-y|^2 = 0$ and so, $x = y$.

Next, consider

$$\begin{aligned}
d(x,y) &= |x-y|^2 + \frac{|x|}{n} + \frac{|y|}{m} \\
&\le (|x-z| + |z-y|)^2 + \frac{|x|}{n} + \frac{|y|}{m} \\
&\le |x-z|^2 + 2|x-z|\cdot|z-y| + |z-y|^2 + \frac{|x|}{n} + \frac{|y|}{m} \\
&\le 2(|x-z|^2 + |z-y|^2) + \frac{|x|}{n} + \frac{|z|}{m} + \frac{|z|}{n} + \frac{|y|}{m} \\
&\le s[d(x,z) + d(z,y)],
\end{aligned}$$

where $s = 2$.

$$\begin{aligned}
d(\frac{1}{2},\frac{1}{4}) &= |\frac{1}{2} - \frac{1}{4}|^2 + \frac{|\frac{1}{2}|}{n} + \frac{|\frac{1}{4}|}{m} \\
&= \frac{1}{16} + \frac{1}{2n} + \frac{1}{4m} \\
&= \frac{324}{5184} + \frac{3}{6n} + \frac{4}{12m} \\
&> \frac{180}{5184} + \frac{5}{6n} + \frac{7}{12m} \\
&= \frac{1}{36} + \frac{1}{2n} + \frac{1}{3m} + \frac{1}{144} + \frac{1}{3n} + \frac{1}{4m} \\
&= |\frac{1}{2} - \frac{1}{3}|^2 + \frac{|\frac{1}{2}|}{n} + \frac{|\frac{1}{3}|}{m} + |\frac{1}{3} - \frac{1}{4}|^2 + \frac{|\frac{1}{3}|}{n} + \frac{|\frac{1}{4}|}{m} \\
&= d(\frac{1}{2},\frac{1}{3}) + d(\frac{1}{3},\frac{1}{4}),
\end{aligned}$$

where $n, m > 42$.

Example 3.2.4. [30]*Let* $X = \{0,1,2\}$, *and let* $d : X \times X \to \mathbb{R}^+$ *be defined by*

$$d(x,y) = \begin{cases} 2; & x = y = 0, \\ \frac{1}{2}; & x = 0,\ y = 1, \\ 2; & x = 1\ y = 0, \\ \frac{1}{2}; & otherwise. \end{cases}$$

Then (X,d) *is a dislocated quasi b-metric space with the coefficient* $s = 2$, *but since* $d(1,1) \neq 0$, *we have* (X,b) *is not a quasi b-metric space and since* $d(1,2) \neq d(2,1)$, *we have* (X,b) *is not a b-metric-like space. It is obvious that* (X,b) *is not a dislocated quasi-metric space.*

Example 3.2.5. *Let* $X = \mathbb{R}$ *and let*

$$d(x,y) = |x-y|^2 + 3|x|^2 + 2|y|^2.$$

Then (X,d) *is a dislocated quasi b-metric space with the coefficient* $s = 2$, *but since* $d(0,1) \neq d(1,0)$, *we have* (X,b) *is not a b-metric-like space, since* $d(1,1) \neq 0$, *we have* (X,b) *is not a quasi b-metric space. It is obvious that* (X,b) *is not a dislocated*

Example 3.2.6. *Let $X = \mathbb{R}$ and let*

$$d(x,y) = |2x-y|^2 + |2x+y|^2.$$

Then (X,d) is a dislocated quasi b-metric space with the coefficient $s=2$, but since $d(1,1) \neq 0$, we have (X,b) is not a quasi b-metric space. It is obvious that (X,b) is not a dislocated quasi-metric space.

We will introduced dislocated quasi-b-converges sequence, Cauchy sequence and complete of space according to Zoto, Kumari and Hoxha[29].

Definition 3.2.7. *(1) A sequence $(\{x_n\})$ in a dqb-metric space (X,d) dislocated quasi-b-converges (for short, dqb-converges) to $x \in X$ if*

$$\lim_{n\to\infty} d(x_n, x) = 0 = \lim_{n\to\infty} d(x, x_n).$$

In this case x called a dqb-limit of $(\{x_n\})$ and we write $(x_n \to x)$.
(2) A sequence $(\{x_n\})$ in a dqb-metric space (X,d) is call Cauchy if

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0 = \lim_{n,m\to\infty} d(x_m, x_n).$$

(3) A dqb-metric space (X,d) is complete if every Cauchy sequence in it is dqb-convergent in X.

Next, we begin with introducing the concept of a dqb-cyclic-Banach Contraction.

Definition 3.2.8. *Let A and B be nonempty subsets of a dislocated quasi-b-metric spaces. (X,d). A cyclic map $T : A \cup B \to A \cup B$ is said to be a* dqb-cyclic-Banach Contraction *and if there exists $k \in [0,1)$ such that*

$$d(Tx, Ty) \leq kd(x,y). \tag{3.1}$$

for all $x \in A$, $y \in B$ and $s \geq 1$ and $sk \leq 1$.

Now we prove our main results.

Theorem 3.2.9. *Let A and B be nonempty subsets of a complete dislocated quasi-b-metric space (X,d). Let T be a cyclic mapping that satisfies the condition a dqb-cyclic-Banach Contraction. Then, T has a unique fixed point in $A \cap B$.*

Proof. Let $x \in A\,(fix)$ and using contractive condition of theorem, we have

$$\begin{aligned} d(T^2x, Tx) &= d(T(Tx), Tx) \\ &\leq kd(Tx, x), \end{aligned}$$

and

$$\begin{aligned} d(Tx, T^2x) &= d(Tx, T(Tx)) \\ &\leq kd(x, Tx). \end{aligned}$$

So,

$$d(T^2x, Tx) \leq k\alpha \tag{3.2}$$

and

$$d(Tx, T^2x) \leq k\alpha, \tag{3.3}$$

where $\alpha = \max\{d(Tx, x), d(x, Tx)\}$.
By using (3.2) and (3.2), we have $d(T^3x, T^2x) \leq k^2\alpha$, and $d(T^2x, T^3x) \leq k^2\alpha$.
For all $n \in \mathbb{N}$, we get

$$d(T^{n+1}x, T^nx) \leq k^n\alpha,$$

and

$$d(T^nx, T^{n+1}x) \leq k^n\alpha.$$

Let $n, m \in \mathbb{N}$ with $m > n$, by using the triangular inequality, we have,

$$\begin{aligned} d(T^mx, T^nx) &\leq s^{m-n}d(T^mx, T^{m-1}x) + s^{m-n-1}d(T^{m-1}x, T^{m-2}x) + ... + sd(T^{n+1}x, T^nx) \\ &\leq (s^{m-n}k^{m-1} + s^{m-n-1}k^{m-2} + s^{m-n-2}k^{m-3} + ... + s^2k^{n+1} + sk^n)\alpha \\ &= (sk)^{m-n}k^{n-1} + (sk)^{m-n-1}k^{m-2} + (sk)^{m-n-2}k^{n-1} + ... + (sk)^2k^{n-1} + (sk)k^{n-} \\ &\leq (k^{n-1} + k^{n-1} + k^{n-1} + ... + k^{n-1} + k^{n-1})\alpha \\ &= (k^{n-1})(m - n + 1)\alpha \\ &\leq (k^{n-1})\xi\alpha, \end{aligned}$$

