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Strong convergence theorems for generalized mixed equilibrium problems and Bregman totally quasi-asymptotically nonexpansive mappings in reflexive Banach spaces

ผู้วิจัย

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หน้า

ABSTRACT

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In this project, we propose a new iterative algorithm for finding common solutions of generalized mixed equilibrium problems and fixed point problems for Bregman totally quasi-asymptotically nonexpansive mappings in reflexive Banach spaces. Moreover, we also study the strong convergence theorem under suitable control conditions. Furthermore, we are interested in introducing a new iterative algorithm for finding common solutions of mixed equilibrium problems and common fixed point problems for a countable family of Bregman totally quasi-asymptotically nonexpansive mappings in reflexive Banach spaces.

Keywords: Generalized mixed equilibrium problems, Bregman totally quasi-asymptotically nonexpansive mappings, reflexive Banach spaces

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ในโครงการนี้ผู้วิจัยได้สร้างขั้นตอนการทำซ้ำแบบใหม่สำหรับการหาผลเฉลยร่วมของปัญหาสมดุลผสมวางนัยทั่วไปและปัญหาจุดตรึงสำหรับการส่งแบบไม่ขยายกึ่งกำกับเบรกแมนทุกส่วนในปริภูมิบานาคสะท้อน นอกจากนี้ผู้วิจัยยังได้ศึกษาทฤษฎีบทการลู่เข้าอย่างเข้มภายใต้เงื่อนไขควบคุมที่เหมาะสม รวมทั้งผู้วิจัยยังได้สนใจในการแนะนำขั้นตอนการทำซ้ำแบบใหม่สำหรับการหาผลเฉลยร่วมสำหรับปัญหาสมดุลผสมและปัญหาจุดตรึงร่วมสำหรับวงศ์แบบนับได้ของการส่งแบบไม่ขยายกึ่งกำกับเบรกแมนทุกส่วนในปริภูมิบานาคสะท้อน

คำสำคัญ: ปัญหาสมดุลผสมวางนัยทั่วไป, การส่งแบบไม่ขยายกึ่งกำกับเบรกแมนทุกส่วน, ปริภูมิบานาคสะท้อน

CHAPTER I

EXECUTIVE SUMMARY

The fixed point theory of nonexpansive mappings can be applied to solve the solutions of the certain evolution equations and to solve convex feasibility, variational inequality and equilibrium problems. There are, in fact, many papers deal with methods for finding fixed points of nonexpansive and quasi-nonexpansive mappings in Hilbert, uniformly convex and uniformly smooth Banach spaces.

When we try to extend this theory to general Banach spaces we encounter some difficulties, and there are several ways to overcome these difficulties. One of them is to use the Bregman distance instead of the norm, Bregman (quasi-) nonexpansive mappings instead of the (quasi-) nonexpansive mappings and the Bregman projection instead of the metric projection.

In 1967, Bregman discovered an elegant and effective technique for using Bregman distance function in the process of designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which Bregman's technique has been applied in various ways in order to design and analyze iterative algorithms for solving the feasibility and optimization problems, for approximating the variational inequalities and equilibrium problem, for computing the fixed points of nonlinear mappings and so on.

In 2014, Chang et al. used the shrinking projection method introduced by Takahashi, Kubota and Takeuchi to propose an iteration algorithm for Bregman total quasi-ϕ-asymptotically nonexpansive mapping to have the strong convergence under a limit condition only in the framework of reflexive Banach spaces. As applications, they applied their results to a system of equilibrium problems and zero point problem of maximal monotone mappings in reflexive Banach spaces. The results presented in the mentioned paper improved and extended the corresponding results in the literature.

In 2015, Darvish studied a new iterative method for a common fixed point of a finite family of Bragman strongly nonexpansive mappings in the frame work of reflexive real Banach spsces. Moreover, the author proved the strong convergence theorem for finding common fixed points with the solutions of a mixed equilibrium problem.

In 2016, Zhu and Huang proposed a new hybrid iterative scheme for finding a common solution of an equilibrium problem and fixed point of Bregman totally quasi–asymptotically nonexpansive mapping in reflexive Banach spaces. Moreover, they proved some strong convergence theorems under suitable control conditions. Finally, the application to zero point problem of maximal monotone operators was given by the result.

Inspired and motivated by the above results, we are interested in proposing a new iterative algorithm for finding common solutions of generalized mixed equilibrium problems and fixed point problems for Bregman totally quasi–asymptotically nonexpansive mappings in reflexive Banach spaces. Moreover, we also study the strong convergence theorem under suitable control conditions. Furthermore, we are interested in introducing a new iterative algorithm for finding common solutions of mixed equilibrium problems and common fixed point problems for a countable family of Bregman totally quasi–asymptotically nonexpansive mappings in reflexive Banach spaces.

CHAPTER II

OUTPUT

In this project, we obtain two publications that published in the international journal as the followings:

1. Kittisak Jantakarn and Anchalee Kaewcharoen, Strong convergence theorems for mixed equilibrium problems and Bregman relatively nonexpansive mappings in reflexive Banach spaces, Journal of Nonlinear Science and Applications, 14 (2021), no. 2, 63–79 (SCOPUS)

   In this paper, we propose a new iterative method for solving the mixed equilibrium problems and the fixed point problems for a countable family of Bregman relatively nonexpansive mappings in reflexive Banach spaces. We prove that the sequence generated by the proposed iterative algorithm converges strongly to a common solution of the mentioned problems. Further, a numerical example of the iterative algorithm supporting our main result is presented.

2. Kittisak Jantakarn and Anchalee Kaewcharoen, Strong convergence theorems for generalized mixed equilibrium problems and Bregman relatively nonexpansive mappings in reflexive Banach spaces, Journal of Nonlinear and Convex Analysis, Impact Factor 0.710 (ISI), accepted.

   In this paper, we deal with the Bregman iterative methods for finding common solutions of generalized mixed equilibrium problems and fixed point problems for Bregman relatively nonexpansive mappings in reflexive Banach spaces. The strong convergence theorems for the Bregman iterative methods under some mild conditions are proven. Furthermore, we present a numerical example to illustrate the main result.

ภาคผนวก 1

Kittisak Jantakarn and Anchalee Kaewcharoen

Strong convergence theorems for mixed equilibrium problems and Bregman relatively nonexpansive mappings in reflexive Banach spaces

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Strong convergence theorems for mixed equilibrium problems and Bregman relatively nonexpansive mappings in reflexive Banach spaces

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Abstract

In this paper, we propose a new iterative method for solving the mixed equilibrium problems and the fixed point problems for a countable family of Bregman relatively nonexpansive mappings in reflexive Banach spaces. We prove that the sequence generated by the proposed iterative algorithm converges strongly to a common solution of the mentioned problems. Further, a numerical example of the iterative algorithm supporting our main result is presented.

Keywords: Mixed equilibrium problems, Bregman relatively nonexpansive mappings, reflexive Banach spaces.

2020 MSC: 47H10, 54H25.

1. Introduction

Throughout this paper, let C be a nonempty closed convex subset of a reflexive Banach space E and denote the dual sapce of E by E^*. The norm and the dual pair between E and E^* are denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. We denote the set of fixed points of a mapping T on a subset C of E by $F(T) = \{x \in C : Tx = x\}$ and $\mathbb{R}$ is the set of all real numbers. Let $G : C \times C \longrightarrow \mathbb{R}$ be a bifunction and $\psi : C \longrightarrow \mathbb{R}$ be a real-valued function. We consider the following mixed equilibrium problem which is to find $x \in C$ such that

$$G(x, y) + \psi(y) - \psi(x) \geqslant 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of the problem (1.1) is denoted by $MEP(G, \psi)$ and studied by Ceng and Yao [12]. If we set ψ to be the zero mapping, then the mixed equilibrium problem (1.1) becomes the following equilibrium problem, find $x \in C$ such that

$$G(x, y) \geqslant 0, \quad \forall y \in C. \tag{1.2}$$

The solution set of the problem (1.2) is denoted by $EP(G)$ which is introduced and studied by Blum and Oettli [5]. The equilibrium problem provided a very general formulation of variational problems such as:

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(i) minimization problem: find $x \in C$ such that $h(x) \leqslant h(y)$ for all $y \in C$, where $h : C \longrightarrow \mathbb{R}$ is a functional, in this case, we define $G(x,y) = h(y) - h(x)$ for all $x, y \in C$;
(ii) variational inequality: find $x \in C$ such that $\langle A(x), y - x\rangle \geqslant 0$ for all $y \in C$, where $A : C \longrightarrow E^*$ is a mapping, in this case, we define $G(x,y) = \langle A(x), y - x\rangle$ for all $x, y \in C$.

In 2008, Ceng and Yao [12] investigated the problem of finding a common element of the set of solutions of the mixed equilibrium problem (1.1) and the set of common fixed points of finitely many nonexpansive mappings in real Hilbert spaces.

Whenever the researchers attempted to extend this theory to generalized Banach spaces, they discovered some difficulties and there are a lot of ways to overpower these barriers, for instant, using the Bregman distance in place of the norm, Bregman (quasi-) nonexpansive mappings in place of the (quasi-) nonexpansive mappings and the Bregman projection in place of the metric projection.

In 1967, Bregman [6] discovered an elegant and effective technique using the Bregman distance function $D_f(\cdot,\cdot)$ in the process of designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which the Bregman's technique has been applied in various ways in order to design and analyze iterative algorithms for solving the feasibility and optimization problems, for approximating the variational inequalities and equilibrium problems, for computing the fixed points of nonlinear mappings and so on (see, e.g., [7, 14–16, 20, 22, 27] and the references therein).

In 2013, Agarwal et al. [1] proved the strong convergence theorems for finding the common solutions of the equilibrium problem (1.2) and the fixed point problem of a weak Bregman relatively nonexpansive mapping in real reflexive Banach spaces. Recently, Kazmi et al. [18] introduced the following algorithm:

$$\begin{cases} x_1, z_1 \in C, \\ u_n = \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(Tx_n)), \\ z_{n+1} = \mathrm{Res}^f_{G,\phi} u_n, \\ C_n = \{z \in C : D_f(z, z_{n+1}) \leqslant \alpha_n D_f(z, z_n) + (1-\alpha_n) D_f(z, x_n)\}, \\ Q_n = \{z \in C : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n\rangle \leqslant 0\}, \\ x_{n+1} = \mathrm{proj}^f_{C_n \cap Q_n} x_1, \quad \forall n \geqslant 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. They proved a strong convergence theorem for finding a common solution of a generalized equilibrium problem and a fixed point problem for a Bregman relatively nonexpansive mapping in reflexive Banach spaces.

Recall the generalized equilibrium problem which is to find $x \in C$ such that

$$G(x,y) + \phi(x,y) - \phi(x,x) \geqslant 0, \quad \forall y \in C, \tag{1.4}$$

where $\phi : C \times C \longrightarrow \mathbb{R}$ is a bifunction. The solution set of the problem (1.4) is denoted by $GEP(G,\phi)$.

Motivated and inspired by above works, the purpose of this paper is to establish a new iterative method for finding a common solution of the mixed equilibrium problems and the fixed point problems for a countable family of Bregman relatively nonexpansive mappings in reflexive Banach spaces. The strong convergence theorems under suitable control conditions are proven and a numerical example of the iterative algorithm supporting our main result is also illustrated.

2. Preliminaries

Throughout this paper, we let E be a reflexive Banach space and with dual E^*, $f : E \to (-\infty, +\infty]$ be a proper lower semicontinuous convex function. We denote the domain of f by $\mathrm{dom} f$, that is $\mathrm{dom} f = \{x \in E : f(x) < +\infty\}$. The subdifferential of f at $x \in \mathrm{int}(\mathrm{dom} f)$ is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \leqslant f(y), \quad \forall y \in E\},$$

and the Fenchel conjugate of f is the function $f^* : E^* \longrightarrow (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}.$$

We know that the Young-Fenchel inequality holds:

$$\langle x^*, x\rangle \leqslant f(x) + f^*(x^*), \ \ \forall x \in E, \ x^* \in E^*.$$

Furthermore, we know that $x^* \in \partial f(x)$ if and only if $f(x) + f^*(x^*) = \langle x^*, x\rangle$ for all $x \in E$. It is not difficult to check that f^* is a proper convex and lower semicontinuous function. A function f on E is said to be strong coercive if

$$\lim_{\|x\|\to+\infty} \left(\frac{f(x)}{\|x\|}\right) = +\infty.$$

For any $x \in \text{int}(\text{dom}f)$ and $y \in E$, the right-hand derivative of f at x in the direction y is defined by

$$f^0(x, y) := \lim_{t\to0^+} \frac{f(x + ty) - f(x)}{t}. \tag{2.1}$$

The function f is said to be Gâteaux differentiable at x if the limit as $t \longrightarrow 0^+$ in (2.1) exists for any y. In this case, the gradient of f at x is the linear function $\nabla f(x)$, which is defined by $\langle y, \nabla f(x)\rangle := f^0(x, y)$ for all $y \in E$. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at each $x \in \text{int}(\text{dom}f)$. When the limit as $t \longrightarrow 0^+$ in (2.1) is attained uniformly $\|y\| = 1$, we say that f is Frêchet differentiable at x. Finally f is said to be uniform Frêchet differentiable on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

The Legendre function f is defined from a general Banach space E into $(-\infty, +\infty]$, see [4]. It is well known that in reflexive spaces, f is the Legendre function if and only if it satisfies the following conditions:

(L_1) $\text{int}(\text{dom}f) \neq \emptyset$, f is Gâteaux differentiable on $\text{int}(\text{dom}f)$ and $\text{dom}\nabla f = \text{int}(\text{dom}f)$;

(L_2) $\text{int}(\text{dom}f^*) \neq \emptyset$, f^* is Gâteaux differentiable on $\text{int}(\text{dom}f^*)$ and $\text{dom}\nabla f^* = \text{int}(\text{dom}f^*)$.

