

อภิธานการ



รายงานวิจัยฉบับสมบูรณ์

โครงการ : เงื่อนไขค่าเหมาะที่สุดสำหรับผลเฉลยคงทนของปัญหา
ค่าเหมาะที่สุดคงทน

On Optimality Conditions for robust sharp solutions
of Robust Optimization Problems

คณะผู้วิจัย

สังกัด

รองศาสตราจารย์ ดร.รัตนพร วังศิริ คณะวิทยาศาสตร์

สำนักหอสมุด มหาวิทยาลัยนเรศวร
วันลงทะเบียน 14 มิ.ย. 2565
เลขทะเบียน 1052802
เลขเรียกหนังสือ 2 09
467

5
13145
2563

สนับสนุนโดย

งบประมาณรายได้มหาวิทยาลัยนเรศวร

ปีงบประมาณ 2563

กิตติกรรมประกาศ(Acknowledgement)

รายงานการวิจัยฉบับนี้สำเร็จลุล่วงได้ ข้าพเจ้าขอขอบพระคุณทางมหาวิทยาลัย
นเรศวรที่ให้ทุนอุดหนุนการวิจัยจากงบประมาณรายได้ กองทุนวิจัยมหาวิทยาลัยนเรศวร
ประจำปีงบประมาณ พ.ศ. 2563 เป็นจำนวนเงินทั้งสิ้น 220,000 บาท

รัตนาพร วงศ์ศรี



ชื่อโครงการ: เงื่อนไขค่าเหมาะที่สุดสำหรับผลเฉลยคงทนของปัญหาค่าเหมาะที่สุดคงทน
On Optimality Conditions for robust sharp solutions of Robust Optimization Problems

ชื่อผู้วิจัย รศ. ดร. รัตนาพร วงศ์ศิริ
หน่วยงานที่สังกัด ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยนเรศวร
ได้รับทุนอุดหนุนการวิจัยจาก งบประมาณรายได้มหาวิทยาลัยนเรศวร

จำนวนเงิน 220,000 บาท
ระยะเวลาการทำวิจัย 1 ปี

บทคัดย่อ(ภาษาไทย)

ในงานวิจัยนี้ ผู้วิจัยได้ศึกษาปัญหาค่าเหมาะสมคงทนซึ่งมีฟังก์ชันจุดประสงค์เป็นฟังก์ชันที่หาอนุพันธ์ต่อเนื่อง จากนั้นเราได้ศึกษาเงื่อนไข KKT แบบใหม่สำหรับที่จะทำให้จุด KKT เป็นจุดค่าต่ำสุดเหมาะสมคงทนวงกว้างของปัญหาค่าเหมาะสมไม่แน่นอนซึ่งอาจมีผลเฉลยค่าเหมาะสมคงทนเฉพาะที่หลายๆ ค่า โดยเงื่อนไขที่ได้นั้นได้ใช้ประโยชน์จาก underestimators ซึ่งถูกคิดค้นครั้งแรกโดย Jayakumar and Srisatkunarajah [1,2] จากนั้นผู้วิจัยได้ศึกษา the Wolfe type robust duality ของปัญหาค่าเหมาะสมไม่แน่นอนแบบปรับเรียบพร้อมทั้งปัญหาคู่ควบไม่แน่นอนโดยการให้เงื่อนไขที่พอเพียงสำหรับ weak duality และ strong duality ระหว่าง deterministic robust counterpart ของ primal model และ the optimistic counterpart ของ dual problem โดยผลลัพธ์ที่ได้สำหรับทฤษฎีคู่ควบคงทนจะอยู่ในรูปของ underestimators พร้อมทั้งในงานวิจัยนี้เราได้ยกตัวอย่างเพื่อสนับสนุนผลลัพธ์ที่ได้ทั้งหมด

บทคัดย่อ(ภาษาอังกฤษ)

In this research, a robust optimization problem, which features a maximum function of continuously differentiable functions as its objective function, is investigated. Some new conditions for a robust KKT point, which is a robust feasible solution that satisfies the robust KKT condition, to be a global robust optimal solution of the uncertain optimization problem, which may have many local robust optimal solutions that are not global, are established. The obtained conditions make use of underestimators, which were first introduced by Jayakumar and Srisatkunarajah [1,2] of the Lagrangian associated with the problem at the robust KKT point. Furthermore, we also investigate the Wolfe type robust duality between the smooth uncertain optimization problem and its uncertain dual problem by proving the sufficient conditions for a weak duality and a strong duality between the deterministic robust counterpart of the primal model and the optimistic counterpart of its dual problem. The results on robust duality theorems are established in terms of underestimators. Additionally, to illustrate or support this study, some examples are presented.

LIST OF CONTENT

Chapter	Page
1 INTRODUCTION.....	1
2 NOTATIONS AND PRELIMINARIES.....	4
3 MAIN RESULTS.....	9
3.1 The robust KKT sufficiency for global robust optimal solutions via underestimators.....	9
3.2 Duality.....	18
4 CONCLUSION.....	23
REFERENCES.....	24
APPENDIX.....	27

CHAPTER 1

INTRODUCTION

It has been acknowledged that the data of objective function or constraints of the numerous practical programs, including real-world optimization problems, are usually not known exactly beforehand and infrequently uncertain owing to errors (from estimation, prediction or measurement) and asymmetric information [3, 4]. Solving the optimization problems with ignorance of the uncertainty may result in solutions which are suboptimal for even infeasible, so it is essential to take care of uncertain optimization problems, which have data uncertainty within the objective function and/or the constraint; see, e.g., [4, 5, 6, 7, 8, 9, 10, 11]. In line with this fact, it is imperative to review the optimization problems with data uncertainty [3, 8, 12, 13, 14]. A robust optimization, which is its robust counterpart of an uncertain optimization problem [3], has emerged as a useful deterministic approach in [5, 10] to study optimization problems with data uncertainty. The concept of the robust counterpart is minimizing the value of the objective function in the worst case of all scenarios and getting a solution that works well even in the worst-case scenario but is also immunized against the data uncertainty [11]. Furthermore, constraints are enforced for every possible value of the parameters within their prescribed uncertainty set.

On the other hand, seeking a global solution is of fundamental importance in mathematical programming, where problems have many, often infinitely many, possible solutions. Nevertheless, it is known that locating a global solution of a multi-extremal problem, whose several local solutions are not global, is intrinsically tough [15, 16, 17, 18]. Accordingly, improving the criteria for identifying global solutions of optimization problems is of noticeable interest. Over the years, much attention has been targeted on developing criteria for identifying global solutions of

nonconvex optimization problems [19, 20, 21]. The KKT condition, which is, under a certain qualification, a necessary condition for a feasible point to be a locally optimal solution of a mathematical programming problem, is often used to creating criteria for identifying global solutions. For instance, a feasible solution that fulfills the KKT conditions, referred to likewise as the KKT point, is a global minimizer for a convex programming problem. Different criteria for identifying global minimizers of nonconvex programming problems regarding the KKT conditions have been given in the literature. Nonetheless, a significant part of work in this area the KKT sufficiency frequently necessitates that the Lagrangian associated with the problem fulfills certain generalized convexity conditions [22, 23, 24, 25, 26, 27].

In [1], the authors employed underestimators of the Lagrangian functions for developing criteria for a KKT point to be a global minimizer of a mathematical programming problem as well as presented also sufficient conditions for weak and strong duality results in terms of underestimators. Furthermore, in [2], the authors presented geometric criteria, which is established in terms of underestimators of the Lagrangian of a mathematical programming problem, for a KKT point to be a global minimizer of such problem with or without bounds on the variables. The idea of dealing with a problem by requiring its Lagrangian to admit underestimator with certain generalized convexity conditions, which was given [1, 2], is very effective and interesting. It leads us to interesting : “How does one obtain sufficient optimality conditions for a robust KKT point to be a global robust optimal solution of a smooth nonconvex optimization problem with data uncertainty in terms of underestimators?” and also, How does one derive robust duality results between the uncertain optimization problem and its dual uncertain optimization problem?”. These questions motivate us to handle with a smooth nonconvex uncertain optimization problem that features with data uncertainty in both objective and constraints for obtaining global optimality conditions and duality theorems for robust optimal solutions.

In this research, we employ underestimators of the Lagrangian function for developing criteria for a robust KKT point to be a global robust optimal solution of a smooth nonconvex optimization problem with data uncertainty in both objective function and constraints. We establish sufficient optimality conditions by requiring an underestimator, rather the Lagrangian, to satisfy a geometric condition. Here, we present that a robust KKT point is a global robust optimal solution if the Lagrangian associated with the uncertain problem admits an underestimator that fulfills certain conditions (the convexity, or the pseudo-convexity, or the property that every stationary point is a global minimizer). In special, by using the fact that the biconjugate function of the Lagrangian is a convex underestimator at a point whenever it coincides with the Lagrangian at that point, we derive the sufficient optimality conditions as well. We also obtain results on sufficient conditions for a robust weak duality and a robust strong duality between the smooth uncertain optimization problem and its conventional Wolf type dual optimization problem, by proving the sufficient conditions for a weak duality and a strong duality between the deterministic robust counterpart of the primal model and the optimistic counterpart of its conventional Wolf type dual model, in terms of underestimators of the Lagrangian function.

The research is organized as follows. In Chapter 2, we recall some notions and give some preliminary results. In section 3.1, several sufficient optimality conditions for a robust KKT conditions to be a global robust optimal solution in terms of underestimators are presented, and also, some examples for illustrating the applications of our results to nonconvex optimization problem with data uncertainty are given. In section 3.2, we present some results on sufficient conditions for a robust weak duality and a robust strong duality between the smooth uncertain optimization problem and its uncertain dual problem. Finally, in Chapter 4, we conclude our study.

CHAPTER 2

NOTATIONS AND PRELIMINARIES

Let us first recall some notation and preliminary results which is able to used throughout this paper. First of all, let $\mathbb{R}^n, n \in \mathbb{N}$, be the n -dimensional Euclidean space and for $x, y \in \mathbb{R}^n$, the notation $x^T y$ stands for the inner product of x and y .

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a real-valued function, it is said to be continuously differentiable if the derivative $f'(x)$, where $x \in \mathbb{R}^n$, exists and is itself a continuous function. Although the derivative of a differentiable function never features a jump discontinuity, it is possible for the derivative to possess an essential discontinuity. Besides, a point $x \in \mathbb{R}^n$ is called a stationary point of the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ if f is differentiable at x and $\nabla f(x) = 0$, where ∇f denotes the usual gradient of f . Simultaneously, the function f is said to be a convex function if for any $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

For a convex function, its local minimizer is also a global minimizer, and if the convex function is differentiable, then its stationary point is its global minimizer as well. Furthermore, let C be a given subset of \mathbb{R}^n and the function f be (Gâteaux) differentiable at a point $x \in C$, then f is said to be pseudoconvex at x if

$$\nabla f(x)^T(y - x) \geq 0, y \in C \Rightarrow f(y) \geq f(x).$$

The term pseudoconvex is employed to explain the very fact that such functions share many properties of convex functions, particularly with regards to derivative properties and finding local extrema. Note, however, that pseudoconvexity is strictly weaker than convexity as every convex function is pseudoconvex though one easily checks that $f(x) = x + x^3$ is pseudoconvex and non-convex. The Legendre-

Fenchel conjugate function (for short, conjugate function) of $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is $f^* : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$f^*(x^*) := \sup_{x \in \mathbb{R}^n} \{\langle x^*, x \rangle - f(x)\}$$

for all $x \in \mathbb{R}^n$. The function f^* is lower semicontinuous convex irrespective of the nature of f but for f^* to be proper, we need f to be a proper convex function. By reiterating the operation $f \rightarrow f^{**}$ on f , we get the biconjugate of f , defined for all $x \in \mathbb{R}^n$ by

$$f^{**}(x) := \sup_{x^* \in \mathbb{R}^n} \{\langle x, x^* \rangle - f(x^*)\}.$$

Clearly, if f is identically equal to $+\infty$, then f^{**} is identically equal to $-\infty$. Note that $f^{**} : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ is a convex function and it coincides with f when f is convex.

Next, we recall the concepts of underestimators, introduced in [1, 2] for developing conditions for a KKT point to be a global minimizer of a standard mathematical programming problem. First of all, please consider the following mathematical programming problem:

$$\begin{aligned} & \text{Minimize} && f(x) \\ & \text{subject to} && x \in \Omega, g_i(x) \leq 0, i = 1, \dots, m, \end{aligned} \tag{P}$$

where Ω is a closed and convex subset of \mathbb{R}^n and $f, g_i : \mathbb{R}^n \rightarrow \mathbb{R}, i = 1, \dots, m$, are continuously differentiable functions. Let $A := \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, \dots, m\}$ and let the feasible set $K := \Omega \cap A$.

The following concepts of underestimators for developing conditions for a KKT point to be a global minimizer of (P) can be found in [1, 2].

Definition 2.1. A function $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be

- (i) an underestimator of the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at \bar{x} over the feasible set K if, for each $x \in K$, $f(x) \geq \rho(x)$, and $f(\bar{x}) = \rho(\bar{x})$;

- (ii) a minimizing underestimator of the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at \bar{x} over the feasible set K if it is an underestimator of f at \bar{x} over K and it attains its minimum over K at \bar{x} ;
- (iii) a smooth underestimator of the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at \bar{x} over the feasible set K if it is an underestimator of f at \bar{x} over K that is differentiable at \bar{x} and $\nabla f(\bar{x}) = \nabla \rho(\bar{x})$.

It is easy to check from the definition of biconjugate function of f that, for each $x \in \mathbb{R}^n$, $f(x) \geq f^{**}(x)$. So, if $f(\bar{x}) = f^{**}(\bar{x})$, then we get from Definition 2.1 that f^{**} is a convex underestimator of f at \bar{x} over \mathbb{R}^n .

A feasible point $\bar{x} \in K$ is said to be a local minimizer of (P) if there is a neighborhood U of \bar{x} such that for each $x \in K \cap U$, $f(x) \geq f(\bar{x})$. Specially, if $U = \mathbb{R}^n$, \bar{x} is called a global minimizer of (P). If a feasible point \bar{x} is a local minimizer of (P) and if a certain constraint qualification holds then the following KKT conditions hold at \bar{x} with multiplier $\lambda \in \mathbb{R}_+^m$:

$$\sum_{i=1}^m \lambda_i g_i(\bar{x}) = 0 \quad \text{and} \quad \nabla L(\bar{x}, \lambda)^T (x - \bar{x}) \geq 0, \quad \forall x \in \Omega.$$

Here $L(\bar{x}, \lambda) := f(\bar{x}) + \sum_{i=1}^m \lambda_i g_i(\bar{x})$ is the Lagrangian associated with (P) and \mathbb{R}_+^m is the set of all nonnegative vectors of \mathbb{R}^m .

