รูปแบบชัดแจ้งของเวกเตอร์เฉพาะและตัวผกผันของเมทริกซ์เลสลีย์สองชั้น

Explicit Eigenvectors Formula and Inverse of a Doubly Leslie Matrix

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ABSTRACT

The special form of Schur complement is extended to have a Schur's formula to obtains the explicit formula of determinant, inverse, and eigenvector formula of the doubly Leslie matrix which is the generalized forms of the Leslie matrix. It is also a generalized form of the doubly companion matrix, and the companion matrix, respectively. The doubly Leslie matrix is a nonderogatory matrix.

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บทคัดย่อ

ส่วนเติมเต็มของชูร์(Schur complement) สามารถนำไปประยุกต์ใช้ในการหารูปแบบชัดแจ้งของตัวกำหนด ตัวผกผัน และเวกเตอร์ลักษณะเฉพาะของเมทริกซ์เลสลีย์สองชั้น (doubly Leslie matrix) ซึ่งเมทริกซ์เลสลีย์สองชั้นเป็นนัยทั่วไปของเมทริกซ์เลสลีย์ เมทริกซ์คอมแพเนียนสองชั้น และเมทริกซ์คอมแพเนียน เมทริกซ์เลสลีย์สองชั้นเป็นเมทริกซ์ที่มีพหุนามลักษณะเฉพาะเท่ากับพหุนามต่ำสุด (nonderogatory matrix)

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CHAPTER I

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

Let $\mathbb{R}$ and $\mathbb{C}$ be the field of real numbers and complex numbers respectively. The set of all polynomials in x over $\mathbb{C}$ is denoted by $\mathbb{C}[x]$. For a positive integer n, let M_n be the set of all $n \times n$ matrices over $\mathbb{C}$. The set of all complex vectors, or $n \times 1$ matrices over $\mathbb{C}$ is denoted by $\mathbb{C}^n$. A nonzero vector $\mathrm{v} \in \mathbb{C}^n$ is called an eigenvector of $A \in M_n$ corresponding to a scalar $\lambda \in \mathbb{C}$ if $A\mathrm{v} = \lambda\mathrm{v}$, and the scalar λ is an eigenvalue of the matrix A. The set of eigenvalues of A is call the spectrum of A and is denoted by $\sigma(A)$. Eigenvectors and eigenvalues are used widely in science and engineering.

Let

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

be a polynomial with coefficients over an arbitrary field. As is well known, the matrix

$$C = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

has the property that

$$\det(xI - C) = p(x).$$

The matrix A, or some of its modifications, is being called companion matrix of the polynomial $p(x)$ since its characteristic polynomial is $p(x)$.

Companion matrix appear in literature in several forms. To illustrate, consider the companion matrix

$$C_1 := \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & -a_n \\ 1 & 0 & \cdots & 0 & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & 0 & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_2 \\ 0 & 0 & \cdots & 0 & 1 & -a_1 \end{bmatrix} \in M_n$$

using the permutation matrix P of order n, the "backward identity" permutation. Since $P = P^T = P^{-1}$,

$$P^{-1}C_1P = \begin{bmatrix} 0 & 0 & \cdot & 0 & 1 \\ 0 & 0 & \cdot & 1 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & -a_n \\ 1 & 0 & \cdots & 0 & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & 0 & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_2 \\ 0 & 0 & \cdots & 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{1.1.1}$$

The companion matrix C_1 is seen to be similar to the following matrix

$$C_2 := \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 & 0 \\ -a_2 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n-2} & 0 & 0 & \cdots & 1 & 0 \\ -a_{n-1} & 0 & 0 & \cdots & 0 & 1 \\ -a_n & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in M_n.$$

Moreover, any matrix is similar to it transpose (see, e.g. [6, pp.134-135]), thus the following companion matrices

$$C_3 := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix} \in M_n,$$

$$C_4 := \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \in M_n$$

similar to the matrix C_1, C_2 respectively. Therefore the companion matrices C_i, $i = 1, 2, 3, 4$ have the same characteristic polynomial:

$$\begin{aligned} p(x) &= \det(xI - C_i) \\ &= t^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n. \end{aligned}$$

Butcher and Wright [4, p.363] defined a doubly companion matrix for the pair of polynomials $\alpha(x) = x^n - \alpha_1x^{n-1} - \alpha_2x^{n-2} - \cdots - \alpha_n$ and $\beta(x) = x^n - \beta_1x^{n-1} - \beta_2x^{n-2} - \cdots - \beta_n$, as $C \in M_n$ given by

$$C = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & \ldots & -\alpha_{n-1} & -\alpha_n - \beta_n \\ 1 & 0 & 0 & \ldots & 0 & -\beta_{n-1} \\ 0 & 1 & 0 & \ldots & 0 & -\beta_{n-2} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -\beta_2 \\ 0 & 0 & 0 & \ldots & 1 & -\beta_1 \end{bmatrix}, \qquad (1.1.2)$$

that is, a $n \times n$ matrix C with $n > 1$ is called a doubly companion matrix if its entries c_{ij} satisfy $c_{ij} = 1$ for all entries in the sub-maindiagonal of C and else $c_{ij} = 0$ for $i \neq 1$ and $j \neq n$, which is a special case of unreduced upper Hessenberg matrix. Butcher and Wright used the doubly companion matrices as a tool for analyzing various extension of classical methods with inherent Runge-Kutta stability. The doubly companion matrices is important for application in some certain matrix equations, numerical and linear methods.

Let $\alpha(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$ and $\beta(x) = x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$ be two monic polynomials over complex numbers, we prefer to consider the corresponding lower doubly companion matrix of $\alpha(x)$ and $\beta(x)$ as,

$$L(\alpha, \beta) = \begin{bmatrix} -b_{n-1} & 1 & \dots & 0 & 0 \\ -b_{n-2} & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -b_1 & 0 & \dots & 0 & 1 \\ -b_0 - a_0 & -a_1 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix}. \tag{1.1.3}$$

Malešević, Todorić, Jovović, and Telebaković in [?, Lemma 3.3] studied the sum of its principal minors of order k containing the first column $(1 \leq k \leq n)$ of the lower doubly companion matrix for using in the second step of reduction process for linear system of first order operator equations.

We define the corresponding upper doubly companion matrix of $\alpha(x)$ and $\beta(x)$ as,

$$U(\alpha, \beta) = \begin{bmatrix} -b_{n-1} & -b_{n-2} & \dots & -b_1 & -a_0 - b_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -a_{n-2} \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}. \tag{1.1.4}$$

From (1.1.3), if $b_0 = b_1 = \cdots = b_{n-2} = b_{n-1} = 0$ then the lower doubly companion matrix is become a companion matrix of the form,

$$L(\alpha) = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \qquad (1.1.5)$$

and, if $a_0 = a_1 = \cdots = a_{n-2} = a_{n-1} = 0$ then the matrix in (1.1.3) is become a companion matrix of another form,

$$L(\beta) = \begin{bmatrix} -b_{n-1} & 1 & \dots & 0 & 0 \\ -b_{n-2} & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -b_1 & 0 & \dots & 0 & 1 \\ -b_0 & 0 & \dots & 0 & 0 \end{bmatrix}. \qquad (1.1.6)$$

It is well known that the last two of these companion matrices are nonderogatory. The matrix $U(\alpha, \beta)$ is also nonderogatory, that is the characteristic polynomial $c_{U(\alpha,\beta)}$ is equal to the minimal polynomial $m_{U(\alpha,\beta)}$, see [12] for more details.

One of the most popular models of population growth is a matrix-based model, first introduced by P. H. Leslie. In 1945, he published his most famous article in Biometrika, a journal. The article was entitled, On the use of matrices in certain population mathematics [1, pp. 117—120]. The Leslie model describes the growth of the female portion of a population which is assumed to have a maximum lifespan. The females are divided into age classes all of which span an equal number of years. Using data about the average birthrates and survival probabilities of each class, the model is then able to determine the growth of the population over time, [11, 7].

Chen and Li in [5] asserted that, Leslie matrix models are discrete models for the development of age-structured populations. It is known that eigenvalues of a Leslie matrix are important in describing the asymptotic behavior of the corresponding population model. It is also known that the ratio of the spectral radius and the second largest (subdominant) eigenvalue in modulus of a non-periodic Leslie matrix determines the rate of convergence of the corresponding population distributions to a stable age distribution.

A Leslie matrix arises in a discrete, age-dependent model for population growth. It is a matrix of the form

$$L = \begin{bmatrix} r_1 & r_2 & r_3 & \dots & r_{n-1} & r_n \\ s_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & s_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \dots & s_{n-1} & 0 \end{bmatrix}, \tag{1.1.7}$$

where $r_j \geq 0$, $0 < s_j \leq 1$, $j = 1, 2, \dots, n-1$.

We define a doubly Leslie matrix analogous as the doubly companion matrix by replacing the subdiagonal of the doubly companion matrix by $s_1, s_2, \dots, s_{n-1}$ where s_j, $j = 1, 2, \dots, n-1$, respectively, and denoted by $\mathbf{L}$, that is, a doubly Leslie matrix is defined to be a matrix as follows

$$\mathbf{L} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \dots & -a_{n-1} & -a_n - b_n \\ s_1 & 0 & 0 & \dots & 0 & -b_{n-1} \\ 0 & s_2 & 0 & \dots & 0 & -b_{n-2} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -b_2 \\ 0 & 0 & 0 & \dots & s_{n-1} & -b_1 \end{bmatrix}, \tag{1.1.8}$$

where $a_j, b_j \in \mathbb{R}$, the real numbers, $j = 1, 2, \dots, n$. As the Leslie matrix, we restriction only $s_j > 0$, $j = 1, 2, \dots, n-1$.

For convenience, we can be written the matrix $\mathbf{L}$ in a partitioned form as

$$\mathbf{L} = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}_{(n,n)} \quad \textit{where} \quad \mathbf{p} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \end{bmatrix},$$

and $\Lambda = \text{diag}(s_1, s_2, \ldots, s_{n-1})$ is a diagonal matrix of order $n-1$.