Remark 2.1 ([4]). If E is a reflexive Banach space and $f : E \longrightarrow (-\infty, +\infty]$ is the Legendre function, then all of the following conditions are true:

(a) f is the Legendre function if and only if f^* is the Legendre function;

(b) $(\partial f)^{-1} = \partial f^*$;

(c) $\nabla f = (\nabla f^*)^{-1}$, $\text{ran}\nabla f = \text{dom}\nabla f^* = \text{int}(\text{dom}f^*)$, $\text{ran}\nabla f^* = \text{dom}\nabla f = \text{int}(\text{dom}f)$;

(d) the functions f and f^* are strictly convex on the interior of respective domains.

Example 2.2 ([4]). Let E be a smooth and strictly convex Banach space. One important and interesting Legendre function is $\frac{1}{p}\|\cdot\|^p (1 < p < \infty)$. In this case, the gradient ∇f of f is coincident with the generalized duality mapping of E, i.e., $\nabla f = J_p$ $(1 < p < \infty)$. In particular, $\nabla f = I$ the identity mapping in Hilbert spaces.

Definition 2.3 ([6]). Let $f : E \longrightarrow (-\infty, +\infty]$ be a Gâteaux differentiable and convex function. The Bregman distance with respect to f is the bifunction $D_f : \text{dom}f \times \text{int}(\text{dom}f) \longrightarrow [0, +\infty)$ defined by

$$D_f(y, x) := f(y) - f(x) - \langle\nabla f(x), y - x\rangle. \tag{2.2}$$

Remark 2.4 ([23]). The Bregman distance D_f is not a distance in the usual sense because D_f is not symmetric and does not satisfy the triangle inequality. However, D_f satisfies the three point identity:

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle\nabla f(z) - \nabla f(y), x - y\rangle,$$

for any $x \in \text{dom}f$ and $y, z \in \text{int}(\text{dom}f)$.

Definition 2.5 ([6]). Let C be a nonempty closed convex subset of $\text{int}(\text{dom}f)$, $f : E \longrightarrow (-\infty, +\infty]$ be a Gâteaux differentiable and convex function. The Bregman projection with respect to f of $x \in \text{int}(\text{dom}f)$ onto C is defined as the necessarily unique vector $\text{proj}^f_C(x) \in C$, which satisfies

$$D_f(\text{proj}^f_C(x), x) = \inf\{D_f(y, x) : y \in C\}. \tag{2.3}$$

Remark 2.6 ([1]). In Example 2.2, if $f(x) = \frac{1}{2}\|x\|^2$, $\forall x \in E$, then we have $\nabla f = J$, where J is the normalized duality mapping from E to 2^{E^*}, and hence $D_f(x,y)$ is reduced to the Lyapunov function defined by $\Phi(x,y) = \|y\|^2 - 2\langle Jx, y\rangle + \|x\|^2$, $\forall x, y \in E$, which is introduced by Alber [2], and so we obtain that the Bregman projection $\mathrm{proj}_C^f(x)$ is reduced to the generalized projection $\prod_C(x)$, which is defined by

$$\Phi(\prod_C(x), x) = \min_{y \in C} \Phi(y, x).$$

Moreover, in Hilbert spaces, the Bregman projection $\mathrm{proj}_C^f(x)$ is reduced to the metric projection of x onto C.

Definition 2.7 ([8]). Let $f : E \longrightarrow (-\infty, +\infty]$ be a Gâteaux differentiable and convex function, $\nu_f : \mathrm{int(dom}f) \times [0, +\infty) \longrightarrow [0, +\infty)$, define the modulus of total convexity of the function f at x by

$$\nu_f(x,t) := \inf\{D_f(y,x) : y \in \mathrm{dom}f, \|y - x\| = t\}.$$

Then the function f is called to be

(a) totally convex at a point $x \in \mathrm{int(dom}f)$, if the modulus of total convexity of the function f at x is positive, $\nu_f(x,t) > 0$ whenever $t > 0$;

(b) totally convex, if it is totally convex at every point $x \in \mathrm{int(dom}f)$, let B be a nonempty bounded subset of E, define the modulus of total convexity of the function f on the set B by

$$\nu_f(B,t) := \inf\{\nu_f(x,t) : x \in B \cap \mathrm{dom}f\};$$

(c) totally convex on bounded sets, if the modulus of total convexity of the function f on the set B is positive, $\nu_f(B,t) > 0$ for any nonempty bounded subset B of E and $t > 0$.

Lemma 2.8 ([9]). *Let $f : E \longrightarrow (-\infty, +\infty]$ be a Legendre function. Then, the function f is totally convex on bounded sets if and only if f is uniformly convex on bounded subsets of E.*

Lemma 2.9 ([29]). *Let $f : E \longrightarrow \mathbb{R}$ be a strong coercive and uniformly convex on bounded subsets of E, then f^* is bounded and uniformly Frêchet differentiable on bounded subsets of E^*.*

Lemma 2.10 ([21]). *Let C be a bounded subset of a reflexive Banach space E and $f : E \to (-\infty, +\infty]$ be uniformly Frêchet differentiable and bounded on $C \subset E$. Then, f is uniformly continuous on $C \subset E$ and ∇f is uniformly continuous on a bounded subset C from the strong topology of E to the strong topology of E^*.*

Definition 2.11 ([22]). The function $f : E \longrightarrow (-\infty, +\infty]$ is called sequentially consistent, if for any two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathrm{int(dom}f)$ and $\mathrm{dom}f$, respectively such that sequence $\{x_n\}$ is bounded, then

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \text{ implies } \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Lemma 2.12 ([11]). *If $f : E \longrightarrow (-\infty, +\infty]$ is a convex function whose domain contains at least two points, then, f is totally convex on bounded sets if and only if it is sequentially consistent.*

Let $f : E \longrightarrow \mathbb{R}$ be a Legendre and Gâteaux differentiable function. We make use of the function $V_f : E \times E^* \longrightarrow [0, +\infty)$ associated with f, which is defined by

$$V_f(x, x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*), \quad \forall x \in E, x^* \in E^*.$$

Then V_f is nonnegative and

$$V_f(x, x^*) = D_f(x, \nabla f(x^*)), \quad \forall x \in E, x^* \in E^*.$$

Moreover, by the subdifferential inequality,

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \leqslant V_f(x, x^* + y^*), \quad \forall x \in E, x^*, y^* \in E^*,$$

(for more details see [2]).

Lemma 2.13 ([19]). *Let $f : E \longrightarrow (-\infty, +\infty]$ be a proper lower semicontinuous and convex function, then $f^* : E^* \longrightarrow (-\infty, +\infty]$ is proper weak* lower semicntinuous and convex. Hence, V_f is convex in the second variable. Thus, for all $z \in E$, we have*

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)\right) \leqslant \sum_{i=1}^{N} t_i D_f(z, x_i), \tag{2.4}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$.

Lemma 2.14 ([22]). *Let $f : E \longrightarrow (-\infty, +\infty]$ be a Gâteaux differentiable and totally convex function. If $x_1 \in E$ and the sequence $\{D_f(x_n, x_1)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 2.15 ([25]). *Let $f : E \longrightarrow (-\infty, +\infty]$ be a Legendre function such that ∇f^* is bounded on bounded subsets of $\text{int}(\text{dom} f)$. If $x_1 \in E$ and $\{D_f(x_1, x_n)\}$ is bounded, then the sequence $\{x_n\}$ is bounded.*

Lemma 2.16 ([11]). *Let $f : E \longrightarrow (-\infty, +\infty]$ be a Gâteaux differentiable and totally convex function on $\text{int}(\text{dom} f)$. Let $x \in \text{int}(\text{dom} f)$ and $C \subset \text{int}(\text{dom} f)$ be a nonempty closed convex set. If $z \in C$, then the following conditions are equivalent:*

(i) *the vector $z \in C$ is the Bregman projection of x onto C with respect to f, i.e., $z = \text{proj}_C^f(x)$;*
(ii) *the vector $z \in C$ is the unique solution of the variational inequality:*

$$\langle \nabla f(x) - \nabla f(z), z - y \rangle \geqslant 0, \quad \forall y \in C;$$

(iii) *the vector z is the unique solution of the inequality:*

$$D_f(y, z) + D_f(z, x) \leqslant D_f(y, x), \quad \forall y \in C. \tag{2.5}$$

Definition 2.17 ([20]). Let T be a mapping from C into itself. A point $\hat{x} \in C$ is said to be an asymptotic fixed point of T if there exists a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup \hat{x}$ and $\|x_n - Tx_n\| \longrightarrow 0$. We denote the set of asymptotic fixed points of T by $\hat{F}(T)$.

Definition 2.18 ([13]). Let $T : C \longrightarrow \text{int}(\text{dom} f)$ be a mapping. Then

(a) T is said to be Bregman quasi-nonexpansive if

$$F(T) \neq \emptyset \text{ and } D_f(p, Tx) \leqslant D_f(p, x), \quad \forall x \in C, p \in F(T);$$

(b) T is said to be Bregman relatively nonexpansive if

$$\hat{F}(T) = F(T) \neq \emptyset \text{ and } D_f(p, Tx) \leqslant D_f(p, x), \quad \forall x \in C, p \in F(T);$$

(c) T is said to be Bregman firmly nonexpansive if

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leqslant D_f(Tx, y) + D_f(Ty, x), \quad \forall x, y \in C.$$

Assumption 2.19. Let $G : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying the following assumptions:

(i) $G(x, x) = 0$ for all $x \in C$;
(ii) G is monotone, i.e., $G(x, y) + G(y, x) \leqslant 0$ for all $x, y \in C$;
(iii) for each $x, y, z \in C$, $\limsup_{t \to 0^+} G(tz + (1-t)x, y) \leqslant G(x, y)$;
(iv) for each $x \in C$, $G(x, \cdot)$ is convex and lower semicontinuous.

Assumption 2.20. The function $\psi : C \longrightarrow \mathbb{R}$ satisfies the following assumptions:

(i) ψ is lower semicontinuous;

(ii) ψ is convex.

Lemma 2.21 ([17]). *Let $f : E \to (-\infty, +\infty]$ be a strong coercive Legendre function and C be a nonempty closed convex subset of* $\mathrm{int}(\mathrm{dom} f)$. *Let $G : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 2.19 and $\psi : C \longrightarrow \mathbb{R}$ satisfying Assumption 2.20. For $x \in E$ and define a mapping $\mathrm{Res}^f_{G,\psi} : E \to 2^C$ as follows:*

$$\mathrm{Res}^f_{G,\psi}(x) = \{z \in C : G(z,y) + \psi(y) - \psi(z) + \langle \nabla f(z) - \nabla f(x), y - z\rangle \geqslant 0, \forall y \in C\}.$$

Then the following statements are true:

(1) $\mathrm{Res}^f_{G,\psi}$ is single-valued and $\mathrm{dom}(\mathrm{Res}^f_{G,\psi}) = E$;
(2) $\mathrm{Res}^f_{G,\psi}$ is Bregman firmly nonexpansive;
(3) $MEP(G,\psi)$ is a closed convex subset of C and $MEP(G,\psi) = F(\mathrm{Res}^f_{G,\psi})$;
(4) for all $x \in E, u \in F(\mathrm{Res}^f_{G,\psi})$,

$$D_f(u, \mathrm{Res}^f_{G,\psi}x) + D_f(\mathrm{Res}^f_{G,\psi}x, x) \leqslant D_f(u,x). \tag{2.6}$$

Let $CB(C)$ denote the family of nonempty closed bounded subsets of C.

Lemma 2.22 ([26]). *Let E be a reflexive Banach space, and let $f : E \longrightarrow \mathbb{R}$ be uniformly Frêchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty closed and convex subset of* $\mathrm{int}(\mathrm{dom} f)$ *and $T : C \longrightarrow CB(C)$ be a Bregman relatively nonexpansive mapping. Then $F(T)$ is closed and convex.*

Lemma 2.23 ([22]). *Let $f : E \longrightarrow (-\infty, +\infty]$ be a Gâteaux differentiable and totally convex function, x_1 be an element in E and C be a nonempty closed convex subset of E. Suppose that the sequence $\{x_n\}$ is bounded and the weak limits of any subsequence of a sequence $\{x_n\}$ belong to $C \subset E$. If $D_f(x_n, x_1) \leqslant D_f(\mathrm{proj}^f_C(x_1), x_1)$ for any $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to $\mathrm{proj}^f_C(x_1)$.*

3. Main Result

In this section, we prove the strong convergence theorems for the common solutions of the mixed equilibrium problems and the common fixed points for a countable family of Bregman relatively nonexpansive mappings in reflexive Banach spaces.

