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Symmetry classes of polynomial and o–basis

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สนับสนุนโดย

งบประมาณรายได้มหาวิทยาลัยนเรศวร

ปีงบประมาณ 2558

ชื่อโครงการ คลาสของพหุนามเชิงสมมาตรและฐานเชิงตั้งฉากปรกติ

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บทคัดย่อ(ภาษาไทย)

ในงานวิจัยนี้ ผู้วิจัยได้ค้นหาการมีอยู่ของฐานเชิงตั้งฉากปรกติของคลาสพหุนามเชิงสมมาตร ในลักษณะที่คล้ายคลึงกับกรณีของฐานเชิงตั้งฉากปรกติสำหรับคลาสของเทนเซอร์ ผู้วิจัยได้ค้นพบเงื่อนไขของการมีอยู่ของฐานดังกล่าวของพหุนามสำหรับกรุปจำกัด และยังได้สำรวจเงื่อนไขสำหรับการมีอยู่ของฐานเชิงตั้งฉากปรกติของคลาสของพหุนามเชิงสมมาตรสำหรับกรุปสมมาตรและ คาร์แรกเตอร์ลดทอนไม่ได้บางตัวของกรุปสมมาตรนั้นด้วย

บทคัดย่อ(ภาษาอังกฤษ)

In this project, we discuss the existence of orthogonal *-basis of the symmetry classes of polynomials. Analogously to the orthogonal *-basis of symmetry classes of tensor, some criteria for the existence of the basis for finite groups have been provided. We also investigate a condition for the existence of such basis of symmetry classes of polynomials associated to symmetric groups and some irreducible characters.

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สารบัญ

Chapter 1
Introduction

One of the classical areas of algebra, the theory of symmetric polynomials is well-known because of its role in branches of algebra, such as Galois Theory, representation theory and algebraic combinatorics. For a review of the theory of symmetric polynomials, one can see the book of Macdonald, [6]. The relative symmetric polynomials as a generalization of symmetric polynomials are introduced by M. Shahryari in [11]. In fact, he used the idea of symmetry classes of tensors to introduce such notions.

One of the most interesting topics about symmetry classes of tensors is the issues of finding a necessary condition for the existence of an orthogonal $*$-basis for the symmetry classes of tensors associated with a finite group and an irreducible character. Many researchers pay a lot of attention to investigate condition stated above. For example, M.R. Pournaki, [8], gave such a necessary condition for the irreducible constituents of the permutation character of the finite groups in which he extended a result of R.R. Holmes, [2]. Also, M. Shahryari provided an excellent condition for the existence of such basis in [10]. Furthermore, the existence of the special basis for particular groups have been discussed by many authors, see, for example, [3, 4, 13]. Similar questions concerning about the existence of an orthogonal $*$-basis arise in the context of relative symmetric polynomials as well, see, for example [9, 14, 15]. The general criterion is still an open problem, [11].

In this project, we provide some criteria for the existence of the special basis of symmetry classes of polynomials for finite groups and some corresponding permutation characters which are parallel to those of M.R. Pournaki in [8], R.R. Holmes in [2] and M. Shahryari in [10]. We also investigate some condition for the existence of such basis of symmetry classes of polynomials associated to symmetric groups and some irreducible characters, which are similar to the results of Y. Zamani in [12].

Chapter 2
Notations and Background

Let G be a subgroup of the full symmetric group S_m and χ be an irreducible character of G. Let $H_d[x_1, ..., x_m]$ be the complex space of homogenous polynomials of degree d with the independent commuting variables $x_1, ..., x_m$. Let $\Gamma^+_{m,d}$ be the set of all m-tuples of non-negative integers $\alpha = (\alpha_1, ..., \alpha_m)$, such that $\sum_{i=1}^m \alpha_i = d$. For any $\alpha \in \Gamma^+_{m,d}$, let X^α be the monomial $x_1^{\alpha_1} x_2^{\alpha_2} ... x_m^{\alpha_m}$. Then the set $\{X^\alpha | \alpha \in \Gamma^+_{m,d}\}$ is a basis of $H_d[x_1, ..., x_m]$. An inner product on $H_d[x_1, ..., x_m]$ is defined by

(0.1) $$\langle X^\alpha, X^\beta \rangle = \delta_{\alpha,\beta}.$$

The group G, as a subgroup of the full symmetric group S_m, acts on $H_d[x_1, ..., x_m]$ by (for $\sigma \in G$),

$$q^\sigma(x_1, ..., x_m) = q(x_{\sigma^{-1}(1)}, ..., x_{\sigma^{-1}(m)}).$$

It also acts on $\Gamma^+_{m,d}$ by

$$\sigma\alpha = (\alpha_{\sigma(1)}, ..., \alpha_{\sigma(m)}).$$

Let Δ be a set of representatives of orbits of $\Gamma^+_{m,d}$ under the action of G. Now consider the symmetrizer associated with G and χ

(0.2) $$T(G, \chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma)\sigma$$

It is well known that $T(G, \chi)^2 = T(G, \chi)$ and $T(G, \chi)^* = T(G, \chi)$. The image of $H_d[x_1, ..., x_m]$ under the map $T(G, \chi)$ is called the *symmetry class of polynomials of degree* d with respect to G and χ and it is denoted by $H_d(G; \chi)$.

For any $q \in H_d[x_1, ..., x_m]$,

$$q^*_\chi = T(G, \chi)(q)$$

is called a symmetrized polynomial with respect to G and χ. Note that

$$H_d(G; \chi) = \langle X^{\alpha,*}_\chi ; \alpha \in \Gamma^+_{m,d} \rangle.$$

We write $X^{\alpha,*}$ instead of $X^{\alpha,*}_\chi$ unless it is necessary to avoid confusion.

Definition 0.1. *An* orthogonal $*$-basis *(*o-basis, *for short) of a subspace U of $H_d(G; \chi)$ is an orthogonal basis of U of the form $\{X^{\alpha_1,*}, X^{\alpha_2,*}, ..., X^{\alpha_t,*}\}$ for some $\alpha_i \in \Gamma^+_{m,d}$.*

Since the set $\{T(G, \chi) : \chi \in Irr(G)\}$ is a complete set of orthogonal idempotents, where $Irr(G)$ is the set of irreducible complex characters of G, we have the following orthogonal direct sum decomposition (cf. Remark 2.3 in [11])

(0.3) $$H_d[x_1, ..., x_m] = \bigoplus_{\chi \in Irr(G)} H_d(G; \chi).$$

Note that $X^{\alpha,*}$ is a generator of $H_d(G; \chi)$ if $X^{\alpha,*} \neq 0$, which can be checked from $(\chi, 1)_{G_\alpha}$, where $(\chi, \phi)_K$ is the inner product of characters χ and ϕ of an arbitrary group K, i.e. $(\chi, \phi)_K = \frac{1}{|K|} \sum_{\sigma \in K} \chi(\sigma)\psi(\sigma^{-1})$. Namely, (see, [9, 11]),

(0.4) $$X^{\alpha,*} \neq 0 \text{ if and only if } (\chi, 1)_{G_\alpha} \neq 0.$$

Also, for the induced inner product on $H_d(G;\chi)$, we have (see, [9, 11]).

$$\langle X^{\sigma_1\alpha,*}, X^{\sigma_2\beta,*}\rangle = \begin{cases} 0, & \text{if } \alpha \notin \mathrm{Orb}(\beta)\ ; \\ \frac{\chi(1)}{|G|}\sum_{\sigma\in\sigma_2 G_\alpha \sigma_1^{-1}} \chi(\sigma), & \text{if } \alpha = \beta, \end{cases} \tag{0.5}$$

where $\mathrm{Orb}(\beta)$ is the orbit of β in $\Gamma^+_{m,d}$ under the action of G. Then the norm of $X^{\alpha,*}$, with respect to the induced inner product, is given by

$$\| X^{\alpha,*} \|^2 = \chi(1)\frac{(\chi,1)_{G_\alpha}}{[G:G_\alpha]}. \tag{0.6}$$