Note: If we define the doubly Leslie matrix in an another form such as $\mathbf{L} = \begin{bmatrix} \mathbf{p}^T & a_n + b_n \\ \Lambda & \mathbf{q} \end{bmatrix}_{(n,n)}$, where all symbols are as above, then some consequence productions will be complicates forms.

1.2 Preliminaries

Let M be a matrix partitioned into four blocks

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1.2.1}$$

where the submatrix C is assumed to be square and nonsingular. Brezinski in [3, p.232] asserted that, the Schur complement of C in M, denoted by (M/C), is defined by

$$(M/C) = B - AC^{-1}D, \tag{1.2.2}$$

which is related to Gaussian elimination by

$$M = \begin{bmatrix} I & AC^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & (M/C) \\ C & D \end{bmatrix}. \tag{1.2.3}$$

Suppose that B and C are $k \times k$ and $(n-k) \times (n-k)$ matrices, respectively, $k < n$, and C is nonsingular, as in [8, p.39] we have the following theorem.

Theorem 1.2.1 (Schur's formula). Let M be a square matrix of order $n \times n$ partitioned as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where B and C are $k \times k$ and $(n-k) \times (n-k)$ matrices, respectively, $k < n$. If C is nonsingular, then

$$\det M = (-1)^{(n+1)k} \det C \det(M/C). \tag{1.2.4}$$

Proof. From the (1.2.3)

$$M = \begin{bmatrix} I & AC^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & (M/C) \\ C & D \end{bmatrix}.$$

The identity (1.2.4) follows by taking the determinant of both sides. Then,

$$\det M = \det \begin{bmatrix} I & AC^{-1} \\ 0 & I \end{bmatrix} \det \begin{bmatrix} 0 & (M/C) \\ C & D \end{bmatrix}.$$

Since $\det \begin{bmatrix} I & AC^{-1} \\ 0 & I \end{bmatrix} = 1$. Therefore

$$\det M = \det \begin{bmatrix} 0 & (M/C) \\ C & D \end{bmatrix}.$$

By Laplace's theorem, expansion of $\det \begin{bmatrix} 0 & (M/C) \\ C & D \end{bmatrix}$ by the first k rows i.e., rows $\{1, 2, \ldots, k\}$. We have

$$\det \begin{bmatrix} 0 & (M/C) \\ C & D \end{bmatrix} = (-1)^{(n+1)k} \det C \det(M/C).$$

Therefore

$$\det M = (-1)^{(n+1)k} \det C \det(M/C).$$

This completes the proof. □

The following useful formula, presents the inverse of a matrix in terms of Schur complements, analogous as in [14, p. 19], we obtain.

Theorem 1.2.2. Let M be partitioned as in (1.2.1) and suppose both M and C are nonsingular. Then (M/C) is nonsingular and

$$M^{-1} = \begin{bmatrix} -C^{-1}D\,(M/C)^{-1} & C^{-1} + \left(C^{-1}D\,(M/C)^{-1}\,AC^{-1}\right) \\ (M/C)^{-1} & -(M/C)^{-1}\,AC^{-1} \end{bmatrix}. \tag{1.2.5}$$

Proof. The Schur complements (M/C) is nonsingular by virtue of (1.2.4). Under the given hypotheses, from (1.2.3) one checks that

$$\begin{aligned} M &= \begin{bmatrix} I & AC^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & (M/C) \\ C & D \end{bmatrix} \\ &= \begin{bmatrix} I & AC^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & (M/C) \\ C & 0 \end{bmatrix} \begin{bmatrix} I & C^{-1}D \\ 0 & I \end{bmatrix}. \end{aligned}$$

Inverting both sides yields

$$\begin{aligned} M^{-1} &= \begin{bmatrix} I & C^{-1}D \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & (M/C) \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} I & AC^{-1} \\ 0 & I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} I & -C^{-1}D \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & C^{-1} \\ (M/C)^{-1} & 0 \end{bmatrix} \begin{bmatrix} I & -AC^{-1} \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} -C^{-1}D\,(M/C)^{-1} & C^{-1} \\ (M/C)^{-1} & 0 \end{bmatrix} \begin{bmatrix} I & -AC^{-1} \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} -C^{-1}D\,(M/C)^{-1} & C^{-1} + \left(C^{-1}D\,(M/C)^{-1}\,AC^{-1}\right) \\ (M/C)^{-1} & -(M/C)^{-1}\,AC^{-1} \end{bmatrix}, \end{aligned}$$

from which the identity (1.2.5) follows. □

We recall some well-known results from linear algebra and matrix analysis.

Definition 1.2.3. [6, Definition 1.3.1]. A matrix $B \in M_n$ is said to be *similar* to a matrix $A \in M_n$ if there exists a nonsingular matrix $S \in M_n$ such that $B = S^{-1}AS$.

Theorem 1.2.4. [6, Theorem 1.4.8]. Let $A, B \in M_n$, if $\mathbf{x} \in \mathbb{C}^n$ is an eigenvector corresponding to $\lambda \in \sigma(B)$ and if B is similar to A via S, then $S\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue λ.

Theorem 1.2.5. [6, Theorem 3.3.15]. A matrix $A \in M_n$ is similar to the companion matrix of its characteristic polynomial if and only if the minimal and characteristic polynomial of A are identical.

Definition 1.2.6. [9, p. 644] A matrix $A \in M_n$ for which the characteristic polynomial $\Delta_A(x)$ equal to the minimum polynomial $m_A(x)$ are said to be *nonderogatory* matrix.

In this present report we give explicit determinant, inverse matrix, and eigenvector formulae for the doubly Leslie matrix and give some related topics.

CHAPTER II

MAIN RESULTS

According to Schur's formula in Theorem 1.2.1 Let M be a square matrix of order $n \times n$ partitioned as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where B and C are $k \times k$ and $(n-k) \times (n-k)$ matrices, respectively, $k < n$. If C is nonsingular, then

$$\det M = (-1)^{(n+1)k} \det C \det(M/C).$$

It well known that any square matrix M is invertible if and only if $\det M \neq 0$, we have the following results.

2.1 Inverse Formula of Doubly Leslie Matrix

The following theorem is follows from Theorem 1.2.1.

Theorem 2.1.1 (Determinant of doubly Leslie matrix). Let $\mathbf{L}$ be a doubly Leslie matrix as in (1.1.8) with partitioned as $\mathbf{L} = \begin{bmatrix} -\mathrm{p}^T & -a_n - b_n \\ \Lambda & -\mathrm{q} \end{bmatrix}_{(n,n)}$, where $\mathrm{p} = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}^T$, $\mathrm{q} = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_1 \end{bmatrix}^T$, and $\Lambda = \mathrm{diag}(s_1, s_2, \dots, s_{n-1})$, $s_j > 0$, $j = 1, 2, \dots, n-1$ is a diagonal matrix of order $n-1$, then

$$\det \mathbf{L} = (-1)^n \left((a_n + b_n) + \sum_{i=1}^{n-1} \frac{a_i b_{n-i}}{s_i} \right) \prod_{i=1}^{n-1} s_i.$$

Proof. Since Λ is a $(n-1) \times (n-1)$ submatrix of the matrix $\mathbf{L}$. Then we apply the Schur' formula (1.2.4),

$$\det \mathbf{L} = (-1)^{(n+1)\times 1} \det \Lambda \det(\mathbf{L}/\Lambda) \tag{2.1.1}$$

As in (1.2.2), the Schur complement of Λ in $\mathbf{L}$, denoted by $(\mathbf{L}/\Lambda)$, is a 1×1 matrix or a scalar

$$\begin{aligned}
(\mathbf{L}/\Lambda) &= (-a_n - b_n) - (-\mathbf{p}^T)\Lambda^{-1}(-\mathbf{q}) \\
&= (-a_n - b_n) - \begin{bmatrix} -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \\
&\quad \times \begin{bmatrix} 1/s_1 & 0 & \dots & 0 \\ 0 & 1/s_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1/s_{n-1} \end{bmatrix} \begin{bmatrix} -b_{n-1} \\ -b_{n-2} \\ \vdots \\ -b_2 \\ -b_1 \end{bmatrix} \\
&= -(a_n + b_n) - \left(\frac{a_1 b_{n-1}}{s_1} + \frac{a_2 b_{n-2}}{s_2} + \dots + \frac{a_{n-1} b_1}{s_{n-1}} \right) \\
&= -(a_n + b_n) - \sum_{i=1}^{n-1} \frac{a_i b_{n-i}}{s_i} \\
&= -\left((a_n + b_n) + \sum_{i=1}^{n-1} \frac{a_i b_{n-i}}{s_i} \right). \qquad (2.1.2)
\end{aligned}$$

Now, from (2.1.1) it is easy to see that $\det \Lambda = \prod_{i=1}^{n-1} s_i$. Therefore

$$\begin{aligned}
\det \mathbf{L} &= (-1)^{(n+1)} \prod_{i=1}^{n-1} s_i \left\{ -\left((a_n + b_n) + \sum_{i=1}^{n-1} \frac{a_i b_{n-i}}{s_i} \right) \right\} \\
&= (-1)^n \left((a_n + b_n) + \sum_{i=1}^{n-1} \frac{a_i b_{n-i}}{s_i} \right) \prod_{i=1}^{n-1} s_i.
\end{aligned}$$

This completes the proof. □

Immediately, we have the following corollaries.