Theorem 3.1. *Let E be a reflexive Banach space with dual E^* and C be a nonempty closed convex subset of E such that $C \subset \mathrm{int}(\mathrm{dom} f)$. Let $f : E \longrightarrow (-\infty, +\infty]$ be a strong coercive Legendre function which is bounded uniformly Frêchet differentiable and totally convex on bounded subsets of E, $G : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying the Assumption 2.19 and $\psi : C \longrightarrow \mathbb{R}$ satisfy the Assumption 2.20. Let $\{T_i : C \longrightarrow C\}_{i=1}^N$ be a countable family of Bregman relatively nonexpansive mappings. Assume that $\Omega := \bigcap_{i=1}^N F(T_i) \cap MEP(G,\psi) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by the iterative scheme:*

$$\begin{cases} x_1 \in C, T_i x_1 = z_1^i \in C; \\ u_n^i = \nabla f^*(\alpha_n \nabla f(z_n^i) + (1-\alpha_n)\nabla f(T_i x_n)); \\ z_{n+1}^i = \mathrm{Res}^f_{G,\psi}(u_n^i); \\ C_n^i = \{z \in C : D_f(z, z_{n+1}^i) \leqslant \alpha_n D_f(z, z_n^i) + (1-\alpha_n) D_f(z, x_n)\}; \\ C_n = \bigcap_{i=1}^N C_n^i; \\ Q_n = \{z \in C : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n\rangle \leqslant 0\}; \\ x_{n+1} = \mathrm{proj}^f_{C_n \cap Q_n} x_1, \quad \forall n \geqslant 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. Then, $\{x_n\}$ converges strongly to $\mathrm{proj}^f_\Omega x_1$ where $\mathrm{proj}^f_\Omega x_1$ is the Bregman projection of C onto Ω.

Proof. The proof is separated into seven steps.

Step 1: We will show that Ω is closed and convex. By the result of Lemma 2.22, we obtain that $F(T_i)$ is closed and convex for all $i = 1, 2, \ldots, N$ which implies that $\bigcap_{i=1}^{N} F(T_i)$ is also and follows from Lemma 2.21 (3), we have $MEP(G, \psi)$ is closed and convex and hence $\Omega := \bigcap_{i=1}^{N} F(T_i) \cap MEP(G, \psi)$ is closed and convex.

Step 2: We will prove that $C_n \cap Q_n$ is closed and convex for all n. First, we will show that Q_n is convex for all $n \geqslant 1$. Let $a, b \in Q_n$ and $t \in [0, 1]$, setting $w = ta + (1-t)b$. Then

$$\langle \nabla f(x_1) - \nabla f(x_n), a - x_n \rangle \leqslant 0 \tag{3.2}$$

and

$$\langle \nabla f(x_1) - \nabla f(x_n), b - x_n \rangle \leqslant 0. \tag{3.3}$$

Multiplying t and $(1-t)$ on both sides of (3.2) and (3.3), respectively, we obtain that

$$\langle \nabla f(x_1) - \nabla f(x_n), ta + (1-t)b - x_n \rangle \leqslant 0,$$

implies that

$$\langle \nabla f(x_1) - \nabla f(x_n), w - x_n \rangle \leqslant 0.$$

Therefore, $w \in Q_n$ and so Q_n is convex. Let $\{v_m\}$ be a sequence in Q_n with $v_m \longrightarrow v$ as $m \longrightarrow \infty$. From the definition of Q_n, we have

$$\langle \nabla f(x_1) - \nabla f(x_n), v_m - x_n \rangle \leqslant 0,$$

implies that

$$\langle \nabla f(x_1) - \nabla f(x_n), v_m - v \rangle + \langle \nabla f(x_1) - \nabla f(x_n), v - x_n \rangle \leqslant 0.$$

Taking $m \longrightarrow \infty$, we obtain

$$\langle \nabla f(x_1) - \nabla f(x_n), v - x_n \rangle \leqslant 0.$$

Hence $v \in Q_n$, this shows that Q_n is closed for all $n \geqslant 1$. Next, we will show that C_n is closed for all $n \geqslant 1$. Let $\{s_m\}$ be a sequence in C_n with $s_m \longrightarrow s$ as $m \longrightarrow \infty$. Then $\{s_m\}$ is a sequence in C_n^i for all $i = 1, 2, \ldots, N$, by the definition of C_n^i, we have

$$D_f(s_m, z_{n+1}^i) \leqslant \alpha_n D_f(s_m, z_n^i) + (1-\alpha_n) D_f(s_m, x_n), \quad \forall i = 1, 2, \ldots, N. \tag{3.4}$$

By the equation (2.2), definition of the Bregman distance $D_f(\cdot, \cdot)$, we obtain that

$$\begin{aligned} f(s_m) - f(z_{n+1}^i) - \langle \nabla f(z_{n+1}^i), s_m - z_{n+1}^i \rangle &\leqslant \alpha_n \left(f(s_m) - f(z_{n+1}^i) - \langle \nabla f(z_n^i), s_m - z_n^i \rangle \right) \\ &\quad + (1-\alpha_n) \left(f(s_m) - f(x_n) - \langle \nabla f(x_n), s_m - x_n \rangle \right), \end{aligned} \tag{3.5}$$

it follows that

$$\begin{aligned} \alpha_n \langle \nabla f(z_n^i), s_m - z_n^i \rangle + (1-\alpha_n) \langle \nabla f(x_n), s_m - x_n \rangle - \langle \nabla f(z_{n+1}^i), s_m - z_{n+1}^i \rangle \\ \leqslant f(z_{n+1}^i) - \alpha_n f(z_n^i) - (1-\alpha_n) f(x_n). \end{aligned} \tag{3.6}$$

This implies that

$$\begin{aligned} \alpha_n (\langle f(z_n^i), s_m - s \rangle + \langle \nabla f(z_n^i), s - z_n^i \rangle) + (1-\alpha_n) \left(\langle \nabla f(x_n), s_m - s \rangle + \langle \nabla f(x_n), s - x_n \rangle \right) \\ - \langle \nabla f(z_{n+1}^i), s_m - s \rangle - \langle \nabla f(z_{n+1}^i), s - z_{n+1}^i \rangle \leqslant f(z_{n+1}^i) - \alpha_n f(z_n^i) - (1-\alpha_n) f(x_n). \end{aligned}$$

Taking $m \longrightarrow \infty$, we obtain that

$$\alpha_n \langle \nabla f(z_n^i), s - z_n^i \rangle + (1-\alpha_n) \langle \nabla f(x_n), s - x_n \rangle - \langle \nabla f(z_{n+1}^i), s - z_{n+1}^i \rangle$$

$$\leqslant f(z^i_{n+1}) - \alpha_n f(z^i_n) - (1-\alpha_n) f(x_n), \quad \forall i = 1,2,\ldots,N,$$

which implies that $s \in C^i_n$ for all $i = 1,2,\ldots,N$. Therefore, $s \in C_n$ and C_n is closed. For any $a, b \in C_n$, we have $a, b \in C^i_n$ for all $i = 1,2,\ldots,N$ and $a, b \in C$. Since C is convex, $w = ta + (1-t)b \in C$ for $t \in [0,1]$. By the definition of C_n, we have

$$D_f(a, z^i_{n+1}) \leqslant \alpha_n D_f(a, z^i_n) + (1-\alpha_n) D_f(a, x_n)$$

and

$$D_f(b, z^i_{n+1}) \leqslant \alpha_n D_f(b, z^i_n) + (1-\alpha_n) D_f(b, x_n).$$

It follows from (3.4), (3.5), and (3.6), we observe that the above two inequalities are equivalent to

$$\alpha_n \langle \nabla f(z^i_n), a - z^i_n \rangle + (1-\alpha_n) \langle \nabla f(x_n), a - x_n \rangle - \langle \nabla f(z^i_{n+1}), a - z^i_{n+1} \rangle \leqslant f(z^i_{n+1}) - f(z^i_n) - (1-\alpha_n) f(x_n) \tag{3.7}$$

and

$$\alpha_n \langle \nabla f(z^i_n), b - z^i_n \rangle + (1-\alpha_n) \langle \nabla f(x_n), b - x_n \rangle - \langle \nabla f(z^i_{n+1}), b - z^i_{n+1} \rangle \leqslant f(z^i_{n+1}) - f(z^i_n) - (1-\alpha_n) f(x_n). \tag{3.8}$$

Multiplying t and $(1-t)$ on both sides of (3.7) and (3.8), respectively, we obtain that

$$\alpha_n \langle \nabla f(z^i_n), ta + (1-t)b - z^i_n \rangle + (1-\alpha_n) \langle \nabla f(x_n), ta + (1-t)b - x_n \rangle - \langle \nabla f(z^i_{n+1}), ta + (1-t)b - z^i_{n+1} \rangle \leqslant f(z^i_{n+1}) - f(z^i_n) - (1-\alpha_n) f(x_n), \quad \forall i = 1,2,\ldots,N.$$

From the above inequality, we can rewrite that

$$\alpha_n \langle \nabla f(z^i_n), w - z^i_n \rangle + (1-\alpha_n) \langle \nabla f(x_n), w - x_n \rangle - \langle \nabla f(z^i_{n+1}), w - z^i_{n+1} \rangle \leqslant f(z^i_{n+1}) - f(z^i_n) - (1-\alpha_n) f(x_n),$$

which implies that $w \in C^i_n$ for all $i = 1,2,\ldots,N$ and hence $w \in C_n$. It follows that C_n is closed and convex for all $n \geqslant 1$. Therefore, $C_n \cap Q_n$ is closed and convex for all $n \geqslant 1$.

Step 3: We show that $\Omega \subset C_n \cap Q_n$ for all $n \geqslant 1$. Let $p \in \Omega$ be given. Since $\mathrm{Res}^f_{G,\psi}$ is single-valued, $\mathrm{Res}^f_{G,\psi}(u^i_n) = z^i_{n+1}$ for all $i = 1,2,\ldots,N$. Then, by the results of Lemma 2.21 (3) and (2.4), we obtain that

$$\begin{aligned}
D_f(p, z^i_{n+1}) &= D_f(p, \mathrm{Res}^f_{G,\psi}(u^i_n)) \\
&\leqslant D_f(p, u^i_n) - D_f(\mathrm{Res}^f_{G,\psi}(u^i_n), u^i_n) \\
&\leqslant D_f(p, u^i_n) \\
&= D_f(p, \nabla f^*(\alpha_n \nabla f(z^i_n) + (1-\alpha_n) \nabla f(T_i x_n))) \\
&\leqslant \alpha_n D_f(p, z^i_n) + (1-\alpha_n) D_f(p, T_i x_n) \\
&\leqslant \alpha_n D_f(p, z^i_n) + (1-\alpha_n) D_f(p, x_n), \quad \forall i = 1,2,\ldots,N.
\end{aligned} \tag{3.9}$$

This implies that $p \in C^i_n$ for all $i = 1,2,\ldots,N$ and hence $p \in C_n = \bigcap_{i=1}^N C^i_n$. Therefore, $\Omega \subset C_n$ for all $n \geqslant 1$. Next, we show by induction that $\Omega \subset C_n \cap Q_n$ for all $n \geqslant 1$. By the definition of Q_n, we obtain that $Q_1 = C$, implies that $\Omega \subset C_1 \cap Q_1$. Suppose that $\Omega \subset C_k \cap Q_k$ for some $k > 0$. Since $C_k \cap Q_k$ is closed and convex, it follows from (2.3), definition of Bregman projection, there exists $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = \mathrm{proj}^f_{C_k \cap Q_k}(x_1)$. From Lemma 2.16 (ii), we have

$$\langle \nabla f(x_1) - \nabla f(x_{k+1}), x_{k+1} - z \rangle \geqslant 0, \quad \forall z \in C_k \cap Q_k.$$

Since $\Omega \subset C_k \cap Q_k$,

$$\langle \nabla f(x_1) - \nabla f(x_{k+1}), x_{k+1} - p \rangle \geqslant 0, \quad \forall p \in \Omega,$$

and hence $p \in Q_{k+1}$. Since $\Omega \subset C_n$ for all $n \geqslant 1$, $\Omega \subset C_{k+1} \cap Q_{k+1}$. Therefore, we have $\Omega \subset C_n \cap Q_n$, for all $n \geqslant 1$ and hence $x_{n+1} = \mathrm{proj}^f_{C_n \cap Q_n}(x_1)$ is well-defined for all $n \geqslant 1$. This means that $\{x_n\}$ is well-defined.

Step 4: We will prove that the sequences $\{x_n\}$, $\{z_n^i\}_{n=1}^{\infty}$ and $\{T_i x_n\}_{n=1}^{\infty}$ are bounded for all $i = 1, 2, \dots, N$. It follows from the definition of Q_n and Lemma 2.6 that $x_n = \mathrm{proj}^f_{Q_n}(x_1)$. By using (2.5), we have

$$D_f(x_n, x_1) = D_f(\mathrm{proj}^f_{Q_n}(x_1), x_1) \leqslant D_f(p, x_1) - D_f(p, \mathrm{proj}^f_{Q_n}(x_1)) \leqslant D_f(p, x_1), \quad \forall p \in \Omega \subset Q_n.$$

Hence $\{D_f(x_n, x_1)\}$ is bounded. Therefore by Lemma 2.14, $\{x_n\}$ is bounded. On the orther hand, we have

$$D_f(p, x_n) = D_f(p, \mathrm{proj}^f_{C_{n-1}\cap Q_{n-1}}(x_1)) \leqslant D_f(p, x_1) - D_f(x_n, x_1) \leqslant D_f(p, x_1),$$

implies that $\{D_f(p, x_n)\}$ is bounded. Now, it follows from the fact $D_f(p, T_i x_n) \leqslant D_f(p, x_n)$ for all $p \in \Omega$, $i = 1, 2, \dots, N$, which implies that $\{D_f(p, T_i x_n)\}_{n=1}^{\infty}$ is bounded for all $i = 1, 2, \dots, N$. Since f is strong coercive, f^* and ∇f^* are bounded on bounded subsets. It follows from Lemma 2.15, we obtain that $\{T_i x_n\}_{n=1}^{\infty}$ is bounded for all $i = 1, 2, \dots, N$. Since $\{D_f(p, x_n)\}$ is bounded, there exists $M > 0$ such that $D_f(p, x_n) \leqslant M$. It follows from (3.9), we obtain that

$$D_f(p, z_{n+1}^i) \leqslant \alpha_n D_f(p, z_n^i) + (1 - \alpha_n) M.$$

Let $K = \max\{D_f(p, z_1^i), M\}$. Clearly that $D_f(p, z_1^i) \leqslant K$ for all $i = 1, 2, \dots, N$. Let $D_f(p, z_n^i) \leqslant K$ for some n, then it follows from above inequality, we get that

$$D_f(p, z_{n+1}^i) \leqslant \alpha_n K + (1 - \alpha_n) K \leqslant K, \quad \forall i = 1, 2, \dots, N.$$

It follows that $\{D_f(p, z_n^i)\}_{n=1}^{\infty}$ is bounded, for all $i = 1, 2, \dots, N$. Again, by Lemma 2.15, we have $\{z_n^i\}_{n=1}^{\infty}$ is also bounded for all $i = 1, 2, \dots, N$.

