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# รายงานวิจัยฉบับสมบูรณ์

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ทฤษฎีบทจุดตรึงร่วมพีพีเอฟแบบไม่อิสระสำหรับฟังก์ชันใน  
ปริภูมิบานาค

PPF dependent common fixed point theorems for mappings in  
Banach spaces

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PPF dependent common fixed point theorems for  
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ผู้วิจัย

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สนับสนุนโดยมหาวิทยาลัยนเรศวร จากงบประมาณแผ่นดินมหาวิทยาลัยนเรศวร  
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- A generalized system of nonlinear variational inequalities in Banach spaces	

## ABSTRACT

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We introduce a new generalized system of nonlinear variational inequality problem (GSNVIP) by using the generalized projection method. Moreover, we introduce an iterative scheme for finding a solution to this problem. Moreover, some existence and strong convergence theorems are established in uniformly smooth and strictly convex Banach spaces under suitable conditions.

Keywords: generalized projection mappings, generalized system of nonlinear variational inequalities, uniformly smooth Banach spaces, normalized duality mappings.

## บทคัดย่อ

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ชื่อโครงการ : ทฤษฎีบทจุดตรึงร่วมพีทีเอฟแบบไม่มีอิสระสำหรับฟังก์ชันในปริภูมิบานาค

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ผู้วิจัยได้แนะนำระบบการวางนัยทั่วไปรูปแบบใหม่ของปัญหาอสมการการแปรผันไม่เชิงเส้นโดยใช้วิธีการฉายภาพวางนัยทั่วไป ยิ่งไปกว่านั้นเรายังแนะนำแบบแผนกระบวนการทำซ้ำสำหรับการหาคำตอบของปัญหานี้ ผลจากกระบวนการดังกล่าว ทำให้เราได้ทฤษฎีบทการ มีอยู่จริงและการลู่เข้าในปริภูมิบานาคราบเรียบเอกรูปที่เป็นคอนเว็กซ์โดยแท้ภายใต้เงื่อนไขที่เป็นไปได้

คำสำคัญ: การส่งภาพฉายวางนัยทั่วไป, ระบบการวางนัยทั่วไปของอสมการการแปรผันไม่เชิงเส้น, ปริภูมิบานาคราบเรียบเอกรูป, การส่งคู่กันปกติ

## CHAPTER I

### EXECUTIVE SUMMARY

Let  $E$  be a real Banach space with dual space  $E^*$ ,  $\langle \cdot, \cdot \rangle$  be the dual pair between  $E$  and  $E^*$ ,  $K$  a nonempty closed convex subset of  $E$ . The normalized duality mapping  $J : E \rightarrow 2^{E^*}$  is defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \text{ for all } x \in E.$$

A Banach space  $E$  is said to be strictly convex if  $\frac{\|x+y\|}{2} < 1$  for all  $x, y \in U = \{z \in E : \|z\| = 1\}$  with  $x \neq y$ .  $E$  is said to be uniformly convex if for each  $\epsilon \in (0, 2]$ , there exists  $\delta > 0$  such that  $\frac{\|x+y\|}{2} < 1 - \delta$  for all  $x, y \in U$  with  $\|x - y\| \geq \epsilon$ .  $E$  is said to be smooth if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t},$$

exists for all  $x, y \in U$ .  $E$  is said to be uniformly smooth if the above limit exists uniformly in  $x, y \in U$ .

Assume that  $E$  is a smooth, strictly convex and reflexive Banach space and  $K$  is a nonempty closed convex subset of  $E$ ,  $\phi : E \times E \rightarrow \mathbb{R}^+ := [0, \infty)$  to denote the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \text{ for all } x, y \in E.$$

Following Alber, the generalized projection  $\Pi_K : E \rightarrow K$  is defined by  $\Pi_K x = z$ , where  $z$  is the unique solution to the minimization problem

$$\phi(z, x) = \min_{y \in K} \phi(y, x).$$

The existence and uniqueness of the mapping  $\Pi_K$  follow from the property of the function  $\phi(x, y)$  and the strict monotonicity of the mapping  $J$ .

Let  $F : K \times K \rightarrow \mathbb{R}$  be a bifunction. The equilibrium problem is to find  $x \in K$  such that

$$F(x, y) \geq 0 \text{ for all } y \in K. \quad (1)$$

The set of solutions of (1) is denoted by  $EP(F)$ .

Let  $A : K \rightarrow E^*$  be a mapping. The classical variational inequality, denoted by  $VI(A, K)$ , is to find  $x^* \in K$  such that

$$\langle Ax^*, v - x^* \rangle \geq 0 \text{ for all } v \in K.$$

Let  $F : K \times K \rightarrow \mathbb{R}$  be a bifunction,  $\psi : K \rightarrow \mathbb{R}$  be a real-valued function, and  $A : K \rightarrow E^*$  be a nonlinear mapping. The generalized mixed equilibrium problem is to find  $u \in K$  such that

$$F(u, y) + \langle Au, y - u \rangle + \psi(y) - \psi(u) \geq 0, \forall y \in K. \quad (2)$$

The set of solutions of (2) is denoted by  $GMEP(F, A, \psi)$ .

Variational inequality theory has become a very effective and powerful tool for studying a wide range of problems arising in pure and applied sciences which include work on differential equations, general equilibrium problems in economics, mechanics, control problems and transportation. In 2005, Verma introduced a general model for two-step projection methods and applied it to the approximation solvability of a system of nonlinear variational inequality problems in a Hilbert space. Based on the convergence of projection methods, Chang et al. introduced and studied the approximate solvability of a generalized system for relaxed cocoercive nonlinear variational inequality in Hilbert spaces. Recently, Chang et al. introduced a system of generalized nonlinear variational inequality and an iterative scheme for finding a solution to a system of generalized nonlinear variational inequality problem by using the generalized projection method. Moreover, they proved some existence and strong convergence theorems in uniformly smooth and strictly convex Banach spaces.

In this work, we introduce a generalized system of nonlinear variational inequality problem (GSNVIP) by using the generalized projection approach to introduce an iterative scheme for finding a solution to this problem. Finally, we prove some existence and strong convergence theorems in uniformly smooth and strictly convex Banach spaces under suitable conditions.

## CHAPTER II

### CONTENTS OF RESEARCH

In this project, we obtain one publications that published in the international journals as the following:

1. Prapairat Junlouchai, Anchalee Kaewcharoen, and Somyot Plubtieng, A generalized system of nonlinear variational inequalities in Banach spaces, Abstract and Applied Analysis, Volume 2014, Article ID 869372, 10 pages.

We assume that  $E$  is a real Banach space with dual space  $E^*$ ,  $K$  is a nonempty closed convex subset of  $E$ . Let  $T_1, \dots, T_N : K^N \rightarrow E^*$  be nonlinear mappings and  $f : K \rightarrow E$  a mapping. The generalized system of nonlinear variational inequality problem (GSNVIP) is to find  $x_1^*, \dots, x_N^*$  such that for all  $x \in K$ ,

$$\begin{cases} \langle f(x) - f(x_1^*), T_1(x_2^*, x_3^*, \dots, x_N^*, x_1^*) \rangle \geq 0 \\ \langle f(x) - f(x_2^*), T_2(x_3^*, x_4^*, \dots, x_N^*, x_1^*, x_2^*) \rangle \geq 0 \\ \vdots \\ \langle f(x) - f(x_N^*), T_N(x_1^*, x_2^*, \dots, x_{N-1}^*, x_N^*) \rangle \geq 0. \end{cases} \quad (1)$$

If  $N = 3$ ,  $f = I$  and  $T_1, T_2, T_3 : K^3 \rightarrow E^*$  are nonlinear mappings, then the generalized system of nonlinear variational inequality problem (GSNVIP) reduces to the following problem: is to find  $x_1^*, x_2^*, x_3^*$  such that for all  $x \in K$ ,

$$\begin{cases} \langle x - x_1^*, T_1(x_2^*, x_3^*, x_1^*) \rangle \geq 0 \\ \langle x - x_2^*, T_2(x_3^*, x_1^*, x_2^*) \rangle \geq 0 \\ \langle x - x_3^*, T_3(x_1^*, x_2^*, x_3^*) \rangle \geq 0. \end{cases} \quad (2)$$