Corollary 2.1.2. Let L be a Leslie matrix defined as in (1.1.7) with partitioned as $L = \begin{bmatrix} -\mathbf{p}^T & -a_n \\ \Lambda & \mathbf{0} \end{bmatrix}_{(n,n)}$, where $\mathbf{p} = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}^T$, $-a_j \geq 0$, $j = 1, 2, \dots, n.$ and $\Lambda = \operatorname{diag}(s_1, s_2, \dots, s_{n-1})$, $s_j > 0$, $j = 1, 2, \dots, n-1$ is a diagonal matrix of order $n-1$, then

$$\det L = (-1)^n a_n \prod_{i=1}^{n-1} s_i.$$

Corollary 2.1.3. Let $\mathbf{C} = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ I_{n-1} & -\mathbf{q} \end{bmatrix}_{(n,n)}$ be a doubly companion matrix, where $\mathbf{p} = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}^T$, and $\mathbf{q} = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_1 \end{bmatrix}^T$, then

$$\det \mathbf{C} = (-1)^n \left((a_n + b_n) + \sum_{i=1}^{n-1} a_i b_{n-i} \right).$$

Corollary 2.1.4. Let $\mathbf{C} = \begin{bmatrix} -\mathbf{p}^T & -a_n \\ I_{n-1} & 0 \end{bmatrix}_{(n,n)}$ be a companion matrix, where $\mathbf{p} = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}^T$, then $\det \mathbf{C} = (-1)^n a_n$.

Now we wish to find the inverse of doubly Leslie matrix.

Theorem 2.1.5. Let $\mathbf{L} = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}_{(n,n)}$ be a doubly Leslie matrix, where $\mathbf{p} = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}^T$, $\mathbf{q} = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_1 \end{bmatrix}^T$, and $\Lambda = \operatorname{diag}(s_1, s_2, \dots, s_{n-1})$, where $s_j > 0$, $j = 1, 2, \dots, n-1$ is a diagonal matrix of order $n-1$. If $\det \mathbf{L} \neq 0$ then

$$\mathbf{L}^{-1} = (\mathbf{L}/\Lambda)^{-1} \begin{bmatrix} \Lambda^{-1}\mathbf{q} & (\mathbf{L}/\Lambda)\,\Lambda^{-1} + \left(\Lambda^{-1}\mathbf{q}\mathbf{p}^T\Lambda^{-1}\right) \\ 1 & \mathbf{p}^T\Lambda^{-1} \end{bmatrix}_{(n,n)},$$

where $(\mathbf{L}/\Lambda) = -\left((a_n + b_n) + \sum_{i=1}^{n-1} \frac{a_i b_{n-i}}{s_i}\right)$, as in (2.1.2), and $\Lambda^{-1} = \operatorname{diag}(\frac{1}{s_1}, \frac{1}{s_2}, \dots, \frac{1}{s_{n-1}})$.

Proof. Apply the identity (1.2.5) to the matrix L, we have

$$\begin{aligned} \mathbf{L}^{-1} &= \begin{bmatrix} -\Lambda^{-1}(-\mathbf{q})\,(\mathbf{L}/\Lambda)^{-1} & \Lambda^{-1} + \left(\Lambda^{-1}(-\mathbf{q})\,(\mathbf{L}/\Lambda)^{-1}\,(-\mathbf{p}^T)\Lambda^{-1}\right) \\ (\mathbf{L}/\Lambda)^{-1} & -\,(\mathbf{L}/\Lambda)^{-1}\,(-\mathbf{p}^T)\Lambda^{-1} \end{bmatrix}_{(n,n)} \\ &= \begin{bmatrix} \Lambda^{-1}\mathbf{q}\,(\mathbf{L}/\Lambda)^{-1} & \Lambda^{-1} + \left(\Lambda^{-1}\mathbf{q}\,(\mathbf{L}/\Lambda)^{-1}\,\mathbf{p}^T\Lambda^{-1}\right) \\ (\mathbf{L}/\Lambda)^{-1} & (\mathbf{L}/\Lambda)^{-1}\,\mathbf{p}^T\Lambda^{-1} \end{bmatrix}_{(n,n)}. \end{aligned}$$

The Schur complement of Λ in $\mathbf{L}$ is $(\mathbf{L}/\Lambda)$, in (2.1.2) showed that $(\mathbf{L}/\Lambda)$ is a scalar.

Then

$$\mathbf{L}^{-1} = (\mathbf{L}/\Lambda)^{-1}\begin{bmatrix} \Lambda^{-1}\mathbf{q} & (\mathbf{L}/\Lambda)\,\Lambda^{-1} + \left(\Lambda^{-1}\mathbf{q}\mathbf{p}^{T}\Lambda^{-1}\right) \\ 1 & \mathbf{p}^{T}\Lambda^{-1} \end{bmatrix}_{(n,n)}.$$

□

Immediately, we have the following corollaries.

Corollary 2.1.6. Let L be a Leslie matrix defined in Corollary 2.1.2. If $\det L \neq 0$ then

$$L^{-1} = (L/\Lambda)^{-1}\begin{bmatrix} 0 & (L/\Lambda)\Lambda^{-1} \\ 1 & \mathbf{p}^{T}\Lambda^{-1} \end{bmatrix}_{(n,n)},$$

where $(L/\Lambda) = -a_n$.

Corollary 2.1.7. Let C be a doubly companion matrix defined in Corollary 2.1.3. If $\det \mathrm{C} \neq 0$ then

$$\mathrm{C}^{-1} = (\mathrm{C}/I_{n-1})^{-1}\begin{bmatrix} \mathrm{q} & (\mathrm{C}/I_{n-1}) + \mathrm{q}\mathrm{p}^{T} \\ 1 & \mathrm{p}^{T} \end{bmatrix}_{(n,n)},$$

where $(\mathrm{C}/I_{n-1}) = -\left((a_n + b_n) + \sum\limits_{i=1}^{n-1} a_i b_{n-i}\right)$.

Corollary 2.1.8. Let C be a companion matrix defined in Corollary 2.1.4. If $\det \mathrm{C} \neq 0$ then

$$\mathrm{C}^{-1} = -\frac{1}{a_n}\begin{bmatrix} 0 & -a_n I_{n-1} \\ 1 & \mathrm{p}^{T} \end{bmatrix}_{(n,n)}.$$

2.2 Explicit Minimum Polynomial of Doubly Leslie Matrix

The author in [12, Theorem 3.3] asserted that, the doubly companion matrix is nonderogatory. Now, we wish to show that any doubly Leslie matrix $\mathbf{L}$ in (1.1.8) is similar to a companion matrix, that is, it is a nonderogatory.

Theorem 2.2.1. The doubly Leslie matrix matrix $\mathbf{L}$ defined in (1.1.8) is nonderogatory and the characteristic polynomial and the explicit minimum polynomial is

$$\begin{aligned} m_{\mathbf{L}}(x) &= x^n + (c_1 + d_1)x^{n-1} + (\sum_{i+j=2} c_i d_j + c_2 + d_2)x^{n-2} + \dots \\ &\quad +(\sum_{i+j=n-1} c_i d_j + c_{n-1} + d_{n-1})x + (\sum_{i+j=n} c_i d_j + c_n + d_n), \end{aligned}$$

where $c_1 = a_1$, $c_i = a_i \prod_{k=1}^{i-1} s_k$, and $d_1 = b_1$, $d_i = b_i \prod_{k=n+1-i}^{n-1} s_k$, for $i = 2, 3, \dots, n$.

Proof. Let

$$\mathrm{L} = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n - b_n \\ s_1 & 0 & \dots & 0 & -b_{n-1} \\ 0 & s_2 & \dots & 0 & -b_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & s_{n-1} & -b_1 \end{bmatrix}.$$

Firstly to show that L is similar to a doubly companion matrix.

By a similarity transformation with a diagonal matrix

$$D = \mathrm{diag}\left(\frac{1}{s_1 s_2 \dots s_{n-1}}, \frac{1}{s_2 s_3 \dots s_{n-1}}, \dots, \frac{1}{s_{n-2} s_{n-1}}, \frac{1}{s_{n-1}}, 1\right),$$

L can be transformed to a doubly companion matrix,

$$D^{-1}\mathrm{L}D = \begin{bmatrix} -a_1 & -a_2 s_1 & \dots & -a_{n-1}(s_1 s_2 \dots s_{n-2}) & -(a_n + b_n)(s_1 s_2 s_3 \dots s_{n-1}) \\ 1 & 0 & \dots & 0 & -b_{n-1}(s_2 s_3 \dots s_{n-1}) \\ 0 & 1 & \dots & 0 & -b_{n-2}(s_3 \dots s_{n-1}) \\ \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & -b_2 s_{n-1} \\ 0 & 0 & \dots & 1 & -b_1 \end{bmatrix}.$$

For convenient, let us denote the doubly companion matrix $D^{-1}\mathrm{L}D$ by

$$D^{-1}\mathrm{L}D = \begin{bmatrix} -c_1 & -c_2 & -c_3 & \dots & -c_{n-1} & -c_n - d_n \\ 1 & 0 & 0 & \dots & 0 & -d_{n-1} \\ 0 & 1 & 0 & \dots & 0 & -d_{n-2} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -d_2 \\ 0 & 0 & 0 & \dots & 1 & -d_1 \end{bmatrix}, \qquad (2.2.1)$$

where $c_1 = a_1$, $c_i = a_i \prod_{k=1}^{i-1} s_k$, and $d_1 = b_1$, $d_i = b_i \prod_{k=n+1-i}^{n-1} s_k$, for $i = 2, 3, \dots, n$.

Let J be the backward identity matrix of order $n \times n$(or reversal matrix of order $n \times n$), J $(= J^{-1})$, which showing that

$$J^{-1}(D^{-1}\mathrm{L}D)J = \begin{bmatrix} -d_1 & 1 & \dots & 0 & 0 \\ -d_2 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -d_{n-1} & 0 & \dots & 0 & 1 \\ -c_n - d_n & -c_{n-1} & \dots & -c_2 & -c_1 \end{bmatrix} =: \Gamma. \qquad (2.2.2)$$

To show that the matrix Γ is similar to a companion matrix. We shall prove by explicit construction the existence of an invertible matrix M such that $M^{-1}\Gamma M$ is a companion matrix. Now, chosen a matrix M of size $n \times n$,

$$M = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ d_1 & 1 & 0 & \ddots & \vdots \\ d_2 & d_1 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ d_{n-1} & \dots & d_2 & d_1 & 1 \end{bmatrix}.$$

Then M is nonsingular matrix. In fact the matrix M is an lower triangular Toeplitz matrix with diagonal-constant 1, and

$$M^{-1} = \begin{bmatrix} \mathbf{e}_1 & \Gamma\mathbf{e}_1 & \Gamma^2\mathbf{e}_1 & \dots & \Gamma^{n-1}\mathbf{e}_1 \end{bmatrix}^T,$$

where $\mathbf{e}_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T \in \mathbb{R}^n$ is the unit column vector.