Step 5: We will show that $\lim_{n\to\infty} \|x_n - z_{n+1}^i\| = 0$, $\lim_{n\to\infty} \|x_n - u_n^i\| = 0$ and $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i = 1, 2, \dots, N$. We know that $x_{n+1} = \mathrm{proj}^f_{C_n \cap Q_n}(x_1)$ and $x_n = \mathrm{proj}^f_{Q_n}(x_1)$, we have

$$D_f(x_n, x_1) \leqslant D_f(x_{n+1}, x_1), \quad \forall n \geqslant 1.$$

It follows that $\{D_f(x_n, x_1)\}$ is nondecreasing. Sine $\{D_f(x_n, x_1)\}$ is bounded, $\lim_{n\to\infty} D_f(x_n, x_1)$ exists. Further, the inequality

$$D_f(x_{n+1}, x_n) = D_f(x_{n+1}, \mathrm{proj}^f_{Q_n}(x_1)) \leqslant D_f(x_{n+1}, x_1) - D_f(\mathrm{proj}^f_{Q_n}(x_1), x_1) = D_f(x_{n+1}, x_1) - D_f(x_n, x_1),$$

implies that

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0. \tag{3.10}$$

Since f is totally convex on bounded sets, f is sequentially consistent. It follows from Lemma 2.11 and above equality, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.11}$$

It follows from the three point identity of the Bregman distance, we have

$$D_f(x_{n+1}, z_n^i) = \langle \nabla f(z_n^i) - \nabla f(x_{n+1}), p - x_{n+1} \rangle + D_f(p, z_n^i) - D_f(p, x_{n+1}).$$

Since f is bounded on bounded subsets of E, ∇f is also bounded on bounded subsets of E. It follows from boundedness of $\{x_n\}$, $\{z_n^i\}_{n=1}^{\infty}$ and $\{T_i x_n\}_{n=1}^{\infty}$, we obtain that the sequences $\{\nabla f(x_n)\}$, $\{\nabla f(z_n^i)\}_{n=1}^{\infty}$ and $\{\nabla f(T_i x_n)\}_{n=1}^{\infty}$ are bounded in E^* for all $i = 1, 2, \dots, N$, which implies that $\{D_f(x_{n+1}, z_n^i)\}_{n=1}^{\infty}$ is bounded. It follows from $x_{n+1} = \mathrm{proj}^f_{C_n \cap Q_n}(x_1) \in C_n$ and the definition of C_n, we have

$$D_f(x_{n+1}, z_{n+1}^i) \leqslant \alpha_n D_f(x_{n+1}, z_n^i) + (1 - \alpha_n) D_f(x_{n+1}, x_n), \quad \forall i = 1, 2, \dots, N.$$

Since $\{D_f(x_{n+1}, z_n^i)\}_{n=1}^{\infty}$ is bounded and $\lim_{n\to\infty} \alpha_n = 0$, it follows from the above inequality and (3.10), we obtain that

$$\lim_{n\to\infty} D_f(x_{n+1}, z_{n+1}^i) = 0, \ \ \forall i = 1, 2, \dots, N.$$

Since f is totally convex on bounded subsets, again using Lemma 2.11, we have

$$\lim_{n\to\infty} \|x_{n+1} - z_{n+1}^i\| = 0, \ \ \forall i = 1, 2, \dots, N. \tag{3.12}$$

Taking into account

$$\|x_n - z_{n+1}^i\| \leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - z_{n+1}^i\|,$$

it follows from (3.11) and (3.12), we get

$$\lim_{n\to\infty} \|x_n - z_{n+1}^i\| = 0, \ \ \forall i = 1, 2, \dots, N. \tag{3.13}$$

It follows from Lemma 2.10, we have f and ∇f are uniformly continuous since f is uniformly Frêchet differentiable on bounded subsets. Therefore,

$$\lim_{n\to\infty} |f(x_n) - f(z_{n+1}^i)| = 0, \ \ \forall i = 1, 2, \dots, N \tag{3.14}$$

and

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(z_{n+1}^i)\|, \ \ \forall i = 1, 2, \dots, N. \tag{3.15}$$

We next consider the following inequality, for each $i = 1, 2, \dots, N$,

$$\begin{aligned} &D_f(p, x_n) - D_f(p, z_{n+1}^i) \\ &\quad = f(p) - f(x_n) - \langle \nabla f(x_n), p - x_n \rangle - \big(f(p) - f(z_{n+1}^i) - \langle \nabla f(z_{n+1}^i), p - z_{n+1}^i \rangle\big) \\ &\quad = f(z_{n+1}^i) - f(x_n) + \langle \nabla f(z_{n+1}^i), p - x_n \rangle + \langle \nabla f(z_{n+1}^i), x_n - z_{n+1}^i \rangle - \langle \nabla f(x_n), p - x_n \rangle \\ &\quad = f(z_{n+1}^i) - f(x_n) + \langle \nabla f(z_{n+1}^i) - \nabla f(x_n), p - x_n \rangle + \langle \nabla f(z_{n+1}^i), x_n - z_{n+1}^i \rangle. \end{aligned} \tag{3.16}$$

Since $\{z_{n+1}^i\}_{n=1}^{\infty}$ and $\{\nabla f(z_{n+1}^i)\}_{n=1}^{\infty}$ are bounded for all $i = 1, 2, \dots, N$, it follows from (3.13), (3.14), (3.15), and (3.16) that

$$\lim_{n\to\infty} \|D_f(p, x_n) - D_f(p, z_{n+1}^i)\| = 0, \ \ \forall i = 1, 2, \dots, N. \tag{3.17}$$

Moreover, it follows from (2.6) and Lemma 2.13, we obtain that, for each $i = 1, 2, \dots, N$,

$$\begin{aligned} D_f(z_{n+1}^i, u_n^i) &\leqslant D_f(p, u_n^i) - D_f(p, z_{n+1}^i) \\ &= D_f(p, \nabla f^*(\alpha_n \nabla f(z_n^i) + (1 - \alpha_n) \nabla f(T_i x_n))) - D_f(p, z_{n+1}^i) \\ &\leqslant \alpha_n D_f(p, z_n^i) + (1 - \alpha_n) D_f(p, T_i x_n) - D_f(p, z_{n+1}^i) \\ &\leqslant \alpha_n D_f(p, z_n^i) + (1 - \alpha_n) D_f(p, x_n) - D_f(p, z_{n+1}^i) \\ &= \alpha_n \left(D_f(p, z_n^i) - D_f(p, x_n)\right) + D_f(p, x_n) - D_f(p, z_{n+1}^i). \end{aligned} \tag{3.18}$$

Since $\{D_f(p, x_n)\}$ and $\{D_f(p, z_n^i)\}_{n=1}^{\infty}$ are bounded for all $i = 1, 2, \dots, N$, it follows from (3.17), (3.18), and $\lim_{n\to\infty} \alpha_n = 0$,

$$\lim_{n\to\infty} D_f(z_{n+1}^i, u_n^i) = 0, \ \ \forall i = 1, 2, \dots, N,$$

so, we have

$$\lim_{n\to\infty} \|z_{n+1}^i - u_n^i\| = 0, \ \ \forall i = 1, 2, \dots, N. \tag{3.19}$$

Taking into account

$$\|x_n - u_n^i\| \leqslant \|x_n - z_{n+1}^i\| + \|z_{n+1}^i - u_n^i\|,$$

and using (3.13) and (3.19), we get that

$$\lim_{n\to\infty}\|x_n - u_n^i\| = 0, \quad \forall i = 1,2,\ldots,N. \tag{3.20}$$

Since f is uniformly Frêchet differentiable and by Lemma 2.10, ∇f is uniformly continuous on bounded sets. It follows from (3.19) and (3.20), we obtain that

$$\lim_{n\to\infty}\|\nabla f(z_{n+1}^i) - \nabla f(u_n^i)\| = 0, \quad \forall i = 1,2,\ldots,N, \tag{3.21}$$

and

$$\lim_{n\to\infty}\|\nabla f(x_n) - \nabla f(u_n^i)\| = 0, \quad \forall i = 1,2,\ldots,N. \tag{3.22}$$

Furthermore, for each $i = 1,2,\ldots,N$, we now consider the following inequality

$$\begin{aligned}\|\nabla f(x_n) - \nabla f(u_n^i)\| &= \|\nabla f(x_n) - \nabla f\left(\nabla f^*\left(\alpha_n \nabla f(z_n^i) + (1-\alpha_n)\nabla f(T_i x_n)\right)\right)\| \\ &= \|\nabla f(x_n) - \alpha_n \nabla f(z_n^i) - (1-\alpha_n)\nabla f(T_i x_n)\| \\ &= \|\alpha_n\left(\nabla f(x_n) - \nabla f(z_n^i)\right) + (1-\alpha_n)\left(\nabla f(x_n) - \nabla f(T_i x_n)\right)\| \\ &\geqslant (1-\alpha_n)\|\nabla f(x_n) - \nabla f(T_i x_n)\| - \alpha_n\|\nabla f(x_n) - \nabla f(z_n^i)\|,\end{aligned}$$

which implies that

$$(1-\alpha_n)\|\nabla f(x_n) - \nabla f(T_i x_n)\| \leqslant \|\nabla f(x_n) - \nabla f(u_n^i)\| + \alpha_n\|\nabla f(x_n) - \nabla f(z_n^i)\|. \tag{3.23}$$

Since $\{\nabla f(x_n)\}$ and $\{\nabla f(z_n^i)\}_{n=1}^{\infty}$ are bounded for all $i = 1,2,\ldots,N$, it follows from (3.22), (3.23) and $\lim_{n\to\infty} \alpha_n = 0$, we have

$$\lim_{n\to\infty}\|\nabla f(x_n) - \nabla f(T_i x_n)\| = 0, \quad \forall i = 1,2,\ldots,N.$$

It follows from f is the Legendre function and f^* is uniformly Frêchet differentiable on bounded subsets, the above inequality yields that

$$\lim_{n\to\infty}\|x_n - T_i x_n\| = 0, \quad \forall i = 1,2,\ldots,N. \tag{3.24}$$

Step 6: We show that $x^* \in \Omega$. By the boundedness of the sequence $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^* \in C$ as $k \longrightarrow \infty$. It follows from (3.13) and (3.19), there exist subsequences $\{u_{n_k}^i\}$ of $\{u_n^i\}$ and $\{z_{n_k}^i\}$ of $\{z_n^i\}$ such that $u_{n_k}^i \rightharpoonup x^*$ and $z_{n_k}^i \rightharpoonup x^*x$ as $k \longrightarrow \infty$, for all $i = 1,2,\ldots,N$, respectively. The consequence of (3.24) is

$$\lim_{n\to\infty}\|x_{n_k} - T_i x_{n_k}\| = 0, \quad \forall i = 1,2,\ldots,N.$$

Since $x_{n_k} \rightharpoonup x^*$ and using the above equality, it follows from the definition of asymptotic fixed points, we have $x^* \in \hat{F}(T_i)$ for all $i = 1,2,\ldots,N$. Since $\{T_i\}_{i=1}^N$ is a countable family of Bregman relatively nonexpansive mappings, $x^* \in F(T_i)$ for all $i = 1,2,\ldots,N$, implies that $x^* \in \bigcap_{i=1}^{N} F(T_i)$. Next, we show that x^* is the solution of the mixed equilibrium problem. Since $z_{n+1}^i = \mathrm{Res}_{G,\psi}^f(u_n^i)$, for each $i = 1,2,\ldots,N$

$$G(z_{n_k+1}^i, y) + \psi(y) - \psi(z_{n_k+1}^i) + \langle \nabla f(z_{n_k+1}^i) - \nabla f(u_{n_k}^i), y - z_{n_k+1}^i\rangle \geqslant 0, \quad \forall y \in C.$$

Using the Assumption 2.19 (ii), we obtain that

$$\begin{aligned}\psi(y) - \psi(z_{n_k+1}^i) + \langle \nabla f(z_{n_k+1}^i) - \nabla f(u_{n_k}^i), y - z_{n_k+1}^i\rangle &\geqslant -G(z_{n_k+1}^i, y) \\ &\geqslant G(y, z_{n_k+1}^i), \quad \forall y \in C, \quad i = 1,2,\ldots,N.\end{aligned}$$

For any $y \in C$ and $t \in (0,1]$, we let $y_t = ty + (1-t)x^* \in C$. This implies that

$$\psi(y_t) - \psi(u^i_{n_k}) + \langle \nabla f(z^i_{n_k+1}) - \nabla f(u^i_{n_k}), y_t - z^i_{n_k+1}\rangle \geqslant G(y_t, z^i_{n_k+1}).$$

Using the Assumption 2.19 (iv) and the Assumption 2.20 (i), $G(x,\cdot)$ and ψ are lower semicontinuous, it follows from (3.21) and above inequality, this yields

$$\liminf_{k\to\infty}(G(y_t, z^i_{n_k+1}) - \psi(y_t) + \psi(z^i_{n_k+1})) \leqslant \liminf_{k\to\infty}\langle \nabla f(z^i_{n_k+1}) - \nabla f(u^i_{n_k}), y_t - z^i_{n_k+1}\rangle, \quad \forall i = 1,2,\dots,N.$$

This implies that

$$G(y_t, x^*) - \psi(y_t) + \psi(x^*) \leqslant 0.$$

Furthermore, we next consider the following inequality,

$$\begin{aligned}
0 &= G(y_t, y_t) + \psi(y_t) - \psi(y_t)\\
&= G(y_t, ty + (1-t)x^*) + \psi(ty + (1-t)x*) - \psi(y_t)\\
&\leqslant tG(y_t, y) + (1-t)G(y_t, x^*) + t\psi(y) + (1-t)\psi(x^*) - t\psi(y_t) - (1-t)\psi(y_t)\\
&= t\,(G(y_t, y) + \psi(y) - \psi(y_t)) + (1-t)\,(G(y_t, x^*) + \psi(x^*) - \psi(y_t))\\
&\leqslant t\,(G(y_t, y) + \psi(y) - \psi(y_t)),
\end{aligned}$$

which implies that

$$G(y_t, y) + \psi(y) - \psi(y_t) \geqslant 0.$$

It follows from the Assumption 2.19 (iii), we have

$$\begin{aligned}
0 &\leqslant \limsup_{t\to 0^+}(G(y_t, y) + \psi(y) - \psi(y_t))\\
&= \limsup_{t\to 0^+}(G(ty + (1-t)x^*, y) + \psi(y) - \psi(ty + (1-t)x^*)) \leqslant G(x^*, y) + \psi(y) - \psi(x^*).
\end{aligned}$$

This implies that x^* is a solution of the mixed equilibrium problem and hence $x^* \in MEP(G,\psi)$. To sum up, we have $x^* \in \Omega := \bigcap_{i=1}^{N} F(T_i) \cap MEP(G,\psi)$.