- (a) **Lemma** : Let  $E$  be a smooth, strictly convex and reflexive Banach space,  $K$  be a nonempty closed convex subset of  $E$ . Let  $T_1, \dots, T_N : K^N \rightarrow E^*$  be mappings,  $f : K \rightarrow K$  be a bijective mapping and  $\rho_1, \dots, \rho_N$  be any positive real numbers. Then  $(x_1^*, \dots, x_N^*) \in K^N$  is a solution to problem (1) if and only if  $(x_1^*, \dots, x_N^*) \in K^N$  is a

solution to the following system of operator equations:

$$\begin{cases} x_1^* = f^{-1} \prod_K J^{-1}(Jf(x_1^*) - \rho_1 T_1(x_2^*, x_3^*, \dots, x_N^*, x_1^*)), \\ x_2^* = f^{-1} \prod_K J^{-1}(Jf(x_2^*) - \rho_2 T_2(x_3^*, x_4^*, \dots, x_N^*, x_1^*, x_2^*)), \\ \vdots \\ x_{N-1}^* = f^{-1} \prod_K J^{-1}(Jf(x_{N-1}^*) \\ \quad - \rho_{N-1} T_{N-1}(x_N^*, x_1^*, x_2^*, \dots, x_{N-2}^*, x_{N-1}^*)), \\ x_N^* = f^{-1} \prod_K J^{-1}(Jf(x_N^*) - \rho_N T_N(x_1^*, x_2^*, \dots, x_N^*)). \end{cases} \quad (3)$$

(b) **Algorithm** : For any given initial points  $x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(N)} \in K$ , compute the sequences  $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \dots, \{x_n^{(N)}\}$  by the iterative processes

$$\begin{cases} x_{n+1}^{(N)} = f^{-1}(J^{-1}((1 - \alpha_n^{(N)})Jf(x_n^{(N)}) \\ \quad + \alpha_n^{(N)}J(\prod_K J^{-1}(Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}))))), \\ x_{n+1}^{(N-1)} = f^{-1}(J^{-1}((1 - \alpha_n^{(N-1)})Jf(x_n^{(N-1)}) + \alpha_n^{(N-1)}J(\prod_K J^{-1}(Jf(x_n^{(N-1)}) \\ \quad - \rho_{N-1} T_{N-1}(x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N-2)}, x_n^{(N-1)}))))), \\ \vdots \\ x_{n+1}^{(2)} = f^{-1}(J^{-1}((1 - \alpha_n^{(2)})Jf(x_n^{(2)}) \\ \quad + \alpha_n^{(2)}J(\prod_K J^{-1}(Jf(x_n^{(2)}) - \rho_2 T_2(x_{n+1}^{(3)}, x_{n+1}^{(4)}, \dots, x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}))))), \\ x_{n+1}^{(1)} = f^{-1}(J^{-1}((1 - \alpha_n^{(1)})Jf(x_n^{(1)}) \\ \quad + \alpha_n^{(1)}J(\prod_K J^{-1}(Jf(x_n^{(1)}) - \rho_1 T_1(x_{n+1}^{(2)}, x_{n+1}^{(3)}, \dots, x_{n+1}^{(N)}, x_n^{(1)}))))), \quad n \geq 0, \end{cases} \quad (4)$$

where  $\prod_K$  is the generalized projection and  $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \dots, \{\alpha_n^{(N)}\}$  are sequences in  $[0, 1]$ .

(c) **Theorem** : Let  $E$  be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property,  $K$  be a nonempty closed and convex subset of  $E$  with  $\theta \in K$ . Let  $f : K \rightarrow K$  be an isometry mapping,  $T_1, \dots, T_N : K^N \rightarrow E^*$  be continuous mappings and  $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \dots, \{\alpha_n^{(N)}\}$  be the sequences in  $(a, b)$  with  $0 < a < b < 1$  satisfying the following conditions:

(a) there exist a compact subset  $C \subset E^*$  and constants  $\rho_1 > 0, \rho_2 > 0, \dots, \rho_N > 0$  such that

$$(J(K) - \rho_N T_N(K^N)) \cup (J(K) - \rho_{N-1} T_{N-1}(K^N)) \cup \dots \cup (J(K) - \rho_1 T_1(K^N)) \subset C,$$

where  $J(x_1, x_2, \dots, x_N) = Jx_N$ ,  $\forall (x_1, x_2, \dots, x_N) \in K^N$  and

$$\begin{cases} \langle T_1(x_1, x_2, \dots, x_N), J^{-1}(Jx_N - \rho_1 T_1(x_1, x_2, \dots, x_N)) \rangle \geq 0, \\ \langle T_2(x_1, x_2, \dots, x_N), J^{-1}(Jx_N - \rho_2 T_2(x_1, x_2, \dots, x_N)) \rangle \geq 0, \\ \vdots \\ \langle T_N(x_1, x_2, \dots, x_N), J^{-1}(Jx_N - \rho_N T_N(x_1, x_2, \dots, x_N)) \rangle \geq 0, \end{cases} \quad (5)$$

for all  $x_1, x_2, \dots, x_N \in K$ .

- (b)  $\lim_{n \rightarrow \infty} \alpha_n^{(1)} = d_1 \in (a, b)$ ,  $\lim_{n \rightarrow \infty} \alpha_n^{(2)} = d_2 \in (a, b)$ , ...,  $\lim_{n \rightarrow \infty} \alpha_n^{(N)} = d_N \in (a, b)$ . Let  $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \dots, \{x_n^{(N)}\}$  be the sequences defined by (4).

Then the problem (1) has a solution  $(x_1^*, x_2^*, \dots, x_N^*) \in K^N$  and the sequences  $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \dots, \{x_n^{(N)}\}$  converge strongly to  $x_1^*, x_2^*, \dots, x_N^*$ , respectively.

- (d) Corollary : Let  $E$  be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property,  $K$  be a nonempty closed and convex subset of  $E$  with  $\theta \in K$ . Let  $T_1, T_2, T_3 : K^3 \rightarrow E^*$  be continuous mappings and  $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \{\alpha_n^{(3)}\}$  be the sequences in  $(a, b)$  with  $0 < a < b < 1$  satisfying the following conditions:

- (a) there exist a compact subset  $C \subset E^*$  and constants  $\rho_1 > 0, \rho_2 > 0, \rho_3 > 0$  such that

$$(J(K) - \rho_3 T_3(K^3)) \cup (J(K) - \rho_2 T_2(K^3)) \cup (J(K) - \rho_1 T_1(K^3)) \subset C,$$

where  $J(x_1, x_2, x_3) = Jx_3$ ,  $\forall (x_1, x_2, x_3) \in K^3$  and

$$\begin{cases} \langle T_1(x_1, x_2, x_3), J^{-1}(Jx_3 - \rho_1 T_1(x_1, x_2, x_3)) \rangle \geq 0, \\ \langle T_2(x_1, x_2, x_3), J^{-1}(Jx_3 - \rho_2 T_2(x_1, x_2, x_3)) \rangle \geq 0, \\ \langle T_3(x_1, x_2, x_3), J^{-1}(Jx_3 - \rho_3 T_3(x_1, x_2, x_3)) \rangle \geq 0, \end{cases} \quad (6)$$

for all  $x_1, x_2, x_3 \in K$ .

- (b)  $\lim_{n \rightarrow \infty} \alpha_n^{(1)} = d_1 \in (a, b)$ ,  $\lim_{n \rightarrow \infty} \alpha_n^{(2)} = d_2 \in (a, b)$ ,  $\lim_{n \rightarrow \infty} \alpha_n^{(3)} = d_3 \in (a, b)$ .

Let  $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \{x_n^{(3)}\}$  be the sequences defined by

$$\begin{cases} x_{n+1}^{(3)} = J^{-1}((1 - \alpha_n^{(3)})Jf(x_n^{(3)}) \\ \quad + \alpha_n^{(3)}J(\prod_K J^{-1}(Jf(x_n^{(3)}) - \rho_3 T_3(x_n^{(1)}, x_n^{(2)}, x_n^{(3)})))), \\ x_{n+1}^{(2)} = J^{-1}((1 - \alpha_n^{(2)})Jf(x_n^{(2)}) \\ \quad + \alpha_n^{(2)}J(\prod_K J^{-1}(Jf(x_n^{(2)}) - \rho_2 T_2(x_{n+1}^{(3)}, x_n^{(1)}, x_n^{(2)})))), \\ x_{n+1}^{(1)} = J^{-1}((1 - \alpha_n^{(1)})Jf(x_n^{(1)}) \\ \quad + \alpha_n^{(1)}J(\prod_K J^{-1}(Jf(x_n^{(1)}) - \rho_1 T_1(x_{n+1}^{(2)}, x_{n+1}^{(3)}, x_n^{(1)})))), \\ n \geq 0. \end{cases} \quad (7)$$

Then the problem (2) has a solution  $(x_1^*, x_2^*, x_3^*) \in K^3$  and the sequences  $\{x_n^{(1)}\}, \{x_n^{(2)}\}$  and  $\{x_n^{(3)}\}$  converge strongly to  $x_1^*, x_2^*$  and  $x_3^*$ , respectively.