Computation shows that

$$M^{-1}\Gamma M = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -\gamma_n & -\gamma_{n-1} & \dots & -\gamma_2 & -\gamma_1 \end{bmatrix} =: C,$$

where

$$\begin{aligned} \gamma_1 &= c_1 + d_1, \\ \gamma_2 &= \sum_{i+j=2} c_i d_j + c_2 + d_2, \\ &\vdots \\ \gamma_{n-1} &= \sum_{i+j=n-1} c_i d_j + c_{n-1} + d_{n-1}, \\ \gamma_n &= \sum_{i+j=n} c_i d_j + c_n + d_n. \end{aligned} \tag{2.2.3}$$

The matrix

$$C = M^{-1}\Gamma M = M^{-1}J^{-1}(D^{-1}\mathbf{L}DJ)M = (DJM)^{-1}\mathbf{L}(DJM) \tag{2.2.4}$$

is the desired companion matrix. Then, we have the doubly Leslie matrix $\mathbf{L}$ is similar to the companion matrix C. By Theorem 1.2.5, the characteristic polynomial $\Delta_{\mathbf{L}}(x)$ equal to the minimum polynomial $m_{\mathbf{L}}(x)$, we have

$$m_{\mathbf{L}}(x) = x^n + \gamma_1 x^{n-1} + \gamma_2 x^{n-2} + \cdots + \gamma_{n-1}x + \gamma_n.$$

That is

$$\begin{aligned} m_{\mathbf{L}}(x) &= x^n + (c_1 + d_1)x^{n-1} + (\sum_{i+j=2} c_i d_j + c_2 + d_2)x^{n-2} + \dots \\ &\quad + (\sum_{i+j=n-1} c_i d_j + c_{n-1} + d_{n-1})x + (\sum_{i+j=n} c_i d_j + c_n + d_n), \end{aligned}$$

where $c_1 = a_1$, $c_i = a_i \prod_{k=1}^{i-1} s_k$, and $d_1 = b_1$, $d_i = b_i \prod_{k=n+1-i}^{n-1} s_k$, for $i = 2, 3, \dots, n$. $\square$

2.3 Explicit Eigenvector Formula of Doubly Leslie Matrix

Now analogous as eigenvector of a companion matrix in [2, pp.630−631] and in [10, p.6], we obtain.

Theorem 2.3.1. Let λ be an eigenvalue of a doubly Leslie matrix $\mathbf{L}$ defined in (1.1.8). Then

$$\mathbf{v} = \begin{bmatrix} \frac{1}{s_1 s_2 \dots s_{n-1}} \left(\lambda^{n-1} + d_1\lambda^{n-2} + d_2\lambda^{n-3} + \cdots + d_{n-2}\lambda + d_{n-1}\right) \\ \frac{1}{s_2 s_3 \dots s_{n-1}} \left(\lambda^{n-2} + d_1\lambda^{n-3} + \cdots + d_{n-3}\lambda + d_{n-2}\right) \\ \vdots \\ \frac{1}{s_{n-1}} \left(\lambda + d_1\right) \\ 1 \end{bmatrix}$$

is an eigenvector of $\mathbf{L}$ corresponding to the eigenvalue λ, where $d_1 = b_1$, $d_i = b_i \prod\limits_{k=n+1-i}^{n-1} s_k$, for $i = 2, 3, \dots, n$.

Proof. From Theorem 2.2.1, $\mathbf{L}$ is similar to the companion matrix C as in (2.2.4). Then they have the same eigenvalues in common. Let λ be an eigenvalue of $\mathbf{L}$, then λ also an eigenvalue of C. Since λ is a root of the characteristic polynomial $\Delta_{\mathbf{L}}(x)$, we have

$$\begin{aligned} \Delta_{\mathbf{L}}(\lambda) &= \lambda^n + (c_1 + d_1)\lambda^{n-1} + (\sum_{i+j=2} c_i d_j + c_2 + d_2)\lambda^{n-2} + \dots \\ &\quad + (\sum_{i+j=n-1} c_i d_j + c_{n-1} + d_{n-1})\lambda + (\sum_{i+j=n} c_i d_j + c_n + d_n) = 0. \end{aligned}$$

From (2.2.3), we have,

$$\Delta_{\mathbf{L}}(\lambda) = \lambda^n + \gamma_1\lambda^{n-1} + \gamma_2\lambda^{n-2} + \cdots + \gamma_{n-11}\lambda + \gamma_n = 0.$$

Therefore

$$\lambda^n = -\left(\gamma_1\lambda^{n-1} + \gamma_2\lambda^{n-2} + \cdots + \gamma_{n-11}\lambda + \gamma_n\right).$$

Then, we put a vector $\mathbf{u} = \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix}$. We must show that this vector $\mathbf{u}$ is an eigenvector of C corresponding to the eigenvalue λ. Form equation (2.2.4), $C = (DJM)^{-1}\mathbf{L}(DJM)$, we have

$$\begin{aligned} C\mathbf{u} &= \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -\gamma_n & -\gamma_{n-1} & \dots & -\gamma_2 & -\gamma_1 \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix} \\ &= \begin{bmatrix} \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \\ -\gamma_n - \gamma_{n-1}\lambda - \dots - \gamma_2\lambda^{n-2} - \gamma_1\lambda^{n-1} \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \\ \lambda^n \end{bmatrix} \\ &= \lambda \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix} = \lambda\mathbf{u}, \end{aligned}$$

it is easy to see that the first component in the vector $\mathbf{u}$ cannot be zero, the vector $\mathbf{u}$ is not a zero-vector, it is an eigenvector of C corresponding to λ.

Since $(DJM)^{-1}\mathbf{L}(DJM) = C$. Theorem 1.2.4 asserted that $(DJM)\mathbf{u}$ is an eigenvector of $\mathbf{L}$ corresponding to the eigenvalue λ. Hence, the explicit form of an eigenvector corresponding to an eigenvalue λ of the matrix $\mathbf{L}$ is

$$\begin{aligned}
\mathbf{v} &:= (DJM)\mathbf{u} \\
&= \begin{bmatrix} \frac{1}{s_1 s_2 \ldots s_{n-1}} & 0 & \ldots & 0 & 0 & 0 \\ 0 & \frac{1}{s_2 s_3 \ldots s_{n-1}} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \frac{1}{s_{n-2} s_{n-1}} & 0 & 0 \\ 0 & 0 & \ldots & 0 & \frac{1}{s_{n-1}} & 0 \\ 0 & 0 & \ldots & 0 & 0 & 1 \end{bmatrix} \\
&\times \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 & 1 \\ 0 & 0 & \ldots & 0 & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 1 & \ldots & 0 & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ d_1 & 1 & \ddots & \vdots & \vdots \\ d_2 & d_1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & 1 & 0 \\ d_{n-1} & \ldots & d_2 & d_1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix},
\end{aligned}$$

that is

$$\begin{aligned}
\mathbf{v} &= \begin{bmatrix} \frac{d_{n-1}}{s_1 s_2 \ldots s_{n-1}} & \frac{d_{n-2}}{s_1 s_2 \ldots s_{n-1}} & \frac{d_{n-3}}{s_1 s_2 \ldots s_{n-1}} & \cdots & \frac{d_1}{s_1 s_2 \ldots s_{n-1}} & \frac{1}{s_1 s_2 \ldots s_{n-1}} \\ \frac{d_{n-2}}{s_2 s_3 \ldots s_{n-1}} & \frac{d_{n-3}}{s_2 s_3 \ldots s_{n-1}} & \iddots & \frac{d_1}{s_2 s_3 \ldots s_{n-1}} & \frac{1}{s_2 s_3 \ldots s_{n-1}} & 0 \\ \frac{d_{n-3}}{s_3 s_4 \ldots s_{n-1}} & \iddots & \iddots & \frac{1}{s_3 s_4 \ldots s_{n-1}} & 0 & \vdots \\ \vdots & \iddots & \iddots & 0 & \vdots & 0 \\ \frac{d_1}{s_{n-1}} & \frac{1}{s_{n-1}} & \iddots & \vdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \mathbf{u} \\
&= \begin{bmatrix} \frac{1}{s_1 s_2 \ldots s_{n-1}} \ \lambda^{n-1} + d_1 \lambda^{n-2} + d_2 \lambda^{n-3} + \cdots + d_{n-2}\lambda + d_{n-1} \\ \frac{1}{s_2 s_3 \ldots s_{n-1}} \ \lambda^{n-2} + d_1 \lambda^{n-3} + \cdots + d_{n-3}\lambda + d_{n-2} \\ \vdots \\ \frac{1}{s_{n-2} s_{n-1}} \ \lambda^2 + d_1 \lambda + d_2 \\ \frac{1}{s_{n-1}} (\lambda + d_1) \\ 1 \end{bmatrix},
\end{aligned}$$

it is easy to see that the last component in the vector $\mathbf{v}$ cannot be zero, which proves the assertion. □

The following corollaries are particular case of Theorem 2.3.1.

If $b_1 = b_2 = \cdots = b_n = 0$, then the matrix become a Leslie matrix, we have the following corollary.