To conclude this section, we recall some important conceptions which will be used in the sequel. Consider the following parameterized program, which is an analog of the deterministic program (P) if the objective, as well as the constraints, are uncertain:

$$\begin{aligned} & \text{Minimize} && f(x, u) \\ & \text{subject to} && x \in \Omega, \quad g_i(x, v_i) \leq 0, i = 1, \dots, m. \end{aligned} \tag{UP}$$

Here u and v_i are uncertain parameters belonging to compact convex uncertainty sets $\mathcal{U} \subseteq \mathbb{R}^p$, and $\mathcal{V}_i \subseteq \mathbb{R}^q, i = 1, \dots, m$, respectively. $f : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$ and

$g_i : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}$ are continuously differentiable function on an open subsets of $\mathbb{R}^n \times \mathbb{R}^p$, and $\mathbb{R}^n \times \mathbb{R}^q$, respectively. By enforcing the constraints for all possible uncertainty within $\mathcal{V}_i, i = 1, \dots, m$, the problem (UP) becomes an uncertain semi-infinite program, which is the robust (worst-case) counterpart of (UP):

$$\begin{aligned} & \text{Minimize} && \max_{u \in \mathcal{U}} f(x, u) \\ & \text{subject to} && x \in \Omega, g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \dots, m. \end{aligned} \quad (\text{RP})$$

In section 3.1 and section 3.2, we investigate the smooth uncertain programming problem (UP) by using examine (RP). The following definition is of the set, is termed as the robust feasible set of (UP).

Definition 2.2. The set of all robust feasible solutions of (UP), equivalently the set of all feasible solution of (RP), is called the robust feasible set of (UP) and is defined by

$$\mathcal{K} := \{x \in \Omega : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \dots, m\}.$$

To avoid triviality in (UP), we always assume that $\mathcal{K} \neq \emptyset$.

The below notion commonly referred to as robust optimal solution or robust minimax solution, can be found in [3]. This concept has been studied extensively by many authors, see, e.g., [8, 12].

Definition 2.3. A robust feasible solution $\bar{x} \in \mathcal{K}$ is said to be a local robust optimal solution for (UP) if it is a global optimal solution of (RP), i.e., there exists a neighborhood U of \bar{x} such that for each $x \in \mathcal{K} \cap U$,

$$\max_{u \in \mathcal{U}} f(x, u) \geq \max_{u \in \mathcal{U}} f(\bar{x}, u).$$

Specially, if $U = \mathbb{R}^n$, then $\bar{x} \in \mathcal{K}$ is said to be a global robust optimal solution for (UP), equivalently, a global optimal solution of (RP).

The Lagrangian associated with (UP), denoted by $L(x, u, v, \lambda)$, is given as follows:

$$L(x, u, v, \lambda) = f(x, u) + \sum_{i=1}^m \lambda_i g_i(x, v_i),$$

where $u \in \mathcal{U} \subseteq \mathbb{R}^p$, $v := (v_1, \dots, v_m)$, $v_i \in \mathcal{V}_i \subseteq \mathbb{R}^q$, and $\lambda := (\lambda_1, \dots, \lambda_m)$, $\lambda_i \in \mathbb{R}$, $i = 1, \dots, m$. If \bar{x} is a local robust optimal solution of (UP), i.e., a local optimal solution of (RP), and if a certain constraint qualification holds at that point, then the following robust KKT conditions hold: $\exists \bar{u} \in \mathcal{U}, \exists \bar{v} \in \mathcal{V}, \exists \bar{\lambda} \in \mathbb{R}_m^+$ such that

$$f(\bar{x}, \bar{u}) = \max_{u \in \mathcal{U}} f(\bar{x}, u), \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0, \text{ and } \nabla L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda})^T (x - \bar{x}) \geq 0, \forall x \in \Omega \quad (2.1)$$

where $\mathcal{V} := \mathcal{V}_1 \times \dots \times \mathcal{V}_m$, $\mathcal{V}_i \subseteq \mathbb{R}^q$, $i = 1, \dots, m$. Whenever the robust KKT conditions hold at a robust feasible solution $\bar{x} \in \mathcal{K}$, the point \bar{x} is said to be a robust KKT point.

CHAPTER 3

MAIN RESULTS

3.1 The robust KKT sufficiency for global robust optimal solutions via underestimators

In this section, we establish that a robust KKT point is a global robust optimal solution of (UP) if the Lagrangian associated with (UP) admits an underestimator, which is convex or, more generally, has the property that every stationary point is its global minimizer. In addition, by using the fact that the biconjugate function of the Lagrangian is a convex underestimator at a point whenever it coincides with the Lagrangian at that point, we establish sufficient optimality conditions for the point to be a global robust optimal solution of (UP).

The following lemmas are used for obtaining our later results.

Lemma 3.1. Let $\bar{x} \in \mathcal{K}$ and the robust KKT conditions hold at \bar{x} with the uncertain parameters \bar{u}, \bar{v} and the multiplier $\bar{\lambda}$. Suppose that the Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with (UP) admits a minimizing underestimator at \bar{x} over the robust feasible set \mathcal{K} . Then, \bar{x} is a global robust optimal solution of (UP).

Proof. Let ρ be a minimizing underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} over \mathcal{K} and let $x \in \mathcal{K}$ be arbitrary. Then, for the uncertain parameters \bar{u}, \bar{v} and the multiplier $\bar{\lambda}$,

we have $\sum_{i=1} \bar{\lambda}_i g_i(x, \bar{v}_i) \leq 0$, and

$$\begin{aligned}
\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\bar{x}, u) &= \max_{u \in \mathcal{U}} f(x, u) - f(\bar{x}, \bar{u}) \\
&\geq f(x, \bar{u}) - f(\bar{x}, \bar{u}) \\
&\geq f(x, \bar{u}) - f(\bar{x}, \bar{u}) + \sum_{i=1}^m \lambda_i g_i(x, \bar{v}_i) \\
&= f(x, \bar{u}) + \sum_{i=1}^m \lambda_i g_i(x, \bar{v}_i) - \left[f(\bar{x}, \bar{u}) + \sum_{i=1}^m \lambda_i g_i(\bar{x}, \bar{v}_i) \right] \\
&\geq L(x, \bar{u}, \bar{v}, \bar{\lambda}) - L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda}) \\
&\geq \rho(x) - \rho(\bar{x}) \\
&\geq 0.
\end{aligned}$$

Since x was arbitrary, this inequality yields $\max_{u \in \mathcal{U}} f(x, u) \geq \max_{u \in \mathcal{U}} f(\bar{x}, u)$ for all $x \in \mathcal{K}$. Thus, \bar{x} is a global optimal solution of (RP), and so it is a global robust optimal solution of (UP) as desired. \square

The following theorem shows that every smooth underestimator at \bar{x} over Ω , which is pseudo-convex, is a minimizing underestimator at that point over Ω .

Theorem 3.2. Let $\bar{x} \in \mathcal{K}$ and the robust KKT conditions hold at \bar{x} with the uncertain parameters \bar{u}, \bar{v} and the multiplier $\bar{\lambda}$. Suppose that $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth underestimator of Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with (UP) at \bar{x} over Ω and ρ is pseudo-convex at \bar{x} over Ω . Then, the function ρ is a minimizing underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} over Ω as well as \bar{x} is a global robust optimal solution of (UP).

Proof. By the definition of the smooth underestimator of a function at a point, we have $\nabla \rho(\bar{x}) = \nabla L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda})$. Hence, from the robust KKT conditions, we have

$$\nabla \rho(\bar{x})^T (x - \bar{x}) = \nabla L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda})^T (x - \bar{x}) \geq 0, \quad \forall x \in \Omega.$$

It then follows from the pseudo-convexity of ρ that $\rho(x) \geq \rho(\bar{x})$, for all $x \in \Omega$. This means ρ is a minimizing underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with (UP) at \bar{x}

over $\mathcal{K} \subseteq \Omega$. Then, by Lemma 3.1, the point \bar{x} is a global robust optimal solution of (UP). \square

Remark 3.3. In the special case that there is no uncertainty in the objective function and constraint function, i.e., \mathcal{U} and \mathcal{V} are singletons, it can be seen easily that

1. the problem (UP) becomes $\min\{f(x) : g_i(x) \leq 0, i = 1, \dots, m, x \in \Omega\}$, which was studied in [2];
2. Lemma 3.1 provides the same results presented in [2, Lemma 2.1];
3. Theorem 3.2 provides the same results presented in [2, Theorem 2.1].

Lemma 3.4. If ρ is a convex underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with (UP) at \bar{x} over Ω and ρ is differentiable at \bar{x} , then ρ is a pseudo-convex underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} over Ω .

Proof. Since ρ is convex over Ω and is differentiable at \bar{x} , we have from the gradient inequality that

$$\rho(x) \geq \rho(\bar{x}) + \nabla \rho(\bar{x})^T (x - \bar{x}), \quad \forall x \in \Omega.$$

Hence, for each $x \in \Omega$, if $\rho(\bar{x})^T (x - \bar{x}) \geq 0$ then $\rho(x) \geq \rho(\bar{x})$. This means ρ is pseudo-convex at \bar{x} over Ω , and so the conclusion holds. \square

The following result follows from Theorem 3.2 and Lemma 3.4.

Corollary 3.5. Let $\bar{x} \in \mathcal{K}$ and the robust KKT conditions hold at \bar{x} with the uncertain parameters \bar{u}, \bar{v} and the multiplier $\bar{\lambda}$. Suppose that $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth convex underestimator of Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with (UP) at \bar{x} over Ω . Then, the function ρ is a minimizing underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} over Ω as well as \bar{x} is a global robust optimal solution of (UP).

Proof. Since ρ is a smooth underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} over Ω , it is differentiable at \bar{x} . Then, from Lemma 3.4, ρ is pseudo-convex at \bar{x} over Ω . Hence, it follows from Theorem 3.2 that \bar{x} is a global robust optimal solution of (UP). \square

The following results (Theorem 3.6, Corollary 3.9, Corollary 3.12 and Corollary 3.14) are applicable for the problem (UP) with $\Omega = \mathbb{R}^n$. The sufficient conditions for a robust KKT point to be a global robust optimal solution of (UP) are presented in the following theorem.

Theorem 3.6. For the uncertain optimization problem (UP), let $\Omega = \mathbb{R}^n$. Let $\bar{x} \in \mathcal{K}$ and the robust KKT conditions hold at \bar{x} with the uncertain parameters \bar{u}, \bar{v} and the multiplier $\bar{\lambda}$. Suppose that the Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with (UP) admits an underestimator at \bar{x} and this underestimator is differentiable at \bar{x} . If every stationary point of the underestimator is its global minimizer, then \bar{x} is a global robust optimal solution of (UP).

Proof. Let ρ be an underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} , and let $d \in \mathbb{R}^n$ be arbitrary. Then, for each $x \in \mathbb{R}^n$, $\rho(x) \leq L(x, \bar{u}, \bar{v}, \bar{\lambda})$. So, for any $\mu > 0$, we have

$$\frac{\rho(\bar{x} + \mu d) - \rho(\bar{x})}{\mu} \leq \frac{L(\bar{x} + \mu d, \cdot, \bar{u}, \bar{v}, \bar{\lambda}) - L(\bar{x}, \cdot, \bar{u}, \bar{v}, \bar{\lambda})}{\mu}.$$

Taking $\mu \rightarrow 0$, we obtain $\nabla \rho(\bar{x})^T d \leq \nabla L(\bar{x}, \cdot, \bar{u}, \bar{v}, \bar{\lambda})^T d = 0$. Hence, $\nabla \rho(\bar{x}) \leq 0$ for each $d \in \mathbb{R}^n$. Therefore, $\nabla \rho(\bar{x}) = 0$, and so the point \bar{x} is a stationary point of ρ . By the assumption, it then follows that, $\rho(x) \geq \rho(\bar{x})$ for all $x \in \mathbb{R}^n$ and hence for all $x \in \mathcal{K} \subseteq \mathbb{R}^n$, $\rho(x) \geq \rho(\bar{x})$ as well. Clearly, ρ is an underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} over \mathcal{K} and it attains its minimum over the set \mathcal{K} at \bar{x} . That is to say, ρ is a minimizing underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} over \mathbb{R}^n and so, by Lemma 3.1, \bar{x} is a global robust optimal solution of (UP). \square

Let us give some examples of multi-extremal nonconvex optimization problems that have some local robust optimal solutions such that the robust KKT

conditions hold at each of the solutions. In each example, the Lagrangian admits a nonconvex underestimator at only one local robust optimal solution with the property that every local minimizer of the underestimator is its global minimizer.

Example 3.7. Consider the nonconvex problem:

$$\begin{aligned} & \text{Minimize} && ux^4 - 2x^3u \\ & \text{subject to} && x \in \Omega := \mathbb{R}, \frac{1}{v}x^2 - 16 \leq 0, \end{aligned} \quad (\text{UP}_1)$$

where $u \in \mathcal{U} := [1, 2], v \in \mathcal{V} := [1, 2]$. The robust counterpart of (UP_1) is the following robust optimization problem:

$$\begin{aligned} & \text{Minimize} && \max_{u \in \mathcal{U}} ux^4 - 2x^3u \\ & \text{subject to} && x \in \Omega, \frac{1}{v}x^2 - 16 \leq 0, \forall v \in \mathcal{V}. \end{aligned} \quad (\text{RP}_1)$$

It is not hard to see that the set of all robust feasible solutions of (UP_1) is $\mathcal{K} = [-4, 4]$. Indeed, $\mathcal{K} = \{x \in \mathbb{R} : \frac{1}{v}x^2 - 16 \leq 0, \forall v \in [1, 2]\} = \cap_{v \in [1, 2]} \{x \in \mathbb{R} : \frac{1}{v}x^2 - 16 \leq 0\} = \{x \in \mathbb{R} : x^2 - 16 \leq 0\} = [-4, 4]$. Besides, the robust optimization (RP_1) can be written as follows: $\min\{\phi(x) : x \in [-4, 4]\}$ where

$$\phi(x) = \begin{cases} x^4 - 2x^3; & 0 \leq x \leq \frac{3}{2}, \\ 2x^4 - 4x^3; & \text{otherwise.} \end{cases}$$

Then, 0 and $\frac{3}{2}$ are local robust optimal solutions of (UP_1) and the robust KKT

conditions are satisfied at both points with $\bar{u} = 1, \bar{\lambda} = 0$ and any $\bar{v} \in \mathcal{V}$. Further-

more, the Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with the problem (UP_1) is given by

$L(x, \bar{u}, \bar{v}, \bar{\lambda}) = x^4 - 2x^3$ for all $x \in \mathbb{R}$. So, the function $\rho : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\rho(x) = \begin{cases} x^4 - 2x^3; & x \leq \frac{3}{2}, \\ \frac{27}{16}; & \text{otherwise} \end{cases}$$

is a (nonconvex) underestimator of the Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at $\bar{x} = \frac{3}{2}$. In addi-

tion, we also obtain

$$\rho'(x) = \begin{cases} 2x^2(2x - 3) \leq 0; & x \leq \frac{3}{2}, \\ 0; & \text{otherwise.} \end{cases}$$

It can be seen that, all the stationary points of ρ are its global minimizers and the point $\bar{x} = \frac{3}{2}$ is a global robust optimal solution of (UP₁).