(e) **Corollary :** Let  $H$  be a real Hilbert space,  $K$  be a nonempty closed and convex subset of  $H$ . Let  $f : K \rightarrow K$  be an isometry mapping and  $T_1, \dots, T_N : K^N \rightarrow H$  be continuous mappings and  $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \dots, \{\alpha_n^{(N)}\}$  are sequences in  $(a, b)$  with  $0 < a < b < 1$  satisfying the following conditions:

(a) there exist a compact subset  $C \subset H$  and constants  $\rho_1 > 0, \rho_2 > 0, \dots, \rho_N > 0$  such that

$$(I(K) - \rho_N T_N(K^N)) \cup (I(K) - \rho_{N-1} T_{N-1}(K^N)) \cup \dots \cup (I(K) - \rho_1 T_1(K^N)) \subset C,$$

where  $(x_1, x_2, \dots, x_N) = x_N, \forall (x_1, x_2, \dots, x_N) \in K^N$  and

$$\begin{cases} \langle T_1(x_1, x_2, \dots, x_N), x_N - \rho_1 T_1(x_1, x_2, \dots, x_N) \rangle \geq 0, \\ \langle T_2(x_1, x_2, \dots, x_N), x_N - \rho_2 T_2(x_1, x_2, \dots, x_N) \rangle \geq 0, \\ \vdots \\ \langle T_N(x_1, x_2, \dots, x_N), x_N - \rho_N T_N(x_1, x_2, \dots, x_N) \rangle \geq 0, \end{cases} \quad (8)$$

for all  $x_1, x_2, \dots, x_N \in K$ .

(b)  $\lim_{n \rightarrow \infty} \alpha_n^{(1)} = d_1 \in (a, b)$ ,  $\lim_{n \rightarrow \infty} \alpha_n^{(2)} = d_2 \in (a, b)$ ,  $\dots$ ,  $\lim_{n \rightarrow \infty} \alpha_n^{(N)} = d_N \in (a, b)$ .

Let  $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \dots, \{x_n^{(N)}\}$  be the sequences defined by

$$\begin{cases} x_{n+1}^{(N)} = f^{-1}((1 - \alpha_n^{(N)})f(x_n^{(N)}) + \alpha_n^{(N)}P_K(f(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}))), \\ x_{n+1}^{(N-1)} = f^{-1}((1 - \alpha_n^{(N-1)})f(x_n^{(N-1)}) + \alpha_n^{(N-1)}P_K(f(x_n^{(N-1)}) - \rho_{N-1} T_{N-1}(x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N-2)}, x_n^{(N-1)}))), \\ \vdots \\ x_{n+1}^{(2)} = f^{-1}((1 - \alpha_n^{(2)})f(x_n^{(2)}) + \alpha_n^{(2)}P_K(f(x_n^{(2)}) - \rho_2 T_2(x_{n+1}^{(3)}, x_{n+1}^{(4)}, \dots, x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}))), \\ x_{n+1}^{(1)} = f^{-1}((1 - \alpha_n^{(1)})f(x_n^{(1)}) + \alpha_n^{(1)}P_K(f(x_n^{(1)}) - \rho_1 T_1(x_{n+1}^{(2)}, x_{n+1}^{(3)}, \dots, x_{n+1}^{(N)}, x_n^{(1)}))), \quad n \geq 0, \end{cases} \quad (9)$$

where  $P_K$  is a metric projection on  $H$  to  $K$ .

Then the problem (1) has a solution  $(x_1^*, x_2^*, \dots, x_N^*) \in K^N$  and the sequences  $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \dots, \{x_n^{(N)}\}$  converge strongly to  $x_1^*, x_2^*, \dots, x_N^*$ , respectively.

## CHAPTER III

### OUTPUT

ผลลัพธ์จากโครงการวิจัยที่ได้รับทุนจากงบประมาณแผ่นดินมหาวิทยาลัยนเรศวร  
ประจำปี 2557

#### 1. ผลงานวิจัยตีพิมพ์ในวารสารวิชาการนานาชาติ

- Prapairat Junlouchai, Anchalee Kaewcharoen, and Somyot Plubtieng,

A generalized system of nonlinear variational inequalities in Banach spaces, Abstract  
and Applied Analysis, Volume 2014, Article ID 869372, 10 pages.

#### 2. การนำผลงานวิจัยไปใช้ประโยชน์

มีการนำไปใช้ประโยชน์ทั้งเชิงวิชาการและเชิงสาธารณะโดยทำให้มีการพัฒนาการเรียน  
การสอน รวมทั้งมีการสร้างเครือข่ายความร่วมมือในการทำวิจัย

## Research Article

# A Generalized System of Nonlinear Variational Inequalities in Banach Spaces

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We introduce a new generalized system of nonlinear variational inequality problems (GSNVIP) by using the generalized projection method. Moreover, we introduce an iterative scheme for finding a solution to this problem. Moreover, some existence and strong convergence theorems are established in uniformly smooth and strictly convex Banach spaces under suitable conditions. The results presented in the paper improve and extend some recent results.

## 1. Introduction

Variational inequality theory has become a very effective and powerful tool for studying a wide range of problems arising in pure and applied sciences which include work on differential equations, general equilibrium problems in economics and mechanics, control problems, and transportation. In 2005, Verma [1] introduced a general model for two-step projection methods and applied it to the approximation solvability of a system of nonlinear variational inequality problems in a Hilbert space. Based on the convergence of projection methods, Chang et al. [2] introduced and studied the approximate solvability of a generalized system for relaxed cocoercive nonlinear variational inequalities in Hilbert spaces (see, for instance, [3–5] and the references therein). Recently, Chang et al. [6] introduced a system of generalized nonlinear variational inequalities and an iterative scheme for finding a solution to a system of generalized nonlinear variational inequality problems by using the generalized projection method. Moreover, they proved some existence and strong convergence theorems in uniformly smooth and strictly convex Banach spaces.

In this paper, we introduce a generalized system of nonlinear variational inequality problems (GSNVIP) by using the generalized projection approach to introduce an iterative scheme for finding a solution to this problem. Finally, we

prove some existence and strong convergence theorems in uniformly smooth and strictly convex Banach spaces under suitable conditions.

## 2. Preliminaries

Let  $E$  be a real Banach space with dual space  $E^*$ ,  $\langle \cdot, \cdot \rangle$  the dual pair between  $E$  and  $E^*$ , and  $K$  a nonempty closed convex subset of  $E$ . The normalized duality mapping  $J : E \rightarrow 2^{E^*}$  is defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \quad \forall x \in E. \quad (1)$$

A Banach space  $E$  is said to be strictly convex if  $\|x + y\|/2 < 1$  for all  $x, y \in U = \{z \in E : \|z\| = 1\}$  with  $x \neq y$ .  $E$  is said to be uniformly convex if for each  $\epsilon \in (0, 2)$  there exists  $\delta > 0$  such that  $\|x + y\|/2 < 1 - \delta$  for all  $x, y \in U$  with  $\|x - y\| \geq \epsilon$ .  $E$  is said to be smooth if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2)$$

exists for all  $x, y \in U$ .  $E$  is said to be uniformly smooth if the above limit exists uniformly in  $x, y \in U$ .

**Remark 1** (see [7]). (i) If  $E$  is a uniformly smooth Banach space, then the normalized duality mapping  $J$  is uniformly continuous on each bounded subset of  $E$ .

(ii) If  $E$  is a smooth, strictly convex and reflexive Banach space, then the normalized duality mapping  $J : E \rightarrow 2^{E^*}$  is a single valued bijective mapping.

(iii) If  $E$  is a smooth, strictly convex and reflexive Banach space and  $J^* : E^* \rightarrow E$  is the duality mapping in  $E^*$ , then  $J^{-1} = J^*$ ,  $JJ^* = I_{E^*}$ , and  $J^*J = I_E$ .

(iv) If  $E$  is a strictly convex and reflexive Banach space, then  $J^{-1}$  is hemicontinuous; that is,  $J^{-1}$  is norm-weak-continuous.

(v)  $E$  is uniformly smooth if and only if  $E^*$  is uniformly convex.

(vi) If  $E$  is a uniformly smooth and strictly convex Banach space with the Kadec-Klee property (i.e., for any sequence  $\{x_n\} \subset E$ , if  $x_n \rightarrow x \in E$  and  $\|x_n\| \rightarrow \|x\|$ , then  $x_n \rightarrow x$ ), then both the normalized duality mappings  $J : E \rightarrow E^*$  and  $J^* = J^{-1} : E^* \rightarrow E$  are continuous.

(vii) Each uniformly convex Banach space  $E$  has the Kadec-Klee property.