Corollary 2.3.2. Let λ be an eigenvalue of a Leslie matrix L defined in Corollary 2.1.2. Then

$$\mathbf{v} = \begin{bmatrix} \frac{1}{s_1 s_2 \ldots s_{n-1}} \lambda^{n-1} \\ \frac{1}{s_2 s_3 \ldots s_{n-1}} \lambda^{n-2} \\ \vdots \\ \frac{1}{s_{n-1}} \lambda \\ 1 \end{bmatrix}$$

is an eigenvector of L corresponding to the eigenvalue λ. A nonzero scalar multiple of $\mathbf{v}$ namely

$$\mathbf{w} := (s_1 s_2 \ldots s_{n-1}/\lambda^{n-1})\mathbf{v} = \begin{bmatrix} 1 \\ s_1/\lambda \\ s_1 s_2/\lambda^2 \\ \vdots \\ s_1 s_2 \ldots s_{n-1}/\lambda^{n-1} \end{bmatrix}$$

is also an eigenvector of L corresponding to the eigenvalue λ.

If $s_1 = s_2 = \cdots = s_n = 1$, then we have the following corollary, as in [13, pp. 270—272].

Corollary 2.3.3. Let λ be an eigenvalue of a doubly companion matrix C defined in Corollary 2.1.3. Then

$$\mathbf{v} = \begin{bmatrix} \lambda^{n-1} + b_1\lambda^{n-2} + b_2\lambda^{n-3} + \cdots + b_{n-2}\lambda + b_{n-1} \\ \lambda^{n-2} + b_1\lambda^{n-3} + \cdots + b_{n-3}\lambda + b_{n-2} \\ \vdots \\ \lambda + b_1 \\ 1 \end{bmatrix}$$

is an eigenvector of C corresponding to the eigenvalue λ.

Corollary 2.3.4. Let λ be an eigenvalue of a companion matrix C defined in Corollary 2.1.4. Then

$$\mathrm{v} = \begin{bmatrix} \lambda^{n-1} \\ \lambda^{n-2} \\ \vdots \\ \lambda \\ 1 \end{bmatrix}$$

is an eigenvector of C corresponding to the eigenvalue λ.

CHAPTER III

Conclusion

The doubly Leslie matrix is a nonderogatory matrix. This report has explored a special form of a Schur complement to obtained the determinant, inverse, and explicit eigenvector formulas of the doubly Leslie matrix which is the generalized forms of the Leslie matrix. It is also a generalized form of the doubly companion matrix, and the companion matrix, respectively.

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APPENDIX

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EXPLICIT MINIMUM POLYNOMIAL, EIGENVECTOR, AND INVERSE FORMULA OF DOUBLY LESLIE MATRIX

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Abstract. The special form of Schur complement is extended to have a Schur's formula to obtains the explicit formula of determinant, inverse, and eigenvector formula of the doubly Leslie matrix which is the generalized forms of the Leslie matrix. It is also a generalized form of the doubly companion matrix, and the companion matrix, respectively. The doubly Leslie matrix is a nonderogatory matrix.

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1. Introduction

One of the most popular models of population growth is a matrix-based model, first introduced by P. H. Leslie. In 1945, he published his most famous article in *Biometrika*, a journal. The article was entitled, *On the use of matrices in certain population mathematics* [1, pp. 117–120]. The Leslie model describes the growth of the female portion of a population which is assumed to have a maximum lifespan. The females are divided into age classes all of which span an equal number of years. Using data about the average birthrates and survival probabilities of each class, the model is then able to determine the growth of the population over time, [11, 7].

Chen and Li in [5] asserted that, Leslie matrix models are discrete models for the development of age-structured populations. It is known that eigenvalues of a Leslie matrix are important in describing the asymptotic behavior of the corresponding population model. It is also known that the ratio of the spectral

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radius and the second largest (subdominant) eigenvalue in modulus of a non-periodic Leslie matrix determines the rate of convergence of the corresponding population distributions to a stable age distribution.

A *Leslie matrix* arises in a discrete, age-dependent model for population growth. It is a matrix of the form

$$L=\begin{bmatrix} r_1 & r_2 & r_3 & \dots & r_{n-1} & r_n \\ s_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & s_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \dots & s_{n-1} & 0 \end{bmatrix}, \tag{1}$$

where $r_j \geq 0$, $0 < s_j \leq 1$, $j = 1, 2, \dots, n-1$.

For a given field $\mathbb{F}$, the set of all polynomials in x over $\mathbb{F}$ is denoted by $\mathbb{F}[x]$. For a positive integer n, let $M_n(\mathbb{F})$ be the set of all $n \times n$ matrices over $\mathbb{F}$. The set of all vectors, or $n \times 1$ matrices over $\mathbb{F}$ is denoted by $\mathbb{F}^n$. A nonzero vector $\mathbf{v} \in \mathbb{F}^n$ is called an eigenvector of $A \in M_n(\mathbb{F})$ corresponding to a scalar $\lambda \in \mathbb{F}$ if $A\mathbf{v} = \lambda\mathbf{v}$, and the scalar λ is an eigenvalue of the matrix A. The set of eigenvalues of A is call the spectrum of A and is denoted by $\sigma(A)$. In the most common case in which $\mathbb{F} = \mathbb{C}$, the complex numbers, $M_n(\mathbb{C})$ is abbreviated to M_n.

Doubly companion matrices $\mathbf{C} \in M_n$ were first introduced by Butcher and Chartier in [4, pp. 274–276], given by

$$\mathbf{C}=\begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & \dots & -\alpha_{n-1} & -\alpha_n-\beta_n \\ 1 & 0 & 0 & \dots & 0 & -\beta_{n-1} \\ 0 & 1 & 0 & \dots & 0 & -\beta_{n-2} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -\beta_2 \\ 0 & 0 & 0 & \dots & 1 & -\beta_1 \end{bmatrix}, \tag{2}$$

that is, a $n \times n$ matrix $\mathbf{C}$ with $n > 1$ is called a *doubly companion matrix* if its entries c_{ij} satisfy $c_{ij} = 1$ for all entries in the sub-maindiagonal of $\mathbf{C}$ and else $c_{ij} = 0$ for $i \neq 1$ and $j \neq n$.

We define a *doubly Leslie matrix* analogous as the doubly companion matrix by replacing the subdiagonal of the doubly companion matrix by $s_1, s_2, \dots, s_{n-1}$ where s_j, $j = 1, 2, \dots, n-1$, respectively, and denoted by $\mathbf{L}$, that is, a doubly

Leslie matrix is defined to be a matrix as follows

$$\mathbf{L} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \dots & -a_{n-1} & -a_n - b_n \\ s_1 & 0 & 0 & \dots & 0 & -b_{n-1} \\ 0 & s_2 & 0 & \dots & 0 & -b_{n-2} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -b_2 \\ 0 & 0 & 0 & \dots & s_{n-1} & -b_1 \end{bmatrix}, \tag{3}$$

where $a_j, b_j \in \mathbb{R}$, the real numbers, $j = 1, 2, \dots, n$. As the Leslie matrix, we restriction only $s_j > 0$, $j = 1, 2, \dots, n-1$.

For convenience, we can be written the matrix $\mathbf{L}$ in a partitioned form as

$$\mathbf{L} = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}_{(n,n)} \quad \mathit{where} \quad \mathbf{p} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \end{bmatrix},$$

and $\Lambda = \mathrm{diag}(s_1, s_2, \dots, s_{n-1})$ is a diagonal matrix of order $n-1$.

Note: If we define the doubly Leslie matrix in an another form such as $\mathbf{L} = \begin{bmatrix} \mathbf{p}^T & a_n + b_n \\ \Lambda & \mathbf{q} \end{bmatrix}_{(n,n)}$, where all symbols are as above, then some consequence productions will be complicates forms.

We recall some well-known results from linear algebra and matrix analysis.

Definition 1.1. [6, Definition 1.3.1]. *A matrix $B \in M_n$ is said to be similar to a matrix $A \in M_n$ if there exists a nonsingular matrix $S \in M_n$ such that $B = S^{-1}AS$.*

Theorem 1.1. [6, Theorem 1.4.8]. *Let $A, B \in M_n$, if $\mathrm{x} \in \mathbb{C}^n$ is an eigenvector corresponding to $\lambda \in \sigma(B)$ and if B is similar to A via S, then $S\mathrm{x}$ is an eigenvector of A corresponding to the eigenvalue λ.*

Theorem 1.2. [6, Theorem 3.3.15]. *A matrix $A \in M_n$ is similar to the companion matrix of its characteristic polynomial if and only if the minimal and characteristic polynomial of A are identical.*

Definition 1.2. [9, p. 644] *A matrix $A \in M_n$ for which the characteristic polynomial $\Delta_A(x)$ equal to the minimum polynomial $m_A(x)$ are said to be* non-derogatory *matrix.*

In the present paper we give explicit determinant, inverse matrix, and eigenvector formulae for the doubly Leslie matrix and give some related topics.

2. Some Properties of Schur Complement

Let M be a matrix partitioned into four blocks

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{4}$$

where the submatrix C is assumed to be square and nonsingular. Brezinski in [3, p.232] asserted that, the Schur complement of C in M, denoted by (M/C), is defined by

$$(M/C) = B - AC^{-1}D, \tag{5}$$

which is related to Gaussian elimination by

$$M = \begin{bmatrix} I & AC^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & (M/C) \\ C & D \end{bmatrix}. \tag{6}$$

Suppose that B and C are $k \times k$ and $(n-k) \times (n-k)$ matrices, respectively, $k < n$, and C is nonsingular, as in [8, p.39] we have the following theorem.

Theorem 2.1 (Schur's formula). *Let M be a square matrix of order $n \times n$ partitioned as*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where B and C are $k \times k$ and $(n-k) \times (n-k)$ matrices, respectively, $k < n$. If C is nonsingular, then

$$\det M = (-1)^{(n+1)k} \det C \det(M/C). \tag{7}$$

Proof. From the (6)

$$M = \begin{bmatrix} I & AC^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & (M/C) \\ C & D \end{bmatrix}.$$

The identity (7) follows by taking the determinant of both sides. Then,

$$\det M = \det \begin{bmatrix} I & AC^{-1} \\ 0 & I \end{bmatrix} \det \begin{bmatrix} 0 & (M/C) \\ C & D \end{bmatrix}.$$

Since $\det \begin{bmatrix} I & AC^{-1} \\ 0 & I \end{bmatrix} = 1$. Therefore

$$\det M = \det \begin{bmatrix} 0 & (M/C) \\ C & D \end{bmatrix}.$$

By Laplace's theorem, expansion of $\det \begin{bmatrix} 0 & (M/C) \\ C & D \end{bmatrix}$ by the first k rows i.e., rows $\{1, 2, \ldots, k\}$. We have

$$\det \begin{bmatrix} 0 & (M/C) \\ C & D \end{bmatrix} = (-1)^{(n+1)k} \det C \det(M/C).$$

Therefore

$$\det M = (-1)^{(n+1)k} \det C \det(M/C).$$

This completes the proof. □

The following useful formula, presents the inverse of a matrix in terms of Schur complements, analogous as in [14, p. 19], we obtain.