According to (0.4), let $\Omega = \{\alpha \in \Gamma^{+_{m,d}} : (\chi,1)_{G_\alpha} \neq 0\}$. Since

$$H_d[x_1,\dots,x_m] = \bigoplus_{\alpha\in\Delta}\langle X^{\sigma\alpha} : \sigma \in G\rangle,$$

we have the orthogonal direct sum

$$H_d(G;\chi) = \bigoplus_{\alpha\in\overline{\Delta}} H_d^{\alpha,*}(\chi), \tag{0.7}$$

where $\overline{\Delta} = \Delta \cap \Omega$ and $H_d^{\alpha,*}(\chi) = \langle X^{\sigma\alpha,*} | \sigma \in G\rangle$. The dimension of $H_d^{\alpha,*}(\chi)$ can be calculated by using Freese's Theorem (see, e.g. [1], [9])

$$\dim H_d^{\alpha,*}(\chi) = \chi(1)(\chi,1)_{G_\alpha} = \frac{\chi(1)}{|G_\alpha|}\sum_{\sigma\in G_\alpha}\chi(\sigma). \tag{0.8}$$

As an immediate consequence of (0.7) and (0.8),

$$\dim H_d(G;\chi) = \chi(1)\sum_{\alpha\in\overline{\Delta}}(\chi,1)_{G_\alpha}. \tag{0.9}$$

In particular, if χ is linear, then the set $\{X^{\alpha,*} : \alpha \in \overline{\Delta}\}$ is an orthogonal basis of $H_d(G;\chi)$ and $\dim H_d(G;\chi) = |\overline{\Delta}|$. Thus, the orthogonal $*$-basis for $H_d(G;\chi)$ exists for any abelian group G.

Chapter 3
Main Results

According to the notations in the previous section, $H_d(G;\chi)$ denotes the relative symmetry classes of polynomials of degree d with respect to G and χ. This class is equipped with the induced inner product as in (0.5). Let Λ be a set of m elements. Suppose G acts faithfully on Λ. So, we consider $\{f_\sigma \mid \sigma \in G\}$ as the group G, where $f_\sigma : \Lambda \to \Lambda$ defined by $f_\sigma(\lambda) = \sigma \cdot \lambda$, for all $\lambda \in \Lambda$. Namely, G can be viewed as a subgroup of S_m in this way. We also denote the permutation character of G by θ. It is well known that $\theta(\sigma) = |\{\lambda \in \Lambda \mid \sigma \cdot \lambda = \lambda\}|$, for each $\sigma \in G$. The similar criterion as in the main theorem of [8] is shown below.

Theorem 0.2. *Let G be a finite group and let Λ be a set of m elements, $m > 1$. Assume that G acts transitively and faithfully on Λ. Let χ be an irreducible constituent of permutation character θ of G. If $\chi(1)(\chi,\theta)_G > \frac{m}{2}$, then $H_d(G;\chi)$ does not have an orthogonal $*$-basis.*

Proof. Suppose $H_d(G;\chi)$ has an orthogonal $*$-basis. Then, by (0.7), $H_d^{\alpha,*}(\chi)$ has an o-basis for each $\alpha \in \overline{\Delta}$. We now consider $\alpha = (d,0,0,...,0) \in \Gamma^+_{m,d}$ and choose Δ to be the set of representatives of orbits of $\Gamma^+_{m,d}$ under the action of G in which $\alpha \in \Delta$. We can assume without loss of generality that $\Lambda = \{1,2,...,m\}$ and thus $G_\alpha = G_1$, where G_α refers to the stabilizer subgroup of α (when G acts on $\Gamma^+_{m,d}$) and G_1 refers to the stabilizer subgroup of 1 (when G acts on Λ). Since G acts transitively on Λ, $(1_{G_\alpha})^G = (1_{G_1})^G = \theta$, by Lemma 5.14 of [5]. Hence, by (0.8) and Frobenius reciprocity, we have that

$$\begin{aligned}\sum_{\sigma \in G_\alpha} \chi(\sigma) &= |G_\alpha|(\chi, 1_{G_\alpha})_{G_\alpha} \\ &= |G_\alpha|(\chi, (1_{G_\alpha})^G)_G \\ &= |G_\alpha|(\chi,\theta)_G.\end{aligned}$$

Since χ is an irreducible constituent of permutation character θ of G, $(\chi,\theta)_G \neq 0$ and $\sum_{\sigma\in G_\alpha} \chi(\sigma) \neq 0$. Thus $\alpha \in \overline{\Delta}$. So, $H_d^{\alpha,*}(\chi)$ has an o-basis.

By orbit-stabilizer theorem and transitive action of G on Λ, we have that $m = |\Lambda| = \mathrm{Orb}(1) = [G : G_1] = [G : G_\alpha]$. So, $G = \bigcup_{i=1}^m \sigma_i G_\alpha$, where $\{\sigma_1, \sigma_2, ..., \sigma_m\}$ is a system of distinct representatives of left cosets of G_α in G. Let

$$\dim H_d^{\alpha,*}(\chi) = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma) = \chi(1)(\chi,\theta)_G := t.$$

We can assume that $\{X^{\sigma_1\alpha,*}, X^{\sigma_2\alpha,*}, ..., X^{\sigma_t\alpha,*}\}$ is an o-basis for $H_d^{\alpha,*}(\chi)$. Define the $m\times m$ complex matrix $D = [D_{ij}]$ by $D_{ij} := \langle X^{\sigma_i\alpha,*}, X^{\sigma_j\alpha,*}\rangle$. Note that D is idempotent. In fact,

$$\begin{aligned}
(D)^2_{ij} &= \sum_{k=1}^{m} D_{ik}D_{kj} \\
&= \sum_{k=1}^{m} \langle X^{\sigma_i\alpha,*}, X^{\sigma_k\alpha,*}\rangle\langle X^{\sigma_k\alpha,*}, X^{\sigma_j\alpha,*}\rangle \\
&= \sum_{k=1}^{m}\left(\frac{\chi(1)}{|G|}\sum_{\sigma\in\sigma_k G_\alpha\sigma_i^{-1}}\chi(\sigma)\right)\left(\frac{\chi(1)}{|G|}\sum_{\tau\in\sigma_j G_\alpha\sigma_k^{-1}}\chi(\tau)\right) \\
&= \frac{\chi(1)^2}{|G|^2}\sum_{k=1}^{m}\sum_{\sigma\in G_\alpha}\sum_{\tau\in G_\alpha}\chi(\sigma_k\sigma\sigma_i^{-1})\chi(\sigma_j\tau\sigma_k^{-1}) \\
&= \frac{\chi(1)^2}{|G|^2}\sum_{k=1}^{m}\sum_{\lambda\in\sigma_k G_\alpha}\sum_{\mu\in G_\alpha\sigma_k^{-1}}\chi(\lambda\sigma_i^{-1})\chi(\sigma_j\mu).
\end{aligned}$$

Now, let $\mu\lambda = \delta\in G_\alpha$. Then $\mu = \delta\lambda^{-1}$ and we have

$$\begin{aligned}
(D)^2_{ij} &= \frac{\chi(1)^2}{|G|^2}\sum_{k=1}^{m}\sum_{\lambda\in\sigma_k G_\alpha}\sum_{\delta\in G_\alpha}\chi(\lambda\sigma_i^{-1})\chi(\sigma_j\delta\lambda^{-1}) \\
&= \frac{\chi(1)}{|G|}\sum_{\delta\in G_\alpha}\left(\frac{\chi(1)}{|G|}\sum_{\lambda\in G}\chi(\lambda\sigma_i^{-1})\chi(\sigma_j\delta\lambda^{-1})\right) \\
&= \frac{\chi(1)}{|G|}\sum_{\delta\in G_\alpha}\left(\frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)\chi(\sigma_j\delta\sigma_i^{-1}\sigma^{-1})\right).
\end{aligned}$$

By orthogonal relations of irreducible character, we have

$$(D)^2_{ij} = \frac{\chi(1)}{|G|}\sum_{\delta\in G_\alpha}\chi(\sigma_j\delta\sigma_i^{-1}),$$

which shows that $D^2 = D$.