Assume that  $E$  is a smooth, strictly convex and reflexive Banach space and  $K$  is a nonempty closed convex subset of  $E$ ;  $\phi : E \times E \rightarrow \mathbb{R}^+ := [0, \infty)$  to denote the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \quad (3)$$

Following Alber [8], the generalized projection  $\Pi_K : E \rightarrow K$  is defined by  $\Pi_K x = z$ , where  $z$  is the unique solution to the minimization problem

$$\phi(z, x) = \min_{y \in K} \phi(y, x). \quad (4)$$

The existence and uniqueness of the mapping  $\Pi_K$  follow from the property of the function  $\phi(x, y)$  and the strict monotonicity of the mapping  $J$ .

**Lemma 2** (see [8]). *Let  $E$  be a smooth, strictly convex and reflexive Banach space and  $K$  a nonempty closed convex subset of  $E$ . Then the following conclusions hold:*

(a) if  $x \in E$  and  $z \in K$ , then

$$z = \Pi_K x \iff \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in K; \quad (5)$$

(b)  $\Pi_K$  is a continuous mapping from  $E$  onto  $K$ .

**Remark 3.** If  $E$  is a real Hilbert space, then  $J = I$  (identity mapping),  $\phi(x, y) = \|x - y\|^2$ , and  $\Pi_K$  is the metric projection  $P_K$  from  $E$  onto  $K$ .

**Lemma 4** (see [9, 10]). *Let  $E$  be a uniformly convex Banach space,  $r > 0$  a positive number, and  $B_r(0) := \{x \in E :$*

$\|x\| \leq r\}$  a closed ball of  $E$ . Then, for any given finite subset  $\{x_1, x_2, \dots, x_N\} \subset B_r(0)$  and for any given positive numbers  $\lambda_1, \lambda_2, \dots, \lambda_N$  with  $\sum_{n=1}^N \lambda_n = 1$ , there exists a continuous, strictly increasing, and convex function  $g : [0, 2r) \rightarrow [0, \infty)$  with  $g(0) = 0$  such that for any  $i, j \in \{1, 2, \dots, N\}$  with  $i < j$  the following holds:

$$\left\| \sum_{n=1}^N \lambda_n x_n \right\|^2 \leq \sum_{n=1}^N \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|). \quad (6)$$

**Lemma 5** (see [11]). *Let  $E$  be a real reflexive, smooth, and strictly convex Banach space. Then the following inequality holds:*

$$\|f + g\|^2 \leq \|f\|^2 + 2\langle g, J^{-1}(f + g) \rangle, \quad \forall f, g \in E^*. \quad (7)$$

**Lemma 6** (see [6]). *Let  $E$  be a real Banach space,  $K$  a nonempty closed convex subset of  $E$  with  $0 \in K$ , and  $\Pi_K : E \rightarrow K$  the generalized projection. Then for each  $x \in E$ , one has  $\|\Pi_K x\| \leq \|x\|$ .*

### 3. Main Results

In this section, we assume that  $E$  is a real Banach space with dual space  $E^*$  and  $K$  is a nonempty closed convex subset of  $E$ . Let  $T_1, \dots, T_N : K^N \rightarrow E^*$  be nonlinear mappings and  $f : K \rightarrow E$  a mapping. The generalized system of nonlinear variational inequality problems (GSNVIP) is to find  $x_1^*, \dots, x_N^*$  such that for all  $x \in K$

$$\begin{aligned} \langle f(x) - f(x_1^*), T_1(x_2^*, x_3^*, \dots, x_N^*, x_1^*) \rangle &\geq 0, \\ \langle f(x) - f(x_2^*), T_2(x_3^*, x_4^*, \dots, x_N^*, x_1^*, x_2^*) \rangle &\geq 0, \\ &\vdots \\ \langle f(x) - f(x_N^*), T_N(x_1^*, x_2^*, \dots, x_{N-1}^*, x_N^*) \rangle &\geq 0. \end{aligned} \quad (8)$$

If  $N = 3$ ,  $f = I$ , and  $T_1, T_2, T_3 : K^3 \rightarrow E^*$  are nonlinear mappings, then the generalized system of nonlinear variational inequality problems (GSNVIP) reduces to the following problem (see [6]) to find  $x_1^*, x_2^*, x_3^*$  such that, for all  $x \in K$ ,

$$\begin{aligned} \langle x - x_1^*, T_1(x_2^*, x_3^*, x_1^*) \rangle &\geq 0, \\ \langle x - x_2^*, T_2(x_3^*, x_1^*, x_2^*) \rangle &\geq 0, \\ \langle x - x_3^*, T_3(x_1^*, x_2^*, x_3^*) \rangle &\geq 0. \end{aligned} \quad (9)$$

If  $N = 2$  and  $T_1, T_2 : K^2 \rightarrow E^*$  are nonlinear mappings and  $f : K \rightarrow E$  is a mapping, then the generalized system of nonlinear variational inequality problems (GSNVIP) reduces to the following problem to find  $x_1^*, x_2^*$  such that, for all  $x \in K$ ,

$$\begin{aligned} \langle f(x) - f(x_1^*), T_1(x_2^*, x_1^*) \rangle &\geq 0, \\ \langle f(x) - f(x_2^*), T_2(x_1^*, x_2^*) \rangle &\geq 0. \end{aligned} \quad (10)$$

If  $T, S : K^2 \rightarrow E^*$  are nonlinear mappings and  $g, f : K \rightarrow E$  are two mappings. Define  $T_1, T_2 : K^2 \rightarrow E^*$  by  $T_1(x_1^*, x_2^*) = \rho_1 T(x_1^*, x_2^*) + g(x_2^*) - g(x_1^*)$  and  $T_2(x_1^*, x_2^*) = \rho_2 S(x_1^*, x_2^*) + g(x_2^*) - g(x_1^*)$ . Then the generalized system of nonlinear variational inequality problems (GSNVIP) reduces to the following problem to find  $x_1^*, x_2^* \in K$  such that, for all  $x \in K$ ,

$$\begin{aligned} \langle f(x) - f(x_1^*), \rho_1 T(x_2^*, x_1^*) + g(x_2^*) - g(x_1^*) \rangle &\geq 0, \\ \langle f(x) - f(x_2^*), \rho_2 S(x_1^*, x_2^*) + g(x_2^*) - g(x_1^*) \rangle &\geq 0, \end{aligned} \quad (11)$$

where  $\rho_1$  and  $\rho_2$  are two positive constants.

**Lemma 7.** Let  $E$  be a smooth, strictly convex, and reflexive Banach space and  $K$  a nonempty closed convex subset of  $E$ . Let  $T_1, \dots, T_N : K^N \rightarrow E^*$  be mappings,  $f : K \rightarrow K$  a bijective mapping, and  $\rho_1, \dots, \rho_N$  any positive real numbers. Then  $(x_1^*, \dots, x_N^*) \in K^N$  is a solution to problem (8) if and only if  $(x_1^*, \dots, x_N^*) \in K^N$  is a solution to the following system of operator equations:

$$\begin{aligned} x_1^* &= f^{-1} \prod_K J^{-1} (Jf(x_1^*) - \rho_1 T_1(x_2^*, x_3^*, \dots, x_N^*, x_1^*)), \\ x_2^* &= f^{-1} \prod_K J^{-1} (Jf(x_2^*) - \rho_2 T_2(x_3^*, x_4^*, \dots, x_N^*, x_1^*, x_2^*)), \\ &\vdots \\ x_{N-1}^* &= f^{-1} \prod_K J^{-1} (Jf(x_{N-1}^*) \\ &\quad - \rho_{N-1} T_{N-1}(x_N^*, x_1^*, x_2^*, \dots, x_{N-2}^*, x_{N-1}^*)), \\ x_N^* &= f^{-1} \prod_K J^{-1} (Jf(x_N^*) - \rho_N T_N(x_1^*, x_2^*, \dots, x_N^*)). \end{aligned} \quad (12)$$