Theorem 2.2. *Let M be partitioned as in (4) and suppose both M and C are nonsingular. Then (M/C) is nonsingular and*

$$M^{-1} = \begin{bmatrix} -C^{-1}D\,(M/C)^{-1} & C^{-1} + \left(C^{-1}D\,(M/C)^{-1}\,AC^{-1}\right) \\ (M/C)^{-1} & -\,(M/C)^{-1}\,AC^{-1} \end{bmatrix}. \tag{8}$$

Proof. The Schur complements (M/C) is nonsingular by virtue of (7). Under the given hypotheses, from (6) one checks that

$$\begin{aligned} M &= \begin{bmatrix} I & AC^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & (M/C) \\ C & D \end{bmatrix} \\ &= \begin{bmatrix} I & AC^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & (M/C) \\ C & 0 \end{bmatrix} \begin{bmatrix} I & C^{-1}D \\ 0 & I \end{bmatrix}. \end{aligned}$$

Inverting both sides yields

$$\begin{aligned} M^{-1} &= \begin{bmatrix} I & C^{-1}D \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & (M/C) \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} I & AC^{-1} \\ 0 & I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} I & -C^{-1}D \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & C^{-1} \\ (M/C)^{-1} & 0 \end{bmatrix} \begin{bmatrix} I & -AC^{-1} \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} -C^{-1}D\,(M/C)^{-1} & C^{-1} \\ (M/C)^{-1} & 0 \end{bmatrix} \begin{bmatrix} I & -AC^{-1} \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} -C^{-1}D\,(M/C)^{-1} & C^{-1} + \left(C^{-1}D\,(M/C)^{-1}\,AC^{-1}\right) \\ (M/C)^{-1} & -\,(M/C)^{-1}\,AC^{-1} \end{bmatrix}, \end{aligned}$$

from which the identity (8) follows. ☐

3. Inverse Formula of Doubly Leslie Matrix

The following theorem is follows from Theorem 2.1.

Theorem 3.1 (Determinant of doubly Leslie matrix). *Let $\mathbf{L}$ be a doubly Leslie matrix as in (3) with partitioned as $\mathbf{L} = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}_{(n,n)}$, where $\mathbf{p} = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}^T$, $\mathbf{q} = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_1 \end{bmatrix}^T$, and $\Lambda = \operatorname{diag}(s_1, s_2, \dots, s_{n-1})$, $s_j > 0$, $j = 1, 2, \dots, n-1$ is a diagonal matrix of order $n-1$, then*

$$\det \mathbf{L} = (-1)^n \left((a_n + b_n) + \sum_{i=1}^{n-1} \frac{a_i b_{n-i}}{s_i} \right) \prod_{i=1}^{n-1} s_i.$$

Proof. Since Λ is a $(n-1) \times (n-1)$ submatrix of the matrix $\mathbf{L}$. Then we apply the Schur' formula (7),

$$\det \mathbf{L} = (-1)^{(n+1)\times 1} \det \Lambda \det(\mathbf{L}/\Lambda) \tag{9}$$

As in (5), the Schur complement of Λ in $\mathbf{L}$, denoted by $(\mathbf{L}/\Lambda)$, is a 1×1 matrix or a scalar

$$\begin{aligned}
(\mathbf{L}/\Lambda) &= (-a_n - b_n) - (-\mathbf{p}^T)\Lambda^{-1}(-\mathbf{q}) \\
&= (-a_n - b_n) - \begin{bmatrix} -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \\
&\quad \times \begin{bmatrix} 1/s_1 & 0 & \dots & 0 \\ 0 & 1/s_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1/s_{n-1} \end{bmatrix} \begin{bmatrix} -b_{n-1} \\ -b_{n-2} \\ \vdots \\ -b_2 \\ -b_1 \end{bmatrix} \\
&= -(a_n + b_n) - \left(\frac{a_1 b_{n-1}}{s_1} + \frac{a_2 b_{n-2}}{s_2} + \dots + \frac{a_{n-1} b_1}{s_{n-1}} \right) \\
&= -(a_n + b_n) - \sum_{i=1}^{n-1} \frac{a_i b_{n-i}}{s_i} \\
&= -\left((a_n + b_n) + \sum_{i=1}^{n-1} \frac{a_i b_{n-i}}{s_i} \right). \qquad (10)
\end{aligned}$$

Now, from (9) it is easy to see that $\det \Lambda = \prod_{i=1}^{n-1} s_i$. Therefore

$$\begin{aligned}
\det \mathbf{L} &= (-1)^{(n+1)} \prod_{i=1}^{n-1} s_i \left\{ -\left((a_n + b_n) + \sum_{i=1}^{n-1} \frac{a_i b_{n-i}}{s_i} \right) \right\} \\
&= (-1)^n \left((a_n + b_n) + \sum_{i=1}^{n-1} \frac{a_i b_{n-i}}{s_i} \right) \prod_{i=1}^{n-1} s_i.
\end{aligned}$$

This completes the proof. □

Immediately, we have the following corollaries.

Corollary 3.2. *Let L be a Leslie matrix defined as in (1) with partitioned as* $L = \begin{bmatrix} -\mathbf{p}^T & -a_n \\ \Lambda & 0 \end{bmatrix}_{(n,n)}$, *where* $\mathbf{p} = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}^T$, $-a_j \geq 0$, $j = 1, 2, \dots, n$. *and* $\Lambda = \operatorname{diag}(s_1, s_2, \dots, s_{n-1})$, $s_j > 0$, $j = 1, 2, \dots, n-1$ *is a diagonal matrix of order* $n-1$, *then*

$$\det L = (-1)^n a_n \prod_{i=1}^{n-1} s_i.$$

Corollary 3.3. *Let* $\mathbf{C} = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ I_{n-1} & -\mathbf{q} \end{bmatrix}_{(n,n)}$ *be a doubly companion matrix, where* $\mathbf{p} = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}^T$, *and* $\mathbf{q} = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_1 \end{bmatrix}^T$, *then*

$$\det \mathbf{C} = (-1)^n \left((a_n + b_n) + \sum_{i=1}^{n-1} a_i b_{n-i} \right).$$

Corollary 3.4. *Let* $\mathrm{C} = \begin{bmatrix} -\mathrm{p}^T & -a_n \\ I_{n-1} & 0 \end{bmatrix}_{(n,n)}$ *be a companion matrix, where* $\mathrm{p} = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} \end{bmatrix}^T$*, then* $\det \mathrm{C} = (-1)^n a_n$.

Now we wish to find the inverse of doubly Leslie matrix.

Theorem 3.5. *Let* $\mathbf{L} = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathrm{q} \end{bmatrix}_{(n,n)}$ *be a doubly Leslie matrix, where* $\mathbf{p} = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} \end{bmatrix}^T$, $\mathbf{q} = \begin{bmatrix} b_{n-1} & b_{n-2} & \cdots & b_1 \end{bmatrix}^T$*, and* $\Lambda = \operatorname{diag}(s_1, s_2, \ldots, s_{n-1})$*, where* $s_j > 0$*,* $j = 1, 2, \ldots, n-1$ *is a diagonal matrix of order* $n-1$*. If* $\det \mathbf{L} \neq 0$ *then*

$$\mathbf{L}^{-1} = (\mathbf{L}/\Lambda)^{-1} \begin{bmatrix} \Lambda^{-1}\mathbf{q} & (\mathbf{L}/\Lambda)\,\Lambda^{-1} + (\Lambda^{-1}\mathbf{q}\mathbf{p}^T\Lambda^{-1}) \\ 1 & \mathbf{p}^T\Lambda^{-1} \end{bmatrix}_{(n,n)},$$

where $(\mathbf{L}/\Lambda) = -\left((a_n + b_n) + \sum_{i=1}^{n-1} \frac{a_i b_{n-i}}{s_i} \right)$*, as in* (10)*, and* $\Lambda^{-1} = \operatorname{diag}(\frac{1}{s_1}, \frac{1}{s_2}, \ldots, \frac{1}{s_{n-1}})$.

Proof. Apply the identity (8) to the matrix $\mathbf{L}$, we have

$$\begin{aligned} \mathbf{L}^{-1} &= \begin{bmatrix} -\Lambda^{-1}(-\mathbf{q})\,(\mathbf{L}/\Lambda)^{-1} & \Lambda^{-1} + \left(\Lambda^{-1}(-\mathbf{q})\,(\mathbf{L}/\Lambda)^{-1}\,(-\mathbf{p}^T)\Lambda^{-1}\right) \\ (\mathbf{L}/\Lambda)^{-1} & -(\mathbf{L}/\Lambda)^{-1}\,(-\mathbf{p}^T)\Lambda^{-1} \end{bmatrix}_{(n,n)} \\ &= \begin{bmatrix} \Lambda^{-1}\mathbf{q}\,(\mathbf{L}/\Lambda)^{-1} & \Lambda^{-1} + \left(\Lambda^{-1}\mathbf{q}\,(\mathbf{L}/\Lambda)^{-1}\,\mathbf{p}^T\Lambda^{-1}\right) \\ (\mathbf{L}/\Lambda)^{-1} & (\mathbf{L}/\Lambda)^{-1}\,\mathbf{p}^T\Lambda^{-1} \end{bmatrix}_{(n,n)}. \end{aligned}$$

The Schur complement of Λ in $\mathbf{L}$ is $(\mathbf{L}/\Lambda)$, in (10) showed that $(\mathbf{L}/\Lambda)$ is a scalar. Then

$$\mathbf{L}^{-1} = (\mathbf{L}/\Lambda)^{-1} \begin{bmatrix} \Lambda^{-1}\mathbf{q} & (\mathbf{L}/\Lambda)\,\Lambda^{-1} + (\Lambda^{-1}\mathbf{q}\mathbf{p}^T\Lambda^{-1}) \\ 1 & \mathbf{p}^T\Lambda^{-1} \end{bmatrix}_{(n,n)}.$$

□

Immediately, we have the following corollaries.