The following example is motivated by Example 2.1, in [1], whose objective and constraint functions are absent of data uncertainty.

Example 3.8. Consider the nonconvex problem:

$$\begin{aligned} & \text{Minimize} && uxe^x(x-1)^2 \\ & \text{subject to} && x \in \Omega := \mathbb{R}, \frac{1}{v}x^2 - 18 \leq 0, \end{aligned} \quad (\text{UP}_2)$$

where $u \in \mathcal{U} := [1, 2], v \in \mathcal{V} := [\frac{1}{2}, 2]$, by examining the following robust optimization problem:

$$\begin{aligned} & \text{Minimize} && \max_{u \in \mathcal{U}} uxe^x(x-1)^2 \\ & \text{subject to} && x \in \Omega, \frac{1}{v}x^2 - 18 \leq 0, \forall v \in \mathcal{V}. \end{aligned} \quad (\text{RP}_2)$$

The robust feasible set of the problem (UP₂) is $\mathcal{K} = [-3, 3]$ and we can rewrite the problem (RP₂) as follows: $\min\{\phi(x) : x \in [-3, 3]\}$ where

$$\phi(x) = \begin{cases} xe^x(x-1)^2; & x \in (-\infty, 0], \\ 2xe^x(x-1)^2; & \text{otherwise.} \end{cases}$$

So, the points $\bar{x} = 1$ and $\bar{x} = -1 - \sqrt{2}$ are robust feasible solutions and also local robust optimal solutions of (UP₂). The robust KKT conditions are satisfied at both points with $\bar{u} = 1, \bar{\lambda} = 0$ and any $\bar{v} \in \mathcal{V}$ and the Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with (UP₂) at $x \in \mathbb{R}$ is given by $L(x, \bar{u}, \bar{v}, \bar{\lambda}) = xe^x(x-1)^2$. Thus, the function $\rho : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\rho(x) = \begin{cases} xe^x(x-1)^2; & x \leq -1 - \sqrt{2}, \\ (14 + 10\sqrt{2})e^{-1-\sqrt{2}}; & \text{otherwise,} \end{cases}$$

is a (nonconvex) underestimator of the Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at $\bar{x} = -1 - \sqrt{2}$. Then, by following the scheme used for solving the problem (P1) in [1, Example 2.1], we can verify that $-1 - \sqrt{2}$ is a global robust optimal solution of (UP₂).

Corollary 3.9. For the uncertain optimization problem (UP), let $\Omega = \mathbb{R}^n$. Let $\bar{x} \in \mathcal{K}$ and the robust KKT conditions hold at \bar{x} with the uncertain parameters \bar{u}, \bar{v} and the multiplier $\bar{\lambda}$. If the Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with (UP) admits a convex underestimator at \bar{x} , then \bar{x} is a global robust optimal solution of (UP).

Proof. Let ρ be the convex underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} over \mathbb{R}^n , and let $d \in \mathbb{R}^n$ be arbitrary. Since for each $x \in \mathbb{R}^n$, $\rho(x) \leq L(x, \bar{u}, \bar{v}, \bar{\lambda})$, we obtain that for each $\mu > 0$,

$$\frac{\rho(\bar{x} + \mu d) - \rho(\bar{x})}{\mu} \leq \frac{L(\bar{x} + \mu d, \bar{u}, \bar{v}, \bar{\lambda}) - L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda})}{\mu}.$$

By taking $\mu \rightarrow 0$, we arrive $\rho'(\bar{x}, d) \leq \nabla L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda})^T d = 0$. Since ρ is convex and $\rho'(\bar{x}, d) \leq 0$ for all $d \in \mathbb{R}^n$, ρ is differentiable at \bar{x} and $\nabla \rho(\bar{x}) = 0$. Thus, \bar{x} is a stationary point of the convex function ρ and so it is a global minimizer of ρ over \mathbb{R}^n . Hence, ρ is a minimizing underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} over $\mathcal{K} \subseteq \mathbb{R}^n$ and then it follows from Lemma 3.1 that \bar{x} is a global robust optimal solution of (UP). \square

The following examples illustrate Corollary 3.9.

Example 3.10. Consider the nonconvex uncertain optimization problem (UP₁) defined as in Example 3.7. The function $\tilde{\rho}: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\tilde{\rho}(x) = \begin{cases} -\frac{27}{16}, & x \leq \frac{3}{2}, \\ (x - \frac{3}{2})^2 - \frac{27}{16}; & \text{otherwise} \end{cases}$$

is a convex underestimator of the Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at $\bar{x} = \frac{3}{2}$, where $L(x, \bar{u}, \bar{v}, \bar{\lambda}) = x^4 - 2x^3$ for all $x \in \mathbb{R}$. It can be seen that, the point $x = \frac{3}{2}$ is a global robust optimal solution of (UP₁).

Example 3.11. Consider the following nonconvex programming problem:

$$\begin{aligned} & \text{Minimize} && x^4 + x^3 - 3x^2 + u \\ & \text{subject to} && x \in \Omega := \mathbb{R}, x^2 - x^3 - v_1 \leq 0, v_2 - x^2 \leq 0, \end{aligned} \quad (\text{UP}_3)$$

where $\mathcal{U} := [-1, 1]$, $\mathcal{V}_1 := [0, 1]$, $\mathcal{V}_2 := [0, 1]$, by examining the following robust optimization problem:

$$\begin{aligned} & \text{Minimize} \quad \max_{u \in \mathcal{U}} x^4 + x^3 - 3x^2 + u \\ & \text{subject to} \quad x \in \mathbb{R}, x^2 - x^3 - v_1 \leq 0, \forall v_1 \in \mathcal{V}_1, v_2 - x^2 \leq 0, \forall v_2 \in \mathcal{V}_2. \quad (\text{RP}_3) \end{aligned}$$

Clearly, the problem (RP_3) becomes $\min\{x^4 + x^3 - 3x^2 + 1 : x \in \mathcal{K}\}$ where

$$\begin{aligned} \mathcal{K} &= \{x \in \mathbb{R} : x^2 - x^3 - v_1 \leq 0, \forall v_1 \in [0, 1] \text{ and } v_2 - x^2 \leq 0, \forall v_2 \in [0, 1]\} \\ &= \{x \in \mathbb{R} : x^2 - x^3 \leq 0, \text{ and } 1 - x^2 \leq 0\} \\ &= [1, \infty). \end{aligned}$$

Direct calculations show that the robust KKT conditions hold at the feasible point $\bar{x} = 1$ with $\bar{u} = 1$, $\bar{v} := (\bar{v}_1, \bar{v}_2) = (0, 1)$ and $\bar{\lambda} := (\bar{\lambda}_1, \bar{\lambda}_2) = (1, 0)$. The Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with (UP_3) is given by $L(x, \bar{u}, \bar{v}, \bar{\lambda}) = (x^2 - 1)^2$. Hence, we obtain the function $\rho : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\rho(x) = \begin{cases} 0; & x \leq 1, \\ L(x, \bar{u}, \bar{v}, \bar{\lambda}); & \text{otherwise} \end{cases}$$

is a convex underestimator of the Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at $\bar{x} = 1$, which is a global robust optimal solution of (UP_3) .

Corollary 3.12. For the uncertain optimization problem (UP) , let $\Omega = \mathbb{R}^n$. Let $\bar{x} \in \mathcal{K}$ and the robust KKT conditions hold at \bar{x} with the uncertain parameters \bar{u}, \bar{v} and the multiplier $\bar{\lambda}$. If $L^{**}(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda}) = L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda})$, then \bar{x} is a global robust optimal solution of (UP) .

Proof. It is clear by the definition of biconjugate for a function that $L^{**}(\cdot, \bar{v}, \bar{\lambda})$ is convex and $L^{**}(x, \bar{u}, \bar{v}, \bar{\lambda}) \leq L(x, \bar{u}, \bar{v}, \bar{\lambda})$ for all $x \in \mathbb{R}^n$. So, it then follows from $L^{**}(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda}) = L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda})$ that $L^{**}(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ is a convex underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} . Hence, by applying Corollary 3.9, \bar{x} is a global robust optimal solution of (UP) as desired. \square

Example 3.13. Consider the following nonconvex minimization problem:

$$\begin{aligned} & \text{Minimize} && u + (x^2 - 4)^2 \\ & \text{subject to} && x \in \Omega := \mathbb{R}, x - 1 - v \leq 0, \end{aligned} \quad (\text{UP}_4)$$

where $\mathcal{U} := [-1, 1]$, $\mathcal{V} := [-1, 1]$, by examining the following robust optimization problem:

$$\begin{aligned} & \text{Minimize} && \max_{u \in [-1, 1]} u + (x^2 - 4)^2 \\ & \text{subject to} && x - 1 - v \leq 0, \forall v \in \mathcal{V}. \end{aligned} \quad (\text{RP}_4)$$

Clearly, the problem (RP_4) becomes $\min\{1 + (x^2 - 4)^2 : x \in K\}$ where the robust feasible set of the problem is $\mathcal{K} = \cap_{v \in [-1, 1]} \{x \in \mathbb{R} : x - 1 - v \leq 0, \forall v \in [0, 1]\} = (-\infty, 0]$. Then, $\bar{x} = -2$ is a local robust optimal solution of (UP_4) and the robust KKT conditions are satisfied at \bar{x} with $\bar{u} = 1, \bar{\lambda} = 0$ and any $\bar{v} \in \mathcal{V}$. Thus, we obtain $L(x, \bar{u}, \bar{v}, \bar{\lambda}) = 1 + (x^2 - 4)^2$ and

$$L^{**}(x, \bar{u}, \bar{v}, \bar{\lambda}) = \begin{cases} 1; & x \in [-2, 2], \\ 1 + (x^2 - 4)^2; & \text{otherwise.} \end{cases}$$

It is clear that $L(x, \bar{u}, \bar{v}, \bar{\lambda}) \geq L^{**}(x, \bar{u}, \bar{v}, \bar{\lambda})$ for all $x \in K$ and $L^{**}(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda}) = L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda}) = L(-2, 1, \bar{v}, 0) = 1$. Thus, $L^{**}(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ is a convex underestimator of the Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with (UP_4) at $\bar{x} = -2$. Besides, it can be seen that, the points -2 is a global robust optimal solutions of (UP_4) .

Corollary 3.14. For the uncertain optimization problem (UP) , let $\Omega = \mathbb{R}^n$. Let $\bar{x} \in \mathcal{K}$ and the robust KKT conditions hold at \bar{x} with the uncertain parameters \bar{u}, \bar{v} and the multiplier $\bar{\lambda}$. If the Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with (UP) is convex, then \bar{x} is a global robust optimal solution of (UP) .

Proof. Clearly, by assumption, $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ admits its convex underestimator $L^{**}(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} . Hence, it follows from Corollary 3.9 that \bar{x} is a global robust optimal solution of (UP) . \square

Remark 3.15. In the special case that there is no uncertainty in the objective function and constraint function, i.e., \mathcal{U} and \mathcal{V} are singletons, it can be seen easily that

1. the problem (UP) with $\Omega = \mathbb{R}^n$ becomes $\min\{f(x) : g_i(x) \leq 0, i = 1, \dots, m\}$, which was studied in [1];
2. the Theorem 3.1 provides the same results presented in [1, Theorem 2.1.];
3. the Corollary 3.9 provides the same results presented in [1, Corollary 2.1];
4. the Corollary 3.12 provides the same results presented in [1, Corollary 2.2];
5. the Corollary 3.14 provides the same results presented in [1, Corollary 2.3].

3.2 Duality

In this section, we study particularly the case that Ω of (UP) is \mathbb{R}^n . By virtue of robust optimization, we formulate a Wolfe type robust dual problem for the uncertain optimization problem and then discuss the robust weak duality and robust strong duality properties for a class of, not necessarily convex, possibly multi-extremal, optimization problems. To begin, let us call the following uncertain optimization problem (UP) with $\Omega = \mathbb{R}^n$ as (UP₀):

$$\begin{aligned} & \text{Minimize}_x \quad f(x, u) \\ & \text{subject to} \quad g_i(x, v_i) \leq 0, \quad i = 1, \dots, m. \end{aligned} \quad (\text{UP}_0)$$

The robust counterpart of (UP₀) is the following problem:

$$\begin{aligned} & \text{Minimize}_x \quad \max_{u \in \mathcal{U}} f(x, u) \\ & \text{subject to} \quad g_i(x, v_i) \leq 0, \quad \forall v_i \in \mathcal{V}_i, i = 1, \dots, m. \end{aligned} \quad (\text{RP}_0)$$

Let $y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}_+^m$. For each fixed $u \in \mathcal{U}$ and $v \in \mathcal{V}$, the conventional Wolfe type dual program of (UP_0) is given by

$$\begin{aligned} & \text{Maximize}_{(y,\lambda)} \quad L(y, u, v, \lambda) = f(y, u) + \sum_{i=1}^m \lambda_i g_i(y, v_i) \\ & \text{subject to} \quad \nabla L(y, u, v, \lambda) = 0, \lambda_i \geq 0, i = 1, \dots, m. \end{aligned} \quad (UD_0)$$

The optimistic counterpart of the uncertain dual (UD_0) , called optimistic dual optimization problem, is a deterministic optimization problem which is given by

$$\begin{aligned} & \text{Maximize}_{(y,u,v,\lambda)} \quad L(y, u, v, \lambda) = f(y, u) + \sum_{i=1}^m \lambda_i g_i(y, v_i) \\ & \text{subject to} \quad \nabla L(y, u, v, \lambda) = 0, u \in \mathcal{U}, v_i \in \mathcal{V}_i, \lambda_i \geq 0, i = 1, \dots, m, \end{aligned} \quad (OD_0)$$

where the maximization is also over all the parameters $u \in \mathcal{U}$ and $v \in \mathcal{V}$. The feasible solution set of (OD_0) is defined by

$$\mathcal{K}_D := \left\{ (y, u, v, \lambda) \in \mathbb{R}^n \times \mathcal{U} \times \mathcal{V} \times \mathbb{R}^m : \nabla L(y, u, v, \lambda) = 0, u \in \mathcal{U}, v_i \in \mathcal{V}_i, \lambda_i \geq 0, i = 1, \dots, m \right\}$$

Remark 3.16. In the special case that there is no uncertainty in the objective function and constraint function, i.e., \mathcal{U} and \mathcal{V} are singletons, (RP_0) becomes $\min\{f(x) : g_i(x) \leq 0, i = 1, \dots, m\}$, and (OD_0) collapses to the following Wolfe type dual problem: $\max\{L(y, \lambda) = f(y) + \sum_{i=1}^m \lambda_i g_i(y) : \nabla L(y, \lambda) = 0, \lambda_i \geq 0, i = 1, \dots, m\}$.