Step 7: We shall show that the sequence $\{x_n\}$ converges strongly to $x^* = \mathrm{proj}^f_\Omega(x_1)$. Since Ω is a nonempty closed convex subset of E, $\mathrm{proj}^f_\Omega(x_1)$ is well-defined. Let $u^* = \mathrm{proj}^f_\Omega(x_1)$ be given. It follows from $x_{n+1} = \mathrm{proj}^f_{C_n\cap Q_n}(x_1)$ and $\mathrm{proj}^f_\Omega(x_1) \in \Omega \subseteq C_n \cap Q_n$, we obtain that

$$D_f(x_{n+1}, x_1) \leqslant D_f(u^*, x_1).$$

Since $\{x_{n_k}\}$ is a weak convergent subsequence of $\{x_n\}$ and follows from Lemma 2.23, we obtain that $\{x_n\}$ converges strongly to u^*. By the uniqueness of the limit, we obtain that the sequence $\{x_n\}$ converges strongly to $x^* = \mathrm{proj}^f_\Omega(x_1)$. This completes the proof. □

If we assume that $T_i = T$ for each $i = 1,2,\dots,N$ and ψ is a zero mapping in Theorem 3.1, then we get the following corollary.

Corollary 3.2. *Let E be a reflexive Banch space with dual E^* and C be a nonempty closed convex subset of E such that $C \subset \mathrm{int}(\mathrm{dom} f)$. Let $f : E \longrightarrow (-\infty, +\infty]$ be a strong coercive Legendre function which is bounded uniformly Frêchet differentiable and totally convex on bounded subsets of E, $G : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying the*

Assumption 2.19. *Let* $T : C \longrightarrow C$ *be a Bregman relatively nonexpansive mapping. Assume that* $F(T) \cap EP(G) \neq \emptyset$. *Let* $\{x_n\}$ *be the sequence generated by the iterative scheme:*

$$\begin{cases} x_1 \in C, Tx_1 = z_1 \in C; \\ u_n = \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(Tx_n)); \\ z_{n+1} = \mathrm{Res}^f_{G,\psi}(u_n); \\ C_n = \{z \in C : D_f(z, z_{n+1}) \leqslant \alpha_n D_f(z, z_n) + (1-\alpha_n) D_f(z, x_n)\}; \\ Q_n = \{z \in C : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n\rangle \leqslant 0\}; \\ x_{n+1} = \mathrm{proj}^f_{C_n \cap Q_n} x_1, \ \forall n \geqslant 1, \end{cases}$$

where $\{\alpha_n\}$ *is a sequence in* $[0,1]$ *such that* $\lim_{n\to\infty} \alpha_n = 0$. *Then,* $\{x_n\}$ *converges strongly to* $\mathrm{proj}^f_{F(T)\cap EP(G)} x_1$.

In Theorem 3.1, if we assume that $MEP(G, \psi) = C$ and using the facts given in Example 2.2 for the generalized duality mapping J_p, then we obtain the following corollary.

Corollary 3.3. *Let* E *be a uniformly smooth and uniformly convex Banach space and* C *be a nonempty closed convex subset of* E *such that* $C \subset \mathrm{int}(\mathrm{dom} f)$. *Let* $f(x) = \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$ *and* $\{T_i : C \longrightarrow C\}_{i=1}^N$ *be a countable family of relatively nonexpansive mappings. Assume that* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. *Let* $\{x_n\}$ *be the sequence generated by the iterative scheme:*

$$\begin{cases} x_1 \in C, T_i x_1 = z_1^i \in C; \\ z_{n+1}^i = J_p^{-1}(\alpha_n J_p(z_n^i) + (1-\alpha_n) J_p(T_i x_n)); \\ C_n^i = \{z \in C : V(z, z_{n+1}^i) \leqslant \alpha_n V(z, z_n^i) + (1-\alpha_n) V(z, x_n)\}; \\ C_n = \bigcap_{i=1}^N C_n^i; \\ Q_n = \{z \in C : \langle J_p(x_1) - J_p(x_n), z - x_n\rangle \leqslant 0\}; \\ x_{n+1} = \mathrm{proj}^f_{C_n \cap Q_n} x_1, \ \forall n \geqslant 1, \end{cases}$$

where $\{\alpha_n\}$ *is a sequence in* $[0,1]$ *such that* $\lim_{n\to\infty} \alpha_n = 0$. *Then,* $\{x_n\}$ *converges strongly to* $\mathrm{proj}^f_{\bigcap_{i=1}^N F(T_i)} x_1$.

4. Applications

Zeros of maximal monotone operators

Let $A : E \longrightarrow 2^{E^*}$ be a set-valued mapping. Denote $G(A)$ by the graph of A, that is $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$. A multi-valued operator A is said to be monotone if $\langle x^* - y^*, x - y\rangle > 0$ for each $(x, x^*), (y, y^*) \in G(A)$. A monotone operator A is said to be maximal if its graph, $G(A)$ is not contained in the graph of any other monotone operators on E. Let $f : E \longrightarrow (-\infty, +\infty]$, then the resolvent of A, $\mathrm{Res}^f_{\lambda A} : E \longrightarrow 2^E$ is defined as follows:

$$\mathrm{Res}^f_{\lambda A}(x) = (\nabla f + \lambda A)^{-1} \circ \nabla f(x), \ \ \lambda > 0.$$

In 2003, Bauschke et al. [3] proved that $\mathrm{Res}^f_{\lambda A}$ is a single-valued and Bregman firmly nonexpansive mapping and $F(\mathrm{Res}^f_{\lambda A}) = A^{-1}(0^*) = \{x \in E : 0^* \in Ax\}$. It is known that if A is maximal monotone, then the set $A^{-1}(0^*)$ is closed and convex. We also define the Yosida approximation $A_\lambda : E \longrightarrow E$ by

$$A_\lambda(x) = \frac{1}{\lambda}(\nabla f - \nabla f \circ \mathrm{Res}^f_{\lambda A})(x), \ \ \forall x \in E, \lambda > 0.$$

It is shown in Reich and Sabach [22] that for any $x \in E$ and $\lambda > 0$, we have

(i) $(\mathrm{Res}^f_{\lambda A}(x), A_\lambda(x)) \in G(A)$;
(ii) $0^* \in Ax$ if and only if $0^* \in A_\lambda(x)$.

In 2011, Reich and Sabach [24] proved that if f is the Legendre function which is bounded uniformly Frêchet differentiable on bounded subsets of E, then $\hat{F}(\mathrm{Res}^f_{\lambda A}) = F(\mathrm{Res}^f_{\lambda A})$. We also know that if $\hat{F}(\mathrm{Res}^f_{\lambda A}) = F(\mathrm{Res}^f_{\lambda A})$, then a Bregman firmly nonexpansive mapping is a Bregman relatively nonexpansive mapping. Furthermore, if we take $MEP(G,\psi) = C$ and $T_i = \mathrm{Res}^f_{\lambda A_i}$ for all $i = 1,2,\ldots,N$ in Theorem 3.1, then we obtain the following consequence.

Theorem 4.1. *Let E be a reflexive Banch space with dual E^* and C be a nonempty closed convex subset of E such that $C \subset \mathrm{int}(\mathrm{dom} f)$. Let $f : E \longrightarrow (-\infty,+\infty]$ be a strong coercive Legendre function which is bounded uniformly Frêchet differentiable and totally convex on bounded subsets of E. Let $\{A_i : E \longrightarrow 2^{E^*}\}_{i=1}^N$ be a countable family of maximal monotone operators. Assume that $\bigcap_{i=1}^N A_i^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by the iterative scheme:*

$$\begin{cases} x_1 \in C, \mathrm{Res}^f_{\lambda A_i}(x_1) = z_1^i \in C; \\ z_{n+1}^i = \nabla f^*(\alpha_n \nabla f(z_n^i) + (1-\alpha_n)\nabla f(\mathrm{Res}^f_{\lambda A_i}(x_n))); \\ C_n^i = \{z \in C : D_f(z, z_{n+1}^i) \leqslant \alpha_n D_f(z, z_n^i) + (1-\alpha_n) D_f(z, x_n)\}; \\ C_n = \bigcap_{i=1}^N C_n^i; \\ Q_n = \{z \in C : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n \rangle \leqslant 0\}; \\ x_{n+1} = \mathrm{proj}^f_{C_n \cap Q_n} x_1, \quad \forall n \geqslant 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. Then, $\{x_n\}$ converges strongly to $\mathrm{proj}^f_{\bigcap_{i=1}^N A_i^{-1}(0)} x_1$.

5. Numerical example

In this section, we present some numerical examples for comparing the values of sequences generated by iteration (1.3) and (3.1) and supporting Theorem 3.1.

Example 5.1. Let $E = \mathbb{R}$, $C = (-\infty, 0]$, let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = \frac{2}{3}x^2$ (f is a strong coercive Legendre function which is bounded uniformly Frêchet differentiable and totally convex on bounded subsets of E, see in the numerical example of [28]). Let $T : C \longrightarrow C$ be defined by $Tx = \frac{1}{3}x$, $G : C \times C \longrightarrow \mathbb{R}$ be defined by $G(x,y) = x - y$ for all $x,y \in C$, $\psi : C \longrightarrow \mathbb{R}$ be defined by $\psi(x) = x^2$ for all $x \in C$. Let $\phi : C \times C \longrightarrow \mathbb{R}$ in the iteration (1.3) be defined by $\phi(x,y) = y - x$ for all $x,y \in C$. By the numerical example section of [18], we obtain that T is a Bregman relatively nonexpansive mapping. It is easy to show that G and ψ satisfy the Assumption 2.19 and the Assumption 2.20, respectively, and ϕ is skew-symmetric, i.e., $\phi(x,x) - \phi(x,y) - \phi(y,x) + \phi(y,y) \geqslant 0$ for all $x,y \in C$, convex in the second argument and continuous. Let $\{x_n\}$ be generated by iteration (1.3) and (3.1). Given initial values $x_1 = -1 = z_1$ and $\alpha_n = \frac{1}{n^3}$ for all $n \geqslant 1$. Then the sequence $\{x_n\}$ converges strongly to 0, where $\mathrm{proj}^f_{GEP(G,\phi)\cap F(T)}(x_1) = 0 = \mathrm{proj}^f_{MEP(G,\psi)\cap F(T)}(x_1)$.

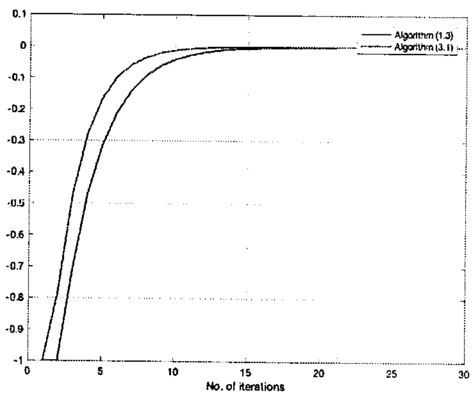

Figure 1: The numerical results for comparing Algorithm (3.1) and Algorithm (1.3).

We now illustrate the example supporting our main result.