*Proof.* By Lemma 2, we have that  $(x_1^*, \dots, x_N^*) \in K^N$  is a solution of problem (8),

$$\Leftrightarrow \begin{cases} \langle f(x) - f(x_1^*), \\ \rho_1 T_1(x_2^*, x_3^*, \dots, x_N^*, x_1^*) \rangle \geq 0, \\ \langle f(x) - f(x_2^*), \\ \rho_2 T_2(x_3^*, x_4^*, \dots, x_N^*, x_1^*, x_2^*) \rangle \geq 0, \\ \vdots \\ \langle f(x) - f(x_{N-1}^*), \\ \rho_{N-1} T_{N-1}(x_N^*, x_1^*, x_2^*, \dots, x_{N-2}^*, x_{N-1}^*) \rangle \geq 0, \\ \langle f(x) - f(x_N^*), \\ \rho_N T_N(x_1^*, x_2^*, \dots, x_N^*) \rangle \geq 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} \langle f(x) - f(x_1^*), Jf(x_1^*) - Jf(x_1^*) \\ + \rho_1 T_1(x_2^*, x_3^*, \dots, x_N^*, x_1^*) \rangle \geq 0, \\ \langle f(x) - f(x_2^*), Jf(x_2^*) - Jf(x_2^*) \\ + \rho_2 T_2(x_3^*, x_4^*, \dots, x_N^*, x_1^*, x_2^*) \rangle \geq 0, \\ \vdots \\ \langle f(x) - f(x_{N-1}^*), Jf(x_{N-1}^*) - Jf(x_{N-1}^*) \\ + \rho_{N-1} T_{N-1}(x_N^*, x_1^*, x_2^*, \dots, x_{N-2}^*, x_{N-1}^*) \rangle \geq 0, \\ \langle f(x) - f(x_N^*), Jf(x_N^*) \\ - Jf(x_N^*) + \rho_N T_N(x_1^*, x_2^*, \dots, x_N^*) \rangle \geq 0, \\ \langle f(x) - f(x_1^*), Jf(x_1^*) \\ - J(J^{-1}(Jf(x_1^*) \\ - \rho_1 T_1(x_2^*, x_3^*, \dots, x_N^*, x_1^*))) \rangle \geq 0, \\ \langle f(x) - f(x_2^*), Jf(x_2^*) \\ - J(J^{-1}(Jf(x_2^*) \\ - \rho_2 T_2(x_3^*, x_4^*, \dots, x_N^*, x_1^*, x_2^*))) \rangle \geq 0, \\ \vdots \\ \langle f(x) - f(x_{N-1}^*), Jf(x_{N-1}^*) \\ - J(J^{-1}(Jf(x_{N-1}^*) - \rho_{N-1} T_{N-1} \\ \times (x_N^*, x_1^*, x_2^*, \dots, x_{N-2}^*, x_{N-1}^*))) \rangle \geq 0, \\ \langle f(x) - f(x_N^*), Jf(x_N^*) \\ - J(J^{-1}(Jf(x_N^*) \\ - \rho_N T_N(x_1^*, x_2^*, \dots, x_N^*))) \rangle \geq 0, \end{cases} \quad (13)$$

for all  $x \in K$ ,

$$\begin{aligned} &f(x_1^*) \\ &= \prod_K J^{-1} (Jf(x_1^*) - \rho_1 T_1(x_2^*, x_3^*, \dots, x_N^*, x_1^*)), \\ &f(x_2^*) \\ &= \prod_K J^{-1} (Jf(x_2^*) \\ &\quad - \rho_2 T_2(x_3^*, x_4^*, \dots, x_N^*, x_1^*, x_2^*)), \\ &\vdots \\ &f(x_{N-1}^*) \\ &= \prod_K J^{-1} (Jf(x_{N-1}^*) \\ &\quad - \rho_{N-1} T_{N-1} \\ &\quad \times (x_N^*, x_1^*, x_2^*, \dots, x_{N-2}^*, x_{N-1}^*)), \\ &f(x_N^*) \\ &= \prod_K J^{-1} (Jf(x_N^*) - \rho_N T_N(x_1^*, x_2^*, \dots, x_N^*)), \end{aligned} \quad (14)$$

for any  $\rho_1 > 0, \dots, \rho_N > 0$ ,

$$\begin{aligned} x_{n+1}^{(2)} &= f^{-1} \\ &\times \left( J^{-1} \left( (1 - \alpha_n^{(2)}) Jf(x_n^{(2)}) + \alpha_n^{(2)} J \right. \right. \\ &\times \left( \prod_K J^{-1} (Jf(x_n^{(2)}) - \rho_2 T_2 \right. \\ &\times (x_{n+1}^{(3)}, x_{n+1}^{(4)}, \dots, x_{n+1}^{(N)}, \\ &\left. \left. \left. x_n^{(1)}, x_n^{(2)} \right) \right) \right) \right), \\ x_1^* &= f^{-1} \prod_K J^{-1} (Jf(x_1^*) \\ &\quad - \rho_1 T_1(x_2^*, x_3^*, \dots, x_N^*, x_1^*)), \\ x_2^* &= f^{-1} \prod_K J^{-1} (Jf(x_2^*) \\ &\quad - \rho_2 T_2(x_3^*, x_4^*, \dots, x_N^*, x_1^*, x_2^*)), \\ &\vdots \\ x_{N-1}^* &= f^{-1} \prod_K J^{-1} (Jf(x_{N-1}^*) - \rho_{N-1} T_{N-1} \\ &\quad \times (x_N^*, x_1^*, x_2^*, \dots, x_{N-2}^*, x_{N-1}^*)), \\ x_N^* &= f^{-1} \prod_K J^{-1} (Jf(x_N^*) \\ &\quad - \rho_N T_N(x_1^*, x_2^*, \dots, x_N^*)). \end{aligned} \quad (15)$$

**Algorithm 8.** For any given initial points  $x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(N)} \in K$ , compute the sequences  $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \dots, \{x_n^{(N)}\}$  by the iterative processes

$$\begin{aligned} x_{n+1}^{(N)} &= f^{-1} \\ &\times \left( J^{-1} \left( (1 - \alpha_n^{(N)}) Jf(x_n^{(N)}) + \alpha_n^{(N)} J \right. \right. \\ &\times \left( \prod_K J^{-1} (Jf(x_n^{(N)}) - \rho_N T_N \right. \\ &\times (x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)})) \right) \right), \end{aligned} \quad n \geq 0, \quad (16)$$

$$\begin{aligned} x_{n+1}^{(N-1)} &= f^{-1} \\ &\times \left( J^{-1} \left( (1 - \alpha_n^{(N-1)}) Jf(x_n^{(N-1)}) + \alpha_n^{(N-1)} J \right. \right. \\ &\times \left( \prod_K J^{-1} (Jf(x_n^{(N-1)}) - \rho_{N-1} T_{N-1} \right. \\ &\times (x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}, \dots, \\ &\quad \left. \left. \left. x_n^{(N-2)}, x_n^{(N-1)} \right) \right) \right) \right), \\ &\vdots \end{aligned}$$

where  $\prod_K$  is the generalized projection and  $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \dots, \{\alpha_n^{(N)}\}$  are sequences in  $[0, 1]$ .

**Theorem 9.** Let  $E$  be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property and  $K$  a nonempty closed and convex subset of  $E$  with  $\theta \in K$ . Let  $f : K \rightarrow K$  be an isometry mapping,  $T_1, \dots, T_N : K^N \rightarrow E^*$  continuous mappings, and  $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \dots, \{\alpha_n^{(N)}\}$  the sequences in  $(a, b)$  with  $0 < a < b < 1$  satisfying the following conditions:

- (i) there exist a compact subset  $C \subset E^*$  and constants  $\rho_1 > 0, \rho_2 > 0, \dots, \rho_N > 0$  such that

$$\begin{aligned} &(J(K) - \rho_N T_N(K^N)) \cup (J(K) - \rho_{N-1} T_{N-1}(K^N)) \\ &\cup \dots \cup (J(K) - \rho_1 T_1(K^N)) \subset C, \end{aligned} \quad (17)$$

where  $J(x_1, x_2, \dots, x_N) = Jx_N$ , for all  $(x_1, x_2, \dots, x_N) \in K^N$ , and

Similarly, we note that

$$\begin{aligned}
 & \langle T_1(x_1, x_2, \dots, x_N), \\
 & \quad J^{-1}(Jx_N - \rho_1 T_1(x_1, x_2, \dots, x_N)) \rangle \geq 0, \\
 & \langle T_2(x_1, x_2, \dots, x_N), \\
 & \quad J^{-1}(Jx_N - \rho_2 T_2(x_1, x_2, \dots, x_N)) \rangle \geq 0, \quad (18) \\
 & \vdots \\
 & \langle T_N(x_1, x_2, \dots, x_N), \\
 & \quad J^{-1}(Jx_N - \rho_N T_N(x_1, x_2, \dots, x_N)) \rangle \geq 0, \\
 & \text{for all } x_1, x_2, \dots, x_N \in K;
 \end{aligned}$$

$$\begin{aligned}
 & \|Jf(x_n^{(N-1)}) - \rho_{N-1} T_{N-1}(x_{n+1}^{(N)}, x_n^{(1)}, \dots, x_n^{(N-2)}, x_n^{(N-1)})\|^2 \\
 & \leq \|f(x_n^{(N-1)})\|^2, \\
 & \|Jf(x_n^{(N-2)}) \\
 & \quad - \rho_{N-2} T_{N-2}(x_{n+1}^{(N-1)}, x_{n+1}^{(N)}, x_n^{(1)}, \dots, x_n^{(N-3)}, x_n^{(N-2)})\|^2 \\
 & \leq \|f(x_n^{(N-2)})\|^2, \\
 & \vdots \\
 & \|Jf(x_n^{(2)}) - \rho_2 T_2(x_{n+1}^{(3)}, x_{n+1}^{(4)}, \dots, x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)})\|^2 \\
 & \leq \|f(x_n^{(2)})\|^2, \\
 & \|Jf(x_n^{(1)}) - \rho_1 T_1(x_{n+1}^{(2)}, x_{n+1}^{(3)}, \dots, x_{n+1}^{(N)}, x_n^{(1)})\|^2 \\
 & \leq \|f(x_n^{(1)})\|^2.
 \end{aligned} \quad (20)$$