We note that $m = \theta(1) = \sum_{\chi\in\Theta}\chi(1)(\chi,\theta)$, where Θ is the set of all irreducible constituents of the permutation character θ. Since $m > 1$, $|\Theta| > 1$ and hence $m > t$. We can now write D in the form

$$\begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix},$$

where D_1, D_2, D_3 and D_4 are matrices of sizes $t\times t, t\times(m-t), (m-t)\times t$ and $(m-t)\times(m-t)$ respectively. On the matrix D_1, we have, by (0.5), that, for $1\le i,j\le t$,

$$(D_1)_{ij} = \langle X^{\sigma_i\alpha,*}, X^{\sigma_j\alpha,*}\rangle = \begin{cases} 0, & \text{if } i\neq j; \\ \frac{\chi(1)}{|G|}\sum_{\sigma\in G_\alpha}\chi(\sigma), & \text{if } i=j \end{cases} = \begin{cases} 0, & \text{if } i\neq j; \\ \frac{t}{m}, & \text{if } i=j \end{cases} = \left(\frac{t}{m}I_t\right)_{ij},$$

where I_t is the $t \times t$ identity matrix. So, $D = \begin{bmatrix} \left(\frac{t}{m}\right) I_t & D_2 \\ D_3 & D_4 \end{bmatrix}$. Now, using $D^2 = D$, we get

$$D_2 D_3 = \left(\frac{t}{m} - \frac{t^2}{m^2}\right) I_t.$$

Since $t < m$, $\left(\frac{t}{m} - \frac{t^2}{m^2}\right) \neq 0$ and hence D_2D_3 is invertible. This means that if $H_d^{\alpha,*}(\chi)$ has an o-basis, then

$$t = \operatorname{rank} D_2D_3 \leq \min\{\operatorname{rank} D_2, \operatorname{rank} D_3\} \leq \min\{t, m-t\} \leq m-t.$$

Therefore, if $\chi(1)(\chi,\theta)_G = t > \frac{m}{2}$, then $H_d(G;\chi)$ does not have an orthogonal $*$-basis, by (0.7). □

We also obtain a similar results of Holmes in [2].

Corollary 0.3. *(cf. [2, 8]) Let G be a 2-transitive subgroup of S_m, $m > 2$. Let $\chi = \theta - 1_G$, where θ is the permutation character of G. Then $H_d(G;\chi)$ does not have an orthogonal $*$-basis.*

Proof. Note that G has a canonical transitive an faithful action on the set $\Lambda = \{1, 2, ..., m\}$, given by $\sigma \cdot i := \sigma(i)$ for each $\sigma \in G \leq S_m$ and $i \in \Lambda$. Since G acts 2-transitively on Λ with permutation character θ, by Corollary 5.17 in [5], $\chi = \theta - 1_G$ is an irreducible constituent of θ. We compute that

$$\chi(1)(\chi,\theta)_G = \chi(1)(\theta - 1_G, \theta)_G = \chi(1)[(\theta,\theta)_G - (\theta, 1_G)_G] = \chi(1)[2-1] = m-1.$$

Since $m > 2$, $m - 1 > \frac{m}{2}$ and hence $\chi(1)(\chi,\theta)_G > \frac{m}{2}$. Thus, by Theorem 0.2, the result follows. □

Example 0.4. *(cf. [8]) Let $G = \Lambda = A_4$ be the alternating group of degree 4. We know that G acts transitively and faithfully on Λ by left multiplication. Then we can view G as a subgroup of S_{12}. Note that G has an irreducible character, χ, of degree 3 and the permutation character θ of G is regular. Thus χ is an irreducible constituent of θ of multiplicity 3. Hence $\chi(1)(\chi,\theta)_G = 9 > \frac{12}{2} = \frac{|\Lambda|}{2}$ and then $H_d(A_4;\chi)$ does not have an orthogonal $*$-basis, by Theorem 0.2. In this example, however, the action G on Λ is not 2-transitive.*

By using the same technique as in the proof of Theorem 0.2, we also obtain an analogous criterion of Shahryari in [10].

Theorem 0.5. *Let G be a permutation group of degree m and χ be a non-linear irreducible character of G. If there is $\alpha \in \Gamma^+_{m,d}$ such that*

$$\frac{\sqrt{2}}{2} < \| X_\chi^{\alpha,*} \| < 1,$$

then $H_d(G;\chi)$ does not have an orthogonal $*$-basis.

Proof. Let $\alpha \in \Gamma^+_{m,d}$. Suppose the orbit of α under the action of G is $\operatorname{Orb}(\alpha) = \{\sigma_1\alpha, \sigma_2\alpha, ..., \sigma_r\alpha\}$. Then, by orbit-stabilizer theorem, $r = [G : G_\alpha]$ and $G = \bigcup_{i=1}^r \sigma_i G_\alpha$ is a partition. Now, we construct $r \times r$ matrix $D = [D_{ij}]$ by $D_{ij} := \langle X^{\sigma_i\alpha,*}, X^{\sigma_j\alpha,*}\rangle$ which is idempotent as before. Next, suppose χ is a non-linear irreducible character of G and

$\alpha \in \overline{\Delta}$ and assume also that $\{X^{\sigma_1\alpha,*}, X^{\sigma_2\alpha,*}, ..., X^{\sigma_t\alpha,*}\}$ is an o-basis for $H_d^{\alpha,*}(\chi)$ in which $t < r$, where $t = \dim H_d^{\alpha,*}(\chi)$. So, the matrix D has the block partition form

$$D = \begin{bmatrix} \left(\frac{t}{r}\right) I_t & D_2 \\ D_3 & D_4 \end{bmatrix},$$

where D_2, D_3 and D_4 are matrices of sizes $t \times (r-t), (r-t) \times t$ and $(r-t) \times (r-t)$ respectively. By the same arguments as in the proof of Theorem 0.2, we reach to the conclusion that $t \leq r - t$ or $t \leq \frac{r}{2}$. Thus if $t < r$ and $t > \frac{r}{2}$, then $H_d^{\alpha,*}(\chi)$ does not have o-basis. Substituting $r = [G : G_\alpha]$, $t = \chi(1)(\chi, 1)_{G_\alpha}$ in the inequality $\frac{r}{2} < t < r$ and using (0.6) and (0.7), the result follows. □

Symmetric groups

It is well known that there is a standard one-to-one correspondence between the complex irreducible characters of the symmetric group S_m and the partitions of m. Here, a partition π of m of length t, denoted by $\pi \vdash m$, means an unordered collection of t positive integers that sum to m. In this article, we use the same symbol to denote an irreducible character of S_m and the partition of m corresponding to it. Typically, we represent the partition by a sequence $\pi = [\pi_1, \pi_2, ..., \pi_t]$ in which $\pi_1 \geq \pi_2 \geq ... \geq \pi_t > 0$. A partition $\pi = [\pi_1, \pi_2, ..., \pi_t]$ is usually represented by a collection of m boxes arranged in t rows such that the number of boxes of row i is equal to π_i, for $i = 1, 2, ..., t$. This collection is called the *Young diagram* associated with π and denoted by $[\pi]$. The *Young subgroup* corresponding to $\pi \vdash m$ is the internal direct product

$$S_\pi = S_{\pi_1} \times S_{\pi_2} \times \cdots \times S_{\pi_t}.$$

We write $1_{S_\pi} = 1_\pi$ for the principle character of S_π. Note that $1_\pi^{S_m}$ is a character of S_m, so there must exist integers $K_{\mu,\pi}$ such that

$$1_\pi^{S_m} = \sum_{\mu \vdash m} K_{\mu,\pi}\mu.$$

The numbers $K_{\mu,\pi} = \left(1_\pi^{S_m}, \mu\right)_{S_m}$ are called *Kostka coefficients*. By Corollary 4.54 in [7], the Kostka coefficient $K_{\pi,\pi} = 1$ for all $\pi \vdash m$.