Example 5.2. Let $E = \mathbb{R}$, $C = (-\infty, 0]$, and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = \frac{2}{3}x^2$. Let $\{T_i : C \longrightarrow C\}_{i=1}^5$ be defined by $T_i x = \frac{1}{i+1}x$, and let $G : C \times C \longrightarrow \mathbb{R}$ be defined by $G(x, y) = x - y$ for all $x, y \in C$, $\psi : C \longrightarrow \mathbb{R}$ be defined by $\psi(x) = x^2$ for all $x \in C$. Setting $\alpha_n = \frac{1}{n}$ for all $n \geqslant 1$. Let $\{x_n\}$ be the sequence generated by the iterative scheme. Given initial values $x_1, T_i x_1 = z_1 \in C$ for $i = 1, 2, \ldots, 5$,

$$\begin{cases} u_n^i = \nabla f^*(\alpha_n \nabla f(z_n^i) + (1-\alpha_n)\nabla f(T_i x_n)); \\ z_{n+1}^i = \mathrm{Res}_{G,\psi}^f(u_n^i); \\ C_n^i = \{z \in C : D_f(z, z_{n+1}^i) \leqslant \alpha_n D_f(z, z_n^i) + (1-\alpha_n) D_f(z, x_n)\}; \\ C_n = \bigcap_{i=1}^{5} C_n^i; \\ Q_n = \{z \in C : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n \rangle \leqslant 0\}; \\ x_{n+1} = \mathrm{proj}_{C_n \cap Q_n}^f x_1, \quad \forall n \geqslant 1. \end{cases} \tag{5.1}$$

It follows from Example 5.1, we know that f is a strong coercive Legendre function which is bounded uniformly Frêchet differentiable and totally convex on bounded subsets of $\mathbb{R}$ such that $\nabla f(x) = \frac{4}{3}x$ and G, ψ satisfy the Assumption 2.19 and the Assumption 2.20, respectively. Since $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}$, $f^*(z) = \frac{3}{8}z^2$ such that $\nabla f^* = \frac{3}{4}z$. Next, we show that T_i is a Bregman relatively nonexpansive mapping for all $i = 1, 2, \ldots, 5$. Clearly $F(T_i) = 0 = \hat{F}(T_i)$ for all $i = 1, 2, \ldots, 5$. Furthermore, we obtain that

$$\begin{aligned} D_f(0, T_i x) &= f(0) - f(T_i x) - \langle 0 - T_i x, \nabla f(T_i x)\rangle \\ &= 0 - \frac{2}{3(i+1)}x^2 - \langle -\frac{1}{i+1}x, \frac{4}{3(i+1)}x\rangle = \frac{4}{3(i+1)^2}x^2 - \frac{2}{3(i+1)}x^2 = \frac{2}{3}\left(\frac{1-i}{(i+1)^2}\right), \end{aligned}$$

and

$$D_f(0, x) = f(0) - f(x) - \langle 0 - x, \nabla f(x)\rangle = 0 - \frac{2}{3}x^2 - \langle -x, \frac{4}{3}x\rangle = \frac{4}{3}x^2 - \frac{2}{3}x^2 = \frac{2}{3}x^2.$$

Since $\frac{1-i}{(i+1)^2} \leqslant 0$ for all $i = 1, 2, \ldots, 5$, $D_f(0, T_i x) \leqslant D_f(0, x)$ for all $i = 1, 2, \ldots, 5$. It follows that $\{T_i\}_{i=1}^5$ is a countable family of Bregman relatively nonexpansive mappings. We also know that

$$G(0, y) + \psi(y) - \psi(0) = (0 - y) + y^2 - 0 = y(y-1) \geqslant 0, \quad \forall y \in C,$$

this implies that $0 \in MEP(G, \psi)$ and $\Omega = \bigcap_{i=1}^5 F(T_i) \cap MEP(G, \psi) = \{0\}$. It follows from iteration (5.1), we have

$$\begin{aligned} u_n^i &= \alpha_n z_n^i + (1-\alpha_n)\left(\frac{1}{i+1}\right) x_n; \\ z_{n+1}^i &= \frac{4}{7} u_n^i; \\ C_n^i &= [e_n^i, \infty), \quad \text{where } e_n^i = \frac{(z_{n+1}^i)^2 + (\alpha_n - 1)x_n^2 - \alpha_n (z_n^i)^2}{2\left(z_{n+1}^i - \alpha_n z_n^i + (\alpha_n - 1)x_n\right)}; \\ C_n &= \bigcap_{i=1}^{5} C_n^i; \\ Q_n &= [x_n, \infty); \\ x_{n+1} &= \mathrm{proj}_{C_n \cap Q_n}^f x_1, \quad \forall n \geqslant 1, i = 1, 2, \ldots, 5. \end{aligned}$$

Then the sequence $\{x_n\}$ generated by (5.1) converges strongly to $x^* = 0 \in \Omega$ as $n \longrightarrow \infty$. The Figure 2 shows the comparision of the values of the sequence $\{x_n\}$. Given initial values $x_1 = -5$, let $x_n(i)$ denote by the values of the sequence $\{x_n\}$ for $i = 1, 2, \ldots, 5$.

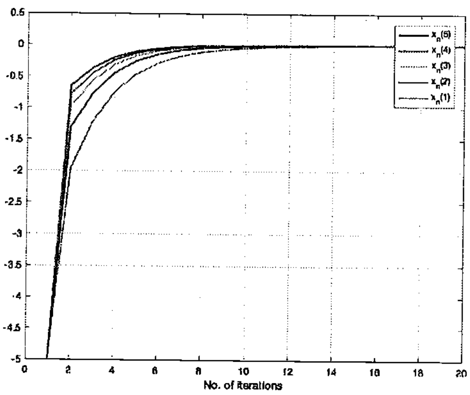

Figure 2: The numerical results for different $i = 1, 2, \ldots, 5$.

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References

[1] R. P. Agarwal, J.-W. Chen, Y. J. Cho, *Strong convergence theorems for equilibrium problems and weak Bregman relatively nonexpansive mappings in Banach spaces*, J. Inequal. Appl., **2013** (2013), 16 pages. 1, 2.6

[2] Y. I. Alber, *Metric and generalized projection operators in Banach spaces: Properties and applications*, In: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, **1996** (1996), 15–50. 2.6, 2

[3] H. H. Bauschke, J. M. Borwein, P. L. Combettes, *Bregman monotone optimization algorithms*, SIAM J. Control Optim., **42** (2003), 596–636. 4

[4] H. H. Bauschke, P. L. Combettes, J. M. Borwein, *Essential Smoothness, Essential Strict Convexity, and Legendre functions in Banach Spaces*, Commun. Contemp. Math., **3** (2001), 615–647. 2, 2.1, 2.2

[5] E. Blum, W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student, **63** (1994), 123–145. 1

[6] L. M. Bregman, *The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming*, USSR Comput. Math. Phys., **7** (1967), 200–217. 1, 2.3, 2.5

[7] R. E. Bruck, S. Reich, *Nonexpansive projections and resolvents of accretive operators in Banch spaces*, Houston J. Math., **3** (1977), 459–470. 1

[8] D. Butnariu, A. N. Iusem, *Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization*, Kluwer Academic Publ., Dordrecht, (2000). 2.7

[9] D. Butnariu, A. N. Iusem, C. Zalineacu, *On uniform convexity, total convexity and convergence of the proximal point and outer Bregman projection algorithms in Banach spaces*, J. Convex Anal., **10** (2003), 35–61. 2.8

[10] D. Butnariu, S. Reich, A. J. Zaslavski, *Asymptotic behavior of relatively nonexpansive operators in Banach spaces*, J. Appl. Anal., **7** (2001), 151–174.

[11] D. Butnariu, E. Resmerita, *Bregman distances, totally convex functions, and a method for solving operator equations in Banach spaces*, Abstr. Appl. Anal., **2006** (2006), 39 pages. 2.12, 2.16

[12] L.-C. Ceng, J.-C. Yao, *A hybrid iterative scheme for mixed equilibrium problems and fixed point problems*, J. Comput. Appl. Math., **214** (2008), 186–201. 1, 1

[13] J. W. Chen, Z. P. Wan, L. Y. Yuan, Y. Zheng, *Approximation of fixed points of weak Bregman relatively nonexpansive mappings in Banach spaces*, Int. J. Math. Math. Sci., **2011** (2011), 23 pages. 2.18

[14] P. Cholamjiak, Y. J. Cho, S. Suantai, *Composite iterative schemes for maximal monotone operators in reflexive Banach spaces*, Fixed Point Theory Appl., **2011** (2011), 10 pages. 1

[15] W. Cholamjiak, P. Cholamjiak, S. Suantai, *Convergence of iterative schemes for solving fixed point of multi-valued nonself mappings and equilibrium problems*, J. Nonlinear Sci. Appl., **8** (2015), 1245–1256.

[16] P. Cholamjiak, S. Suantai, *Iterative methods for solving equilibrium problems, variational inequalities and fixed points of nonexpansive semigroups*, J. Global Optim., **57** (2013), 1277–1297. 1

[17] V. Darvish, *A new algorithm for mixed equilibrium problem and Bregman strongly nonexpansive mapping in Banach spaces*, arXiv, **2015** (2015), 20 pages. 2.21

[18] K. R. Kazmi, R. Ali, S. Yousuf, *Generalized equilibrium and fixed point problems for Bregman relatively nonexpansive mappings in Banach spaces*, J. Fixed Point Theory Appl., **20** (2018), 21 pages. 1, 5.1

[19] R. P. Phelps, *Convex Functions, Monotone Operators, and Differentiability*, Springer-Verlag, Berlin, (1993). 2.13

[20] S. Reich, *A weak convergence theorem for the alternating method with Bregman distances*, In: Theory and Applications of Nonlinear Operators, **1996** (1996), 313–318. 1, 2.17

[21] S. Reich, S. Sabach, *A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces*, J. Nonlinear Convex Anal., **10** (2009), 471–485. 2.10

[22] S. Reich, S. Sabach, *Two strong convergence theorems for a proximal method in reflexive Banach spaces*, Numer. Funct. Anal. Optim., **31** (2010), 22–44. 1, 2.11, 2.14, 2.23, 4

[23] S. Reich, S. Sabach, *A projection method for solving nonlinear problems in reflexive Banach spaces*, J. Fixed Point Theory Appl., **9** (2011), 101–116. 2.4

[24] S. Reich, S. Sabach, *Existence and approximation of fixed points of Bregman firmly nonexpansive mappings in reflexive Banach spaces*, in: Fixed-point algorithms for inverse problems in science and engineering, **2011** (2011), 301–316. 4

[25] S. Sabach, *Products of finitely many resolvents of maximal monotone mappings in reflexive Banach spaces*, SIAM J. Optim., **21** (2011), 1289–1308. 2.15

[26] N. Shahzad, H. Zegeye, *Convergence theorem for common fixed points of a finite family of multi-valued Bregman relatively nonexpansive mappings*, Fixed Point Theory Appl., **2014** (2014), 14 pages. 2.22

[27] S. Suantai, Y. J. Cho, P. Cholamjiak, *Halpern's iteration for Bregman strongly nonexpansive mappings in reflexive Banach spaces*, Comput. Math. Appl., **64** (2012), 489–499. 1

[28] G. C. Ugwunnadi, B. Ali, I. Idris, M. S. Minjibir, *Strong convergence theorem for quasi-Bregman strictly pseudocontractive mappings and equilibrium problems in Banach spaces*, Fixed Point Theory Appl., **231** (2014), 1–16. 5.1

[29] C. Zalinescu, *Convex Analysis in General Vector Spaces*, World Scientific Publishing Co., River Edge, (2002). 2.9

ภาคผนวก 2

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Strong convergence theorems for generalized mixed equilibrium problems and Bregman relatively nonexpansive mappings in reflexive Banach spaces

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STRONG CONVERGENCE THEOREMS FOR GENERALIZED MIXED EQUILIBRIUM PROBREM AND BREGMAN RELATIVELY NONEXPANSIVE MAPPINGS IN REFLEXIVE BANACH SPACES

KITTISAK JANTAKARN AND ANCHALEE KAEWCHAROEN

ABSTRACT. In this paper, we deal with the Bregman iterative methods for finding common solutions of generalized mixed equilibrium problems and fixed point problems for Bregman relatively nonexpansive mappings in reflexive Banach spaces. The strong convergence theorems for the Bregman iterative methods under some mild conditions are proven. Furthermore, we present a numerical example to illustrate the main result.

1. INTRODUCTION

Let C be a nonempty closed and convex subset of the reflexive Banach space E with dual space E^*. Suppose that $G : C \times C \to \mathbb{R}$ is a bifunction. The equilibrium problem (EP) is to find $z \in C$ such that

$$G(z, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of the equilibrium problem is denoted by $EP(G)$. The equilibrium problem is a generalization of many mathematical models such as variational inequalities, fixed point problems and optimization problems. In 2006, Takahashi and Takahashi [28] introduced another iterative scheme for finding a common element of the set of solutions of equilibrium problem and fixed point problem of a nonexpansive mapping in a real Hilbert space. Their results extended and improved the corresponding results in [14, 18, 27].

Ceng and Yao [13] introduced the mixed equilibrium problem (MEP) which is to find $z \in C$ such that

$$G(z, y) + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C, \tag{1.2}$$

where $\varphi : C \to \mathbb{R}$ is a real valued function. The solution set of the mixed equilibrium problem is denoted by $MEP(G)$. In 2008, Ceng and Yao [13] investigated the problem of finding a common solution of mixed equilibrium problem and fixed point problem of finite family of nonexpansive mappings in Hilbert spaces.

Now we consider the following generalized mixed equilibrium problem (GMEP) which is to find $z \in C$ such that

$$G(z, y) + \varphi(y) - \varphi(z) + \langle \Psi(z), y - z \rangle \geq 0, \quad \forall y \in C, \tag{1.3}$$

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where $\Psi : C \to E^*$ is a nonlinear mapping. The solution set of GMEP is denoted by $GMEP(G)$. It is well-known that GMEP is a generalization of the mixed equilibrium and the equilibrium problem by considering $\Psi = 0$ and $\varphi = 0$, respectively.