for some $\xi > m - n + 1$.
Take $n \to \infty$, we get $d(T^mx, T^nx) \to 0$.
Similarly, let $n, m \in \mathbb{N}$ with $m > n$, by using the triangular inequality, we have,

$$d(T^nx, T^mx) = (k^{n-1})\xi\alpha.$$

Take $n \to \infty$, we get $d(T^nx, T^mx) \to 0$. Thus T^nx is a Cauchy sequence.
Since(X, d) is complete, we have $\{T^nx\}$ converges to some $z \in X$.
We note, that $\{T^{2n}x\}$ is a sequence in A and $\{T^{2n-1}x\}$ is a sequence in B in a way that both sequences tend to same limit z.
Since A and B are closed, we have $z \in A \cap B$, and then $A \cap B \neq \emptyset$.
Now, we will to show that $Tz = z$.
By using (3.1), Consider

$$\begin{aligned} d(T^nx, Tz) &= d(T(T^{n-1}x), Tz) \\ &\leq kd(T^{n-1}x, z) \\ &\leq d(T^{n-1}x, z). \end{aligned}$$

Taking limit as $n \to \infty$ in above inequality, we have

$$d(z, Tz) \leq kd(z, Tz) \leq d(z, Tz).$$

And so, $d(z, Tz) = kd(z, Tz)$, where $0 \leq k < 1$. This implies that $d(z, Tz) = 0$.
Similarly considering form (3.1), we get

$$\begin{aligned} d(Tz, T^nx) &= d(Tz, T(T^{n-1}x)) \\ &\leq kd(z, T^{n-1}x) \\ &\leq d(z, T^{n-1}x) \end{aligned}$$

Taking limit as $n \to \infty$ in above inequality, we have

$$d(Tz, z) \leq kd(Tz, z) \leq d(Tz, z).$$

And so, $d(Tz, z) = kd(Tz, z)$, where $0 \leq k < 1$. This implies that $d(Tz, z) = 0$.
Hence $d(z, Tz) = d(Tz, z) = 0$, this implies that $Tz = z$ that is z is a fixed point of T.
Finally, to prove the uniqueness of fixed point, let $z^* \in X$ be another fixed point of T such that $Tz^* = z^*$.
Then, we have

$$d(z, z^*) = d(Tz, Tz^*) \leq kd(z, z^*). \tag{3.4}$$

On the other hand,

$$d(z^*, z) = d(Tz^*, Tz) \leq kd(z^*, z). \tag{3.5}$$

By form (3.4) and (3.5), we obtain that $d(z, z^*) = d(z^*, z) = 0$, this implies that $z^* = z$.
Therefore z is a unique fixed point of T. The complete prove. □

Example 3.2.10. *Let $X = [-1, 1]$ and $T : A \cup B \to A \cup B$ defined by $Tx = \frac{-x}{5}$. Suppose that $A = [-1, 0]$ and $B = [0, 1]$. Defined the function $d : X^2 \to [0, \infty)$ by*

$$d(x, y) = |x - y|^2 + \frac{|x|}{10} + \frac{|y|}{11}.$$

We see that d is a dislocated quasi-b-metric on X.
Now, Consider Let $x \in A$. Then $-1 \leq x \leq 0$. So, $0 \leq \frac{-x}{5} \leq \frac{1}{5}$. Thus, $Tx \in B$.
On the other hand, let $x \in B$. Then $0 \leq x \leq 1$. So, $\frac{-1}{5} \leq \frac{-x}{5} \leq 0$. Thus, $Tx \in A$.
Hence the map T is cyclic on X, because $T(A) \subset B$ and $T(B) \subset A$.
Next, we consider

$$\begin{aligned} d(Tx, Ty) &= |Tx - Ty|^2 + 3|Tx| + 2|Ty| \\ &= |\frac{-x}{5} - \frac{-y}{5}|^2 + \frac{1}{10}|\frac{-x}{5}| + \frac{1}{11}|\frac{-y}{5}| \\ &= \frac{1}{25}|x - y|^2 + \frac{1}{50}|x| + \frac{2}{55}|y| \\ &\leq \frac{1}{5}[|x - y|^2 + \frac{1}{10}|x| + \frac{1}{11}|y|] \\ &\leq kd(x, y), \end{aligned}$$

so for $\frac{1}{5} \leq k < 1$.
Thus T satisfies dqb-cyclic-Banach Contraction of theorem 3.3 and 0 is the unique fixed point of T.

Finally, we begin with introducing the concept of a dqb-cyclic-Kannan mapping.

Definition 3.2.11. *Let A and B be nonempty subsets of a dislocated quasi-b-metric spaces. (X, d). A cyclic map $T : A \cup B \to A \cup B$ is called a* dqb-cyclic-Kannan mapping *if there exists $r \in [0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq r(d(x, Tx) + d(x, Ty)). \tag{3.6}$$

In the next theorem, we will prove fixed point theorem for cyclic-Kannan mapping in dislocated quasi-b-metric space.

Theorem 3.2.12. *Let A and B be nonempty subsets of a complete dislocated quasi-b-metric space (X, d). Let T be a cyclic mapping that satisfies the condition a dqb-cyclic-Kannan mapping Then, T has a unique fixed point in $A \cap B$.*

Proof. Let $x \in A\,(fix)$ and using contractive condition of theorem, we have and

$$\begin{aligned} d(Tx, T^2x) &= d(Tx, T(Tx)) \\ &\leq rd(x, Tx) + rd(Tx, T^2x), \end{aligned}$$

so,

$$d(Tx, T^2x) \leq \frac{r}{1-r} d(x, Tx). \tag{3.7}$$

And from (3.7),

$$\begin{aligned} d(T^2x, Tx) &= d(T(Tx), Tx) \\ &\leq rd(Tx, T^2x) + rd(x, Tx) \\ &\leq \frac{r}{1-r} d(x, Tx) + rd(x, Tx) \\ &\leq (\frac{r}{1-r} + \frac{r}{1-r}) d(x, Tx) \\ &\leq \frac{r}{1-r} 2d(x, Tx), \end{aligned}$$

so,

$$d(Tx, T^2x) \leq \frac{r}{1-r} \beta, \tag{3.8}$$

where $\beta = 2d(x, Tx)$.
By using (3.7) and (3.8), we have

$$d(T^3x, T^2x) \leq (\frac{r}{1-r})^2 \beta,$$

and

$$d(T^2x, T^3x) \leq (\frac{r}{1-r})^2 \beta.$$

For all $n \in \mathbb{N}$, we get

$$d(T^{n+1}x, T^nx) \leq (\frac{r}{1-r})^n \beta,$$

and

$$d(T^nx, T^{n+1}x) \leq (\frac{r}{1-r})^n \beta.$$

Let $n, m \in \mathbb{N}$ with $m > n$, by using the triangular inequality, we have,