It is also worth to note that by Wolfe type robust strong duality we understand the situation when the optimal value of (RP_0) equals the optimal value of (OD_0) and both the minimum of (RP_0) , denoted by $\min(RP_0)$, and the maximum of (OD_0) , denoted by $\max(OD_0)$, are attained.

The following result is the robust weak duality theorem between (RP_0) and (OD_0) .

Theorem 3.17. (Robust Weak Duality) For the problems (RP_0) and (OD_0) , suppose that at each $(y, u, v, \lambda) \in \mathcal{K}_D$, the Lagrangian $L(\cdot, u, v, \lambda)$ associated with

(UP₀) admits an underestimator $\hat{L}(\cdot, u, v, \lambda)$, which is differentiable at that point (y, u, v, λ) , and every stationary point of the function $\hat{L}(\cdot, u, v, \lambda)$ is its global minimizer. Then, for each feasible solution x of (RP₀),

$$\max_{u \in \mathcal{U}} f(x, u) \geq L(y, u, v, \lambda)$$

and hence $\min(\text{RP}_0) \geq \max(\text{OD}_0)$.

Proof. Let (y, u, v, λ) be feasible for (OD₀), i.e., $(y, u, v, \lambda) \in \mathcal{K}_D$ and let $d \in \mathbb{R}^n$ and $\mu > 0$ be arbitrary. By assumption, we have

$$\frac{\hat{L}(y + \mu d, u, v, \lambda) - \hat{L}(y, u, v, \lambda)}{\mu} \leq \frac{L(y + \mu d, u, v, \lambda) - L(y, u, v, \lambda)}{\mu}.$$

Taking $\mu \rightarrow 0$, we obtain $\nabla \hat{L}(y, u, v, \lambda)^T d \leq \nabla L(y, u, v, \lambda)^T d = 0$. Hence, $\nabla \hat{L}(y, u, v, \lambda) = 0$, and so y is a stationary point of $\hat{L}(\cdot, u, v, \lambda)$. It follows from assumption that $\hat{L}(y, u, v, \lambda) \leq \hat{L}(z, u, v, \lambda)$ for all $z \in \mathbb{R}^n$. Let $x \in \mathbb{R}^n$ be an arbitrary feasible solution of (RP₀), then one has

$$\begin{aligned} \max_{x \in \mathcal{U}} f(x, u) - L(y, u, v, \lambda) &\geq f(x, u) - L(y, u, v, \lambda) \\ &\geq f(x, u) + \sum_{i=1}^m \lambda_i g_i(x, v_i) - L(y, v, \lambda) \\ &= L(x, u, v, \lambda) - L(y, u, v, \lambda) \\ &\geq \hat{L}(x, u, v, \lambda) - L(y, u, v, \lambda) \\ &= \hat{L}(x, u, v, \lambda) - \hat{L}(y, u, v, \lambda) \\ &\geq 0, \end{aligned}$$

which yields $\max_{x \in \mathcal{U}} f(x, u) \geq L(y, u, v, \lambda)$. Furthermore, since x and (y, u, v, λ) are arbitrary solutions of (RP₀) and (OD₀), respectively, we arrive $\min(\text{RP}_0) \geq \max(\text{OD}_0)$ as desired. \square

Corollary 3.18. For the problems (RP₀) and (OD₀), suppose that at each $(y, u, v, \lambda) \in \mathcal{K}_D$, the Lagrangian $L(\cdot, u, v, \lambda)$ associated with (UP₀) admits a convex underestimator $\hat{L}(\cdot, u, v, \lambda)$. Then $\min(\text{RP}_0) \geq \max(\text{OD}_0)$.

Proof. Let $d \in \mathbb{R}^n$ and $\mu > 0$ be arbitrary. By assumption, we have

$$\frac{\widehat{L}(y + \mu d, u, v, \lambda) - \widehat{L}(y, u, v, \lambda)}{\mu} \leq \frac{L(y + \mu d, u, v, \lambda) - L(y, u, v, \lambda)}{\mu}.$$

Taking $\mu \rightarrow 0$, we obtain $(\widehat{L})'((y, u, v, \lambda), d) \leq \nabla L(y, u, v, \lambda)^T d = 0$. It then follows from the convexity of \widehat{L} and $(\widehat{L})'((y, u, v, \lambda), d) \leq 0$, for each $d \in \mathbb{R}^n$ that \widehat{L} is differentiable at (y, u, v, λ) and $\nabla \widehat{L}(y, u, v, \lambda) = 0$. That is (y, u, v, λ) is a stationary point of \widehat{L} . Since every stationary point of a convex function is its global minimizer, we obtain $\widehat{L}(y, u, v, \lambda) \leq \widehat{L}(z, u, v, \lambda)$ for all $z \in \mathbb{R}^n$. Then, following the scheme for proving the desired inequality in Theorem 3.17, we arrive the conclusion of this corollary. \square

Corollary 3.19. For the problems (RP_0) and (OD_0) , if at each $(y, u, v, \lambda) \in \mathcal{K}_D$, $L(\cdot, u, v, \lambda) = L^{**}(\cdot, u, v, \lambda)$, then $\min(\text{RP}_0) \geq \max(\text{OD}_0)$.

Proof. Clearly, by assumption and the definition of biconjugate of a function, for each feasible $(y, u, v, \lambda) \in \mathcal{K}_D$, the Lagrangian $L(\cdot, u, v, \lambda)$ associated with (UP_0) admits a convex underestimator $L^{**}(\cdot, u, v, \lambda)$. Then, the conclusion follows from Corollary 3.18. \square

Theorem 3.20. (Strong Duality) For the problems (RP_0) and (OD_0) , suppose that at each $(y, u, v, \lambda) \in \mathcal{K}_D$, the Lagrangian $L(\cdot, u, v, \lambda)$ associated with (UP_0) admits an underestimator $\widehat{L}(\cdot, u, v, \lambda)$, which is differentiable at that point (y, u, v, λ) , and every stationary point of the function $\widehat{L}(\cdot, u, v, \lambda)$ is its global minimizer. If the robust KKT conditions hold at a robust optimal solution \bar{x} of (UP_0) , then $\min(\text{RP}_0) = \max(\text{OD}_0)$.

Proof. Let \bar{x} be a robust optimal solution of (UP_0) , then it is an optimal solution of (RP_0) . Since the KKT conditions hold at \bar{x} , there exist $\bar{u} \in \mathcal{U}$, $\bar{v}_i \in \mathcal{V}_i$ and $\bar{\lambda}_i \geq 0, i = 1, \dots, m$ such that $f(\bar{x}, \bar{u}) = \max_{u \in \mathcal{U}} f(\bar{x}, u)$, $\sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0$ and $\nabla L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda}) = 0$. Thus, $(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda}) \in \mathcal{K}_D$. It then follows from the robust duality

theorem (Theorem 3.17) that

$$\begin{aligned}
 \min(\text{RP}_0) &\geq \max(\text{OD}_0) \\
 &\geq L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda}) \\
 &= f(\bar{x}, \bar{u}) + \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}, \bar{\lambda}_i) \\
 &= \max_{u \in \mathcal{U}} f(\bar{x}, u) \\
 &= \min(\text{RP}_0).
 \end{aligned}$$

Hence, the conclusion of this theorem holds. \square

Corollary 3.21. For the problems (RP_0) and (OD_0) , suppose that at each $(y, u, v, \lambda) \in \mathcal{K}_D$, the Lagrangian $L(\cdot, u, v, \lambda)$ associated with (UP_0) admits a convex underestimator $\hat{L}(\cdot, u, v, \lambda)$. If the robust KKT conditions hold at a robust optimal solution \bar{x} of (UP_0) , then $\min(\text{RP}_0) = \max(\text{OD}_0)$.

Proof. The scheme for proving this corollary is the same as in the proof of Theorem 3.20, but Corollary 3.18 is used instead of Theorem 3.17. \square

Corollary 3.22. For the problems (RP_0) and (OD_0) , suppose that at each $(y, u, v, \lambda) \in \mathcal{K}_D$, the equality: $L(y, u, v, \lambda) = L^{**}(y, u, v, \lambda)$ holds. If the robust KKT conditions hold at a robust optimal solution \bar{x} of (UP_0) , then $\min(\text{RP}_0) = \max(\text{OD}_0)$.

Proof. The conclusion of this corollary follows easily from Corollary 3.21 and Theorem 3.20. \square

CHAPTER 4

CONCLUSION

In this research, a robust optimization problem, which has a maximum function of continuously differentiable functions as its objective function, is investigated. We present new conditions for a robust KKT point to be a global robust optimal solution of such uncertain optimization problems which may have many local robust optimal solutions that are not global. The obtained conditions make use of underestimators, which were first introduced by Jeyakumar and Srisatkunrajah [1, 2] of the Lagrangian at that robust KKT point. We also investigate Wolfe type robust duality between the smooth uncertain optimization problem and its uncertain dual problem by proving the sufficient conditions for the weak and strong duality between the deterministic robust counterpart of the primal model and the optimistic counterpart of its dual problem. The obtained conditions for duality results are established in terms of underestimators. Also, to illustrate or support this study, some examples are presented.

References

- [1] V. Jeyakumar and S. Srisatkunarakjah, New sufficiency for global optimality and duality of mathematical programming problems via underestimators. *J. Optim. Theory. Appl.* 140 (2009) 239–247.
 - [2] V. Jeyakumar and S. Srisatkunarakjah, Geometric conditions for Kuhn–Tucker sufficiency of global optimality in mathematical programming. *European Journal of Operational Research.* 194 (2009) 363–367.
 - [3] A. Ben-Tal, L.E. Ghaoui and A. Nemirovski, *Robust Optimization*. Princeton University Press, Princeton (2009)
 - [4] D. Bertsimas, D.B. Brown and C. Caramanis, Theory and applications of robust optimization. *SIAM Rev.* 53 (2011) 464–501.
 - [5] A. Ben-Tal and A. Nemirovski, Robust convex optimization. *Math. Oper. Res.* 23 (1998) 769–805.
 - [6] V. Jeyakumar and G.Y. Li, Strong duality in robust convex programming: complete characterizations. *SIAM J. Optim.* 20 (2010) 3384–3407.
 - [7] R.I. Bot, V. Jeyakumar and G.Y. Li, Robust duality in parametric convex optimization. *Set-Valued Var. Anal.* 21 (2013) 177–189.
-
- [8] A. Beck and A. Ben-Tal, Duality in robust optimization: primal worst equals dual best. *Oper. Res. Lett.* 37 (2009) 1–6.
 - [9] D. Kuroiwa and G.M. Lee, On robust convex multiobjective optimization. *J. Nonlinear Convex Anal.* 15 (2014) 1125–1136.
 - [10] A. Ben-Tal and A. Nemirovski, Robust optimization-methodology and applications. *Math. Program. Ser. B.* 92 (2002) 453–480.



- [11] D. Bertsimas and D.B. Brown, Constructing uncertainty sets for robust optimization. Oper. Res. 57, (2009) 1483–1495.
- [12] V. Jeyakumar, G.M. Lee and G.Y. Li, Characterizing robust solution sets of convex programs under data uncertainty. J. Optim. Theory Appl. 164 (2015) 407–435.
- [13] X.K. Sun, Z.Y. Peng and X.L. Guo, Some characterizations of robust optimal solutions for uncertain convex optimization problems. Optim. Lett. 10 (2016) 1463–1478.
- [14] X.B. Li and S. Wang, Characterizations of robust solution set of convex programs with uncertain data. Optim. Lett. 12 (2018) 1387–1402.
- [15] C.A. Floudas, Deterministic Global Optimization: Theory, Methods and Applications. Kluwer Academic, Dordrecht (2000)
- [16] R. Horst and P. Pardalos (eds.): Handbook of Global Optimization. Kluwer Academic, Dordrecht (1994)
- [17] A. Migdalas, P.M. Pardalos and P. Varbrand, From Local to Global Optimization, Nonconvex Optimization and Its Applications. Kluwer Academic, Dordrecht (2001)
-
- [18] P. Pardalos and H. Romeijn, Handbook in Global Optimization-2. Kluwer Academic, Dordrecht (2002)
- [19] N.Q. Huy, V. Jeyakumar, G.M. Lee, Sufficient global optimality conditions for multi-extremal smooth minimization problems with bounds and linear matrix inequality constraints. Asia Pac. J. Oper. Res. 47, (2006) 439–450.
- [20] V. Jeyakumar, A.M. Rubinov and Z.Y. Wu, Sufficient global optimality constraints for non-convex quadratic minimization problems with box constraints. J. Glob. Optim. 36 (2006) 461–468.