In 1967, Bregman [6] discovered an elegant and effective technique for using the Bregman distance function $D_f(\cdot,\cdot)$ in the process of designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which Bregman's technique has been applied in various ways in order to design and analyze iterative algorithms for solving the feasibility and optimization problems, for approximating the variational inequalities and equilibrium problems, for computing the fixed points of nonlinear mappings and so on (see, e.g., [7, 20, 22] and the references therein).

In 2013, Agarwal et al. [2] proved the strong convergence theorems for finding the common solutions of the equilibrium problem (1.1) and the fixed point problem of weak Bregman relatively nonexpansive mappings in real reflexive Banach spaces. Recently, Kazmi et al. [17] introduced the following algorithm:

$$(1.4)\quad \begin{cases} x_1, z_1 \in C, \\ u_n = \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(Tx_n)), \\ z_{n+1} = Res^f_{G,\phi} u_n, \\ C_n = \{z \in C : D_f(z, z_{n+1}) \leq \alpha_n D_f(z, z_n) \\ \qquad\qquad + (1-\alpha_n) D_f(z, x_n)\}, \\ Q_n = \{z \in C : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n\rangle \leq 0\}, \\ x_{n+1} = proj^f_{C_n \cap Q_n} x_1, \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. They proved a strong convergence theorem for finding a common solution of a generalized equilibrium problem and a fixed point problem for a Bregman relatively nonexpansive mapping in reflexive Banach spaces.

Recall the generalized equilibrium problem (GEP) which is to find $z \in C$ such that

$$(1.5)\qquad G(z,y) + \phi(z,y) - \phi(z,z) \geq 0, \quad \forall y \in C,$$

where $\phi : C \times C \to \mathbb{R}$ is a bifunction. The solution set of the GEP is denoted by $GEP(G)$.

In this paper, motivated and inspired by the above literatures, we consider a new algorithm for finding a common solution of generalized mixed equilibrium problem and fixed point problem of Bregman relatively nonexpansive mappings in reflexive Banach spaces. That is, we are interested in considering the following problem: let E be a reflexive Banach space and C be a nonempty closed convex subset of E. Let $T : C \to C$ be a Bregman relatively nonexpansive mapping and G, φ, Ψ satisfy the suitable control conditions. We consider the problem of finding a solution $p \in C$ such that

$$p \in F(T) \cap GMEP(G) := \Omega,$$

where $F(T)$ is the fixed point set of T and $\Omega \neq \emptyset$. Under some suitable control conditions, the strong convergence theorem will be provided.

2. Preliminaries

We now provide some basic concepts, definitions and lemmas which will be used in the sequel. We let E be a reflexive Banach space with dual E^* and $f : E \to (-\infty, +\infty]$ be a proper lower semicontinuous convex function. We denote the domain of f by $domf$, that is $domf = \{x \in E : f(x) < +\infty\}$. The subdifferential of f at $x \in int(domf)$ is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \leq f(y), \quad \forall y \in E\},$$

and the Fenchel conjugate of f is the function $f^* : E^* \longrightarrow (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}.$$

We know that the Young-Fenchel inequality holds:

$$\langle x^*, x\rangle \leq f(x) + f^*(x^*), \quad \forall x \in E, \ x^* \in E^*.$$

Furthermore, we know that $x^* \in \partial f(x)$ if and only if $f(x) + f^*(x^*) = \langle x^*, x\rangle$ for all $x \in E$. It is not difficult to check that f^* is a proper convex and lower semicontinuous function.

Definition 2.1. The function $f : E \to (-\infty, +\infty]$ is called;

(1) cofinite if $domf^* = E^*$;

(2) coercive if $\lim_{\|x\| \to +\infty} f(x) = +\infty$;

(3) strongly coercive if $\lim_{\|x\| \to +\infty} \dfrac{f(x)}{\|x\|} = +\infty$.

Definition 2.2. Let $x \in int(domf)$ and $y \in E$, we define the right-hand derivative of f at x in the direction y by

$$f^0(x, y) = \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}.$$

The function f is called to be

(i) Gâteaux differentiable at x if $\lim_{t \to 0^+} \dfrac{f(x + ty) - f(x)}{t}$ exists for any y;

(ii) Gâteaux differentiable if it is Gâteaux differentiable for any $x \in int(domf)$;

(iii) Frêchet differentiable at x if this limit is attained uniformly in $\|y\| = 1$;

(iv) uniformly Frêchet differentiable on a subset C of E if the above limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

In this case, the gradient of f at x is the linear function $\nabla f(x)$, which is defined by $\langle y, \nabla f(x)\rangle := f^0(x, y)$ for all $y \in E$.

The Legendre function $f : E \to (-\infty, +\infty]$ is defined in [3]. It is well-known that in reflexive spaces, f is the Legendre function if and only if it satisfies the following conditions:

(L1) The interior of the domain of f, $int(domf)$, is nonempty, f is Gâteaux differentiable on $int(domf)$ and $domf = int(domf)$;
(L2) The interior of the domain of f^*, $int(domf^*)$, is nonempty, f^* is Gâteaux differentiable on $int(domf^*)$ and $domf^* = int(domf^*)$.

Since E is reflexive, $(\partial f)^{-1} = \partial f^*$ (see[3]). This, with (L1) and (L2), imply the following equalities:

(i) $\nabla f = (\nabla f^*)^{-1}$;
(ii) $ran\nabla f = dom\nabla f^* = int(domf^*)$;
(iii) $ran\nabla f^* = dom(\nabla f) = int(domf)$.

When the subdifferential of f is single-valued, it coincides with the gradient $\partial f = \nabla f$, for more details see [19]. By Bauschke et al. [3], the conditions (L1) and (L2) also yield that the functions f and f^* are strictly convex on the interior of their respective domains.

Definition 2.3. [6] Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and convex function. The Bregman distance with respect to f is the bifunction $D_f : domf \times int(domf) \to [0, +\infty)$ defined by

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle. \tag{2.1}$$

Moreover, the Bregman distance D_f is not a distance in the usual sense because D_f is not symmetric and does not satisfy the triangle inequality.

Remark 2.4. [23] The Bregman distance D_f satisfies the three point identity:

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle, \tag{2.2}$$

for any $x \in domf$ and $y, z \in int(domf)$.

Definition 2.5. [6] Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The Bregman projection of $x \in int(domf)$ onto the nonempty closed and convex subset $C \subset domf$ is the necessary unique vector $proj_C^f(x)$ satisfying

$$D_f(proj_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}. \tag{2.3}$$

Remark 2.6. ([2]) If $f(x) = \frac{1}{2}\|x\|^2$, $\forall x \in E$, then we have $\nabla f = J$, where J is the normalized duality mapping from E to 2^{E^*}, and hence $D_f(x, y)$ is reduced to the Lyapunov function defined by $\Phi(x, y) = \|y\|^2 - 2\langle Jx, y\rangle + \|x\|^2$, $\forall x, y \in E$, which is introduced by Alber [1], and so we obtain that the Bregman projection $proj_C^f(x)$ is reduced to the generalized projection $\prod_C(x)$, which is defined by

$$\Phi(\Pi_C(x), x) = \min_{y \in C} \Phi(y, x).$$

Moreover, in Hilbert spaces, the Bregman projection $proj_C^f(x)$ is reduced to the metric projection of x onto C for $f(x) = \frac{1}{2}\|x\|^2$.

Definition 2.7. [8] Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and convex function, $v_f : int(domf) \times [0, +\infty) \to [0, +\infty)$, define the modulus of total convexity of the function f at x by

$$v_f(x,t) := \inf\{D_f(y,x) : y \in domf, \|y - x\| = t\}.$$

Then the function f is called to be

(a) totally convex at a point $x \in int(domf)$, if the modulus of total convexity of the function f at x is positive, $v_f(x,t) > 0$ whenever $t > 0$;

(b) totally convex, if it is totally convex at every point $x \in int(domf)$. Let B be a nonempty bounded subset of E, define the modulus of total convexity of the function f on the set B by

$$v_f(B,t) := \inf\{v_f(x,t) : x \in B \cap domf\};$$

(c) totally convex on bounded sets, if the modulus of total convexity of the function f on the set B is positive, $v_f(B,t) > 0$ for any nonempty bounded subset B of E and $t > 0$.

Lemma 2.8. [9] *Let $f : E \to (-\infty, +\infty]$ be a Legendre function. Then, the function f is totally convex on bounded sets if and only if f is uniformly convex on bounded subsets of E.*

Lemma 2.9. [30] *Let $f : E \to \mathbb{R}$ be a strongly coercive and uniformly convex on bounded subsets of E, then f^* is bounded and uniformly Fréchet differentiable on bounded subsets of E^*.*

Lemma 2.10. [21] *Let $f : E \to \mathbb{R}$ be uniformly Fréchet differentiable and bounded on bounded subsets of E. Then, f is uniformly continuous on bounded subsets of E and ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Definition 2.11. [22] The function $f : E \to (-\infty, +\infty]$ is called sequentially consistent, if for any two sequences $\{x_n\}$ and $\{y_n\}$ in $int(domf)$ and $domf$, respectively such that sequence $\{x_n\}$ is bounded, then

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \quad \text{implies} \quad \lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{2.4}$$

Lemma 2.12. [11] *If $f : E \to (-\infty, +\infty]$ is a convex function whose domain contains at least two points. Then, f is totally convex on bounded sets if and only if it is sequentially consistent.*

By using totally convex functions, one can obtain algorithms which are less dependent on the geometry of the Banach space in which they are placed. Total convexity is a property of the modulus of total convexity of the function which ensures that some sequential convergence properties which are true in the uniformly-like structure defined on the space via the Bregman distances with respect to a totally convex function are inherited by the norm topology of the space. Therefore, in order to establish convergence of some algorithms in infinite dimentional settings it is enough to do so with respect to the uniformity-like structure determined by the Bregman distance associated to a totally convex function (for more details see [8]).

The usefulness of the strongly coercive Legendre function which is bounded uniformly Frêchet differentiable and totally convex on bounded subsets of reflexive Banach spaces is to establish the convergence for finding the common solution of fixed point problem and generalized mixed equilibrium problem via Bregman distance.

The following is the example of the strongly coercive Legendre function which is bounded uniformly Frêchet differentiable and totally convex on bounded subsets of reflexive Banach spaces appeared in [3]:

Example 2.13. [3] Suppose E is a Hilbert space, $\gamma > 0$, and

$$f(x) = \frac{1+\gamma}{2}\|x\|^2 - \frac{1}{2}d^2(x, C),$$

where $d(x, C) = \min_{c \in C} \|x - c\| = \|x - Px\|$, P denotes the projection map onto C and $x \in E$. Then $\nabla f(x) = \gamma x + Px$,
$D_f(x, y) = \frac{1}{2}\left(\gamma\|x - y\|^2 + \|x - Py\|^2 - \|z - Px\|^2\right)$ for all $x, y \in E$ and f is the strongly coercive Legendre function which is bounded uniformly Frêchet differentiable and totally convex on bounded subsets of E.

Let $f : E \to \mathbb{R}$ be a Legendre and Gâteaux differentiable function. We make use of the function $V_f : E \times E^* \to [0, +\infty)$ associated with f, which is defined by

$$V_f(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \ \forall x \in E, x^* \in E^*.$$

Then V_f is nonnegative and

$$V_f(x, x^*) = D_f(x, \nabla f(x^*)), \ \forall x \in E, x^* \in E^*.$$

Moreover, by the subdifferential inequality,

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x \rangle \leq V_f(x, x^* + y^*), \ \forall x \in E, x^*, y^* \in E^*,$$

(for more details see [1]).

Lemma 2.14. [19] *Let $f : E \to (-\infty, +\infty]$ be a proper lower semicontinuous and convex function, then $f^* : E^* \to (-\infty, +\infty]$ is proper weak* lower semicontinuous and convex. Hence, V_f is convex in the second variable. Thus, for all $z \in E$, we have*

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^{N} t_i D_f(z, x_i), \tag{2.5}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$.

Lemma 2.15. [22] *Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and totally convex function. If $x_1 \in E$ and the sequence $\{D_f(x_n, x_1)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 2.21. [16] *Let $f : E \to (-\infty, +\infty]$ be a coercive and Gâteaux differentiable function. Let C be a closed and convex subset of E. Assume that $\varphi : C \to \mathbb{R}$ is a lower semicontinuous and convex function, $\Psi : C \to E^*$ is a continuous monotone mapping and the bifunction $G : C \times C \to \mathbb{R}$ satisfies the Assumption 2.19, then $dom(Res^f_{G,\varphi,\Psi}) = E$.*

Lemma 2.22. [16] *Let $f : E \to (-\infty, +\infty]$ be a Legendre function. Let C be a closed and convex subset of E. If the bifunction $G : C \times C \to \mathbb{R}$ satisfies the Assumption 2.19, then*

(i) $Res^f_{G,\varphi,\Psi}$ *is single-valued;*
(ii) $Res^f_{G,\varphi,\Psi}$ *is a BFNE operator;*
(iii) $F\left(Res^f_{G,\varphi,\Psi}\right) = GMEP(G)$;
(iv) $GMEP(G)$ *is closed and convex;*
(v) $D_f\left(p, Res^f_{G,\varphi,\Psi}(x)\right) + D_f\left(Res^f_{G,\varphi,\Psi}(x), x\right) \leq D_f(p, x)$, $\forall p \in F\left(Res^f_{G,\varphi,\Psi}\right), x \in E$.

Let $CB(C)$ denote the family of nonempty closed bounded subsets of C.