For each ordered pair (i, j), $1 \leq i \leq t$, $1 \leq j \leq \pi_i$, there is corresponding a box, B_{ij}, in Young diagram $[\pi]$. Each B_{ij} determines a unique *hook* in $[\pi]$ consisting of B_{ij} itself, all the boxes in row i of $[\pi]$ to the right of B_{ij} and all boxes in column j of $[\pi]$ below B_{ij}. The *hook length*,

$$h_{ij} := (\pi_i - i) + (\pi'_j - j) + 1,$$

where $\pi'_j := |\{k \in \{1, 2, ..., t\} | \pi_k \geq j\}|$ (a j part of *conjugate partition* of π), is the number of boxes in the hook determined by B_{ij}. By the Frame-Robinson-Thrall Hook Length Formula (see, e.g., Theorem 4.60 in [7]), if π is a partition of m, then the degree of the irreducible character of S_m corresponding to $\pi = [\pi_1, \pi_2, ..., \pi_t]$ is

$$\pi(1) = \frac{m!}{\prod_{i=1}^{t}\prod_{j=1}^{\pi_i} h_{ij}}. \tag{0.10}$$

As a consequence of Theorem 0.5, we have an analogous result of Y. Zamani in [12].

Theorem 0.6. *Let π be an irreducible character of S_m of the cycle type;*

I: $\pi = [m-l, l]$, $d \equiv 0 \mod l$, $d \neq 0$ *such* $m \geq 3l$, *or*
II: $\pi = [m-l, l-1, 1]$, $d \equiv r \mod l, 0 < r < l, l > 2$, $d \neq r$ *such* $m > 3l + \frac{4}{l-2}$.

Then $H_d(S_m;\pi)$ does not have an orthogonal $*$-basis.

Proof. For the form I, we set $\alpha = (\underbrace{0,0,\ldots,0}_{m-l}, \underbrace{k,k,\ldots,k}_{l})$, where $k = \frac{d}{l}$. Then $\alpha \in \Gamma^+_{m,d}$. Under the action of S_m on $\Gamma^+_{m,d}$, we choose a system Δ of representatives such that $\alpha \in \Delta$. Since $d \neq 0$, $k \neq 0$ and

$$(S_m)_\alpha \cong S_{m-l} \times S_l = S_\pi,$$

where $(S_m)_\alpha$ is the stabilizer subgroup of α and S_π is the Young subgroup corresponding to $\pi \vdash m$. Hence, by Frobenius Reciprocity Theorem,

$$\begin{aligned}
\frac{1}{|(S_m)_\alpha|}\sum_{\sigma\in(S_m)_\alpha}\pi(\sigma) &= \left(\pi, 1_{(S_m)_\alpha}\right)_{(S_m)_\alpha} \\
&= (\pi, 1_\pi)_{S_\pi} \\
&= \left(\pi, 1_\pi^{S_m}\right)_{S_m} \\
&= \left(\pi, \sum_{\mu\vdash m} K_{\mu,\pi}\mu\right)_{S_m} \\
&= \sum_{\mu\vdash m} K_{\mu,\pi}\,(\pi,\mu)_{S_m} \\
&= K_{\mu,\mu} = 1 \neq 0
\end{aligned}$$

This yields $\alpha \in \overline{\Delta}$ and, moreover, by (0.8), that

$$\dim H_d^{\alpha,*}(\pi) = \frac{\pi(1)}{|(S_m)_\alpha|}\sum_{\sigma\in(S_m)_\alpha}\chi(\sigma) = \pi(1).$$

Now, we compute the product of the hook lengths of $[\pi]$ which we get

$$\begin{aligned}
\prod_{i=1}^{2}\prod_{j=1}^{\pi_i} h_{ij} &= (m-l+1)(m-l)\cdots(m-2l+2)(m-2l)\cdots(2)(1)l(l-1)\cdots(2)(1) \\
&= \frac{(m-l+1)!l!}{(m-2l+1)}
\end{aligned}$$

Hence, by (0.10),

$$\dim H_d^{\alpha,*}(\pi) = \pi(1) = \frac{(m-2l+1)m!}{(m-l+1)!l!}.$$

Now, using (0.6), we have

$$\| X^{\alpha,*} \|^2 = \frac{\dim H_d^{\alpha,*}(\pi)}{[S_m : (S_m)_\alpha]} = \frac{m-2l+1}{m-l+1}.$$

Hence, $\frac{1}{2} < \| X^{\alpha,*} \|^2 < 1$ if and only if $m \geq 3l$. Thus the result for the first form follows from Theorem 0.5.

For the form II, $\pi = [m-l, l-1, 1]$, we set $\alpha = (\underbrace{0,0,...,0}_{m-l}, \underbrace{k,k,...,k}_{l-1}, k+r)$, where $k = \frac{d-r}{l}$. Then $\alpha \in \Gamma^+_{m,d}$. Under the action of S_m on $\Gamma^+_{m,d}$, we choose a system Δ of representatives such that $\alpha \in \Delta$. Since $d \neq r \neq 0$, $k \neq 0$ and $k+r \neq k$ and hence

$$(S_m)_\alpha \cong S_{m-l} \times S_{l-1} \times S_1 = S_\pi.$$

By the same arguments as the first form, we conclude that $\dim H_d^{\alpha,*}(\pi) = \pi(1)$. For the products of the hook lengths, we compute that

$$\begin{aligned}\prod_{i=1}^{3}\prod_{j=1}^{\pi_i} h_{ij} &= (m-l+2)(m-l)\cdots(m-2l+3)(m-2l+1)\cdots(2)(1)l(l-2)(l-3)\cdots(2)(1) \\ &= \frac{(m-l+2)!l!}{(m-l+1)(m-2l+2)(l-1)}.\end{aligned}$$

Then

$$\dim H_d^{\alpha,*}(\pi) = \pi(1) = \frac{(m-l+1)(m-2l+2)(l-1)m!}{(m-l+2)!l!}.$$

Now, using (0.6) again, we have

$$\| X^{\alpha,*} \|^2 = \frac{\dim H_d^{\alpha,*}(\pi)}{[S_m : (S_m)_\alpha]} = \frac{(m-2l+2)(l-1)}{(m-l+2)(l)}.$$

It is now easy to show that $\frac{1}{2} < \| X^{\alpha,*} \|^2 < 1$ if and only if $m > 3l + \frac{4}{l-2}$, because $l > 2$. The result for the second form follows from Theorem 0.5. □

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Appendix

ตารางเปรียบเทียบ

วัตถุประสงค์ที่วางไว้ และ สิ่งที่ได้

วัตถุประสงค์ที่วางไว้	สิ่งที่ได้
To study relative symmetric polynomials associated with finite groups and o-basis.	Understand properties of relative symmetric polynomials associated with finite groups and o-basis
To provide a criterion that makes $H_d[G;\chi]$ does not have an o-basis in the case of finite groups G acting transitively and faithfully on some finite sets.	Obtain a criterion that makes $H_d[G;\chi]$ does not have an o-basis in the case of finite groups G acting transitively and faithfully on some finite sets.
To provide a criterion that makes $H_d[G;\chi]$ does not have an o-basis in the case of permutation groups G.	Obtain a criterion that makes $H_d[G;\chi]$ does not have an o-basis in the case of permutation groups G.
To provide a criterion that makes $H_d[G;\chi]$ does not have an o-basis in the case of some full symmetric groups G.	Obtain a criterion that makes $H_d[G;\chi]$ does not have an o-basis in the case of some full symmetric groups G.