$$\begin{aligned} d(T^mx, T^nx) &\leq s^{m-n} d(T^mx, T^{m-1}x) + s^{m-n-1} d(T^{m-1}x, T^{m-2}x) + \ldots + sd(T^{n+1}x, T^nx) \\ &\leq (s^{m-n}k^{m-1} + s^{m-n-1}k^{m-2} + s^{m-n-2}k^{m-3} + \ldots + s^2k^{n+1} + sk^n)\beta \\ &\leq ((\frac{r}{1-r})^{n-1} + (\frac{r}{1-r})^{n-1} + (\frac{r}{1-r})^{n-1} + \ldots + (\frac{r}{1-r})^{n-1} + (\frac{r}{1-r})^{n-1})\beta \\ &= (\frac{r}{1-r})^{n-1}(m-n+1)\beta \\ &\leq (\frac{r}{1-r})^{n-1}\epsilon\beta \end{aligned}$$

for some $\xi > m - n + 1$. Take $n \to \infty$, we get $d(T^m x, T^n x) \to 0$.
Similarly, let $n, m \in \mathbb{N}$ with $m > n$, by using the triangular inequality, we have,

$$d(T^n x, T^m x) < (\frac{r}{1-r})^{n-1}\xi\beta.$$

Take $n \to \infty$, we get $d(T^n x, T^m x) \to 0$. Thus $T^n x$ is a Cauchy sequence.
Since (X, d) is complete, we have $\{(T^n x)\}$ converges to some $z \in X$.
We note, that $\{T^{2n} x\}$ is a sequence in A and $\{T^{2n-1} x\}$ is a sequence in B in a way that both sequences tend to same limit z.
Since A and B are closed, we have $z \in A \cap B$, and then $A \cap B \neq \emptyset$.
Now, we will to show that $Tz = z$.
By using (3.6), Consider

$$\begin{aligned} d(T^n x, Tz) &= d(T(T^{n-1}x), Tz) \\ &\le rd(T^{n-1}x, T^n x) + rd(z, Tz). \end{aligned}$$

Taking limit as $n \to \infty$ in above inequality, we have

$$d(z, Tz) \le rd(z, Tz)$$

Since $0 \le r < \frac{1}{2}$, we have $d(z, Tz) = 0$.
Similarly considering form (3.6), we get

$$\begin{aligned} d(Tz, T^n x) &= d(Tz, T(T^{n-1}x)) \\ &\le rd(z, Tz) + rd(T^{n-1}x, T^n x). \end{aligned}$$

Taking limit as $n \to \infty$ in above inequality, we have

$$d(Tz, z) \le rd(z, Tz)$$

Since $d(z, Tz) = 0$, we have $d(z, Tz) = 0$.
Hence $d(z, Tz) = d(Tz, z) = 0 \;\Rightarrow\; Tz = z$ and z is a fixed point of T.
Finally, to prove the uniqueness of fixed point, let $z^* \in X$ be another fixed point of T such that $Tz^* = z^*$.
Then, we have $d(z, z) = d(z^*, z^*) = 0$, because by assumption.

$$\begin{aligned} d(z, z^*) &= d(Tz, Tz^*) \\ &\le rd(z, Tz) + rd(z^*, Tz^*) \\ &= rd(z, z) + rd(z^*, z^*) \\ &= 0. \end{aligned} \tag{3.9}$$

On the other hand,

$$\begin{aligned} d(z^*, z) &= d(Tz^*, Tz) \\ &\le rd(z^*, Tz^*) + rd(z, Tz) \\ &= rd(z^*, z^*) + rd(z, z) \\ &= 0. \end{aligned} \tag{3.10}$$

By form (3.9) and (3.10), we obtain that $d(z, z^*) = d(z^*, z) = 0 \;\Rightarrow\; z^* = z$

Chapter 4
Conclusion

In this research, we establish dislocated quasi b-metric spaces and introduce the notion of Geraghty type dqb-cyclic-Banach Contraction, dqb-cyclic-Kannan mapping and derive the existence of fixed point theorems for such space. Our main theorem extends and unifies existing results in the recent literature.

The following results are all main theorems of thisresearch:
1. Let A and B be nonempty subsets of a complete dislocated quasi-b-metric space (X, d). Let T be a cyclic mapping that satisfies the condition a dqb-cyclic-Banach Contraction. Then, T has a unique fixed point in $A \cap B$.

2. Let A and B be nonempty subsets of a complete dislocated quasi-b-metric space (X, d). Let T be a cyclic mapping that satisfies the condition a dqb-cyclic-Kannan mapping Then, T has a unique fixed point in $A \cap B$.

Example 3.2.13. *Let* $X = [-1,1]$ *and* $T : X \to X$ *defined by* $Tx = \frac{-x}{7}$*. Suppose that* $A = [-1,0]$ *and* $B = [0,1]$*. Defined the function* $d : X^2 \to [0,\infty)$ *by*

$$d(x,y) = |x-y|^2 + 3|x| + 2|y|.$$

We see that d *is a dislocated quasi-b-metric on* X*.*
Now, Consider Let $x \in A$*. Then* $-1 \leq x \leq 0$*. So,* $0 \leq \frac{-x}{7} \leq \frac{1}{7}$*. Thus,* $Tx \in B$*.*
On the other hand, let $x \in B$*. Then* $0 \leq x \leq 1$*. So,* $\frac{-1}{7} \leq \frac{-x}{7} \leq 0$*. Thus,* $Tx \in A$*.*
Hence the map T *is cyclic on* X*, because* $T(A) \subset B$ *and* $T(B) \subset A$*.*
Next, we consider

$$\begin{aligned}
d(Tx,Ty) &= |Tx - Ty|^2 + 3|Tx| + 2|Ty| \\
&= |\frac{-x}{7} - \frac{-y}{7}|^2 + 3|\frac{-x}{7}| + 2|\frac{-y}{7}| \\
&= \frac{1}{49}|x-y|^2 + \frac{3}{7}|x| + \frac{2}{7}|y| \\
&\leq \frac{1}{49}(|x| + |y|)^2 + \frac{3}{7}|x| + \frac{2}{7}|y| \\
&\leq \frac{2}{49}|x|^2 + \frac{2}{49}|y|^2 + \frac{3}{7}|x| + \frac{2}{7}|y| \\
&\leq \frac{2}{23}([\frac{64}{49}|x|^2 + \frac{23}{7}|x|] + [\frac{64}{49}|y|^2 + \frac{23}{7}|y|]) \\
&= \frac{2}{23}([\frac{64}{49}|x|^2 + \frac{23}{7}|x|] + [\frac{64}{49}|y|^2 + \frac{23}{7}|y|]) \\
&= \frac{2}{23}([|x + \frac{1}{7}x|^2 + 3|x| + 2|\frac{1}{7}x|] + [|y + \frac{1}{7}y|^2 + 3|y| + 2|\frac{1}{7}y|]) \\
&= \frac{2}{23}([|x - Tx|^2 + 3|x| + 2|Tx|] + [|y - Ty|^2 + 3|y| + 2|Ty|]) \\
&= r(d(x,Tx) + d(y,Ty)),
\end{aligned}$$

so for $\frac{2}{23} \leq r < \frac{1}{2}$*.*
Thus T *satisfies dqb-cyclic-Banach Contraction of theorem 3.3 and* 0 *is the unique fixed point of* T*.*

Bibliography

[1] MA. Alghamdi, N. Hussain, P. Salimi, *Fixed point and coupled fixed point theorems on b-metric-like spaces,* J. Inequal. Appl., **402** (2013).

[2] I. A. Bakhtin, *The contraction principle in quasimetric spaces,* Functional Analysis., **30** (1989) 2637.