(ii)  $\lim_{n \rightarrow \infty} \alpha_n^{(1)} = d_1 \in (a, b)$ ,  $\lim_{n \rightarrow \infty} \alpha_n^{(2)} = d_2 \in (a, b)$ ,  $\dots$ ,  $\lim_{n \rightarrow \infty} \alpha_n^{(N)} = d_N \in (a, b)$ . Let  $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \dots, \{x_n^{(N)}\}$  be the sequences defined by (16). Then the problem (8) has a solution  $(x_1^*, x_2^*, \dots, x_N^*) \in K^N$  and the sequences  $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \dots, \{x_n^{(N)}\}$  converge strongly to  $x_1^*, x_2^*, \dots, x_N^*$ , respectively.

By Lemma 6, we obtain that

*Proof.*

*Step 1.* We first show that the sequences  $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \dots, \{x_n^{(N)}\}$  are bounded in  $K$ . It follows from Lemma 5 where  $J$  is bijective and condition (18) that

$$\begin{aligned}
 & \|Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)})\|^2 \\
 & \leq \|Jf(x_n^{(N)})\|^2 \\
 & \quad - 2\rho_N \langle T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}), \\
 & \quad \quad J^{-1}(Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)})) \rangle \\
 & \leq \|Jf(x_n^{(N)})\|^2 = \|f(x_n^{(N)})\|^2. \quad (19)
 \end{aligned}$$

$$\begin{aligned}
 & \|f(x_{n+1}^{(N)})\| \\
 & = \|Jf^{-1} \\
 & \quad \times \left( J^{-1} \left( (1 - \alpha_n^{(N)}) Jf(x_n^{(N)}) \right. \right. \\
 & \quad \left. \left. + \alpha_n^{(N)} J \left( \prod_K J^{-1} \right. \right. \right. \\
 & \quad \left. \left. \left. \times (Jf(x_n^{(N)}) - \rho_N T_N \right. \right. \right. \\
 & \quad \left. \left. \left. \times (x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)})) \right) \right) \right) \| \\
 & = \|J^{-1} \left( (1 - \alpha_n^{(N)}) Jf(x_n^{(N)}) \right. \\
 & \quad \left. + \alpha_n^{(N)} J \left( \prod_K J^{-1} \right. \right. \\
 & \quad \left. \left. \times (Jf(x_n^{(N)}) - \rho_N T_N \right. \right. \\
 & \quad \left. \left. \times (x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)})) \right) \right) \|
 \end{aligned}$$

$$\begin{aligned}
&= \left\| \left( (1 - \alpha_n^{(N)}) Jf(x_n^{(N)}) \right. \right. \\
&\quad \left. \left. + \alpha_n^{(N)} J \left( \prod_K J^{-1} \right. \right. \right. \\
&\quad \left. \left. \left. \times (Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)})) \right) \right) \right\| \\
&\leq (1 - \alpha_n^{(N)}) \|Jf(x_n^{(N)})\| \\
&\quad + \alpha_n^{(N)} \left\| J \left( \prod_K J^{-1} \right. \right. \\
&\quad \left. \left. \times (Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)})) \right) \right\| \\
&\leq (1 - \alpha_n^{(N)}) \|Jf(x_n^{(N)})\| \\
&\quad + \alpha_n^{(N)} \|J J^{-1} (Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}))\| \\
&= (1 - \alpha_n^{(N)}) \|f(x_n^{(N)})\| \\
&\quad + \alpha_n^{(N)} \|Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)})\| \\
&\leq (1 - \alpha_n^{(N)}) \|f(x_n^{(N)})\| + \alpha_n^{(N)} \|f(x_n^{(N)})\| \\
&= \|f(x_n^{(N)})\|.
\end{aligned}
\tag{21}$$

Since  $f$  is an isometry mapping, we have  $\|x_{n+1}^{(N)}\| \leq \|x_n^{(N)}\|$ . By the same argument method as given above, we have  $\|x_{n+1}^{(N-1)}\| \leq \|x_n^{(N-1)}\|, \dots, \|x_{n+1}^{(1)}\| \leq \|x_n^{(1)}\|$ . Therefore, we note that  $\lim_{n \rightarrow \infty} \|x_n^{(1)}\|, \dots, \lim_{n \rightarrow \infty} \|x_n^{(N)}\|$  exist and hence the sequences  $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \dots, \{x_n^{(N)}\}$  are bounded in  $K$ .

**Step 2.** By Lemmas 4 and 6, where  $f$  is an isometry mapping and (19), it follows that there exists a continuous strictly increasing and convex function  $g : [0, 2r) \rightarrow [0, \infty)$  with  $g(0) = 0$  such that

$$\begin{aligned}
&\|f(x_{n+1}^{(N)})\|^2 \\
&\leq (1 - \alpha_n^{(N)}) \|Jf(x_n^{(N)})\|^2 \\
&\quad + \alpha_n^{(N)} \left\| J \prod_K J^{-1} (Jf(x_n^{(N)}) \right. \\
&\quad \left. - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)})) \right\|^2
\end{aligned}$$

$$\begin{aligned}
&- (1 - \alpha_n^{(N)}) \alpha_n^{(N)} g \\
&\times (\|Jf(x_n^{(N)})\| \\
&\quad + J \prod_K J^{-1} (Jf(x_n^{(N)}) \\
&\quad - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}))) \| \\
&\leq (1 - \alpha_n^{(N)}) \|f(x_n^{(N)})\|^2 \\
&\quad + \alpha_n^{(N)} \|Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)})\|^2 \\
&\quad - (1 - \alpha_n^{(N)}) \alpha_n^{(N)} \\
&\quad \times g(\|Jf(x_n^{(N)})\| \\
&\quad + J \prod_K J^{-1} (Jf(x_n^{(N)}) \\
&\quad - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}))) \| \\
&\leq (1 - \alpha_n^{(N)}) \|f(x_n^{(N)})\|^2 + \alpha_n^{(N)} \|f(x_n^{(N)})\|^2 \\
&\quad - (1 - \alpha_n^{(N)}) \alpha_n^{(N)} \\
&\quad \times g(\|Jf(x_n^{(N)})\| \\
&\quad + J \prod_K J^{-1} (Jf(x_n^{(N)}) \\
&\quad - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}))) \| \\
&= \|f(x_n^{(N)})\|^2 - (1 - \alpha_n^{(N)}) \alpha_n^{(N)} \\
&\quad \times g(\|Jf(x_n^{(N)})\| \\
&\quad + J \prod_K J^{-1} (Jf(x_n^{(N)}) \\
&\quad - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}))) \|).
\end{aligned}
\tag{22}$$

This implies that

$$\begin{aligned}
&(1 - \alpha_n^{(N)}) \alpha_n^{(N)} g(\|Jf(x_n^{(1)})\| \\
&\quad + J \prod_K J^{-1} (Jf(x_n^{(N)}) \\
&\quad - \rho_N T_N \\
&\quad \times (x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}))) \|) \\
&\leq \|f(x_n^{(N)})\|^2 - \|f(x_{n+1}^{(N)})\|^2.
\end{aligned}
\tag{23}$$