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13	Journal of Pure and Applied Algebra	j	Q1	1,165	38	173	653	3.255	375	629	0,59	18,82	NL
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1 - 50 of 78 << First | < Previous | Next > | Last >>

ORTHOGONAL $*$-BASIS OF SYMMETRY CLASSES OF POLYNOMIALS*

KIJTI RODTES†

Abstract. In this note, we discuss the existence of orthogonal $*$-basis of the symmetry classes of polynomials. Analogously to the orthogonal $*$-basis of symmetry classes of tensor, some criteria for the existence of the basis for finite groups have been provided . We also investigate a condition for the existence of such basis of symmetry classes of polynomials associated to symmetric groups and some irreducible characters.

Key words. Symmetry classes of polynomials, Orthogonal $*$- basis

AMS subject classifications. Primary 05E05 ; Secondary 15A69

1. Introduction. One of the classical areas of algebra, the theory of symmetric polynomials is well-known because of its role in branches of algebra, such as Galois Theory, representation theory and algebraic combinatorics. For a review of the theory of symmetric polynomials, one can see the book of Macdonald, [6]. The relative symmetric polynomials as a generalization of symmetric polynomials are introduced by M. Shahryari in [11]. In fact, he used the idea of symmetry classes of tensors to introduce such notions.

One of the most interesting topics about symmetry classes of tensors is the issues of finding a necessary condition for the existence of an orthogonal $*$-basis for the symmetry classes of tensors associated with a finite group and an irreducible character. Many researchers pay a lot of attention to investigate condition stated above. For example, M.R. Pournaki, [8], gave such a necessary condition for the irreducible constituents of the permutation character of the finite groups in which he extended a result of R.R. Holmes, [2]. Also, M. Shahryari provided an excellent condition for the existence of such basis in [10]. Furthermore, the existence of the special basis for particular groups have been discussed by many authors, see, for example, [3, 4, 13]. Similar questions concerning about the existence of an orthogonal $*$-basis arise in the context of relative symmetric polynomials as well, see, for example [9, 14, 15]. The general criterion is still an open problem, [11].

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In this article, we provide some criteria for the existence of the special basis of symmetry classes of polynomials for finite groups and some corresponding permutation characters which are parallel to those of M.R. Pournaki in [8], R.R. Holmes in [2] and M. Shahryari in [10]. We also investigate some condition for the existence of such basis of symmetry classes of polynomials associated to symmetric groups and some irreducible characters, which are similar to the results of Y. Zamani in [12].

2. Notations and Background. Let G be a subgroup of the full symmetric group S_m and χ be an irreducible character of G. Let $H_d[x_1, ..., x_m]$ be the complex space of homogenous polynomials of degree d with the independent commuting variables $x_1, ..., x_m$. Let $\Gamma^+_{m,d}$ be the set of all m-tuples of non-negative integers $\alpha = (\alpha_1, ..., \alpha_m)$, such that $\sum_{i=1}^m \alpha_i = d$. For any $\alpha \in \Gamma^+_{m,d}$, let X^α be the monomial $x_1^{\alpha_1} x_2^{\alpha_2} ... x_m^{\alpha_m}$. Then the set $\{X^\alpha | \alpha \in \Gamma^+_{m,d}\}$ is a basis of $H_d[x_1, ..., x_m]$. An inner product on $H_d[x_1, ..., x_m]$ is defined by

$$\langle X^\alpha, X^\beta \rangle = \delta_{\alpha,\beta}. \tag{2.1}$$

The group G, as a subgroup of the full symmetric group S_m, acts on $H_d[x_1, ..., x_m]$ by (for $\sigma \in G$),

$$q^\sigma(x_1, ..., x_m) = q(x_{\sigma^{-1}(1)}, ..., x_{\sigma^{-1}(m)}).$$

It also acts on $\Gamma^+_{m,d}$ by

$$\sigma\alpha = (\alpha_{\sigma(1)}, ..., \alpha_{\sigma(m)}).$$

Let Δ be a set of representatives of orbits of $\Gamma^+_{m,d}$ under the action of G. Now consider the symmetrizer associated with G and χ

$$T(G, \chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma)\sigma \tag{2.2}$$

It is well known that $T(G,\chi)^2 = T(G,\chi)$ and $T(G,\chi)^* = T(G,\chi)$. The image of $H_d[x_1, ..., x_m]$ under the map $T(G,\chi)$ is called the *symmetry class of polynomials of degree d* with respect to G and χ and it is denoted by $H_d(G;\chi)$.

For any $q \in H_d[x_1, ..., x_m]$,

$$q^*_\chi = T(G,\chi)(q)$$

is called a symmetrized polynomial with respect to G and χ. Note that

$$H_d(G;\chi) = \langle X^{\alpha,*}_\chi ; \alpha \in \Gamma^+_{m,d} \rangle.$$

We write $X^{\alpha,*}$ instead of $X^{\alpha,*}_{\chi}$ unless it is necessary to avoid confusion.

Definition 2.1. *An* orthogonal $*$-basis *(*o-basis, *for short) of a subspace U of $H_d(G;\chi)$ is an orthogonal basis of U of the form $\{X^{\alpha_1,*}, X^{\alpha_2,*}, ..., X^{\alpha_t,*}\}$ for some $\alpha_i \in \Gamma^+_{m,d}$.*

Since the set $\{T(G,\chi) : \chi \in Irr(G)\}$ is a complete set of orthogonal idempotents, where $Irr(G)$ is the set of irreducible complex characters of G, we have the following orthogonal direct sum decomposition (cf. Remark 2.3 in [11])

$$H_d[x_1, ..., x_m] = \bigoplus_{\chi \in Irr(G)} H_d(G;\chi). \tag{2.3}$$

Note that $X^{\alpha,*}$ is a generator of $H_d(G;\chi)$ if $X^{\alpha,*} \neq 0$, which can be checked from $(\chi, 1)_{G_\alpha}$, where $(\chi,\phi)_K$ is the inner product of characters χ and ϕ of an arbitrary group K, i.e. $(\chi,\phi)_K = \frac{1}{|K|}\sum_{\sigma\in K}\chi(\sigma)\psi(\sigma^{-1})$. Namely, (see, [9, 11]),

$$X^{\alpha,*} \neq 0 \text{ if and only if } (\chi, 1)_{G_\alpha} \neq 0. \tag{2.4}$$

Also, for the induced inner product on $H_d(G;\chi)$, we have (see, [9, 11]).

$$\langle X^{\sigma_1\alpha,*}, X^{\sigma_2\beta,*}\rangle = \begin{cases} 0, & \text{if } \alpha \notin \mathrm{Orb}(\beta)\ ; \\ \frac{\chi(1)}{|G|}\sum_{\sigma\in\sigma_2 G_\alpha \sigma_1^{-1}}\chi(\sigma), & \text{if } \alpha = \beta, \end{cases} \tag{2.5}$$

where $\mathrm{Orb}(\beta)$ is the orbit of β in $\Gamma^+_{m,d}$ under the action of G. Then the norm of $X^{\alpha,*}$, with respect to the induced inner product, is given by

$$\| X^{\alpha,*} \|^2 = \chi(1)\frac{(\chi,1)_{G_\alpha}}{[G:G_\alpha]}. \tag{2.6}$$