[3] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales,* Fund. Math.,**(3)** (1922) 133-181.

[4] S.S. Basha, P. Veeramani, *Best proximity pair theorems for multifunctions with open fibres,* J. Approx. Theory **103** (2000) 119-129.

[5] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces,* Atti Sem. Mat. Univ. Modena, **46** (1998) 263-276.

[6] Y. Enjouji, M. Nakanishi and T. Suzuki, *A generalization of Kannan's fixed point theorem,* Fixed Point Theory Appl. **2009** (2009) 1-10. Article ID 192872.

[7] P. Hitzler, A. Seda, *Dislocated topologies,* J. Electr. Eng., **51** (2000) 3-7.

[8] R. Kannan, *Some results on fixed points-II,* Amer. Math. Monthly.,**76** (1969) 405-408.

[9] E. Karapinar and I. M. Erhan, *Best Proximity on Different Type Contractions,* Applied Mathematics and Information Science., (2010).

[10] E. Karapinar and P. Salimi, *Dislocated metric space to metric spaces with some fixed point theorems,* Fixed point theory and applications., (2013).

[11] W.A. Kirk, P.S. Srinivasan and P. Veeramani, *Fixed Points for mapping satsifying Cyclic contractive conditions,* Fixed Point Theory., **4** (2003) 79-89.

[12] M. Kikkawa and T. Suzuki, *Some similarity between contractions and Kannan mappings,* Fixed Point Theory Appl. **2008** (2008) 1-8. Article ID 649749.

[13] S. B. Nadler, Jr., *Multi-valued contraction mappings,* Pacific J. Math., **30** (1969) 475-488.

[14] M. Nakanishi and T. Suzuki, *An observation on Kannan mappings,* Cent. Eur. J. Math., **8** (2010) 170-178.

[15] M. H. Shah and N. Hussain, *Nonlinear Contractions in partially ordered quasi b-metric spaces,* Commun. Korean Math. Soc., **27(1)** (2012) 117-128.

[16] S. Reich, *Kannan's fixed point theorem,* Boll. Un. Mat. Ital., **4** (1971) 1-11.

[17] N. Shioji, T. Suzuki and W. Takahashi, *Contractive mappings, Kannan mappings and metric completeness,* Proc. Amer. Mat. Soc., **126** (1998) 3117-3124. http://dx.doi.org/10.1090/S0002-9939-98-04605-X.

[18] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings,* J. Math. Anal. Appl., **340** (2008) 1088-1095.

[19] W. A. Wilson, *on quasi-metric spaces,* American Journal of Mathematics., **53** (1931) 3.

[20] K. Włodarczyk, R. Plebaniak and A. Banach, *Best proximity points for cyclic and noncyclic set-valued relatively quasi-asymptotic contractions in uniform spaces,* Nonlinear Anal., **70** (2009) 3332-3341.

[21] K. Włodarczyk, R. Plebaniak and A. Banach, *Best proximity points for cyclic and noncyclic set-valued relatively quasi-asymptotic contractions in uniform spaces,* Nonlinear Analysis: Theory, Methods and Applications., **71** (2009).

[22] K. Włodarczyk, R. Plebaniak and C. Obczyłski, *Convergence theorems, best approximation and best proximity for set-valued dynamic systems of relatively quasi-asymptotic contractions in cone uniform spaces,* Nonlinear Anal., **72** (2010) 794-805.

[23] K. Włodarczyk and R. Plebaniak, *Kannan-type contractions and fixed points in uniform spaces,* Fixed Point Theory Appl., **90** (2011) 1-24.

[24] K. Włodarczyk and R. Plebaniak, *Contractions of Banach, Tarafdar, Meir-Keller, Ćirić-Jachymski-Matkowski and Suzuki types and fixed points in uniform spaces with generalized pseudodistances,* J. Math. Anal. Appl., **404** (2013) 338-350.

[25] K. Włodarczyk and R. Plebaniak, *Asymmetric structures, discontinuous contractions and iterative approximation of fixed and periodic points,* Fixed Point Theory and Applications., **128** (2013) 1-18.

[26] K. Włodarczyk, *Hausdorff quasi-distances, periodic and fixed points for Nadler type set-valued contractions in quasi-gauge spaces,* Fixed Point Theory and Applications., **239** (2013) 1-27.

[27] K. Włodarczyk and R. Plebaniak, *Dynamic processes, fixed points, endpoints, asymmetric structures and investigations related to Caristi, Nadler and Banach in uniform spaces,* Abstract and Applied Analysis., 2015 (2015), Article ID 942814, 1-16. 3585-3586.

[28] F.M. Zeyada, G.H. Hassan and M.A. Ahmad, *A generalization of fixed point theorem due to Hitzler and Seda in dislocated quasi-metric space,* Arabian J. Sci. Engg., **31** (2005) 111-114.

[29] K. Zoto, P. S. Kumari, E. Hoxha, *Some fixed point theorems and cyclic contractions in dislocated and dislocated quasi-metric spaces,* American Journal of

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Klin-eam and Suanoom *Fixed Point Theory and Applications* (2015) 2015:74
DOI 10.1186/s13663-015-0325-2

Fixed Point Theory and Applications
a SpringerOpen Journal

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Dislocated quasi-b-metric spaces and fixed point theorems for cyclic contractions

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Abstract

In this paper, we establish dislocated quasi-b-metric spaces and introduce the notions of Geraghty type dqb-cyclic-Banach contraction and dqb-cyclic-Kannan mapping and derive the existence of fixed point theorems for such spaces. Our main theorem extends and unifies existing results in the recent literature.

MSC: 47H05; 47H10; 47J25

Keywords: fixed points; dqb-cyclic-Banach contraction; dqb-cyclic-Kannan mapping; b-metric spaces; quasi-b-metric spaces; b-metric-like spaces; dislocated quasi-b-metric spaces

1 Introduction and preliminaries

Fixed point theory has been studied extensively, which can be seen from the works of many authors [1–6]. Banach contraction principle was introduced in 1922 by Banach [7] as follows:

(i) Let (X,d) be a metric space and let $T: X \to X$. Then T is called a Banach contraction mapping if there exists $k \in [0,1)$ such that

$$d(Tx,Ty) \leq kd(x,y)$$

for all $x, y \in X$.

The concept of Kannan mapping was introduced in 1969 by Kannan [8] as follows:

(ii) T is called a Kannan mapping if there exists $r \in [0, \frac{1}{2})$ such that

$$d(Tx,Ty) \leq rd(x,Tx) + rd(y,Ty)$$

for all $x, y \in X$.

Now, we recall the definition of cyclic map. Let A and B be nonempty subsets of a metric space (X,d) and $T: A\cup B \to A \cup B$. T is called a cyclic map iff $T(A) \subseteq B$ and $T(B) \subseteq A$.

In 2003, Kirk *et al.* [9] introduced cyclic contraction as follows:

(iii) A cyclic map $T: A\cup B \to A\cup B$ is said to be a cyclic contraction if there exists $a \in [0,1)$ such that

$$d(Tx,Ty) \leq ad(x,y)$$

for all $x \in A$ and $y \in B$.

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In 2010, Karapinar and Erhan [10] introduced Kannan type cyclic contraction as follows:

(iv) A cyclic map $T : A \cup B \to A \cup B$ is called a Kannan type cyclic contraction if there exists $b \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq bd(x, Tx) + bd(y, Ty)$$

for all $x \in A$ and $y \in B$.