Corollary 3.6. *Let* L *be a Leslie matrix defined in Corollary 3.2. If* $\det L \neq 0$ *then*

$$L^{-1} = (L/\Lambda)^{-1} \begin{bmatrix} 0 & (L/\Lambda)\Lambda^{-1} \\ 1 & \mathbf{p}^T\Lambda^{-1} \end{bmatrix}_{(n,n)},$$

where $(L/\Lambda) = -a_n$.

Corollary 3.7. *Let* $\mathbf{C}$ *be a doubly companion matrix defined in Corollary 3.3. If* $\det \mathbf{C} \neq 0$ *then*

$$\mathbf{C}^{-1} = (\mathbf{C}/I_{n-1})^{-1} \begin{bmatrix} \mathbf{q} & (\mathbf{C}/I_{n-1}) + \mathbf{q}\mathbf{p}^T \\ 1 & \mathbf{p}^T \end{bmatrix}_{(n,n)},$$

where $(\mathbf{C}/I_{n-1}) = -\left((a_n + b_n) + \sum_{i=1}^{n-1} a_i b_{n-i}\right)$.

Corollary 3.8. *Let* $\mathbf{C}$ *be a companion matrix defined in Corollary 3.4. If* $\det \mathbf{C} \neq 0$ *then*

$$\mathbf{C}^{-1} = -\frac{1}{a_n} \begin{bmatrix} \mathbf{0} & -a_n I_{n-1} \\ 1 & \mathbf{p}^T \end{bmatrix}_{(n,n)}.$$

4. Explicit Minimum Polynomial of Doubly Leslie Matrix

The author in [12, Theorem 3.3] asserted that, the doubly companion matrix is nonderogatory. Now, we wish to show that any doubly Leslie matrix $\mathbf{L}$ in (3) is similar to a companion matrix, that is, it is a nonderogatory.

Theorem 4.1. *The doubly Leslie matrix matrix* $\mathbf{L}$ *defined in* (3) *is nonderogatory and the characteristic polynomial and the explicit minimum polynomial is*

$$\begin{aligned} m_{\mathbf{L}}(x) &= x^n + (c_1 + d_1)x^{n-1} + \Big(\sum_{i+j=2} c_i d_j + c_2 + d_2\Big)x^{n-2} + \ldots \\ &\quad + \Big(\sum_{i+j=n-1} c_i d_j + c_{n-1} + d_{n-1}\Big)x + \Big(\sum_{i+j=n} c_i d_j + c_n + d_n\Big), \end{aligned}$$

where $c_1 = a_1$, $c_i = a_i \prod_{k=1}^{i-1} s_k$, *and* $d_1 = b_1$, $d_i = b_i \prod_{k=n+1-i}^{n-1} s_k$, *for* $i = 2, 3, \ldots, n$.

Proof. Let

$$\mathbf{L} = \begin{bmatrix} -a_1 & -a_2 & \ldots & -a_{n-1} & -a_n - b_n \\ s_1 & 0 & \ldots & 0 & -b_{n-1} \\ 0 & s_2 & \ldots & 0 & -b_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & s_{n-1} & -b_1 \end{bmatrix}.$$

Firstly to show that $\mathbf{L}$ is similar to a doubly companion matrix.

By a similarity transformation with a diagonal matrix

$$D = \operatorname{diag}\left(\frac{1}{s_1 s_2 \ldots s_{n-1}}, \frac{1}{s_2 s_3 \ldots s_{n-1}}, \ldots, \frac{1}{s_{n-2} s_{n-1}}, \frac{1}{s_{n-1}}, 1\right),$$

$\mathbf{L}$ can be transformed to a doubly companion matrix,

$$D^{-1}\mathbf{L}D = \begin{bmatrix} -a_1 & -a_2 s_1 & \ldots & -a_{n-1}(s_1 s_2 \ldots s_{n-2}) & -(a_n + b_n)(s_1 s_2 s_3 \ldots s_{n-1}) \\ 1 & 0 & \ldots & 0 & -b_{n-1}(s_2 s_3 \ldots s_{n-1}) \\ 0 & 1 & \ldots & 0 & -b_{n-2}(s_3 \ldots s_{n-1}) \\ \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & -b_2 s_{n-1} \\ 0 & 0 & \ldots & 1 & -b_1 \end{bmatrix}.$$

For convenient, let us denote the doubly companion matrix $D^{-1}\mathbf{L}D$ by

$$D^{-1}\mathbf{L}D = \begin{bmatrix} -c_1 & -c_2 & -c_3 & \dots & -c_{n-1} & -c_n - d_n \\ 1 & 0 & 0 & \dots & 0 & -d_{n-1} \\ 0 & 1 & 0 & \dots & 0 & -d_{n-2} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -d_2 \\ 0 & 0 & 0 & \dots & 1 & -d_1 \end{bmatrix}, \tag{11}$$

where $c_1 = a_1$, $c_i = a_i \prod_{k=1}^{i-1} s_k$, and $d_1 = b_1$, $d_i = b_i \prod_{k=n+1-i}^{n-1} s_k$, for $i = 2, 3, \dots, n$.

Let J be the *backward identity* matrix of order $n \times n$(or *reversal* matrix of order $n \times n$), J $(= J^{-1})$, which showing that

$$J^{-1}(D^{-1}\mathbf{L}D)J = \begin{bmatrix} -d_1 & 1 & \dots & 0 & 0 \\ -d_2 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -d_{n-1} & 0 & \dots & 0 & 1 \\ -c_n - d_n & -c_{n-1} & \dots & -c_2 & -c_1 \end{bmatrix} =: \Gamma. \tag{12}$$

To show that the matrix Γ is similar to a companion matrix. We shall prove by explicit construction the existence of an invertible matrix M such that $M^{-1}\Gamma M$ is a companion matrix. Now, chosen a matrix M of size $n \times n$,

$$M = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ d_1 & 1 & 0 & \ddots & \vdots \\ d_2 & d_1 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ d_{n-1} & \dots & d_2 & d_1 & 1 \end{bmatrix}.$$

Then M is nonsingular matrix. In fact the matrix M is an lower triangular Toeplitz matrix with diagonal-constant 1, and

$$M^{-1} = \begin{bmatrix} \mathbf{e}_1 & \Gamma\mathbf{e}_1 & \Gamma^2\mathbf{e}_1 & \dots & \Gamma^{n-1}\mathbf{e}_1 \end{bmatrix}^T,$$

where $\mathbf{e}_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T \in \mathbb{R}^n$ is the unit column vector.

Computation shows that

$$M^{-1}\Gamma M = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -\gamma_n & -\gamma_{n-1} & \dots & -\gamma_2 & -\gamma_1 \end{bmatrix} =: C,$$

where

$$\begin{array}{rcl}\gamma_1 & = & c_1 + d_1, \\ \gamma_2 & = & \sum\limits_{i+j=2} c_i d_j + c_2 + d_2, \\ & \vdots & \\ \gamma_{n-1} & = & \sum\limits_{i+j=n-1} c_i d_j + c_{n-1} + d_{n-1}, \\ \gamma_n & = & \sum\limits_{i+j=n} c_i d_j + c_n + d_n. \end{array} \tag{13}$$

The matrix

$$C = M^{-1}\Gamma M = M^{-1}J^{-1}(D^{-1}\mathbf{L}DJ)M = (DJM)^{-1}\mathbf{L}(DJM) \tag{14}$$

is the desired companion matrix. Then, we have the doubly Leslie matrix $\mathbf{L}$ is similar to the companion matrix C. By Theorem 1.2, the characteristic polynomial $\Delta_{\mathbf{L}}(x)$ equal to the minimum polynomial $m_{\mathbf{L}}(x)$, we have

$$m_{\mathbf{L}}(x) = x^n + \gamma_1 x^{n-1} + \gamma_2 x^{n-2} + \cdots + \gamma_{n-1}x + \gamma_n.$$

That is

$$\begin{array}{rcl} m_{\mathbf{L}}(x) & = & x^n + (c_1 + d_1)x^{n-1} + (\sum\limits_{i+j=2} c_i d_j + c_2 + d_2)x^{n-2} + \ldots \\ & & +(\sum\limits_{i+j=n-1} c_i d_j + c_{n-1} + d_{n-1})x + (\sum\limits_{i+j=n} c_i d_j + c_n + d_n), \end{array}$$

where $c_1 = a_1$, $c_i = a_i \prod\limits_{k=1}^{i-1} s_k$, and $d_1 = b_1$, $d_i = b_i \prod\limits_{k=n+1-i}^{n-1} s_k$, for $i = 2, 3, \ldots, n$.

□

5. Explicit Eigenvector Formula of Doubly Leslie Matrix

Now analogous as eigenvector of a companion matrix in [2, pp.630–631] and in [10, p.6], we obtain.

Theorem 5.1. *Let λ be an eigenvalue of a doubly Leslie matrix $\mathbf{L}$ defined in (3). Then*

$$\mathbf{v} = \begin{bmatrix} \frac{1}{s_1 s_2 \ldots s_{n-1}} \left(\lambda^{n-1} + d_1\lambda^{n-2} + d_2\lambda^{n-3} + \cdots + d_{n-2}\lambda + d_{n-1}\right) \\ \frac{1}{s_2 s_3 \ldots s_{n-1}} \left(\lambda^{n-2} + d_1\lambda^{n-3} + \cdots + d_{n-3}\lambda + d_{n-2}\right) \\ \vdots \\ \frac{1}{s_{n-1}}(\lambda + d_1) \\ 1 \end{bmatrix}$$

is an eigenvector of $\mathbf{L}$ corresponding to the eigenvalue λ, where $d_1 = b_1$, $d_i = b_i \prod\limits_{k=n+1-i}^{n-1} s_k$, for $i = 2, 3, \ldots, n$.