According to (2.4), let $\Omega = \{\alpha \in \Gamma^{+m,d} : (\chi,1)_{G_\alpha} \neq 0\}$. Since

$$H_d[x_1, ..., x_m] = \bigoplus_{\alpha\in\Delta}\langle X^{\sigma\alpha} : \sigma \in G\rangle,$$

we have the orthogonal direct sum

$$H_d(G;\chi) = \bigoplus_{\alpha\in\overline{\Delta}} H^{\alpha,*}_d(\chi), \tag{2.7}$$

where $\overline{\Delta} = \Delta \cap \Omega$ and $H^{\alpha,*}_d(\chi) = \langle X^{\sigma\alpha,*} | \sigma \in G\rangle$. The dimension of $H^{\alpha,*}_d(\chi)$ can be calculated by using Freese's Theorem (see, e.g. [1], [9])

$$\dim H^{\alpha,*}_d(\chi) = \chi(1)(\chi,1)_{G_\alpha} = \frac{\chi(1)}{|G_\alpha|}\sum_{\sigma\in G_\alpha}\chi(\sigma). \tag{2.8}$$

As an immediate consequence of (2.7) and (2.8),

$$\dim H_d(G;\chi) = \chi(1) \sum_{\alpha\in\overline{\Delta}} (\chi, 1)_{G_\alpha}. \tag{2.9}$$

In particular, if χ is linear, then the set $\{X^{\alpha,*} : \alpha \in \overline{\Delta}\}$ is an orthogonal basis of $H_d(G;\chi)$ and $\dim H_d(G;\chi) = |\overline{\Delta}|$. Thus, the orthogonal $*$-basis for $H_d(G;\chi)$ exists for any abelian group G.

3. Main criteria . According to the notations in the previous section, $H_d(G;\chi)$ denotes the relative symmetry classes of polynomials of degree d with respect to G and χ. This class is equipped with the induced inner product as in (2.5). Let Λ be a set of m elements. Suppose G acts faithfully on Λ. So, we consider $\{f_\sigma \mid \sigma \in G\}$ as the group G, where $f_\sigma : \Lambda \to \Lambda$ defined by $f_\sigma(\lambda) = \sigma \cdot \lambda$, for all $\lambda \in \Lambda$. Namely, G can be viewed as a subgroup of S_m in this way. We also denote the permutation character of G by θ. It is well known that $\theta(\sigma) = |\{\lambda \in \Lambda \mid \sigma \cdot \lambda = \lambda\}|$, for each $\sigma \in G$. The similar criterion as in the main theorem of [8] is shown below.

THEOREM 3.1. *Let G be a finite group and let Λ be a set of m elements, $m > 1$. Assume that G acts transitively and faithfully on Λ. Let χ be an irreducible constituent of permutation character θ of G. If $\chi(1)(\chi,\theta)_G > \frac{m}{2}$, then $H_d(G;\chi)$ does not have an orthogonal $*$-basis.*

Proof. Suppose $H_d(G;\chi)$ has an orthogonal $*$-basis. Then, by (2.7), $H_d^{\alpha,*}(\chi)$ has an o-basis for each $\alpha \in \overline{\Delta}$. We now consider $\alpha = (d, 0, 0, ..., 0) \in \Gamma^+_{m,d}$ and choose Δ to be the set of representatives of orbits of $\Gamma^+_{m,d}$ under the action of G in which $\alpha \in \Delta$. We can assume without loss of generality that $\Lambda = \{1, 2, ..., m\}$ and thus $G_\alpha = G_1$, where G_α refers to the stabilizer subgroup of α (when G acts on $\Gamma^+_{m,d}$) and G_1 refers to the stabilizer subgroup of 1 (when G acts on Λ). Since G acts transitively on Λ, $(1_{G_\alpha})^G = (1_{G_1})^G = \theta$, by Lemma 5.14 of [5]. Hence, by (2.8) and Frobenius reciprocity, we have that

$$\begin{aligned}\sum_{\sigma\in G_\alpha} \chi(\sigma) &= |G_\alpha|(\chi, 1_{G_\alpha})_{G_\alpha} \\ &= |G_\alpha|(\chi, (1_{G_\alpha})^G)_G \\ &= |G_\alpha|(\chi, \theta)_G.\end{aligned}$$

Since χ is an irreducible constituent of permutation character θ of G, $(\chi,\theta)_G \neq 0$ and $\sum_{\sigma\in G_\alpha} \chi(\sigma) \neq 0$. Thus $\alpha \in \overline{\Delta}$. So, $H_d^{\alpha,*}(\chi)$ has an o-basis.

By orbit-stabilizer theorem and transitive action of G on Λ, we have that $m = |\Lambda| = \mathrm{Orb}(1) = [G : G_1] = [G : G_\alpha]$. So, $G = \bigcup_{i=1}^m \sigma_i G_\alpha$, where $\{\sigma_1, \sigma_2, ..., \sigma_m\}$ is a system of distinct representatives of left cosets of G_α in G. Let

$$\dim H_d^{\alpha,*}(\chi) = \frac{\chi(1)}{|G_\alpha|}\sum_{\sigma\in G_\alpha}\chi(\sigma) = \chi(1)(\chi,\theta)_G := t.$$

We can assume that $\{X^{\sigma_1\alpha,*}, X^{\sigma_2\alpha,*}, ..., X^{\sigma_t\alpha,*}\}$ is an o-basis for $H_d^{\alpha,*}(\chi)$. Define the $m\times m$ complex matrix $D=[D_{ij}]$ by $D_{ij} := \langle X^{\sigma_i\alpha,*}, X^{\sigma_j\alpha,*}\rangle$. Note that D is idempotent. In fact,

$$\begin{aligned}
(D)^2_{ij} &= \sum_{k=1}^{m} D_{ik}D_{kj}\\
&= \sum_{k=1}^{m}\langle X^{\sigma_i\alpha,*}, X^{\sigma_k\alpha,*}\rangle\langle X^{\sigma_k\alpha,*}, X^{\sigma_j\alpha,*}\rangle\\
&= \sum_{k=1}^{m}\left(\frac{\chi(1)}{|G|}\sum_{\sigma\in\sigma_k G_\alpha\sigma_i^{-1}}\chi(\sigma)\right)\left(\frac{\chi(1)}{|G|}\sum_{\tau\in\sigma_j G_\alpha\sigma_k^{-1}}\chi(\tau)\right)\\
&= \frac{\chi(1)^2}{|G|^2}\sum_{k=1}^{m}\sum_{\sigma\in G_\alpha}\sum_{\tau\in G_\alpha}\chi(\sigma_k\sigma\sigma_i^{-1})\chi(\sigma_j\tau\sigma_k^{-1})\\
&= \frac{\chi(1)^2}{|G|^2}\sum_{k=1}^{m}\sum_{\lambda\in\sigma_k G_\alpha}\sum_{\mu\in G_\alpha\sigma_k^{-1}}\chi(\lambda\sigma_i^{-1})\chi(\sigma_j\mu).
\end{aligned}$$

Now, let $\mu\lambda = \delta\in G_\alpha$. Then $\mu = \delta\lambda^{-1}$ and we have

$$\begin{aligned}
(D)^2_{ij} &= \frac{\chi(1)^2}{|G|^2}\sum_{k=1}^{m}\sum_{\lambda\in\sigma_k G_\alpha}\sum_{\delta\in G_\alpha}\chi(\lambda\sigma_i^{-1})\chi(\sigma_j\delta\lambda^{-1})\\
&= \frac{\chi(1)}{|G|}\sum_{\delta\in G_\alpha}\left(\frac{\chi(1)}{|G|}\sum_{\lambda\in G}\chi(\lambda\sigma_i^{-1})\chi(\sigma_j\delta\lambda^{-1})\right)\\
&= \frac{\chi(1)}{|G|}\sum_{\delta\in G_\alpha}\left(\frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)\chi(\sigma_j\delta\sigma_i^{-1}\sigma^{-1})\right).
\end{aligned}$$

By orthogonal relations of irreducible character, we have

$$(D)^2_{ij} = \frac{\chi(1)}{|G|}\sum_{\delta\in G_\alpha}\chi(\sigma_j\delta\sigma_i^{-1}),$$

which shows that $D^2 = D$.