If (X, d) is a complete metric space, at least one of (i), (ii), (iii) and (iv) holds, then it has a unique fixed point [7–10]. Next, we discuss the development of spaces. The concept of quasi-metric spaces was introduced by Wilson [11] in 1931 as a generalization of metric spaces, and in 2000 Hitzler and Seda [12] introduced dislocated metric spaces as a generalization of metric spaces, [13] generalized the result of Hitzler, Seda and Wilson and introduced the concept of dislocated quasi-metric space. Włodarczyk *et al.* (see [14–21]) created uniform spaces as this is the concept of metric spaces. In 1989, Bakhtin [22] introduced b-metric space as a generalization of metric space. Moreover, Czerwik [23] made the results of Bakhtin known more in 1998. Finally, many other generalized b-metric spaces such as quasi-b-metric spaces [24], b-metric-like spaces [25] and quasi-b-metric-like spaces [26] were introduced.

We begin with the following definition as a recall from [11, 12].

Definition 1.1 [7, 11, 12] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to [0, \infty)$ satisfies the following conditions:

(d_1) $d(x, x) = 0$ for all $x \in X$;
(d_2) $d(x, y) = d(y, x) = 0$ implies $x = y$ for all $x, y \in X$;
(d_3) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d_4) $d(x, y) \leq [d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

If d satisfies conditions (d_1), (d_2) and (d_4), then d is called a *quasi-metric* on X. If d satisfies conditions (d_2), (d_3) and (d_4), then d is called a *dislocated metric* on X. If d satisfies conditions (d_1)-(d_4), then d is called a *metric* on X.

In 2005 the concept of dislocated quasi-metric spaces [13], which is a new generalization of quasi-b-metric spaces and dislocated b-metric spaces, was introduced. By Definition 1.1, if setting conditions (d_2) and (d_4) hold true, then d is called a *dislocated quasi-metric* on X.

Remark 1.2 It is obvious that metric spaces are quasi-metric spaces and dislocated metric spaces, but the converse is not true.

In 1989, Bakhtin [22] introduced the concept of b-metric spaces and investigated some fixed point theorems in such spaces.

Definition 1.3 [22] Let X be a nonempty set. Suppose that the mapping $b : X \times X \to [0, \infty)$ such that the constant $s \geq 1$ satisfies the following conditions:

(b_1) $b(x, y) = b(y, x) = 0 \Leftrightarrow x = y$ for all $x, y \in X$;
(b_2) $b(x, y) = b(y, x)$ for all $x, y \in X$;
(b_3) $b(x, y) \leq s[b(x, z) + b(z, y)]$ for all $x, y, z \in X$.

The pair (X, b) is then called a b-metric space.

Remark 1.4 It is obvious that metric spaces are b-metric spaces, but conversely this is not true.

In 2012, Shah and Hussain [24] introduced the concept of quasi-b-metric spaces and verified some fixed point theorems in quasi-b-metric spaces.

Definition 1.5 [24] Let X be a nonempty set. Suppose that the mapping $q : X \times X \to [0, \infty)$ such that constant $s \geq 1$ satisfies the following conditions:

(q_1) $q(x,y) = q(y,x) = 0 \Leftrightarrow x = y$ for all $x, y \in X$;
(q_2) $q(x,y) \leq s[q(x,z) + q(z,y)]$ for all $x, y, z \in X$.

The pair (X, q) is then called a quasi-b-metric space.

Remark 1.6 It is obvious that b-metric spaces are quasi-b-metric spaces, but conversely this is not true.

Recently, the concept of b-metric-like spaces, which is a new generalization of metric-like spaces, was introduced by Alghamdi *et al.* [25].

Definition 1.7 [25] Let X be a nonempty set. Suppose that the mapping $D : X \times X \to [0, \infty)$ such that constant $s \geq 1$ satisfies the following conditions:

(D_1) $D(x,y) = 0 \Rightarrow x = y$ for all $x, y \in X$;
(D_2) $D(x,y) = D(y,x)$ for all $x, y \in X$;
(D_3) $D(x,y) \leq s[D(x,z) + D(z,y)]$ for all $x, y, z \in X$.

The pair (X, D) is then called a b-metric-like space (or a dislocated b-metric space).

Remark 1.8 It is obvious that b-metric spaces are b-metric-like spaces, but conversely this is not true.

In this paper we introduce dislocated quasi-b-metric spaces which generalize quasi-b-metric spaces and b-metric-like spaces, and we introduce the notions of Geraghty type dqb-cyclic-Banach contraction and dqb-cyclic-Kannan mapping and derive the existence of fixed point theorems for such spaces. Our main theorems extend and unify existing results in the recent literature.

2 Main results

In this section, we begin with introducing the notion of dislocated quasi-b-metric space.

Definition 2.1 Let X be a nonempty set. Suppose that the mapping $d : X \times X \to [0, \infty)$ such that constant $s \geq 1$ satisfies the following conditions:

(d1) $d(x,y) = d(y,x) = 0$ implies $x = y$ for all $x, y \in X$;
(d2) $d(x,y) \leq s[d(x,z) + d(z,y)]$ for all $x, y, z \in X$.

The pair (X, d) is then called a *dislocated quasi-b-metric space (or simply dqb-metric)*. The number s is called the coefficient of (X, d).

Remark 2.2 It is obvious that b-metric spaces, quasi-b-metric spaces and b-metric-like spaces are dislocated quasi-b-metric spaces, but the converse is not true.

Example 2.3 Let $X = \mathbb{R}$ and let

$$d(x,y) = |x-y|^2 + \frac{|x|}{n} + \frac{|y|}{m},$$

where $n, m \in \mathbb{N} \setminus \{1\}$ with $n \neq m$.

Then (X,d) is a dislocated quasi-b-metric space with the coefficient $s = 2$, but since $d(1,1) \neq 0$, we have (X,b) is not a quasi-b-metric space, and since $d(1,2) \neq d(2,1)$, we have (X,b) is not a b-metric-like space. And (X,b) is not a dislocated quasi-metric space. Indeed, let $x,y,z \in X$. Suppose that $d(x,y) = 0$.

Then

$$|x-y|^2 + \frac{|x|}{n} + \frac{|y|}{m} = 0.$$

It implies that $|x-y|^2 = 0$, and so $x = y$.