Since  $\{\|x_n^{(k)}\|\}$  converges for all  $k = 1, 2, \dots, N$ , it follows by letting  $n \rightarrow \infty$  in (23), condition (ii), and the property of  $g$  that

$$\begin{aligned} & \|Jf(x_n^{(N)}) \\ & - J \prod_K J^{-1} (Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}))\| \rightarrow 0, \end{aligned} \quad (24)$$

as  $n \rightarrow \infty$ . By (16) and (24), we have

$$\begin{aligned} & \|Jf(x_{n+1}^{(N)}) - Jf(x_n^{(N)})\| \\ & = \alpha_n^{(N)} \|Jf(x_n^{(N)}) \\ & - J \prod_K J^{-1} (Jf(x_n^{(N)}) \\ & - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}))\| \rightarrow 0, \end{aligned} \quad (25)$$

as  $n \rightarrow \infty$ . Similarly, we can prove that

$$\begin{aligned} & \|Jf(x_{n+1}^{(N-1)}) - Jf(x_n^{(N-1)})\| \\ & = \alpha_n^{(N-1)} \|Jf(x_n^{(N-1)}) - J \prod_K J^{-1} \\ & \times (Jf(x_n^{(N-1)}) \\ & - \rho_{N-1} T_{N-1} \\ & \times (x_{n+1}^{(N)}, x_n^{(1)}, \\ & x_n^{(2)}, \dots, x_n^{(N-2)}, x_n^{(N-1)}))\| \rightarrow 0, \end{aligned}$$

$$\begin{aligned} & \|Jf(x_{n+1}^{(2)}) - Jf(x_n^{(2)})\| \\ & = \alpha_n^{(2)} \|Jf(x_n^{(2)}) - J \prod_K J^{-1} \\ & \times (Jf(x_n^{(2)}) \\ & - \rho_2 T_2 \\ & \times (x_{n+1}^{(3)}, x_{n+1}^{(4)}, \dots, x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}))\| \rightarrow 0, \end{aligned}$$

$$\begin{aligned} & \|Jf(x_{n+1}^{(1)}) - Jf(x_n^{(1)})\| \\ & = \alpha_n^{(1)} \|Jf(x_n^{(1)}) - J \prod_K J^{-1} \\ & \times (Jf(x_n^{(1)}) - \rho_1 T_1 \\ & \times (x_{n+1}^{(2)}, x_{n+1}^{(3)}, \dots, x_{n+1}^{(N)}, x_n^{(1)}))\| \rightarrow 0, \end{aligned} \quad (26)$$

as  $n \rightarrow \infty$ .

Step 3. Since  $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \dots, \{x_n^{(N)}\}$  are bounded and there exists a compact subset  $C \subset E^*$  such that  $(J(K) - \rho_N T_N(K^N)) \subset C$ , there exists a subsequence  $\{x_{n_i}^{(N)}\}$  of  $\{x_n^{(N)}\}$  such that

$$Jf(x_{n_i}^{(N)}) - \rho_N T_N(x_{n_i}^{(1)}, x_{n_i}^{(2)}, \dots, x_{n_i}^{(N)}) \rightarrow h_1 \in E^*. \quad (27)$$

Since  $E$  is uniformly smooth and strictly convex, it follows by Lemma 2 (b) and Remark 1 that  $\prod_K$  and  $J^{-1}$  are continuous. Thus

$$\begin{aligned} & \prod_K J^{-1} (Jf(x_{n_i}^{(N)}) - \rho_N T_N \\ & \times (x_{n_i}^{(1)}, x_{n_i}^{(2)}, \dots, x_{n_i}^{(N)})) \rightarrow \prod_K J^{-1} (h_1) := f(x_N^*), \end{aligned} \quad (28)$$

$$\begin{aligned} & J \prod_K J^{-1} (Jf(x_{n_i}^{(N)}) \\ & - \rho_N T_N(x_{n_i}^{(1)}, x_{n_i}^{(2)}, \dots, x_{n_i}^{(N)})) \rightarrow Jf(x_N^*). \end{aligned} \quad (29)$$

From (24) and (29), we get

$$Jf(x_{n_i}^{(N)}) \rightarrow Jf(x_N^*) \quad (\text{as } n_i \rightarrow \infty). \quad (30)$$

By (25) and (30), we have

$$Jf(x_{n_i}^{(N)}) \rightarrow Jf(x_N^*) \quad (\text{as } n_i \rightarrow \infty). \quad (31)$$

Since  $E$  is strictly convex and reflexive, it follows by Remark 1 (iv) that  $J^{-1}$  is norm-weak-continuous. Therefore, from (30) and (31), we note that

$$f(x_{n_i}^{(N)}) \rightarrow f(x_N^*), \quad f(x_{n_i}^{(N)+1}) \rightarrow f(x_N^*) \quad (32)$$

and

$$\begin{aligned} & \|f(x_{n_i}^{(N)})\| \rightarrow \|f(x_N^*)\|, \\ & \|f(x_{n_i}^{(N)+1})\| \rightarrow \|f(x_N^*)\| \end{aligned} \quad (33)$$

(as  $n_i \rightarrow \infty$ ).

By the Kadec-Klee property, we have

$$f(x_{n_{(N)}}^{(N)}) \rightarrow f(x_N^*), \quad f(x_{n_{(N)}+1}^{(N)}) \rightarrow f(x_N^*) \quad (34)$$

(as  $n_{(N)} \rightarrow \infty$ ).

Since  $f^{-1}$  is continuous, it implies that  $\{x_{n_{(N)}}^{(N)}\}$  is a subsequence of  $\{x_{n_j}^{(N)}\}$  such that  $x_{n_{(N)}}^{(N)} \rightarrow x_N^* \in E$ . Therefore  $x_n^{(N)} \rightarrow x_N^*$  as  $n \rightarrow \infty$ . So, it follows from (16), (30), (34), and condition (ii) that

$$\begin{aligned} Jf(x_N^*) &= \lim_{n \rightarrow \infty} Jf(x_{n+1}^{(N)}) \\ &= \lim_{n \rightarrow \infty} \left\{ (1 - \alpha_n^{(N)}) Jf(x_n^{(N)}) + \alpha_n^{(N)} f^{-1} \prod_K J^{-1} \right. \\ &\quad \times \left. (Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)})) \right\} \quad (35) \\ &= (1 - d_N) Jf(x_N^*) \\ &\quad + d_N J \prod_K J^{-1} (Jf(x_N^*) - \rho_N T_N(x_1^*, x_2^*, \dots, x_N^*)). \end{aligned}$$

Since  $f$  is a bijective mapping, we obtain that

$$x_N^* = f^{-1} \prod_K J^{-1} (Jf(x_N^*) - \rho_N T_N(x_1^*, x_2^*, \dots, x_N^*)). \quad (36)$$

Similarly, we can prove that for every subsequence  $\{x_{n_j}^{(k)}\}$  of  $\{x_n^{(k)}\}$  there exist a subsequence  $\{x_{n_{(k)}}^{(k)}\}$  of  $\{x_{n_j}^{(k)}\}$  and  $x_k^* \in E$  such that

$$f(x_{n_{(k)}}^{(k)}) \rightarrow f(x_k^*) \quad (\text{as } n_{(k)} \rightarrow \infty), \quad (37)$$

$\forall k = 1, 2, \dots, N-1$ .

Since  $f^{-1}$  is a continuous mapping, we note that

$$x_{n_{(k)}}^{(k)} \rightarrow x_k^* \quad (\text{as } n_{(k)} \rightarrow \infty). \quad (38)$$

Hence  $x_n^{(k)} \rightarrow x_k^* \in E$ , for all  $k = 1, 2, \dots, N-1$ . Therefore, we have

$$\begin{aligned} x_{N-1}^* &= f^{-1} \prod_K J^{-1} (Jf(x_{N-1}^*) \\ &\quad - \rho_{N-1} T_{N-1}(x_N^*, x_1^*, \dots, x_{N-2}^*, x_{N-1}^*)) \\ &\quad \vdots \\ x_1^* &= f^{-1} \prod_K J^{-1} (Jf(x_1^*) \\ &\quad - \rho_1 T_1(x_2^*, x_3^*, \dots, x_N^*, x_1^*)). \quad (39) \end{aligned}$$

By Lemma 7, we can conclude that  $(x_1^*, x_2^*, \dots, x_N^*)$  is a solution of (8) and  $x_n^{(1)} \rightarrow x_1^*, x_n^{(2)} \rightarrow x_2^*, \dots, x_n^{(N)} \rightarrow x_N^*$ .  $\square$

Setting  $N = 3$  and  $f = I$  in Theorem 9, we immediately obtain the following result.