We note that $m = \theta(1) = \sum_{\chi\in\Theta}\chi(1)(\chi,\theta)$, where Θ is the set of all irreducible constituents of the permutation character θ. Since $m>1$, $|\Theta|>1$ and hence $m>t$. We can now write D in the form

$$\begin{bmatrix} D_1 & D_2\\ D_3 & D_4\end{bmatrix},$$

where D_1, D_2, D_3 and D_4 are matrices of sizes $t \times t, t \times (m-t), (m-t) \times t$ and $(m-t) \times (m-t)$ respectively. On the matrix D_1, we have, by (2.5), that, for $1 \leq i, j \leq t$,

$$(D_1)_{ij} = \langle X^{\sigma_i \alpha, *}, X^{\sigma_j \alpha, *} \rangle = \begin{cases} 0, & \text{if } i \neq j; \\ \frac{\chi(1)}{|G|} \sum_{\sigma \in G_\alpha} \chi(\sigma), & \text{if } i = j \end{cases}$$
$$= \begin{cases} 0, & \text{if } i \neq j; \\ \frac{t}{m}, & \text{if } i = j \end{cases} = \left(\frac{t}{m} I_t \right)_{ij},$$

where I_t is the $t \times t$ identity matrix. So, $D = \begin{bmatrix} \left(\frac{t}{m}\right) I_t & D_2 \\ D_3 & D_4 \end{bmatrix}$. Now, using $D^2 = D$, we get

$$D_2 D_3 = \left(\frac{t}{m} - \frac{t^2}{m^2} \right) I_t.$$

Since $t < m$, $\left(\frac{t}{m} - \frac{t^2}{m^2} \right) \neq 0$ and hence $D_2 D_3$ is invertible. This means that if $H_{\boldsymbol{d}}^{\boldsymbol{\alpha},*}(\chi)$ has an o-basis, then

$$t = \operatorname{rank} D_2 D_3 \leq \min\{\operatorname{rank} D_2, \operatorname{rank} D_3\} \leq \min\{t, m-t\} \leq m-t.$$

Therefore, if $\chi(1)(\chi, \theta)_G = t > \frac{m}{2}$, then $H_d(G; \chi)$ does not have an orthogonal $*$-basis, by (2.7). □

We also obtain a similar results of Holmes in [2].

Corollary 3.2. *(cf. [2, 8]) Let G be a 2-transitive subgroup of S_m, $m > 2$. Let $\chi = \theta - 1_G$, where θ is the permutation character of G. Then $H_d(G; \chi)$ does not have an orthogonal $*$-basis.*

Proof. Note that G has a canonical transitive an faithful action on the set $\Lambda = \{1, 2, ..., m\}$, given by $\sigma \cdot i := \sigma(i)$ for each $\sigma \in G \leq S_m$ and $i \in \Lambda$. Since G acts 2-transitively on Λ with permutation character θ, by Corollary 5.17 in [5], $\chi = \theta - 1_G$ is an irreducible constituent of θ. We compute that

$$\chi(1)(\chi, \theta)_G = \chi(1)(\theta - 1_G, \theta)_G = \chi(1)[(\theta, \theta)_G - (\theta, 1_G)_G] = \chi(1)[2-1] = m-1.$$

Since $m > 2$, $m - 1 > \frac{m}{2}$ and hence $\chi(1)(\chi, \theta)_G > \frac{m}{2}$. Thus, by Theorem 3.1, the result follows. □

Example 3.1. *(cf. [8]) Let $G = \Lambda = A_4$ be the alternating group of degree 4. We know that G acts transitively and faithfully on Λ by left multiplication. Then we can view G as a subgroup of S_{12}. Note that G has an irreducible character, χ, of degree 3 and the permutation character θ of G is regular. Thus χ is an irreducible constituent of θ of multiplicity 3. Hence $\chi(1)(\chi, \theta)_G = 9 > \frac{12}{2} = \frac{|\Lambda|}{2}$ and then $H_d(A_4; \chi)$ does not* *have an orthogonal $*$-basis, by Theorem 3.1. In this example, however, the action G on Λ is not 2-transitive.* By using the same technique as in the proof of Theorem 3.1, we also obtain an analogous criterion of Shahryari in [10].

THEOREM 3.3. *Let G be a permutation group of degree m and χ be a non-linear irreducible character of G. If there is $\alpha \in \Gamma^+_{m,d}$ such that*

$$\frac{\sqrt{2}}{2} < \| X^{\alpha,*}_{\chi} \| < 1,$$

then $H_d(G;\chi)$ does not have an orthogonal $*$-basis.

Proof. Let $\alpha \in \Gamma^+_{m,d}$. Suppose the orbit of α under the action of G is $\mathrm{Orb}(\alpha) = \{\sigma_1\alpha, \sigma_2\alpha, ..., \sigma_r\alpha\}$. Then, by orbit-stabilizer theorem, $r = [G : G_\alpha]$ and $G = \bigcup_{i=1}^r \sigma_i G_\alpha$ is a partition. Now, we construct $r \times r$ matrix $D = [D_{ij}]$ by $D_{ij} := \langle X^{\sigma_i\alpha,*}, X^{\sigma_j\alpha,*}\rangle$ which is idempotent as before. Next, suppose χ is a non-linear irreducible character of G and $\alpha \in \overline{\Delta}$ and assume also that $\{X^{\sigma_1\alpha,*}, X^{\sigma_2\alpha,*}, ..., X^{\sigma_t\alpha,*}\}$ is an o-basis for $H^{\alpha,*}_d(\chi)$ in which $t < r$, where $t = \dim H^{\alpha,*}_d(\chi)$. So, the matrix D has the block partition form

$$D = \begin{bmatrix} \left(\frac{t}{r}\right) I_t & D_2 \\ D_3 & D_4 \end{bmatrix},$$

where D_2, D_3 and D_4 are matrices of sizes $t \times (r-t), (r-t) \times t$ and $(r-t) \times (r-t)$ respectively. By the same arguments as in the proof of Theorem 3.1, we reach to the conclusion that $t \leq r - t$ or $t \leq \frac{r}{2}$. Thus if $t < r$ and $t > \frac{r}{2}$, then $H^{\alpha,*}_d(\chi)$ does not have o-basis. Substituting $r = [G : G_\alpha]$, $t = \chi(1)(\chi, 1)_{G_\alpha}$ in the inequality $\frac{r}{2} < t < r$ and using (2.6) and (2.7), the result follows. □