Next, consider

$$\begin{aligned} d(x,y) &= |x-y|^2 + \frac{|x|}{n} + \frac{|y|}{m} \\ &\leq \bigl(|x-z| + |z-y|\bigr)^2 + \frac{|x|}{n} + \frac{|y|}{m} \\ &\leq |x-z|^2 + 2|x-z| \cdot |z-y| + |z-y|^2 + \frac{|x|}{n} + \frac{|y|}{m} \\ &\leq 2\bigl(|x-z|^2 + |z-y|^2\bigr) + \frac{|x|}{n} + \frac{|z|}{m} + \frac{|z|}{n} + \frac{|y|}{m} \\ &\leq s\bigl[d(x,z) + d(z,y)\bigr], \end{aligned}$$

where $s = 2$,

$$\begin{aligned} d\left(\frac{1}{2},\frac{1}{4}\right) &= \left|\frac{1}{2} - \frac{1}{4}\right|^2 + \frac{|\frac{1}{2}|}{n} + \frac{|\frac{1}{4}|}{m} = \frac{1}{16} + \frac{1}{2n} + \frac{1}{4m} \\ &= \frac{324}{5{,}184} + \frac{3}{6n} + \frac{4}{12m} > \frac{180}{5{,}184} + \frac{5}{6n} + \frac{7}{12m} \\ &= \frac{1}{36} + \frac{1}{2n} + \frac{1}{3m} + \frac{1}{144} + \frac{1}{3n} + \frac{1}{4m} \\ &= \left|\frac{1}{2} - \frac{1}{3}\right|^2 + \frac{|\frac{1}{2}|}{n} + \frac{|\frac{1}{3}|}{m} + \left|\frac{1}{3} - \frac{1}{4}\right|^2 + \frac{|\frac{1}{3}|}{n} + \frac{|\frac{1}{4}|}{m} \\ &= d\left(\frac{1}{2},\frac{1}{3}\right) + d\left(\frac{1}{3},\frac{1}{4}\right), \end{aligned}$$

where $n, m > 42$.

Example 2.4 [26] Let $X = \{0,1,2\}$, and let $d : X \times X \to \mathbb{R}^+$ be defined by

$$d(x,y) = \begin{cases} 2; & x = y = 0, \\ \frac{1}{2}; & x = 0, y = 1, \\ 2; & x = 1, y = 0, \\ \frac{1}{2}; & \text{otherwise.} \end{cases}$$

Then (X, d) is a dislocated quasi-b-metric space with the coefficient $s = 2$, but since $d(1,1) \neq 0$, we have (X, b) is not a quasi-b-metric space, and since $d(1,2) \neq d(2,1)$, we have (X, b) is not a b-metric-like space. It is obvious that (X, b) is not a dislocated quasi-metric space.

Example 2.5 Let $X = \mathbb{R}$ and let

$$d(x,y) = |x-y|^2 + 3|x|^2 + 2|y|^2.$$

Then (X, d) is a dislocated quasi-b-metric space with the coefficient $s = 2$, but since $d(0,1) \neq d(1,0)$, we have (X, b) is not a b-metric-like space, since $d(1,1) \neq 0$, we have (X, b) is not a quasi-b-metric space. It is obvious that (X, b) is not a dislocated quasi-metric space.

Example 2.6 Let $X = \mathbb{R}$ and let

$$d(x,y) = |2x-y|^2 + |2x+y|^2.$$

Then (X, d) is a dislocated quasi-b-metric space with the coefficient $s = 2$, but since $d(1,1) \neq 0$, we have (X, b) is not a quasi-b-metric space. It is obvious that (X, b) is not a dislocated quasi-metric space.

We will introduce a dislocated quasi-b-convergent sequence, a Cauchy sequence and a complete space according to Zoto *et al.* [27].

Definition 2.7

(1) A sequence $(\{x_n\})$ in a dqb-metric space (X, d) dislocated quasi-b-converges (for short, dqb-converges) to $x \in X$ if

$$\lim_{n\to\infty} d(x_n, x) = 0 = \lim_{n\to\infty} d(x, x_n).$$

In this case x is called a dqb-limit of $(\{x_n\})$, and we write $(x_n \to x)$.

(2) A sequence $(\{x_n\})$ in a dqb-metric space (X, d) is called Cauchy if

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0 = \lim_{n,m\to\infty} d(x_m, x_n).$$

(3) A dqb-metric space (X, d) is complete if every Cauchy sequence in it is dqb-convergent in X.

Next, we begin with introducing the concept of a dqb-cyclic-Banach contraction.

Definition 2.8 Let A and B be nonempty subsets of a dislocated quasi-b-metric space (X, d). A cyclic map $T : A \cup B \to A \cup B$ is said to be a *dqb-cyclic-Banach contraction* if there exists $k \in [0,1)$ such that

$$d(Tx, Ty) \le kd(x,y) \tag{2.1}$$

for all $x \in A$, $y \in B$ and $s \ge 1$ and $sk \le 1$.

Now we prove our main results.

Theorem 2.9 *Let A and B be nonempty subsets of a complete dislocated quasi-b-metric space (X,d). Let T be a cyclic mapping that satisfies the condition of a dqb-cyclic-Banach contraction. Then T has a unique fixed point in $A \cap B$.*

Proof Let $x \in A$(fix) and, using the contractive condition of the theorem, we have

$$\begin{aligned} d(T^2x, Tx) &= d(T(Tx), Tx) \\ &\leq kd(Tx, x) \end{aligned}$$

and

$$\begin{aligned} d(Tx, T^2x) &= d(Tx, T(Tx)) \\ &\leq kd(x, Tx). \end{aligned}$$

So,

$$d(T^2x, Tx) \leq k\alpha \tag{2.2}$$

and

$$d(Tx, T^2x) \leq k\alpha, \tag{2.3}$$

where $\alpha = \max\{d(Tx,x), d(x,Tx)\}$.

By using (2.2) and (2.3), we have $d(T^3x, T^2x) \leq k^2\alpha$, and $d(T^2x, T^3x) \leq k^2\alpha$.

For all $n \in \mathbb{N}$, we get

$$d(T^{n+1}x, T^nx) \leq k^n\alpha$$

and

$$d(T^nx, T^{n+1}x) \leq k^n\alpha.$$

Let $n, m \in \mathbb{N}$ with $m > n$, by using the triangular inequality, we have

$$\begin{aligned} d(T^mx, T^nx) &\leq s^{m-n}d(T^mx, T^{m-1}x) + s^{m-n-1}d(T^{m-1}x, T^{m-2}x) + \cdots + sd(T^{n+1}x, T^nx) \\ &\leq (s^{m-n}k^{m-1} + s^{m-n-1}k^{m-2} + s^{m-n-2}k^{m-3} + \cdots + s^2k^{n+1} + sk^n)\alpha \\ &= ((sk)^{m-n}k^{n-1} + (sk)^{m-n-1}k^{m-2} + (sk)^{m-n-2}k^{n-1} + \cdots \\ &\quad + (sk)^2k^{n-1} + (sk)k^{n-1})\alpha \\ &\leq (k^{n-1} + k^{n-1} + k^{n-1} + \cdots + k^{n-1} + k^{n-1})\alpha \\ &= (k^{n-1})(m-n+1)\alpha \\ &\leq (k^{n-1})\xi\alpha \end{aligned}$$

for some $\xi > m - n + 1$.

Take $n \to \infty$, we get $d(T^m x, T^n x) \to 0$.

Similarly, let $n, m \in \mathbb{N}$ with $m > n$, by using the triangular inequality, we have

$$d(T^n x, T^m x) = (k^{n-1})\xi\alpha.$$

Take $n \to \infty$, we get $d(T^n x, T^m x) \to 0$. Thus $T^n x$ is a Cauchy sequence.

Since (X, d) is complete, we have $\{T^n x\}$ converges to some $z \in X$.

We note that $\{T^{2n} x\}$ is a sequence in A and $\{T^{2n-1} x\}$ is a sequence in B in a way that both sequences tend to the same limit z.

Since A and B are closed, we have $z \in A \cap B$, and then $A \cap B \neq \emptyset$.