- [21] V. Jeyakumar, S. Srisatkunrajah and N.Q. Huy, Kuhn-Tucker sufficiency for global minimum of multiextremal mathematical programming problems. *J. Math. Anal. Appl.* 335 (2007) 779–788.
- [22] M.S. Bazaraa, H.D. Sherali and C.M. Shetty, *Nonlinear Programming: Theory and Algorithms*. Wiley-Interscience, New York (2006)
- [23] B.D. Craven, *Control and Optimization*. Chapman and Hall, London (1995)
- [24] B.D. Craven, Optimization with generalized invexity. *Optimization*. 54 (2005) 595–603.
- [25] M.A. Hanson, On sufficiency of Kuhn-Tucker conditions. *J. Math. Anal. Appl.* 80, (1991) 545–550.
- [26] M.A. Hanson, A generalization of the Kuhn-Tucker sufficiency conditions. *J. Math. Anal. Appl.* 184 (1994) 146–155.
- [27] V. Jeyakumar and B. Mond, On generalized convex mathematical programming. *J. Aust.Math. Soc. Ser.B.* 34 (1992) 43–53.
- [28] J.B. Hiriart-Urruty and C.M. Lemaréchal, *Fundamentals of Convex Analysis*. Springer, Berlin (2001)
- [29] O.L. Mangasarian, *Nonlinear Programming. Classics in Applied Mathematics*. SIAM, Philadelphia (1994)
- [30] B.S. Murdukhovich, *Variational Analysis and Generalized Differentiation, I: Basic Theory*. Springer, Berlin (2006)
- [31] J.B. Hiriart-Urruty, When is a point x satisfying $\nabla f = 0$ a global minimum of f ? *Am. Math. Mon.* 93 (1986) 556–558.



January 19, 2021

Dr. Rattanaorn Wangkeeree
Department of Mathematics, Faculty of Science
Naresuan University
Phitsanulok 65000, Thailand
E-mail: rattanaornw@nu.ac.th

Author(s): Jutamas Kerdkaew, Rabian Wangkeeree, Rattanaorn Wangkeeree
Title: Global optimality conditions and duality theorems for robust optimal solutions of optimization problems with data uncertainty, using underestimators
Received on: March 5, 2020
Revised on: January 18, 2021
Accepted on: January 19, 2021

Dear Dr. Rattanaorn Wangkeeree:

I am very pleased to inform you that your paper entitled as above is accepted for publication in Numerical Algebra, Control and Optimization (NACO).

Very sincerely yours,

田中 環

Tamaki Tanaka

Tamaki Tanaka, Professor

Niigata University,

Niigata 950-2181, Japan

Tel: +81-25-262-7415

E-mail: tamaki@math.sc.niigata-u.ac.jp

Global optimality conditions and duality theorems for robust optimal solutions of optimization problems with data uncertainty, using underestimators

JUTAMAS KERDKAEW¹, RABIAN WANGKEEREE AND RATTANAPORN WANGKEEREE²

Abstract: In this paper, a robust optimization problem, which features a maximum function of continuously differentiable functions as its objective function, is investigated. Some new conditions for a robust KKT point, which is a robust feasible solution that satisfies the robust KKT condition, to be a global robust optimal solution of the uncertain optimization problem, which may have many local robust optimal solutions that are not global, are established. The obtained conditions make use of underestimators, which were first introduced by Jayakumar and Srisatkunrajah [1, 2] of the Lagrangian associated with the problem at the robust KKT point. Furthermore, we also investigate the Wolfe type robust duality between the smooth uncertain optimization problem and its uncertain dual problem by proving the sufficient conditions for a weak duality and a strong duality between the deterministic robust counterpart of the primal model and the optimistic counterpart of its dual problem. The results on robust duality theorems are established in terms of underestimators. Additionally, to illustrate or support this study, some examples are presented.

Keywords: robust KKT conditions, Sufficient optimality conditions, Strong duality, Robust optimization problems, Underestimators, Biconjugate functions

Mathematics Subject Classification: 90C26, 90C30, 90C46

1 Introduction

It has been acknowledged that the data of objective function or constraints of the numerous practical programs, including real-world optimization problems, are usually not known exactly beforehand and infrequently uncertain owing to errors (from estimation, prediction or measurement) and asymmetric information [3, 4]. Solving the optimization problems with ignorance of the uncertainty may result in solutions which are suboptimal for even infeasible, so it is essential to take care of uncertain optimization problems, which have data uncertainty within the objective function and/or the constraint; see, e.g., [4, 5, 6, 7, 8, 9, 10, 11]. In line with this fact, it is imperative to review the optimization problems with data uncertainty [3, 8, 12, 13, 14]. A robust optimization, which is its robust counterpart of an uncertain optimization problem [3], has emerged as a useful deterministic approach in [5, 10] to study optimization problems with data uncertainty. The concept of the robust counterpart is minimizing the value of the objective function in the worst case of all scenarios and getting a solution that works well even in the worst-case scenario but is also immunized against the data uncertainty [11]. Furthermore, constraints are enforced for every possible value of the parameters within their prescribed uncertainty set.

On the other hand, seeking a global solution is of fundamental importance in mathematical programming, where problems have many, often infinitely many, possible solutions. Nevertheless, it is known that locating a global solution of a multi-extremal problem, whose several local solutions are not global, is intrinsically tough [15, 16, 17, 18]. Accordingly, improving the criteria for identifying global solutions of optimization problems is of noticeable interest. Over the years, much attention has been targeted on developing criteria for identifying global solutions of nonconvex optimization problems [19, 20, 21]. The KKT condition, which is, under a certain qualification, a necessary condition for a feasible point to be a locally optimal solution of a mathematical programming problem, is often used to creating criteria for identifying global solutions. For instance, a feasible solution that fulfills the KKT conditions, referred to likewise as the KKT point, is a global minimizer for a convex programming problem. Different criteria for identifying global minimizers of nonconvex programming problems regarding the KKT conditions have been given in the literature. Nonetheless, a significant part of work in this area the KKT sufficiency frequently necessitates that the Lagrangian associated with the problem fulfills certain generalized convexity conditions [22, 23, 24, 25, 26, 27].

¹This research was supported by Naresuan University and the Science Achievement Scholarship of Thailand and the second author was partially supported by the Thailand Research Fund, Grant No. RSA6080077.

²Corresponding author.

In [1], the authors employed underestimators of the Lagrangian functions for developing criteria for a KKT point to be a global minimizer of a mathematical programming problem as well as presented also sufficient conditions for weak and strong duality results in terms of underestimators. Furthermore, in [2], the authors presented geometric criteria, which is established in terms of underestimators of the Lagrangian of a mathematical programming problem, for a KKT point to be a global minimizer of such problem with or without bounds on the variables. The idea of dealing with a problem by requiring its Lagrangian to admit underestimator with certain generalized convexity conditions, which was given [1, 2], is very effective and interesting. It leads us to interesting : “How does one obtain sufficient optimality conditions for a robust KKT point to be a global robust optimal solution of a smooth nonconvex optimization problem with data uncertainty in terms of underestimators?” and also, How does one derive robust duality results between the uncertain optimization problem and its dual uncertain optimization problem?”. These questions motivate us to handle with a smooth nonconvex uncertain optimization problem that features with data uncertainty in both objective and constraints for obtaining global optimality conditions and duality theorems for robust optimal solutions.

In this paper, we employ underestimators of the Lagrangian function for developing criteria for a robust KKT point to be a global robust optimal solution of a smooth nonconvex optimization problem with data uncertainty in both objective function and constraints. We establish sufficient optimality conditions by requiring an underestimator, rather the Lagrangian, to satisfy a geometric condition. Here, we present that a robust KKT point is a global robust optimal solution if the Lagrangian associated with the uncertain problem admits an underestimator that fulfills certain conditions (the convexity, or the pseudo-convexity, or the property that every stationary point is a global minimizer). In special, by using the fact that the biconjugate function of the Lagrangian is a convex underestimator at a point whenever it coincides with the Lagrangian at that point, we derive the sufficient optimality conditions as well. We also obtain results on sufficient conditions for a robust weak duality and a robust strong duality between the smooth uncertain optimization problem and its conventional Wolf type dual optimization problem, by proving the sufficient conditions for a weak duality and a strong duality between the deterministic robust counterpart of the primal model and the optimistic counterpart of it conventional Wolf type dual model, in terms of underestimators of the Lagrangian function.

The paper is organized as follows. In section 2, we recall some notions and give some preliminary results. In section 3, several sufficient optimality conditions for a robust KKT conditions to be a global robust optimal solution in terms of underestimators are presented, and also, some examples for illustrating the applications of our results to nonconvex optimization problem with data uncertainty are given. In section 4, we present some results on sufficient conditions for a robust weak duality and a robust strong duality between the smooth uncertain optimization problem and its uncertain dual problem. Finally, in section 5, we conclude our study.

2 Preliminaries

Let us first recall some notation and preliminary results which is able to used throughout this paper. First of all, let $\mathbb{R}^n, n \in \mathbb{N}$, be the n -dimensional Euclidean space and for $x, y \in \mathbb{R}^n$, the notation $x^T y$ stands for the inner product of x and y .

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a real-valued function, it is said to be *continuously differentiable* if the derivative $f'(x)$, where $x \in \mathbb{R}^n$, exists and is itself a continuous function. Besides, a point $x \in \mathbb{R}^n$ is called a *stationary point* of the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ if f is differentiable at x and $\nabla f(x) = 0$, where ∇f denotes the usual gradient of f . Simultaneously, the function f is said to be a *convex* function if for any $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

For a convex function, its local minimizer is also a global minimizer, and if the convex function is differentiable, then its stationary point is its global minimizer as well. Furthermore, let C be a given subset of \mathbb{R}^n and the function f be (Gâteaux) differentiable at a point $x \in C$, then f is said to be *pseudoconvex* at x if

$$\nabla f(x)^T (y - x) \geq 0, y \in C \Rightarrow f(y) \geq f(x).$$

The term pseudoconvex is employed to explain the very fact that such functions share many properties of convex functions, particularly with regards to derivative properties and finding local extrema. Note, however, that pseudoconvexity is strictly weaker than convexity as every convex function is pseudoconvex though one easily checks that $f(x) = x + x^3$ is pseudoconvex and non-convex. The *Legendre-Fenchel conjugate function* (for short, conjugate function) of $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is $f^* : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$f^*(x^*) := \sup_{x \in \mathbb{R}^n} \{ \langle x^*, x \rangle - f(x) \}$$

for all $x \in \mathbb{R}^n$. The function f^* is lower semicontinuous convex irrespective of the nature of f but for f^* to be proper, we need f to be a proper convex function. By reiterating the operation $f \rightarrow f^*$ on f , we get the

biconjugate of f , defined for all $x \in \mathbb{R}^n$ by

$$f^{**}(x) := \sup_{x^* \in \mathbb{R}^n} \{ \langle x, x^* \rangle - f^*(x^*) \}.$$

Clearly, if f is identically equal to $+\infty$, then f^{**} is identically equal to $-\infty$. Note that $f^{**} : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ is a convex function and it coincides with f when f is proper, lower semicontinuous and convex.

Next, we recall the concepts of underestimators, introduced in [1, 2] for developing conditions for a KKT point to be a global minimizer of a standard mathematical programming problem. First of all, please consider the following mathematical programming problem:

$$\begin{aligned} & \text{Minimize} && f(x) \\ & \text{subject to} && x \in \Omega, \quad g_i(x) \leq 0, \quad i = 1, \dots, m, \end{aligned} \quad (\text{P})$$

where Ω is a closed and convex subset of \mathbb{R}^n and $f, g_i : \mathbb{R}^n \rightarrow \mathbb{R}, i = 1, \dots, m$, are continuously differentiable functions. Let $A := \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, \dots, m\}$ and let the feasible set $K := \Omega \cap A$.

The following concepts of underestimators for developing conditions for a KKT point to be a global minimizer of (P) can be found in [1, 2].

Definition 2.1. A function $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be

- (i) an *underestimator* of the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at \bar{x} over the feasible set K if, for each $x \in K$, $f(x) \geq \rho(x)$, and $f(\bar{x}) = \rho(\bar{x})$;
- (ii) a *minimizing underestimator* of the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at \bar{x} over the feasible set K if it is an underestimator of f at \bar{x} over K and it attains its minimum over K at \bar{x} ;
- (iii) a *smooth underestimator* of the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at \bar{x} over the feasible set K if it is an underestimator of f at \bar{x} over K that is differentiable at \bar{x} and $\nabla f(\bar{x}) = \nabla \rho(\bar{x})$.

It is easy to check from the definition of biconjugate function of f that, for each $x \in \mathbb{R}^n$, $f(x) \geq f^{**}(x)$. So, if $f(\bar{x}) = f^{**}(\bar{x})$, then we get from Definition 2.1 that f^{**} is a convex underestimator of f at \bar{x} over \mathbb{R}^n .

A feasible point $\bar{x} \in K$ is said to be a *local minimizer* of (P) if there is a neighborhood U of \bar{x} such that for each $x \in K \cap U$, $f(x) \geq f(\bar{x})$. Specially, if $U = \mathbb{R}^n$, \bar{x} is called a *global minimizer* of (P). If a feasible point \bar{x} is a local minimizer of (P) and if a certain constraint qualification holds then the following *KKT conditions* hold at \bar{x} with multiplier $\lambda \in \mathbb{R}_+^m$:

$$\sum_{i=1}^m \lambda_i g_i(\bar{x}) = 0 \quad \text{and} \quad \nabla L(\bar{x}, \lambda)^T (x - \bar{x}) \geq 0, \quad \forall x \in \Omega.$$

Here $L(\bar{x}, \lambda) := f(\bar{x}) + \sum_{i=1}^m \lambda_i g_i(\bar{x})$ is the *Lagrangian* associated with (P) and \mathbb{R}_+^m is the set of all nonnegative vectors of \mathbb{R}^m .

To conclude this section, we recall some important conceptions which will be used in the sequel. Consider the following parameterized program, which is an analog of the deterministic program (P) if the objective, as well as the constraints, are uncertain:

$$\begin{aligned} & \text{Minimize} && f(x, u) \\ & \text{subject to} && x \in \Omega, \quad g_i(x, v_i) \leq 0, \quad i = 1, \dots, m. \end{aligned} \quad (\text{UP})$$

Here u and v_i are uncertain parameters belonging to compact convex uncertainty sets $\mathcal{U} \subseteq \mathbb{R}^p$, and $\mathcal{V}_i \subseteq \mathbb{R}^q, i = 1, \dots, m$, respectively. $f : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$ and $g_i : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}$ are continuously differentiable function on an open subsets of $\mathbb{R}^n \times \mathbb{R}^p$, and $\mathbb{R}^n \times \mathbb{R}^q$, respectively. By enforcing the constraints for all possible uncertainty within $\mathcal{V}_i, i = 1, \dots, m$, the problem (UP) becomes an *uncertain semi-infinite program*, which is the robust (worst-case) counterpart of (UP):

$$\begin{aligned} & \text{Minimize} && \max_{u \in \mathcal{U}} f(x, u) \\ & \text{subject to} && x \in \Omega, \quad g_i(x, v_i) \leq 0, \quad \forall v_i \in \mathcal{V}_i, \quad i = 1, \dots, m. \end{aligned} \quad (\text{RP})$$

In section 3 and section 4, we investigate the smooth uncertain programming problem (UP) by using examine (RP). The following definition is of the set, is termed as the robust feasible set of (UP).