Proof. From Theorem 4.1, $\mathbf{L}$ is similar to the companion matrix C as in (14). Then they have the same eigenvalues in common. Let λ be an eigenvalue of $\mathbf{L}$, then λ also an eigenvalue of C. Since λ is a root of the characteristic polynomial $\Delta_{\mathbf{L}}(x)$, we have

$$\begin{aligned}\Delta_{\mathbf{L}}(\lambda) &= \lambda^n + (c_1+d_1)\lambda^{n-1} + (\sum_{i+j=2} c_id_j + c_2 + d_2)\lambda^{n-2} + \dots \\ &\quad +(\sum_{i+j=n-1} c_id_j + c_{n-1} + d_{n-1})\lambda + (\sum_{i+j=n} c_id_j + c_n + d_n) = 0.\end{aligned}$$

From (13), we have,

$$\Delta_{\mathbf{L}}(\lambda) = \lambda^n + \gamma_1\lambda^{n-1} + \gamma_2\lambda^{n-2} + \dots + \gamma_{n-11}\lambda + \gamma_n = 0.$$

Therefore

$$\lambda^n = -\left(\gamma_1\lambda^{n-1} + \gamma_2\lambda^{n-2} + \dots + \gamma_{n-11}\lambda + \gamma_n\right).$$

Then, we put a vector $\mathbf{u} = \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix}$. We must show that this vector $\mathbf{u}$ is an eigenvector of C corresponding to the eigenvalue λ. Form equation (14), $C = (DJM)^{-1}\mathbf{L}(DJM)$, we have

$$\begin{aligned} C\mathbf{u} &= \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -\gamma_n & -\gamma_{n-1} & \dots & -\gamma_2 & -\gamma_1 \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix} \\ &= \begin{bmatrix} \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \\ -\gamma_n - \gamma_{n-1}\lambda - \dots - \gamma_2\lambda^{n-2} - \gamma_1\lambda^{n-1} \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \\ \lambda^n \end{bmatrix} \\ &= \lambda \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix} = \lambda\mathbf{u}, \end{aligned}$$

it is easy to see that the first component in the vector $\mathbf{u}$ cannot be zero, the vector $\mathbf{u}$ is not a zero-vector, it is an eigenvector of C corresponding to λ.

Since $(DJM)^{-1}\mathbf{L}(DJM) = C$. Theorem 1.1 asserted that $(DJM)\mathbf{u}$ is an eigenvector of $\mathbf{L}$ corresponding to the eigenvalue λ. Hence, the explicit form of an eigenvector corresponding to an eigenvalue λ of the matrix $\mathbf{L}$ is

$$
\begin{aligned}
\mathbf{v} \; &:= \; (DJM)\mathbf{u} \\
&= \begin{bmatrix}
\frac{1}{s_1 s_2 \cdots s_{n-1}} & 0 & \cdots & 0 & 0 & 0 \\
0 & \frac{1}{s_2 s_3 \cdots s_{n-1}} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \frac{1}{s_{n-2} s_{n-1}} & 0 & 0 \\
0 & 0 & \cdots & 0 & \frac{1}{s_{n-1}} & 0 \\
0 & 0 & \cdots & 0 & 0 & 1
\end{bmatrix} \\
&\quad \times \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & \cdots & 0 & 1 & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & 1 & \cdots & 0 & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & \cdots & 0 & 0 \\
d_1 & 1 & \ddots & \vdots & \vdots \\
d_2 & d_1 & \ddots & 0 & 0 \\
\vdots & \ddots & \ddots & 0 & 0 \\
\vdots & \ddots & \ddots & 1 & 0 \\
d_{n-1} & \cdots & d_2 & d_1 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1}
\end{bmatrix},
\end{aligned}
$$

that is

$$
\begin{aligned}
\mathbf{v} &= \begin{bmatrix}
\frac{d_{n-1}}{s_1 s_2 \cdots s_{n-1}} & \frac{d_{n-2}}{s_1 s_2 \cdots s_{n-1}} & \frac{d_{n-3}}{s_1 s_2 \cdots s_{n-1}} & \cdots & \frac{d_1}{s_1 s_2 \cdots s_{n-1}} & \frac{1}{s_1 s_2 \cdots s_{n-1}} \\
\frac{d_{n-2}}{s_2 s_3 \cdots s_{n-1}} & \frac{d_{n-3}}{s_2 s_3 \cdots s_{n-1}} & \iddots & \frac{d_1}{s_2 s_3 \cdots s_{n-1}} & \frac{1}{s_2 s_3 \cdots s_{n-1}} & 0 \\
\frac{d_{n-3}}{s_3 s_4 \cdots s_{n-1}} & \iddots & \iddots & \frac{1}{s_3 s_4 \cdots s_{n-1}} & 0 & \vdots \\
\vdots & \iddots & \iddots & 0 & \vdots & 0 \\
\frac{d_1}{s_{n-1}} & \frac{1}{s_{n-1}} & \iddots & \vdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix} \mathbf{u} \\
&= \begin{bmatrix}
\frac{1}{s_1 s_2 \cdots s_{n-1}} \; \lambda^{n-1} + d_1\lambda^{n-2} + d_2\lambda^{n-3} + \cdots + d_{n-2}\lambda + d_{n-1} \\
\frac{1}{s_2 s_3 \cdots s_{n-1}} \; \lambda^{n-2} + d_1\lambda^{n-3} + \cdots + d_{n-3}\lambda + d_{n-2} \\
\vdots \\
\frac{1}{s_{n-2} s_{n-1}} \; \lambda^2 + d_1\lambda + d_2 \\
\frac{1}{s_{n-1}} \; (\lambda + d_1) \\
1
\end{bmatrix},
\end{aligned}
$$

it is easy to see that the last component in the vector $\mathbf{v}$ cannot be zero, which proves the assertion. □

The following corollaries are particular case of Theorem 5.1.

If $b_1 = b_2 = \cdots = b_n = 0$, then the matrix become a Leslie matrix, we have the following corollary.

Corollary 5.2. *Let λ be an eigenvalue of a Leslie matrix L defined in Corollary 3.2. Then*

$$\mathbf{v} = \begin{bmatrix} \frac{1}{s_1 s_2 \dots s_{n-1}}\lambda^{n-1} \\ \frac{1}{s_2 s_3 \dots s_{n-1}}\lambda^{n-2} \\ \vdots \\ \frac{1}{s_{n-1}}\lambda \\ 1 \end{bmatrix}$$

is an eigenvector of L corresponding to the eigenvalue λ. A nonzero scalar multiple of $\mathbf{v}$ namely

$$\mathbf{w} := (s_1 s_2 \dots s_{n-1}/\lambda^{n-1})\mathbf{v} = \begin{bmatrix} 1 \\ s_1/\lambda \\ s_1 s_2/\lambda^2 \\ \vdots \\ s_1 s_2 \dots s_{n-1}/\lambda^{n-1} \end{bmatrix}$$

is also an eigenvector of L corresponding to the eigenvalue λ.

If $s_1 = s_2 = \dots = s_n = 1$, then we have the following corollary, as in [13, pp. 270–272].

Corollary 5.3. *Let λ be an eigenvalue of a doubly companion matrix $\mathbf{C}$ defined in Corollary 3.3. Then*

$$\mathbf{v} = \begin{bmatrix} \lambda^{n-1} + b_1\lambda^{n-2} + b_2\lambda^{n-3} + \dots + b_{n-2}\lambda + b_{n-1} \\ \lambda^{n-2} + b_1\lambda^{n-3} + \dots + b_{n-3}\lambda + b_{n-2} \\ \vdots \\ \lambda + b_1 \\ 1 \end{bmatrix}$$

is an eigenvector of $\mathbf{C}$ corresponding to the eigenvalue λ.

Corollary 5.4. *Let λ be an eigenvalue of a companion matrix C defined in Corollary 3.4. Then*

$$\mathbf{v} = \begin{bmatrix} \lambda^{n-1} \\ \lambda^{n-2} \\ \vdots \\ \lambda \\ 1 \end{bmatrix}$$

is an eigenvector of C corresponding to the eigenvalue λ.

6. Conclusion

The doubly Leslie matrix is a nonderogatory matrix. This paper has explored a special form of a Schur complement to obtained the determinant, inverse, and explicit eigenvector formulas of the doubly Leslie matrix which is the generalized forms of the Leslie matrix. It is also a generalized form of the doubly companion matrix, and the companion matrix, respectively.

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เลขทะเบียน 30

หนังสือยินยอมการเผยแพร่ผลงานทางวิชาการบนเว็บไซต์
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ปีที่พิมพ์ 2557

ข้าพเจ้าขอรับรองว่า ผลงานทางวิชาการเป็นลิขสิทธิ์ของข้าพเจ้า รศ.วิวรรธน์ วณิชาภิชาติ เป็นเจ้าของลิขสิทธิ์ และเพื่อให้ผลงานทางวิชาการของข้าพเจ้าเป็นประโยชน์ต่อการศึกษาและสาธารณชน จึงอนุญาตให้เผยแพร่ผลงาน ดังนี้

☑ อนุญาตให้เผยแพร่

○ ไม่อนุญาตให้เผยแพร่ เนื่องจาก...

....

....

ลงชื่อ ....

(รศ.วิวรรธน์ วณิชาภิชาติ)

วันที่ 4 พ.ย. 58

หมายเหตุ ลิขสิทธิ์ใดๆ ที่ปรากฏอยู่ในผลงานนี้เป็นความรับผิดชอบของเจ้าของผลงาน ไม่ใช่ของสำนักหอสมุด