4. Symmetric groups. It is well known that there is a standard one-to-one correspondence between the complex irreducible characters of the symmetric group S_m and the partitions of m. Here, a partition π of m of length t, denoted by $\pi \vdash m$, means an unordered collection of t positive integers that sum to m. In this article, we use the same symbol to denote an irreducible character of S_m and the partition of m corresponding to it. Typically, we represent the partition by a sequence $\pi = [\pi_1, \pi_2, ..., \pi_t]$ in which $\pi_1 \geq \pi_2 \geq ... \geq \pi_t > 0$. A partition $\pi = [\pi_1, \pi_2, ..., \pi_t]$ is usually represented by a collection of m boxes arranged in t rows such that the number of boxes of row i is equal to π_i, for $i = 1, 2, ..., t$. This collection is called the *Young diagram* associated with π and denoted by $[\pi]$. The *Young subgroup* corresponding to $\pi \vdash m$ is the internal direct product

$$S_\pi = S_{\pi_1} \times S_{\pi_2} \times \cdots \times S_{\pi_t}.$$

We write $1_{S_\pi} = 1_\pi$ for the principle character of S_π. Note that $1_\pi^{S_m}$ is a character of S_m, so there must exist integers $K_{\mu,\pi}$ such that

$$1_\pi^{S_m} = \sum_{\mu \vdash m} K_{\mu,\pi}\mu.$$

The numbers $K_{\mu,\pi} = \left(1_\pi^{S_m}, \mu\right)_{S_m}$ are called *Kostka coefficients.* By Corollary 4.54 in [7], the Kostka coefficient $K_{\pi,\pi} = 1$ for all $\pi \vdash m$.

For each ordered pair (i, j), $1 \le i \le t$, $1 \le j \le \pi_i$, there is corresponding a box, B_{ij}, in Young diagram $[\pi]$. Each B_{ij} determines a unique *hook* in $[\pi]$ consisting of B_{ij} itself, all the boxes in row i of $[\pi]$ to the right of B_{ij} and all boxes in column j of $[\pi]$ below B_{ij}. The *hook length,*

$$h_{ij} := (\pi_i - i) + (\pi'_j - j) + 1,$$

where $\pi'_j := |\{k \in \{1, 2, ..., t\} | \pi_k \ge j\}|$ (a j part of *conjugate partition* of π), is the number of boxes in the hook determined by B_{ij}. By the Frame-Robinson-Thrall Hook Length Formula (see, e.g., Theorem 4.60 in [7]), if π is a partition of m, then the degree of the irreducible character of S_m corresponding to $\pi = [\pi_1, \pi_2, ..., \pi_t]$ is

$$\pi(1) = \frac{m!}{\prod_{i=1}^{t} \prod_{j=1}^{\pi_i} h_{ij}}. \qquad (4.1)$$

As a consequence of Theorem 3.3, we have an analogous result of Y. Zamani in [12].

THEOREM 4.1. *Let π be an irreducible character of S_m of the cycle type;*

I $\pi = [m-l, l]$, $d \equiv 0 \mod l$, $d \ne 0$ *such* $m \ge 3l$, *or*
II $\pi = [m-l, l-1, 1]$, $d \equiv r \mod l, 0 < r < l, l > 2$, $d \ne r$ *such* $m > 3l + \frac{4}{l-2}$.

Then $H_d(S_m; \pi)$ does not have an orthogonal $*$-basis.

Proof. For the form I, we set $\alpha = (\underbrace{0, 0, ..., 0}_{m-l}, \underbrace{k, k, ..., k}_{l})$, where $k = \frac{d}{l}$. Then $\alpha \in \Gamma^+_{m,d}$. Under the action of S_m on $\Gamma^+_{m,d}$, we choose a system Δ of representatives such that $\alpha \in \Delta$. Since $d \ne 0$, $k \ne 0$ and

$$(S_m)_\alpha \cong S_{m-l} \times S_l = S_\pi,$$

where $(S_m)_\alpha$ is the stabilizer subgroup of α and S_π is the Young subgroup corresponding to $\pi \vdash m$. Hence, by Frobenius Reciprocity Theorem,

$$\begin{aligned}\frac{1}{|(S_m)_\alpha|}\sum_{\sigma\in(S_m)_\alpha}\pi(\sigma) &= \left(\pi, 1_{(S_m)_\alpha}\right)_{(S_m)_\alpha}\\ &= (\pi, 1_\pi)_{S_\pi}\\ &= \left(\pi, 1_\pi^{S_m}\right)_{S_m}\\ &= \left(\pi, \sum_{\mu\vdash m}K_{\mu,\pi}\mu\right)_{S_m}\\ &= \sum_{\mu\vdash m}K_{\mu,\pi}(\pi,\mu)_{S_m}\\ &= K_{\mu,\mu} = 1 \neq 0\end{aligned}$$

This yields $\alpha \in \overline{\Delta}$ and, moreover, by (2.8), that

$$\dim H_d^{\alpha,*}(\pi) = \frac{\pi(1)}{|(S_m)_\alpha|}\sum_{\sigma\in(S_m)_\alpha}\chi(\sigma) = \pi(1).$$

Now, we compute the product of the hook lengths of $[\pi]$ which we get

$$\begin{aligned}\prod_{i=1}^{2}\prod_{j=1}^{\pi_i}h_{ij} &= (m-l+1)(m-l)\cdots(m-2l+2)(m-2l)\cdots(2)(1)l(l-1)\cdots(2)(1)\\ &= \frac{(m-l+1)!l!}{(m-2l+1)}\end{aligned}$$

Hence, by (4.1),

$$\dim H_d^{\alpha,*}(\pi) = \pi(1) = \frac{(m-2l+1)m!}{(m-l+1)!l!}.$$

Now, using (2.6), we have

$$\| X^{\alpha,*} \|^2 = \frac{\dim H_d^{\alpha,*}(\pi)}{[S_m : (S_m)_\alpha]} = \frac{m-2l+1}{m-l+1}.$$

Hence, $\frac{1}{2} < \| X^{\alpha,*} \|^2 < 1$ if and only if $m \geq 3l$. Thus the result for the first form follows from Theorem 3.3.

For the form II, $\pi = [m-l, l-1, 1]$, we set $\alpha = (\underbrace{0,0,...,0}_{m-l},\underbrace{k,k,...,k}_{l-1},k+r)$, where $k = \frac{d-r}{l}$. Then $\alpha \in \Gamma_{m,d}^+$. Under the action of S_m on $\Gamma_{m,d}^+$, we choose a system Δ of representatives such that $\alpha \in \Delta$. Since $d \neq r \neq 0$, $k \neq 0$ and $k+r \neq k$ and hence

$$(S_m)_\alpha \cong S_{m-l} \times S_{l-1} \times S_1 = S_\pi.$$

By the same arguments as the first form, we conclude that $\dim H_d^{\alpha,*}(\pi) = \pi(1)$. For the products of the hook lengths, we compute that

$$\begin{aligned}\prod_{i=1}^{3}\prod_{j=1}^{\pi_i} h_{ij} &= (m-l+2)(m-l)\cdots(m-2l+3)(m-2l+1)\cdots(1)l(l-2)(l-3)\cdots(1)\\ &= \frac{(m-l+2)!l!}{(m-l+1)(m-2l+2)(l-1)}.\end{aligned}$$

Then

$$\dim H_d^{\alpha,*}(\pi) = \pi(1) = \frac{(m-l+1)(m-2l+2)(l-1)m!}{(m-l+2)!l!}.$$

Now, using (2.6) again, we have

$$\| X^{\alpha,*} \|^2 = \frac{\dim H_d^{\alpha,*}(\pi)}{[S_m : (S_m)_\alpha]} = \frac{(m-2l+2)(l-1)}{(m-l+2)(l)}.$$

It is now easy to show that $\frac{1}{2} < \| X^{\alpha,*} \|^2 < 1$ if and only if $m > 3l + \frac{4}{l-2}$, because $l > 2$. The result for the second form follows from Theorem 3.3. □

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