Definition 2.2. The set of all robust feasible solutions of (UP), equivalently the set of all feasible solution of (RP), is called the *robust feasible set* of (UP) and is defined by

$$\mathcal{K} := \{x \in \Omega : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \dots, m\}.$$

To avoid triviality in (UP), we always assume that $\mathcal{K} \neq \emptyset$.

The below notion commonly referred to as *robust optimal solution* or *robust minimax solution*, can be found in [3]. This concept has been studied extensively by many authors, see, e.g., [8, 12].

Definition 2.3. A robust feasible solution $\bar{x} \in \mathcal{K}$ is said to be a *local robust optimal solution* for (UP) if it is a global optimal solution of (RP), i.e., there exists a neighborhood U of \bar{x} such that for each $x \in \mathcal{K} \cap U$,

$$\max_{u \in \mathcal{U}} f(x, u) \geq \max_{u \in \mathcal{U}} f(\bar{x}, u).$$

Specially, if $U = \mathbb{R}^n$, then $\bar{x} \in \mathcal{K}$ is said to be a *global robust optimal solution* for (UP), equivalently, a global optimal solution of (RP).

The Lagrangian associated with (UP), denoted by $L(x, u, v, \lambda)$, is given as follows:

$$L(x, u, v, \lambda) = f(x, u) + \sum_{i=1}^m \lambda_i g_i(x, v_i),$$

where $u \in \mathcal{U} \subseteq \mathbb{R}^p$, $v := (v_1, \dots, v_m)$, $v_i \in \mathcal{V}_i \subseteq \mathbb{R}^q$, and $\lambda := (\lambda_1, \dots, \lambda_m)$, $\lambda_i \in \mathbb{R}$, $i = 1, \dots, m$. If \bar{x} is a local robust optimal solution of (UP), i.e., a local optimal solution of (RP), and if a certain constraint qualification holds at that point, then the following robust KKT conditions hold: $\exists \bar{u} \in \mathcal{U}$, $\exists \bar{v} \in \mathcal{V}$, $\exists \bar{\lambda} \in \mathbb{R}_m^+$ such that

$$f(\bar{x}, \bar{u}) = \max_{u \in \mathcal{U}} f(\bar{x}, u), \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0, \text{ and } \nabla L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda})^T (x - \bar{x}) \geq 0, \forall x \in \Omega \quad (2.1)$$

where $\mathcal{V} := \mathcal{V}_1 \times \dots \times \mathcal{V}_m$, $\mathcal{V}_i \subseteq \mathbb{R}^q$, $i = 1, \dots, m$. Whenever the robust KKT conditions hold at a robust feasible solution $\bar{x} \in \mathcal{K}$, the point \bar{x} is said to be a robust KKT point.

3 The robust KKT sufficiency for global robust optimal solutions via underestimators

In this section, we establish that a robust KKT point is a global robust optimal solution of (UP) if the Lagrangian associated with (UP) admits an underestimator, which is convex or, more generally, has the property that every stationary point is its global minimizer. In addition, by using the fact that the bi-conjugate function of the Lagrangian is a convex underestimator at a point whenever it coincides with the Lagrangian at that point, we establish sufficient optimality conditions for the point to be a global robust optimal solution of (UP).

The following lemmas are used for obtaining our later results.

Lemma 3.1. Let $\bar{x} \in \mathcal{K}$ and the robust KKT conditions hold at \bar{x} with the uncertain parameters \bar{u}, \bar{v} and the multiplier $\bar{\lambda}$. Suppose that the Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with (UP) admits a minimizing underestimator at \bar{x} over the robust feasible set \mathcal{K} . Then, \bar{x} is a global robust optimal solution of (UP).

Proof. Let ρ be a minimizing underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} over \mathcal{K} and let $x \in \mathcal{K}$ be arbitrary. Then, for the uncertain parameters \bar{u}, \bar{v} and the multiplier $\bar{\lambda}$, we have $\sum_{i=1}^m \bar{\lambda}_i g_i(x, \bar{v}_i) \leq 0$, and

$$\begin{aligned} \max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\bar{x}, u) &= \max_{u \in \mathcal{U}} f(x, u) - f(\bar{x}, \bar{u}) \\ &\geq f(x, \bar{u}) - f(\bar{x}, \bar{u}) \\ &\geq f(x, \bar{u}) - f(\bar{x}, \bar{u}) + \sum_{i=1}^m \lambda_i g_i(x, \bar{v}_i) \\ &= f(x, \bar{u}) + \sum_{i=1}^m \lambda_i g_i(x, \bar{v}_i) - \left[f(\bar{x}, \bar{u}) + \sum_{i=1}^m \lambda_i g_i(\bar{x}, \bar{v}_i) \right] \\ &= L(x, \bar{u}, \bar{v}, \bar{\lambda}) - L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda}) \\ &\geq \rho(x) - \rho(\bar{x}) \\ &\geq 0. \end{aligned}$$

Since x was arbitrary, this inequality yields $\max_{u \in \mathcal{U}} f(x, u) \geq \max_{u \in \mathcal{U}} f(\bar{x}, u)$ for all $x \in \mathcal{K}$. Thus, \bar{x} is a global optimal solution of (RP), and so it is a global robust optimal solution of (UP) as desired. \square

The following theorem shows that every smooth underestimator at \bar{x} over Ω , which is pseudo-convex, is a minimizing underestimator at that point over Ω .

Theorem 3.2. Let $\bar{x} \in \mathcal{K}$ and the robust KKT conditions hold at \bar{x} with the uncertain parameters \bar{u}, \bar{v} and the multiplier $\bar{\lambda}$. Suppose that $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth underestimator of Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with (UP) at \bar{x} over Ω and ρ is pseudo-convex at \bar{x} over Ω . Then, the function ρ is a minimizing underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} over Ω as well as \bar{x} is a global robust optimal solution of (UP).

Proof. By the definition of the smooth underestimator of a function at a point, we have $\nabla \rho(\bar{x}) = \nabla L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda})$. Hence, from the robust KKT conditions, we have

$$\nabla \rho(\bar{x})^T (x - \bar{x}) = \nabla L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda})^T (x - \bar{x}) \geq 0, \quad \forall x \in \Omega.$$

It then follows from the pseudo-convexity of ρ that $\rho(x) \geq \rho(\bar{x})$, for all $x \in \Omega$. This means ρ is a minimizing underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with (UP) at \bar{x} over $\mathcal{K} \subseteq \Omega$. Then, by Lemma 3.1, the point \bar{x} is a global robust optimal solution of (UP). \square

Remark 3.3. In the special case that there is no uncertainty in the objective function and constraint function, i.e., \mathcal{U} and \mathcal{V} are singletons, it can be seen easily that

1. the problem (UP) becomes $\min\{f(x) : g_i(x) \leq 0, i = 1, \dots, m, x \in \Omega\}$, which was studied in [2];
2. Lemma 3.1 provides the same results presented in [2, Lemma 2.1];
3. Theorem 3.2 provides the same results presented in [2, Theorem 2.1].

Lemma 3.4. If ρ is a convex underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with (UP) at \bar{x} over Ω and ρ is differentiable at \bar{x} , then ρ is a pseudo-convex underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} over Ω .

Proof. Since ρ is convex over Ω and is differentiable at \bar{x} , we have from the gradient inequality that

$$\rho(x) \geq \rho(\bar{x}) + \nabla \rho(\bar{x})^T (x - \bar{x}), \quad \forall x \in \Omega.$$

Hence, for each $x \in \Omega$, if $\nabla \rho(\bar{x})^T (x - \bar{x}) \geq 0$ then $\rho(x) \geq \rho(\bar{x})$. This means ρ is pseudo-convex at \bar{x} over Ω , and so the conclusion holds. \square

The following result follows from Theorem 3.2 and Lemma 3.4.

Corollary 3.5. Let $\bar{x} \in \mathcal{K}$ and the robust KKT conditions hold at \bar{x} with the uncertain parameters \bar{u}, \bar{v} and the multiplier $\bar{\lambda}$. Suppose that $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth convex underestimator of Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with (UP) at \bar{x} over Ω . Then, the function ρ is a minimizing underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} over Ω as well as \bar{x} is a global robust optimal solution of (UP).

Proof. Since ρ is a smooth underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} over Ω , it is differentiable at \bar{x} . Then, from Lemma 3.4, ρ is pseudo-convex at \bar{x} over Ω . Hence, it follows from Theorem 3.2 that \bar{x} is a global robust optimal solution of (UP). \square

The following results (Theorem 3.6, Corollary 3.9, Corollary 3.12 and Corollary 3.14) are applicable for the problem (UP) with $\Omega = \mathbb{R}^n$. The sufficient conditions for a robust KKT point to be a global robust optimal solution of (UP) are presented in the following theorem.

Theorem 3.6. For the uncertain optimization problem (UP), let $\Omega = \mathbb{R}^n$. Let $\bar{x} \in \mathcal{K}$ and the robust KKT conditions hold at \bar{x} with the uncertain parameters \bar{u}, \bar{v} and the multiplier $\bar{\lambda}$. Suppose that the Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with (UP) admits an underestimator at \bar{x} and this underestimator is differentiable at \bar{x} . If every stationary point of the underestimator is its global minimizer, then \bar{x} is a global robust optimal solution of (UP).

Proof. Let ρ be an underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} , and let $d \in \mathbb{R}^n$ be arbitrary. Then, for each $x \in \mathbb{R}^n$, $\rho(x) \leq L(x, \bar{u}, \bar{v}, \bar{\lambda})$. So, for any $\mu > 0$, we have

$$\frac{\rho(\bar{x} + \mu d) - \rho(\bar{x})}{\mu} \leq \frac{L(\bar{x} + \mu d, \bar{u}, \bar{v}, \bar{\lambda}) - L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda})}{\mu}.$$

Taking $\mu \rightarrow 0$, we obtain $\nabla \rho(\bar{x})^T d \leq \nabla L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda})^T d = 0$. Hence, $\nabla \rho(\bar{x}) \leq 0$ for each $d \in \mathbb{R}^n$. Therefore, $\nabla \rho(\bar{x}) = 0$, and so the point \bar{x} is a stationary point of ρ . By the assumption, it then follows that, $\rho(x) \geq \rho(\bar{x})$ for all $x \in \mathbb{R}^n$ and hence for all $x \in \mathcal{K} \subseteq \mathbb{R}^n$, $\rho(x) \geq \rho(\bar{x})$ as well. Clearly, ρ is an underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} over \mathcal{K} and it attains its minimum over the set \mathcal{K} at \bar{x} . That is to say, ρ is a minimizing underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} over \mathbb{R}^n and so, by Lemma 3.1, \bar{x} is a global robust optimal solution of (UP). \square

Let us give some examples of multi-extremal nonconvex optimization problems that have some local robust optimal solutions such that the robust KKT conditions hold at each of the solutions. In each example, the Lagrangian admits a nonconvex underestimator at only one local robust optimal solution with the property that every local minimizer of the underestimator is its global minimizer.

Example 3.7. Consider the nonconvex problem:

$$\begin{aligned} & \text{Minimize} && ux^4 - 2x^3u \\ & \text{subject to} && x \in \Omega := \mathbb{R}, \frac{1}{v}x^2 - 16 \leq 0, \end{aligned} \quad (\text{UP}_1)$$

where $u \in \mathcal{U} := [1, 2], v \in \mathcal{V} := [1, 2]$. The robust counterpart of (UP_1) is the following robust optimization problem:

$$\begin{aligned} & \text{Minimize} && \max_{u \in \mathcal{U}} ux^4 - 2x^3u \\ & \text{subject to} && x \in \Omega, \frac{1}{v}x^2 - 16 \leq 0, \forall v \in \mathcal{V}. \end{aligned} \quad (\text{RP}_1)$$

It is not hard to see that the set of all robust feasible solutions of (UP_1) is $\mathcal{K} = [-4, 4]$. Indeed, $\mathcal{K} = \{x \in \mathbb{R} : \frac{1}{v}x^2 - 16 \leq 0, \forall v \in [1, 2]\} = \cap_{v \in [1, 2]} \{x \in \mathbb{R} : \frac{1}{v}x^2 - 16 \leq 0\} = \{x \in \mathbb{R} : x^2 - 16 \leq 0\} = [-4, 4]$. Besides, the robust optimization (RP_1) can be written as follows: $\min\{\phi(x) : x \in [-4, 4]\}$ where

$$\phi(x) = \begin{cases} x^4 - 2x^3; & 0 \leq x \leq \frac{3}{2}, \\ 2x^4 - 4x^3; & \text{otherwise.} \end{cases}$$

Then, 0 and $\frac{3}{2}$ are local robust optimal solutions of (UP_1) and the robust KKT conditions are satisfied at both points with $\bar{u} = 1, \bar{\lambda} = 0$ and any $\bar{v} \in \mathcal{V}$. Furthermore, the Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with the problem (UP_1) is given by $L(x, \bar{u}, \bar{v}, \bar{\lambda}) = x^4 - 2x^3$ for all $x \in \mathbb{R}$. So, the function $\rho : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\rho(x) = \begin{cases} x^4 - 2x^3; & x \leq \frac{3}{2}, \\ \frac{27}{16}; & \text{otherwise} \end{cases}$$

is a (nonconvex) underestimator of the Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at $\bar{x} = \frac{3}{2}$. In addition, we also obtain

$$\rho'(x) = \begin{cases} 2x^2(2x - 3) \leq 0; & x \leq \frac{3}{2}, \\ 0; & \text{otherwise.} \end{cases}$$

It can be seen that, all the stationary points of ρ are its global minimizers and the point $\bar{x} = \frac{3}{2}$ is a global robust optimal solution of (UP_1) .

The following example is motivated by Example 2.1, in [1], whose objective and constraint functions are absent of data uncertainty.