Now, we will show that $Tz = z$.

By using (2.1), consider

$$\begin{aligned} d(T^n x, Tz) &= d(T(T^{n-1}x), Tz) \\ &\leq kd(T^{n-1}x, z) \\ &\leq d(T^{n-1}x, z). \end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$d(z, Tz) \leq kd(z, Tz) \leq d(z, Tz).$$

And so $d(z, Tz) = kd(z, Tz)$, where $0 \leq k < 1$. This implies that $d(z, Tz) = 0$.

Similarly, considering form (2.1), we get

$$\begin{aligned} d(Tz, T^n x) &= d(Tz, T(T^{n-1}x)) \\ &\leq kd(z, T^{n-1}x) \\ &\leq d(z, T^{n-1}x). \end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$d(Tz, z) \leq kd(Tz, z) \leq d(Tz, z).$$

And so $d(Tz, z) = kd(Tz, z)$, where $0 \leq k < 1$. This implies that $d(Tz, z) = 0$.

Hence $d(z, Tz) = d(Tz, z) = 0$, this implies that $Tz = z$, that is, z is a fixed point of T.

Finally, to prove the uniqueness of a fixed point, let $z^* \in X$ be another fixed point of T such that $Tz^* = z^*$.

Then we have

$$d(z, z^*) = d(Tz, Tz^*) \leq kd(z, z^*). \tag{2.4}$$

On the other hand,

$$d(z^*, z) = d(Tz^*, Tz) \leq kd(z^*, z). \tag{2.5}$$

By forms (2.4) and (2.5), we obtain that $d(z, z^*) = d(z^*, z) = 0$, this implies that $z^* = z$.

Therefore z is a unique fixed point of T. This completes the proof. □

Example 2.10 Let $X = [-1,1]$ and $T: A \cup B \to A \cup B$ defined by $Tx = \frac{-x}{5}$. Suppose that $A = [-1,0]$ and $B = [0,1]$. Define the function $d: X^2 \to [0,\infty)$ by

$$d(x,y) = |x-y|^2 + \frac{|x|}{10} + \frac{|y|}{11}.$$

We see that d is a dislocated quasi-b-metric on X.

Now, let $x \in A$. Then $-1 \leq x \leq 0$. So, $0 \leq \frac{-x}{5} \leq \frac{1}{5}$. Thus, $Tx \in B$.

On the other hand, let $x \in B$. Then $0 \leq x \leq 1$. So, $\frac{-1}{5} \leq \frac{-x}{5} \leq 0$. Thus, $Tx \in A$.

Hence the map T is cyclic on X because $T(A) \subset B$ and $T(B) \subset A$.

Next, we consider

$$\begin{aligned} d(Tx,Ty) &= |Tx-Ty|^2 + 3|Tx| + 2|Ty| \\ &= \left|\frac{-x}{5} - \frac{-y}{5}\right|^2 + \frac{1}{10}\left|\frac{-x}{5}\right| + \frac{1}{11}\left|\frac{-y}{5}\right| \\ &= \frac{1}{25}|x-y|^2 + \frac{1}{50}|x| + \frac{2}{55}|y| \\ &\leq \frac{1}{5}\left[|x-y|^2 + \frac{1}{10}|x| + \frac{1}{11}|y|\right] \\ &\leq kd(x,y), \end{aligned}$$

so for $\frac{1}{5} \leq k < 1$.

Thus T satisfies the dqb-cyclic-Banach contraction of Theorem 2.9 and 0 is the unique fixed point of T.

Finally, we begin with introducing the concept of dqb-cyclic-Kannan mapping.

Definition 2.11 Let A and B be nonempty subsets of a dislocated quasi-b-metric space (X,d). A cyclic map $T: A \cup B \to A \cup B$ is called a *dqb-cyclic-Kannan mapping* if there exists $r \in [0,\frac{1}{2})$ such that

$$d(Tx,Ty) \leq r\big(d(x,Tx) + d(x,Ty)\big) \tag{2.6}$$

for all $x \in A$, $y \in B$ and $s \geq 1$ and $sr \leq \frac{1}{2}$.

In the next theorem, we will prove the fixed point theorem for a cyclic-Kannan mapping in a dislocated quasi-b-metric space.

Theorem 2.12 *Let A and B be nonempty subsets of a complete dislocated quasi-b-metric space (X,d). Let T be a cyclic mapping that satisfies the condition of a dqb-cyclic-Kannan mapping. Then T has a unique fixed point in $A \cap B$.*

Proof Let $x \in A$(fix) and, using the contractive condition of the theorem, we have

$$\begin{aligned} d\big(Tx,T^2x\big) &= d\big(Tx,T(Tx)\big) \\ &\leq rd(x,Tx) + rd\big(Tx,T^2x\big), \end{aligned}$$

so

$$d(Tx, T^2x) \le \frac{r}{1-r}d(x, Tx). \tag{2.7}$$

And from (2.7) we have

$$\begin{aligned} d(T^2x, Tx) &= d(T(Tx), Tx) \\ &\le rd(Tx, T^2x) + rd(x, Tx) \\ &\le \frac{r}{1-r}d(x, Tx) + rd(x, Tx) \\ &\le \left(\frac{r}{1-r} + \frac{r}{1-r}\right)d(x, Tx) \\ &\le \frac{r}{1-r}2d(x, Tx), \end{aligned}$$

so

$$d(Tx, T^2x) \le \frac{r}{1-r}\beta, \tag{2.8}$$

where $\beta = 2d(x, Tx)$.

By using (2.7) and (2.8), we have

$$d(T^3x, T^2x) \le \left(\frac{r}{1-r}\right)^2\beta$$

and

$$d(T^2x, T^3x) \le \left(\frac{r}{1-r}\right)^2\beta.$$

For all $n \in \mathbb{N}$, we get

$$d(T^{n+1}x, T^nx) \le \left(\frac{r}{1-r}\right)^n\beta$$

and

$$d(T^nx, T^{n+1}x) \le \left(\frac{r}{1-r}\right)^n\beta.$$

Let $n, m \in \mathbb{N}$ with $m > n$, by using the triangular inequality, we have

$$\begin{aligned} d(T^mx, T^nx) &\le s^{m-n}d(T^mx, T^{m-1}x) + s^{m-n-1}d(T^{m-1}x, T^{m-2}x) + \cdots + sd(T^{n+1}x, T^nx) \\ &\le (s^{m-n}k^{m-1} + s^{m-n-1}k^{m-2} + s^{m-n-2}k^{m-3} + \cdots + s^2k^{n+1} + sk^n)\beta \\ &\le \left(\left(\frac{r}{1-r}\right)^{n-1} + \left(\frac{r}{1-r}\right)^{n-1} + \left(\frac{r}{1-r}\right)^{n-1} + \cdots \right. \\ &\quad \left. + \left(\frac{r}{1-r}\right)^{n-1} + \left(\frac{r}{1-r}\right)^{n-1}\right)\beta \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{r}{1-r}\right)^{n-1}(m-n+1)\beta \\
&< \left(\frac{r}{1-r}\right)^{n-1}\xi\beta
\end{aligned}
$$

for some $\xi > m - n + 1$. Take $n \to \infty$, we get $d(T^m x, T^n x) \to 0$.