**Corollary 10** (see [6]). *Let  $E$  be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property and  $K$  a nonempty closed and convex subset of  $E$  with  $\theta \in K$ . Let  $T_1, T_2, T_3 : K^3 \rightarrow E^*$  be continuous mappings and  $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}$ , and  $\{\alpha_n^{(3)}\}$  the sequences in  $(a, b)$  with  $0 < a < b < 1$  satisfying the following conditions.*

(i) *There exist a compact subset  $C \subset E^*$  and constants  $\rho_1 > 0, \rho_2 > 0$ , and  $\rho_3 > 0$  such that*

$$\begin{aligned} &(J(K) - \rho_3 T_3(K^3)) \cup (J(K) - \rho_2 T_2(K^3)) \\ &\quad \cup (J(K) - \rho_1 T_1(K^3)) \subset C, \quad (40) \end{aligned}$$

*where  $J(x_1, x_2, x_3) = Jx_3$ , for all  $(x_1, x_2, x_3) \in K^3$ , and*

$$\begin{aligned} &\langle T_1(x_1, x_2, x_3), J^{-1}(Jx_3 - \rho_1 T_1(x_1, x_2, x_3)) \rangle \geq 0, \\ &\langle T_2(x_1, x_2, x_3), J^{-1}(Jx_3 - \rho_2 T_2(x_1, x_2, x_3)) \rangle \geq 0, \quad (41) \\ &\langle T_3(x_1, x_2, x_3), J^{-1}(Jx_3 - \rho_3 T_3(x_1, x_2, x_3)) \rangle \geq 0, \end{aligned}$$

*for all  $x_1, x_2, x_3 \in K$ .*

(ii)  *$\lim_{n \rightarrow \infty} \alpha_n^{(1)} = d_1 \in (a, b)$ ,  $\lim_{n \rightarrow \infty} \alpha_n^{(2)} = d_2 \in (a, b)$ , and  $\lim_{n \rightarrow \infty} \alpha_n^{(3)} = d_3 \in (a, b)$ . Let  $\{x_n^{(1)}\}, \{x_n^{(2)}\}$ , and  $\{x_n^{(3)}\}$  be the sequences defined by*

$$\begin{aligned} x_{n+1}^{(3)} &= J^{-1} \left( (1 - \alpha_n^{(3)}) Jf(x_n^{(3)}) + \alpha_n^{(3)} J \right. \\ &\quad \times \left. \left( \prod_K J^{-1} (Jf(x_n^{(3)}) - \rho_3 T_3(x_n^{(1)}, x_n^{(2)}, x_n^{(3)})) \right) \right), \\ x_{n+1}^{(2)} &= J^{-1} \left( (1 - \alpha_n^{(2)}) Jf(x_n^{(2)}) + \alpha_n^{(2)} J \right. \\ &\quad \times \left. \left( \prod_K J^{-1} (Jf(x_n^{(2)}) - \rho_2 T_2(x_{n+1}^{(3)}, x_n^{(1)}, x_n^{(2)})) \right) \right), \\ x_{n+1}^{(1)} &= J^{-1} \left( (1 - \alpha_n^{(1)}) Jf(x_n^{(1)}) + \alpha_n^{(1)} J \right. \\ &\quad \times \left. \left( \prod_K J^{-1} (Jf(x_n^{(1)}) - \rho_1 T_1(x_{n+1}^{(2)}, x_{n+1}^{(3)}, x_n^{(1)})) \right) \right), \quad n \geq 0. \quad (42) \end{aligned}$$

*Then the problem (9) has a solution  $(x_1^*, x_2^*, x_3^*) \in K^3$  and the sequences  $\{x_n^{(1)}\}, \{x_n^{(2)}\}$  and  $\{x_n^{(3)}\}$  converge strongly to  $x_1^*, x_2^*$ , and  $x_3^*$ , respectively.*

Setting  $E$  as a real Hilbert space in Theorem 9, we have the following result.

**Corollary 11.** Let  $H$  be a real Hilbert space and  $K$  a nonempty closed and convex subset of  $H$ . Let  $f : K \rightarrow K$  be an isometry mapping and  $T_1, \dots, T_N : K^N \rightarrow H$  continuous mappings and  $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \dots, \{\alpha_n^{(N)}\}$  are sequences in  $(a, b)$  with  $0 < a < b < 1$  satisfying the following conditions.

- (i) There exist a compact subset  $C \subset H$  and constants  $\rho_1 > 0, \rho_2 > 0, \dots, \rho_N > 0$  such that

$$\begin{aligned} & (I(K) - \rho_N T_N(K^N)) \\ & \cup (I(K) - \rho_{N-1} T_{N-1}(K^N)) \\ & \cup \dots \cup (I(K) - \rho_1 T_1(K^N)) \subset C, \end{aligned} \quad (43)$$

where  $(x_1, x_2, \dots, x_N) = x_N$ , for all  $(x_1, x_2, \dots, x_N) \in K^N$ , and

$$\begin{aligned} \langle T_1(x_1, x_2, \dots, x_N), x_N - \rho_1 T_1(x_1, x_2, \dots, x_N) \rangle &\geq 0, \\ \langle T_2(x_1, x_2, \dots, x_N), x_N - \rho_2 T_2(x_1, x_2, \dots, x_N) \rangle &\geq 0, \\ &\vdots \\ \langle T_N(x_1, x_2, \dots, x_N), x_N - \rho_N T_N(x_1, x_2, \dots, x_N) \rangle &\geq 0, \end{aligned} \quad (44)$$

for all  $x_1, x_2, \dots, x_N \in K$ .

- (ii)  $\lim_{n \rightarrow \infty} \alpha_n^{(1)} = d_1 \in (a, b)$ ,  $\lim_{n \rightarrow \infty} \alpha_n^{(2)} = d_2 \in (a, b)$ ,  $\dots$ ,  $\lim_{n \rightarrow \infty} \alpha_n^{(N)} = d_N \in (a, b)$ . Let  $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \dots, \{x_n^{(N)}\}$  be the sequences defined by

$$\begin{aligned} x_{n+1}^{(N)} &= f^{-1} \left( (1 - \alpha_n^{(N)}) f(x_n^{(N)}) + \alpha_n^{(N)} P_K \right. \\ &\quad \times \left( f(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}) \right) \Big), \\ x_{n+1}^{(N-1)} &= f^{-1} \left( (1 - \alpha_n^{(N-1)}) f(x_n^{(N-1)}) + \alpha_n^{(N-1)} P_K \right. \\ &\quad \times \left( f(x_n^{(N-1)}) - \rho_{N-1} T_{N-1} \right. \\ &\quad \times \left( x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N-2)}, x_n^{(N-1)} \right) \Big), \\ &\vdots \\ x_{n+1}^{(2)} &= f^{-1} \left( (1 - \alpha_n^{(2)}) f(x_n^{(2)}) + \alpha_n^{(2)} P_K \right. \\ &\quad \times \left( f(x_n^{(2)}) - \rho_2 T_2 \right. \\ &\quad \times \left( x_{n+1}^{(3)}, x_{n+1}^{(4)}, \dots, x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)} \right) \Big), \end{aligned}$$

$$\begin{aligned} x_{n+1}^{(1)} &= f^{-1} \left( (1 - \alpha_n^{(1)}) f(x_n^{(1)}) \right. \\ &\quad + \alpha_n^{(1)} P_K \left( f(x_n^{(1)}) - \rho_1 T_1 \right. \\ &\quad \times \left( x_{n+1}^{(2)}, x_{n+1}^{(3)}, \dots, x_{n+1}^{(N)}, x_n^{(1)} \right) \Big), \end{aligned} \quad (45)$$

$n \geq 0$ ,

where  $P_K$  is a metric projection on  $H$  to  $K$ . Then the problem (8) has a solution  $(x_1^*, x_2^*, \dots, x_N^*) \in K^N$  and the sequences  $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \dots, \{x_n^{(N)}\}$  converge strongly to  $x_1^*, x_2^*, \dots, x_N^*$ , respectively.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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เลขทะเบียน.....25.....

หนังสือยินยอมการเผยแพร่ผลงานทางวิชาการบนเว็บไซต์  
ฐานข้อมูล NU Digital Repository (<http://obj.lib.nu.ac.th/media/>)  
สำนักหอสมุด มหาวิทยาลัยนเรศวร

ตามที่ข้าพเจ้า ศ.ดร.สมยศ พลับเที่ยง (ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์) ได้ส่งผลงานทางวิชาการการรายงานการวิจัย (เรื่อง) รายงานวิจัยฉบับสมบูรณ์โครงการทฤษฎีบทจุดตรึงร่วมพีที่เอฟแบบไม่อิสระสำหรับฟังก์ชันในปริภูมิบานาค

ปีที่พิมพ์ 2557

ข้าพเจ้าขอรับรองว่า ผลงานทางวิชาการเป็นลิขสิทธิ์ของข้าพเจ้า ศ.ดร.สมยศ พลับเที่ยง เป็นเจ้าของลิขสิทธิ์ และเพื่อให้ผลงานทางวิชาการของข้าพเจ้าเป็นประโยชน์ต่อการศึกษาและสาธารณชน จึงอนุญาตให้เผยแพร่ผลงาน ดังนี้

- ☒ อนุญาตให้เผยแพร่  
☐ ไม่อนุญาตให้เผยแพร่ เนื่องจาก.....  
.....  
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ลงชื่อ .....  
(ศ.ดร. สมยศ พลับเที่ยง)  
วันที่ 3 พฤศจิกายน 2558

หมายเหตุ ลิขสิทธิ์ใดๆ ที่ปรากฏอยู่ในผลงานนี้เป็นความรับผิดชอบของเจ้าของผลงาน ไม่ใช่ของสำนักหอสมุด