Example 3.8. Consider the nonconvex problem:

$$\begin{aligned} & \text{Minimize} && uxe^x (x - 1)^2 \\ & \text{subject to} && x \in \Omega := \mathbb{R}, \frac{1}{v}x^2 - 18 \leq 0, \end{aligned} \quad (\text{UP}_2)$$

where $u \in \mathcal{U} := [1, 2], v \in \mathcal{V} := [\frac{1}{2}, 2]$, by examining the following robust optimization problem:

$$\begin{aligned} & \text{Minimize} && \max_{u \in \mathcal{U}} uxe^x (x - 1)^2 \\ & \text{subject to} && x \in \Omega, \frac{1}{v}x^2 - 18 \leq 0, \forall v \in \mathcal{V}. \end{aligned} \quad (\text{RP}_2)$$

The robust feasible set of the problem (UP_2) is $\mathcal{K} = [-3, 3]$ and we can rewrite the problem (RP_2) as follows: $\min\{\phi(x) : x \in [-3, 3]\}$ where

$$\phi(x) = \begin{cases} xe^x (x - 1)^2; & x \in (-\infty, 0], \\ 2xe^x (x - 1)^2; & \text{otherwise.} \end{cases}$$

So, the points $\bar{x} = 1$ and $\bar{x} = -1 - \sqrt{2}$ are robust feasible solutions and also local robust optimal solutions of (UP_2) . The robust KKT conditions are satisfied at both points with $\bar{u} = 1, \bar{\lambda} = 0$ and any $\bar{v} \in \mathcal{V}$ and the Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with (UP_2) at $x \in \mathbb{R}$ is given by $L(x, \bar{u}, \bar{v}, \bar{\lambda}) = xe^x (x - 1)^2$. Thus, the function $\rho : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\rho(x) = \begin{cases} xe^x (x - 1)^2; & x \leq -1 - \sqrt{2}, \\ (14 + 10\sqrt{2})e^{-1-\sqrt{2}}; & \text{otherwise,} \end{cases}$$

is a (nonconvex) underestimator of the Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at $\bar{x} = -1 - \sqrt{2}$. Then, by following the scheme used for solving the problem (P1) in [1, Example 2.1], we can verify that $-1 - \sqrt{2}$ is a global robust optimal solution of (UP_2) .

Corollary 3.9. For the uncertain optimization problem (UP), let $\Omega = \mathbb{R}^n$. Let $\bar{x} \in \mathcal{K}$ and the robust KKT conditions hold at \bar{x} with the uncertain parameters \bar{u}, \bar{v} and the multiplier $\bar{\lambda}$. If the Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with (UP) admits a convex underestimator at \bar{x} , then \bar{x} is a global robust optimal solution of (UP).

Proof. Let ρ be the convex underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} over \mathbb{R}^n , and let $d \in \mathbb{R}^n$ be arbitrary. Since for each $x \in \mathbb{R}^n$, $\rho(x) \leq L(x, \bar{u}, \bar{v}, \bar{\lambda})$, we obtain that for each $\mu > 0$,

$$\frac{\rho(\bar{x} + \mu d) - \rho(\bar{x})}{\mu} \leq \frac{L(\bar{x} + \mu d, \bar{u}, \bar{v}, \bar{\lambda}) - L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda})}{\mu}.$$

By taking $\mu \rightarrow 0$, we arrive $\rho'(\bar{x}, d) \leq \nabla L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda})^T d = 0$. Since ρ is convex and $\rho'(\bar{x}, d) \leq 0$ for all $d \in \mathbb{R}^n$, ρ is differentiable at \bar{x} and $\nabla \rho(\bar{x}) = 0$. Thus, \bar{x} is a stationary point of the convex function ρ and so it is a global minimizer of ρ over \mathbb{R}^n . Hence, ρ is a minimizing underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} over $\mathcal{K} \subseteq \mathbb{R}^n$ and then it follows from Lemma 3.1 that \bar{x} is a global robust optimal solution of (UP). \square

The following examples illustrate Corollary 3.9.

Example 3.10. Consider the nonconvex uncertain optimization problem (UP₁) defined as in Example 3.7. The function $\tilde{\rho}: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\tilde{\rho}(x) = \begin{cases} -\frac{27}{16}, & x \leq \frac{3}{2}, \\ (x - \frac{3}{2})^2 - \frac{27}{16}, & \text{otherwise} \end{cases}$$

is a convex underestimator of the Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at $\bar{x} = \frac{3}{2}$, where $L(x, \bar{u}, \bar{v}, \bar{\lambda}) = x^4 - 2x^3$ for all $x \in \mathbb{R}$. It can be seen that, the point $x = \frac{3}{2}$ is a global robust optimal solution of (UP₁).

Example 3.11. Consider the following nonconvex programming problem:

$$\begin{aligned} &\text{Minimize} && x^4 + x^3 - 3x^2 + u \\ &\text{subject to} && x \in \Omega := \mathbb{R}, x^2 - x^3 - v_1 \leq 0, v_2 - x^2 \leq 0, \end{aligned} \quad (\text{UP}_3)$$

where $\mathcal{U} := [-1, 1]$, $\mathcal{V}_1 := [0, 1]$, $\mathcal{V}_2 := [0, 1]$, by examining the following robust optimization problem:

$$\begin{aligned} &\text{Minimize} && \max_{u \in \mathcal{U}} x^4 + x^3 - 3x^2 + u \\ &\text{subject to} && x \in \mathbb{R}, x^2 - x^3 - v_1 \leq 0, \forall v_1 \in \mathcal{V}_1, v_2 - x^2 \leq 0, \forall v_2 \in \mathcal{V}_2. \end{aligned} \quad (\text{RP}_3)$$

Clearly, the problem (RP₃) becomes $\min\{x^4 + x^3 - 3x^2 + 1 : x \in \mathcal{K}\}$ where

$$\begin{aligned} \mathcal{K} &= \{x \in \mathbb{R} : x^2 - x^3 - v_1 \leq 0, \forall v_1 \in [0, 1] \text{ and } v_2 - x^2 \leq 0, \forall v_2 \in [0, 1]\} \\ &= \{x \in \mathbb{R} : x^2 - x^3 \leq 0, \text{ and } 1 - x^2 \leq 0\} \\ &= [1, \infty). \end{aligned}$$

Direct calculations show that the robust KKT conditions hold at the feasible point $\bar{x} = 1$ with $\bar{u} = 1, \bar{v} := (\bar{v}_1, \bar{v}_2) = (0, 1)$ and $\bar{\lambda} := (\bar{\lambda}_1, \bar{\lambda}_2) = (1, 0)$. The Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with (UP₃) is given by $L(x, \bar{u}, \bar{v}, \bar{\lambda}) = (x^2 - 1)^2$. Hence, we obtain the function $\rho: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\rho(x) = \begin{cases} 0; & x \leq 1, \\ L(x, \bar{u}, \bar{v}, \bar{\lambda}); & \text{otherwise} \end{cases}$$

is a convex underestimator of the Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at $\bar{x} = 1$, which is a global robust optimal solution of (UP₃).

Corollary 3.12. For the uncertain optimization problem (UP), let $\Omega = \mathbb{R}^n$. Let $\bar{x} \in \mathcal{K}$ and the robust KKT conditions hold at \bar{x} with the uncertain parameters \bar{u}, \bar{v} and the multiplier $\bar{\lambda}$. If $L^{**}(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda}) = L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda})$, then \bar{x} is a global robust optimal solution of (UP).

Proof. It is clear by the definition of biconjugate for a function that $L^{**}(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ is convex and $L^{**}(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda}) \leq L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda})$ for all $x \in \mathbb{R}^n$. So, it then follows from $L^{**}(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda}) = L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda})$ that $L^{**}(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ is a convex underestimator of $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} . Hence, by applying Corollary 3.9, \bar{x} is a global robust optimal solution of (UP) as desired. \square

Example 3.13. Consider the following nonconvex minimization problem:

$$\begin{aligned} & \text{Minimize} && u + (x^2 - 4)^2 \\ & \text{subject to} && x \in \Omega := \mathbb{R}, x - 1 - v \leq 0, \end{aligned} \quad (\text{UP}_4)$$

where $\mathcal{U} := [-1, 1]$, $\mathcal{V} := [-1, 1]$, by examining the following robust optimization problem:

$$\begin{aligned} & \text{Minimize} && \max_{u \in [-1, 1]} u + (x^2 - 4)^2 \\ & \text{subject to} && x - 1 - v \leq 0, \forall v \in \mathcal{V}. \end{aligned} \quad (\text{RP}_4)$$

Clearly, the problem (RP_4) becomes $\min\{1 + (x^2 - 4)^2 : x \in K\}$ where the robust feasible set of the problem is $K = \cap_{v \in [-1, 1]} \{x \in \mathbb{R} : x - 1 - v \leq 0, \forall v \in [0, 1]\} = (-\infty, 0]$. Then, $\bar{x} = -2$ is a local robust optimal solution of (UP_4) and the robust KKT conditions are satisfied at \bar{x} with $\bar{u} = 1$, $\bar{\lambda} = 0$ and any $\bar{v} \in \mathcal{V}$. Thus, we obtain $L(x, \bar{u}, \bar{v}, \bar{\lambda}) = 1 + (x^2 - 4)^2$ and

$$L^{**}(x, \bar{u}, \bar{v}, \bar{\lambda}) = \begin{cases} 1; & x \in [-2, 2], \\ 1 + (x^2 - 4)^2; & \text{otherwise.} \end{cases}$$

It is clear that $L(x, \bar{u}, \bar{v}, \bar{\lambda}) \geq L^{**}(x, \bar{u}, \bar{v}, \bar{\lambda})$ for all $x \in K$ and $L^{**}(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda}) = L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda}) = L(-2, 1, \bar{v}, 0) = 1$. Thus, $L^{**}(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ is a convex underestimator of the Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with (UP_4) at $\bar{x} = -2$. Besides, it can be seen that, the points -2 is a global robust optimal solutions of (UP_4) .

Corollary 3.14. For the uncertain optimization problem (UP) , let $\Omega = \mathbb{R}^n$. Let $\bar{x} \in K$ and the robust KKT conditions hold at \bar{x} with the uncertain parameters \bar{u} , \bar{v} and the multiplier $\bar{\lambda}$. If the Lagrangian $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ associated with (UP) is convex, then \bar{x} is a global robust optimal solution of (UP) .

Proof. Clearly, by assumption, $L(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ admits its convex underestimator $L^{**}(\cdot, \bar{u}, \bar{v}, \bar{\lambda})$ at \bar{x} . Hence, it follows from Corollary 3.9 that \bar{x} is a global robust optimal solution of (UP) . \square

Remark 3.15. In the special case that there is no uncertainty in the objective function and constraint function, i.e., \mathcal{U} and \mathcal{V} are singletons, it can be seen easily that

1. the problem (UP) with $\Omega = \mathbb{R}^n$ becomes $\min\{f(x) : g_i(x) \leq 0, i = 1, \dots, m\}$, which was studied in [1];
2. the Theorem 3.1 provides the same results presented in [1, Theorem 2.1.];
3. the Corollary 3.9 provides the same results presented in [1, Corollary 2.1];
4. the Corollary 3.12 provides the same results presented in [1, Corollary 2.2];
5. the Corollary 3.14 provides the same results presented in [1, Corollary 2.3].

4 Duality

In this section, we study particularly the case that Ω of (UP) is \mathbb{R}^n . By virtue of robust optimization, we formulate a Wolfe type robust dual problem for the uncertain optimization problem and then discuss the robust weak duality and robust strong duality properties for a class of, not necessarily convex, possibly multi-extremal, optimization problems. To begin, let us call the following uncertain optimization problem (UP) with $\Omega = \mathbb{R}^n$ as (UP_0) :

$$\begin{aligned} & \text{Minimize}_x && f(x, u) \\ & \text{subject to} && g_i(x, v_i) \leq 0, i = 1, \dots, m. \end{aligned} \quad (\text{UP}_0)$$

The robust counterpart of (UP_0) is the following problem:

$$\begin{aligned} & \text{Minimize}_x && \max_{u \in \mathcal{U}} f(x, u) \\ & \text{subject to} && g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \dots, m. \end{aligned} \quad (\text{RP}_0)$$

Let $y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}_+^m$. For each fixed $u \in \mathcal{U}$ and $v \in \mathcal{V}$, the conventional Wolfe type dual program of (UP_0) is given by

$$\begin{aligned} & \text{Maximize}_{(y, \lambda)} && L(y, u, v, \lambda) = f(y, u) + \sum_{i=1}^m \lambda_i g_i(y, v_i) \\ & \text{subject to} && \nabla L(y, u, v, \lambda) = 0, \lambda_i \geq 0, i = 1, \dots, m. \end{aligned} \quad (\text{UD}_0)$$

The optimistic counterpart of the uncertain dual (UD₀), called optimistic dual optimization problem, is a deterministic optimization problem which is given by

$$\begin{aligned} \text{Maximize}_{(y,u,v,\lambda)} \quad & L(y, u, v, \lambda) = f(y, u) + \sum_{i=1}^m \lambda_i g_i(y, v_i) \\ \text{subject to} \quad & \nabla L(y, u, v, \lambda) = 0, u \in \mathcal{U}, v_i \in \mathcal{V}_i, \lambda_i \geq 0, i = 1, \dots, m, \end{aligned} \quad (\text{OD}_0)$$

where the maximization is also over all the parameters $u \in \mathcal{U}$ and $v \in \mathcal{V}$. The feasible solution set of (OD₀) is defined by

$$\mathcal{K}_D := \{(y, u, v, \lambda) \in \mathbb{R}^n \times \mathcal{U} \times \mathcal{V} \times \mathbb{R}^m : \nabla L(y, u, v, \lambda) = 0, u \in \mathcal{U}, v_i \in \mathcal{V}_i, \lambda_i \geq 0, i = 1, \dots, m\}.$$

Remark 4.1. In the special case that there is no uncertainty in the objective function and constraint function, i.e., \mathcal{U} and \mathcal{V} are singletons, (RP₀) becomes $\min\{f(x) : g_i(x) \leq 0, i = 1, \dots, m\}$, and (OD₀) collapses to the following Wolfe type dual problem: $\max\{L(y, \lambda) = f(y) + \sum_{i=1}^m \lambda_i g_i(y) : \nabla L(y, \lambda) = 0, \lambda_i \geq 0, i = 1, \dots, m\}$.

It is also worth to note that by Wolfe type robust strong duality we understand the situation when the optimal value of (RP₀) equals the optimal value of (OD₀) and both the minimum of (RP₀), denoted by $\min(\text{RP}_0)$, and the maximum of (OD₀), denoted by $\max(\text{OD}_0)$, are attained.

The following result is the robust weak duality theorem between (RP₀) and (OD₀).