Similarly, let $n, m \in \mathbb{N}$ with $m > n$, by using the triangular inequality, we have

$$d(T^n x, T^m x) < \left(\frac{r}{1-r}\right)^{n-1}\xi\beta.$$

Take $n \to \infty$, we get $d(T^n x, T^m x) \to 0$. Thus $T^n x$ is a Cauchy sequence.

Since (X, d) is complete, we have $\{(T^n x)\}$ converges to some $z \in X$.

We note that $\{T^{2n} x\}$ is a sequence in A and $\{T^{2n-1} x\}$ is a sequence in B in a way that both sequences tend to the same limit z.

Since A and B are closed, we have $z \in A \cap B$, and then $A \cap B \neq \emptyset$.

Now, we will show that $Tz = z$.

By using (2.6), consider

$$
\begin{aligned}
d(T^n x, Tz) &= d(T(T^{n-1}x), Tz) \\
&\leq rd(T^{n-1}x, T^n x) + rd(z, Tz).
\end{aligned}
$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$d(z, Tz) \leq rd(z, Tz).$$

Since $0 \leq r < \frac{1}{2}$, we have $d(z, Tz) = 0$.

Similarly, considering form (2.6), we get

$$
\begin{aligned}
d(Tz, T^n x) &= d(Tz, T(T^{n-1}x)) \\
&\leq rd(z, Tz) + rd(T^{n-1}x, T^n x).
\end{aligned}
$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$d(Tz, z) \leq rd(z, Tz).$$

Since $d(z, Tz) = 0$, we have $d(z, Tz) = 0$.

Hence $d(z, Tz) = d(Tz, z) = 0 \Rightarrow Tz = z$ and z is a fixed point of T.

Finally, to prove the uniqueness of a fixed point, let $z^* \in X$ be another fixed point of T such that $Tz^* = z^*$.

Then we have $d(z, z) = d(z^*, z^*) = 0$, because by assumption

$$
\begin{aligned}
d(z, z^*) &= d(Tz, Tz^*) \\
&\leq rd(z, Tz) + rd(z^*, Tz^*) \\
&= rd(z, z) + rd(z^*, z^*) \\
&= 0. \qquad (2.9)
\end{aligned}
$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements
The authors would like to thank Science Achievement Scholarship of Thailand which provides funding for the research.

Received: 5 February 2015 Accepted: 11 May 2015 Published online: 27 May 2015

References

1. Basha, SS, Veeramani, P: Best proximity pair theorems for multifunctions with open fibres. J. Approx. Theory **103**, 119-129 (2000)
2. Enjouji, Y, Nakanishi, M, Suzuki, T: A generalization of Kannan's fixed point theorem. Fixed Point Theory Appl. **2009**, Article ID 192872 (2009)
3. Kikkawa, M, Suzuki, T: Some similarity between contractions and Kannan mappings. Fixed Point Theory Appl. **2008**, Article ID 649749 (2008)
4. Nakanishi, M, Suzuki, T: An observation on Kannan mappings. Cent. Eur. J. Math. **8**, 170-178 (2010)
5. Reich, S: Kannan's fixed point theorem. Boll. Unione Mat. Ital. **4**, 1-11 (1971)
6. Shioji, N, Suzuki, T, Takahashi, W: Contractive mappings, Kannan mappings and metric completeness. Proc. Am. Math. Soc. **126**, 3117-3124 (1998). doi:10.1090/S0002-9939-98-04605-X
7. Banach, S: Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundam. Math. **3**, 133-181 (1922)
8. Kannan, R: Some results on fixed points - II. Am. Math. Mon. **76**, 405-408 (1969)
9. Kirk, WA, Srinivasan, PS, Veeramani, P: Fixed points for mapping satisfying cyclic contractive conditions. Fixed Point Theory **4**, 79-89 (2003)
10. Karapinar, E, Erhan, IM: Best proximity on different type contractions. Appl. Math. Inf. Sci. **5**, 558-569 (2010)
11. Wilson, WA: On quasi-metric spaces. Am. J. Math. **53**(3), 675-684 (1931)
12. Hitzler, P, Seda, A: Dislocated topologies. J. Electr. Eng. **51**, 3-7 (2000)
13. Zeyada, FM, Hassan, GH, Ahmad, MA: A generalization of fixed point theorem due to Hitzler and Seda in dislocated quasi-metric space. Arab. J. Sci. Eng. **31**, 111-114 (2005)
14. Włodarczyk, K, Plebaniak, R, Banach, A: Best proximity points for cyclic and noncyclic set-valued relatively quasi-asymptotic contractions in uniform spaces. Nonlinear Anal. **70**, 3332-3341 (2009)
15. Włodarczyk, K, Plebaniak, R, Banach, A: Erratum to: 'Best proximity points for cyclic and noncyclic set-valued relatively quasi-asymptotic contractions in uniform spaces'. Nonlinear Anal. **71**, 3585-3586 (2009)
16. Włodarczyk, K, Plebaniak, R, Obczyński, C: Convergence theorems, best approximation and best proximity for set-valued dynamic systems of relatively quasi-asymptotic contractions in cone uniform spaces. Nonlinear Anal. **72**, 794-805 (2010)
17. Włodarczyk, K, Plebaniak, R: Kannan-type contractions and fixed points in uniform spaces. Fixed Point Theory Appl. **2011**, 90 (2011)
18. Włodarczyk, K, Plebaniak, R: Contractions of Banach, Tarafdar, Meir-Keller, Ćirić-Jachymski-Matkowski and Suzuki types and fixed points in uniform spaces with generalized pseudodistances. J. Math. Anal. Appl. **404**, 338-350 (2013)
19. Włodarczyk, K, Plebaniak, R: Asymmetric structures, discontinuous contractions and iterative approximation of fixed and periodic points. Fixed Point Theory Appl. **2013**, 128 (2013)
20. Włodarczyk, K: Hausdorff quasi-distances, periodic and fixed points for Nadler type set-valued contractions in quasi-gauge spaces. Fixed Point Theory Appl. **2013**, 239 (2013)
21. Włodarczyk, K, Plebaniak, R: Dynamic processes, fixed points, endpoints, asymmetric structures and investigations related to Caristi, Nadler and Banach in uniform spaces. Abstr. Appl. Anal. **2015**, Article ID 942814 (2015)
22. Bakhtin, IA: The contraction principle in quasimetric spaces. In: Functional Analysis, vol. 30, pp. 26-37 (1989)
23. Czerwik, S: Nonlinear set-valued contraction mappings in b-metric spaces. Atti Semin. Mat. Fis. Univ. Modena **46**, 263-276 (1998)
24. Shah, MH, Hussain, N: Nonlinear contractions in partially ordered quasi b-metric spaces. Commun. Korean Math. Soc. **27**(1), 117-128 (2012)
25. Alghamdi, MA, Hussain, N, Salimi, P: Fixed point and coupled fixed point theorems on b-metric-like spaces. J. Inequal. Appl. **2013**, 402 (2013)
26. Zhu, CX, Chen, CF, Zhang, X: Some results in quasi-b-metric-like spaces. J. Inequal. Appl. **2014**, 437 (2014)
27. Zoto, K, Kumari, PS, Hoxha, E: Some fixed point theorems and cyclic contractions in dislocated and dislocated quasi-metric spaces. Am. J. Numer. Anal. **2**(3), 79-84 (2014)