Theorem 4.2. (*Robust Weak Duality*) For the problems (RP₀) and (OD₀), suppose that at each $(y, u, v, \lambda) \in \mathcal{K}_D$, the Lagrangian $L(\cdot, u, v, \lambda)$ associated with (UP₀) admits an underestimator $\hat{L}(\cdot, u, v, \lambda)$, which is differentiable at that point (y, u, v, λ) , and every stationary point of the function $\hat{L}(\cdot, u, v, \lambda)$ is its global minimizer. Then, for each feasible solution x of (RP₀),

$$\max_{u \in \mathcal{U}} f(x, u) \geq L(y, u, v, \lambda)$$

and hence $\min(\text{RP}_0) \geq \max(\text{OD}_0)$.

Proof. Let (y, u, v, λ) be feasible for (OD₀), i.e., $(y, u, v, \lambda) \in \mathcal{K}_D$ and let $d \in \mathbb{R}^n$ and $\mu > 0$ be arbitrary. By assumption, we have

$$\frac{\hat{L}(y + \mu d, u, v, \lambda) - \hat{L}(y, u, v, \lambda)}{\mu} \leq \frac{L(y + \mu d, u, v, \lambda) - L(y, u, v, \lambda)}{\mu}.$$

Taking $\mu \rightarrow 0$, we obtain $\nabla \hat{L}(y, u, v, \lambda)^T d \leq \nabla L(y, u, v, \lambda)^T d = 0$. Hence, $\nabla \hat{L}(y, u, v, \lambda) = 0$, and so y is a stationary point of $\hat{L}(\cdot, u, v, \lambda)$. It follows from assumption that $\hat{L}(y, u, v, \lambda) \leq \hat{L}(z, u, v, \lambda)$ for all $z \in \mathbb{R}^n$. Let $x \in \mathbb{R}^n$ be an arbitrary feasible solution of (RP₀), then one has

$$\begin{aligned} \max_{x \in \mathcal{U}} f(x, u) - L(y, u, v, \lambda) &\geq f(x, u) - L(y, u, v, \lambda) \\ &\geq f(x, u) + \sum_{i=1}^m \lambda_i g_i(x, v_i) - L(y, u, v, \lambda) \\ &= L(x, u, v, \lambda) - L(y, u, v, \lambda) \\ &\geq \hat{L}(x, u, v, \lambda) - L(y, u, v, \lambda) \\ &= \hat{L}(x, u, v, \lambda) - \hat{L}(y, u, v, \lambda) \\ &\geq 0, \end{aligned}$$

which yields $\max_{x \in \mathcal{U}} f(x, u) \geq L(y, u, v, \lambda)$. Furthermore, since x and (y, u, v, λ) are arbitrary solutions of (RP₀) and (OD₀), respectively, we arrive $\min(\text{RP}_0) \geq \max(\text{OD}_0)$ as desired. \square

Corollary 4.3. For the problems (RP₀) and (OD₀), suppose that at each $(y, u, v, \lambda) \in \mathcal{K}_D$, the Lagrangian $L(\cdot, u, v, \lambda)$ associated with (UP₀) admits a convex underestimator $\hat{L}(\cdot, u, v, \lambda)$. Then $\min(\text{RP}_0) \geq \max(\text{OD}_0)$.

Proof. Let $d \in \mathbb{R}^n$ and $\mu > 0$ be arbitrary. By assumption, we have

$$\frac{\hat{L}(y + \mu d, u, v, \lambda) - \hat{L}(y, u, v, \lambda)}{\mu} \leq \frac{L(y + \mu d, u, v, \lambda) - L(y, u, v, \lambda)}{\mu}.$$

Taking $\mu \rightarrow 0$, we obtain $(\hat{L})'((y, u, v, \lambda), d) \leq \nabla L(y, u, v, \lambda)^T d = 0$. It then follows from the convexity of \hat{L} and $(\hat{L})'((y, u, v, \lambda), d) \leq 0$, for each $d \in \mathbb{R}^n$ that \hat{L} is differentiable at (y, u, v, λ) and $\nabla \hat{L}(y, u, v, \lambda) = 0$. That is (y, u, v, λ) is a stationary point of \hat{L} . Since every stationary point of a convex function is its global minimizer, we obtain $\hat{L}(y, u, v, \lambda) \leq \hat{L}(z, u, v, \lambda)$ for all $z \in \mathbb{R}^n$. Then, following the scheme for proving the desired inequality in Theorem 4.2, we arrive the conclusion of this corollary. \square

Corollary 4.4. For the problems (RP_0) and (OD_0) , if at each $(y, u, v, \lambda) \in \mathcal{K}_D$, $L(\cdot, u, v, \lambda) = L^{**}(\cdot, u, v, \lambda)$, then $\min(RP_0) \geq \max(OD_0)$.

Proof. Clearly, by assumption and the definition of biconjugate of a function, for each feasible $(y, u, v, \lambda) \in \mathcal{K}_D$, the Lagrangian $L(\cdot, u, v, \lambda)$ associated with (UP_0) admits a convex underestimator $L^{**}(\cdot, u, v, \lambda)$. Then, the conclusion follows from Corollary 4.3. \square

Theorem 4.5. (Strong Duality) For the problems (RP_0) and (OD_0) , suppose that at each $(y, u, v, \lambda) \in \mathcal{K}_D$, the Lagrangian $L(\cdot, u, v, \lambda)$ associated with (UP_0) admits an underestimator $\hat{L}(\cdot, u, v, \lambda)$, which is differentiable at that point (y, u, v, λ) , and every stationary point of the function $\hat{L}(\cdot, u, v, \lambda)$ is its global minimizer. If the robust KKT conditions hold at a robust optimal solution \bar{x} of (UP_0) , then $\min(RP_0) = \max(OD_0)$.

Proof. Let \bar{x} be a robust optimal solution of (UP_0) , then it is an optimal solution of (RP_0) . Since the KKT conditions hold at \bar{x} , there exist $\bar{u} \in \mathcal{U}$, $\bar{v}_i \in \mathcal{V}_i$ and $\bar{\lambda}_i \geq 0, i = 1, \dots, m$ such that $f(\bar{x}, \bar{u}) = \max_{u \in \mathcal{U}} f(\bar{x}, u)$, $\sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0$ and $\nabla L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda}) = 0$. Thus, $(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda}) \in \mathcal{K}_D$. It then follows from the robust duality theorem (Theorem 4.2) that

$$\begin{aligned} \min(RP_0) &\geq \max(OD_0) \\ &\geq L(\bar{x}, \bar{u}, \bar{v}, \bar{\lambda}) \\ &= f(\bar{x}, \bar{u}) + \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) \\ &= \max_{u \in \mathcal{U}} f(\bar{x}, u) \\ &= \min(RP_0). \end{aligned}$$

Hence, the conclusion of this theorem holds. \square

Corollary 4.6. For the problems (RP_0) and (OD_0) , suppose that at each $(y, u, v, \lambda) \in \mathcal{K}_D$, the Lagrangian $L(\cdot, u, v, \lambda)$ associated with (UP_0) admits a convex underestimator $\hat{L}(\cdot, u, v, \lambda)$. If the robust KKT conditions hold at a robust optimal solution \bar{x} of (UP_0) , then $\min(RP_0) = \max(OD_0)$.

Proof. The scheme for proving this corollary is the same as in the proof of Theorem 4.5, but Corollary 4.3 is used instead of Theorem 4.2. \square

Corollary 4.7. For the problems (RP_0) and (OD_0) , suppose that at each $(y, u, v, \lambda) \in \mathcal{K}_D$, the equality: $L(y, u, v, \lambda) = L^{**}(y, u, v, \lambda)$ holds. If the robust KKT conditions hold at a robust optimal solution \bar{x} of (UP_0) , then $\min(RP_0) = \max(OD_0)$.

Proof. The conclusion of this corollary follows easily from Corollary 4.6 and Theorem 4.5. \square

5 Conclusion

In this paper, a robust optimization problem, which has a maximum function of continuously differentiable functions as its objective function, is investigated. We present new conditions for a robust KKT point to be a global robust optimal solution of such uncertain optimization problems which may have many local robust optimal solutions that are not global. The obtained conditions make use of underestimators, which were first introduced by Jeyakumar and Srisatkunrajah [1, 2] of the Lagrangian at that robust KKT point. We also investigate Wolfe type robust duality between the smooth uncertain optimization problem and its uncertain dual problem by proving the sufficient conditions for the weak and strong duality between the deterministic robust counterpart of the primal model and the optimistic counterpart of its dual problem. The obtained conditions for duality results are established in terms of underestimators. Also, to illustrate or support this study, some examples are presented.

Acknowledgments

This research was partially supported by Naresuan university and the Science Achievement Scholarship of Thailand.

References

- [1] V. Jeyakumar and S. Srisatkunrajah, New sufficiency for global optimality and duality of mathematical programming problems via underestimators. *J. Optim. Theory. Appl.* 140 (2009) 239–247.
- [2] V. Jeyakumar and S. Srisatkunrajah, Geometric conditions for Kuhn–Tucker sufficiency of global optimality in mathematical programming. *European Journal of Operational Research.* 194 (2009) 363–367.
- [3] A. Ben-Tal, L.E. Ghaoui and A. Nemirovski, *Robust Optimization*. Princeton University Press, Princeton (2009)
- [4] D. Bertsimas, D.B. Brown and C. Caramanis, Theory and applications of robust optimization. *SIAM Rev.* 53 (2011) 464–501.
- [5] A. Ben-Tal and A. Nemirovski, Robust convex optimization. *Math. Oper. Res.* 23 (1998) 769–805.
- [6] V. Jeyakumar and G.Y. Li, Strong duality in robust convex programming: complete characterizations. *SIAM J. Optim.* 20 (2010) 3384–3407.
- [7] R.I. Bot, V. Jeyakumar and G.Y. Li, Robust duality in parametric convex optimization. *Set-Valued Var. Anal.* 21 (2013) 177–189.
- [8] A. Beck and A. Ben-Tal, Duality in robust optimization: primal worst equals dual best. *Oper. Res. Lett.* 37 (2009) 1–6.
- [9] D. Kuroiwa and G.M. Lee, On robust convex multiobjective optimization. *J. Nonlinear Convex Anal.* 15 (2014) 1125–1136.
- [10] A. Ben-Tal and A. Nemirovski, Robust optimization-methodology and applications. *Math. Program. Ser. B.* 92 (2002) 453–480.
- [11] D. Bertsimas and D.B. Brown, Constructing uncertainty sets for robust linear optimization. *Oper. Res.* 57, (2009) 1483–1495.
- [12] V. Jeyakumar, G.M. Lee and G.Y. Li, Characterizing robust solution sets of convex programs under data uncertainty. *J. Optim. Theory Appl.* 164 (2015) 407–435.
- [13] X.K. Sun, Z.Y. Peng and X.L. Guo, Some characterizations of robust optimal solutions for uncertain convex optimization problems. *Optim. Lett.* 10 (2016) 1463–1478.
- [14] X.B. Li and S. Wang, Characterizations of robust solution set of convex programs with uncertain data. *Optim. Lett.* 12 (2018) 1387–1402.
- [15] C.A. Floudas, *Deterministic Global Optimization: Theory, Methods and Applications*. Kluwer Academic, Dordrecht (2000)
- [16] R. Horst and P. Pardalos (eds.): *Handbook of Global Optimization*. Kluwer Academic, Dordrecht (1994)
- [17] A. Migdalas, P.M. Pardalos and P. Varbrand, *From Local to Global Optimization, Nonconvex Optimization and Its Applications*. Kluwer Academic, Dordrecht (2001)
- [18] P. Pardalos and H. Romeijn, *Handbook in Global Optimization-2*. Kluwer Academic, Dordrecht (2002)
- [19] N.Q. Huy, V. Jeyakumar, G.M. Lee, Sufficient global optimality conditions for multi-extremal smooth minimization problems with bounds and linear matrix inequality constraints. *Asia Pac. J. Oper. Res.* 47, (2006) 439–450.
- [20] V. Jeyakumar, A.M. Rubinov and Z.Y. Wu, Sufficient global optimality constraints for non-convex quadratic minimization problems with box constraints. *J. Glob. Optim.* 36 (2006) 461–468.
- [21] V. Jeyakumar, S. Srisatkunrajah and N.Q. Huy, Kuhn–Tucker sufficiency for global minimum of multiextremal mathematical programming problems. *J. Math. Anal. Appl.* 335 (2007) 779–788.
- [22] M.S. Bazaraa, H.D. Sherali and C.M. Shetty, *Nonlinear Programming: Theory and Algorithms*. Wiley-Interscience, New York (2006)
- [23] B.D. Craven, *Control and Optimization*. Chapman and Hall, London (1995)
- [24] B.D. Craven, Optimization with generalized invexity. *Optimization.* 54 (2005) 595–603.
- [25] M.A. Hanson, On sufficiency of Kuhn–Tucker conditions. *J. Math. Anal. Appl.* 80, (1991) 545–550.
- [26] M.A. Hanson, A generalization of the Kuhn–Tucker sufficiency conditions. *J. Math. Anal. Appl.* 184 (1994) 146–155.
- [27] V. Jeyakumar and B. Mond, On generalized convex mathematical programming. *J. Aust. Math. Soc. Ser. B.* 34 (1992) 43–53.

- [28] J.B. Hiriart-Urruty and C.M. Lemaréchal, *Fundamentals of Convex Analysis*. Springer, Berlin (2001)
- [29] O.L. Mangasarian, *Nonlinear Programming. Classics in Applied Mathematics*. SIAM, Philadelphia (1994)
- [30] B.S. Mordukhovich, *Variational Analysis and Generalized Differentiation, I: Basic Theory*. Springer, Berlin (2006)
- [31] J.B. Hiriart-Urruty, When is a point x satisfying $\nabla f = 0$ a global minimum of f ? *Am. Math. Mon.* 93 (1986) 556–558.



JUTAMAS KERDKAEW
Department of Mathematics, Faculty of Science
Naresuan University
Phitsanulok 65000, Thailand
E-mail address: jutamas.jk85@gmail.com

RABIAN WANGKEEREE
Department of Mathematics, Faculty of Science
Naresuan University
Phitsanulok 65000, Thailand

Research center for Academic Excellence in Mathematics
Naresuan University
Phitsanulok 65000, Thailand
E-mail address: rabianw@nu.ac.th

RATTANAPORN WANGKEEREE
Department of Mathematics, Faculty of Science
Naresuan University
Phitsanulok 65000, Thailand
E-mail address: rattanapornw@nu.ac.th