Lemma 2.23. ([26]) *Let E be a reflexive Banach space, and let $f : E \longrightarrow \mathbb{R}$ be uniformly Frêchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty closed and convex subset of $int(domf)$ and $T : C \longrightarrow CB(C)$ be a Bregman relatively nonexpansive mapping. Then $F(T)$ is closed and convex.*

Lemma 2.24. ([22]) *Let $f : E \longrightarrow (-\infty, +\infty]$ be a Gâteaux differentiable and totally convex function, x_1 be an element in E and C be a nonempty closed convex subset of E. Suppose that the sequence $\{x_n\}$ is bounded and the weak limits of any subsequence of a sequence $\{x_n\}$ belong to $C \subset E$. If $D_f(x_n, x_1) \leq D_f(proj^f_C(x_1), x_1)$ for any $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to $proj^f_C(x_1)$.*

3. Main result

In this section, we present our main algorithm and prove the strong convergence theorem for finding a common solution of the generalized mixed equilibrium and the fixed point problem of Bregman relatively nonexpansive mappings in reflexive Banach spaces.

Let E be a reflexive Banach space with dual E^* and C be a nonempty closed convex subset of E such that $C \subset int(domf)$. Let $f : E \to (-\infty, +\infty]$ be a strongly coercive Legendre function which is bounded uniformly Frêchet differentiable and totally convex on bounded subset of E, $G : C \times C \to \mathbb{R}$ be a bifunction satisfying the Assumption 2.19, $\varphi : C \to \mathbb{R}$ be a convex lower semicontinuous mapping and $\Psi : C \to E^*$ be a continuous monotone mapping. Let $T : C \to C$ be a Bregman relatively nonexpansive mapping. We introduce the following algorithm for solving the generalized mixed equilibrium problem and the fixed point problem.

Algorithm 3.1. Choose $x_1, z_1 \in C$. The control parameters α_n, β_n satisfy the following conditions:

$$\alpha_n \in (0,1), \quad \lim_{n\to\infty} \alpha_n = 0,$$
$$\beta_n \in (0,1), \quad \liminf_{n\to\infty} (1-\alpha_n)(1-\beta_n) > 0.$$

Let $\{x_n\}$ be the sequence generated by the iterative scheme:

$$\begin{cases} u_n = \nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(Tx_n)), \\ y_n = \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(u_n)), \\ z_{n+1} = Res^f_{G,\varphi,\Psi}(y_n), \\ C_n = \{z \in C : D_f(z, z_{n+1}) \le \alpha_n D_f(z, z_n) \\ \qquad\qquad + (1-\alpha_n) D_f(z, x_n)\}, \\ Q_n = \{z \in C : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n \rangle \le 0\}, \\ x_{n+1} = proj^f_{C_n \cap Q_n}(x_1); \forall n \ge 1. \end{cases} \tag{3.1}$$

We now present the convergence theorem for the sequence generated by (3.1).

Theorem 3.2. *Assume that $\Omega = F(T) \cap GMEP(G) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $proj^f_\Omega(x_1)$.*

Proof. From Lemma 2.22 and Lemma 2.23, we obtain that $F(T) \cap GMEP(G)$ is a closed and convex subset of E. It is easy to prove that C_n and Q_n are closed and convex. Therefore, $C_n \cap Q_n$ is closed and convex for all $n \ge 1$. Since $Res^f_{G,\varphi,\Psi}$ is single-valued, we get that $Res^f_{G,\varphi,\Psi}(y_n) = z_{n+1}$. Taking $p \in \Omega$ arbitrarily, we obtain that

$$\begin{aligned} D_f(p, z_{n+1}) &= D_f(p, Res^f_{G,\varphi,\Psi}(y_n)) \\ &\le D_f(p, y_n) - D_f(Res^f_{G,\varphi,\Psi}(y_n), y_n) \\ &\le D_f(p, y_n) \\ &= D_f(p, \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(u_n))) \\ &\le \alpha_n D_f(p, z_n) + (1-\alpha_n) D_f(p, u_n). \end{aligned} \tag{3.2}$$

Since

$$\begin{aligned} D_f(p, u_n) &= D_f(p, \nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(Tx_n))) \\ &\le \beta_n D_f(p, x_n) + (1-\beta_n) D_f(p, Tx_n) \\ &\le D_f(p, x_n), \end{aligned} \tag{3.3}$$

by (3.2) and (3.3) imply that

$$D_f(p, z_{n+1}) \le \alpha_n D_f(p, z_n) + (1-\alpha_n) D_f(p, x_n). \tag{3.4}$$

It follows that $p \in C_n$. Therefore, $\Omega \subset C_n$ for all $n \ge 1$. We now show that $\Omega \subset Q_n$ for all $n \ge 1$. Clearly, $\Omega \subset Q_1 = C$. Assume that $\Omega \subset Q_k$ for all $k > 0$. In view of $x_{n+1} = proj^f_{C_k \cap Q_k}(x_1) \in Q_k$, it follows from the result of Lemma 2.17 (2), we have

$$\langle \nabla f(x_1) - \nabla f(x_{k+1}), x_{k+1} - z \rangle \ge 0, \quad \forall z \in Q_k.$$

Moreover, one has

$$\langle \nabla f(x_1) - \nabla f(x_{k+1}), x_{k+1} - p\rangle \geq 0, \quad \forall p \in \Omega,$$

and so, for each $p \in \Omega$,

$$\langle \nabla f(x_1) - \nabla f(x_{k+1}), p - x_{k+1}\rangle \leq 0.$$

This implies that $\Omega \subset Q_{k+1}$. Therefore, $\Omega \subset Q_n$ for all $n \geq 1$. Consequently, $\Omega \subset C_n \cap Q_n$ for all $n \geq 1$. This, together with $\Omega \neq \emptyset$ yields that $C_n \cap Q_n$ is a nonempty closed and convex subset of C for all $n \geq 1$. Moreover, $\{x_n\}$ is well-defined. In view of Lemma 2.17 and the definition of Q_n, we conclude that

$$\begin{aligned} D_f(x_n, x_1) &= D_f(proj^f_{Q_n}(x_1), x_1) \\ &\leq D_f(p, x_1) - D_f(p, proj^f_{Q_n}(x_1)) \\ &\leq D_f(p, x_1), \quad \forall p \in \Omega \subset Q_n. \end{aligned} \tag{3.5}$$

This implies that the sequence $\{D_f(x_n, x_1)\}$ is bounded. In view of Lemma 2.15, we obtain that the sequence $\{x_n\}$ is bounded. On the other hand, we have

$$\begin{aligned} D_f(p, x_n) &= D_f(p, proj^f_{C_{n-1}\cap Q_{n-1}}(x_1)) \\ &\leq D_f(p, x_1) - D_f(x_n, x_1) \\ &\leq D_f(p, x_1), \end{aligned}$$

which implies that $\{D_f(p, x_n)\}$ is bounded. Since T is a Bregman relatively nonexpansive mapping, we obtain that $D_f(p, Tx_n) \leq D_f(p, x_n)$ for all $p \in \Omega$. By the boundedness of $\{D_f(p, x_n)\}$, we get that $\{D_f(p, Tx_n)\}$ is bounded. Since f is strongly coercive, f^* and ∇f^* are bounded on bounded subsets. It follows from Lemma 2.16, we obtain that $\{Tx_n\}$ is bounded. Since $\{D_f(p, x_n)\}$ is bounded, there exists $M > 0$ such that $D_f(p, x_n) \leq M$. In view of (3.4), we obtain that

$$D_f(p, z_{n+1}) \leq \alpha_n D_f(p, z_n) + (1 - \alpha_n)M.$$

Let $K = \max\{D_f(p, z_1), M\}$. Clearly that $D_f(p, z_1) \leq K$. Let $D_f(p, z_n) \leq K$ for some n, then it follows from above inequality, we get that

$$D_f(p, z_{n+1}) \leq \alpha_n K + (1 - \alpha_n)K \leq K.$$

This implies that $\{D_f(p, z_n)\}$ and $\{z_n\}$ are bounded. Moreover,

$$\begin{aligned} D_f(p, y_n) &= D_f(p, \nabla f^*(\alpha_n \nabla f(z_n) + (1 - \alpha_n)\nabla f(u_n))) \\ &\leq \alpha_n D_f(p, z_n) + (1 - \alpha_n)D_f(p, u_n) \\ &\leq \alpha_n D_f(p, z_n) + (1 - \alpha_n)D_f(p, x_n), \end{aligned} \tag{3.6}$$

which implies that $\{D_f(p, y_n)\}$ is also bounded. In view of $x_{n+1} = proj^f_{C_n\cap Q_n}(x_1)$, $x_n = proj^f_{Q_n}(x_1)$ and $C_n \cap Q_n \subset Q_n$, we conclude that

$$D_f(x_n, x_1) \leq D_f(x_{n+1}, x_1), \quad \forall n \geq 1. \tag{3.7}$$

This implies that $\{D_f(x_n, x_1)\}$ is nondecreasing. From the boundedness of $\{D_f(x_n, x_1)\}$, we obtain that $\lim_{n\to\infty} D_f(x_n, x_1)$ exists. In view of Lemma 2.17, we obtain that

$$\begin{aligned} D_f(x_{n+1}, x_n) &= D_f(x_{n+1}, proj^f_{Q_n}(x_1)) \\ &\le D_f(x_{n+1}, x_1) - D_f(proj^f_{Q_n}(x_1), x_1) \\ &= D_f(x_{n+1}, x_1) - D_f(x_n, x_1), \end{aligned}$$

which implies that

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0. \tag{3.8}$$

Since f is totally convex on bounded sets, it follows from Lemma 2.12 that f is sequentially consistent. This, together with (3.8), implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.9}$$

It follows from the three point identity that

$$D_f(x_{n+1}, z_n) = \langle \nabla f(z_n) - \nabla f(x_{n+1}), p - x_{n+1}\rangle + D_f(p, z_n) - D_f(p, x_{n+1}).$$

Since f is bounded on bounded subsets of E, ∇f is also bounded on bounded subsets of E. It follows from the boundedness of $\{x_n\}$ and $\{z_n\}$, we obtain that the sequences $\{\nabla f(x_n)\}$ and $\{\nabla f(z_n)\}$ are bounded in E^*, which implies that $\{D_f(x_{n+1}, z_n)\}$ is bounded. In view of $x_{n+1} = proj^f_{C_n\cap Q_n}(x_1) \in C_n$ with the definition of C_n, we get that

$$D_f(x_{n+1}, z_{n+1}) \le \alpha_n D_f(x_{n+1}, z_n) + (1-\alpha_n) D_f(x_{n+1}, x_n).$$

From the boundedness of $\{D_f(x_{n+1}, z_n)\}$ and $\lim_{n\to\infty} \alpha_n = 0$, it follows from the above inequality together with (3.8), we conclude that

$$\lim_{n\to\infty} D_f(x_{n+1}, z_{n+1}) = 0.$$

This, together with Lemma 2.11, we obtain that

$$\lim_{n\to\infty} \|x_{n+1} - z_{n+1}\| = 0. \tag{3.10}$$

Taking into account

$$\|x_n - z_{n+1}\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_{n+1}\|,$$

it follows from (3.9) and (3.10), we get that

$$\lim_{n\to\infty} \|x_n - z_{n+1}\| = 0. \tag{3.11}$$

Since f is uniformly Frêchet differentiable on bounded subsets of E, it follows from Lemma 2.10 that f and ∇f are uniformly continuous on bounded subsets of E. It follows that

$$\lim_{n\to\infty} |f(x_n) - f(z_{n+1})| = 0 \tag{3.12}$$

and

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(z_{n+1})\| = 0. \tag{3.13}$$

In view of the definition of the Bregman distance, we obtain that

$$\begin{aligned} D_f(p,x_n) - D_f(p,z_{n+1}) &= f(p) - f(x_n) - \langle \nabla f(x_n), p - x_n\rangle \\ &\quad - (f(p) - f(z_{n+1}) - \langle \nabla f(z_{n+1}), p - z_{n+1}\rangle) \\ &= f(z_{n+1}) - f(x_n) + \langle \nabla f(z_{n+1}), p - z_{n+1}\rangle \\ &\quad - \langle \nabla f(x_n), p - x_n\rangle \\ &= f(z_{n+1}) - f(x_n) + \langle \nabla f(z_{n+1}), p - x_n\rangle \\ &\quad + \langle \nabla f(z_{n+1}), x_n - z_{n+1}\rangle - \langle \nabla f(x_n), p - x_n\rangle \\ &= f(z_{n+1}) - f(x_n) + \langle \nabla f(z_{n+1}) - \nabla f(x_n), p - x_n\rangle \\ &\quad + \langle \nabla f(z_{n+1}), x_n - z_{n+1}\rangle. \end{aligned} \tag{3.14}$$

From this, together with (3.11), (3.12) and (3.13), we conclude that

$$\lim_{n\to\infty} |D_f(p,x_n) - D_f(p,z_{n+1})| = 0. \tag{3.15}$$

In view of Lemma 2.14 and 2.22, we obtain that

$$\begin{aligned} D_f(z_{n+1}, y_n) &= D_f(Res^f_{G,\varphi,\Psi}(y_n), y_n) \\ &\leq D_f(p, y_n) - D_f(p, Res^f_{G,\varphi,\Psi}(y_n)) \\ &= D_f(p, \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(u_n))) - D_f(p, z_{n+1}) \\ &\leq \alpha_n D_f(p,z_n) + (1-\alpha_n) D_f(p,u_n) - D_f(p,z_{n+1}) \\ &\leq \alpha_n D_f(p,z_n) + (1-\alpha_n) D_f(p,x_n) - D_f(p,z_{n+1}) \\ &= \alpha_n (D_f(p,z_n) - D_f(p,x_n)) + (D_f(p,x_n) - D_f(p,z_{n+1})). \end{aligned} \tag{3.16}$$

From the boundedness of $\{D_f(p,x_n)\}$ and $\{D_f(p,z_n)\}$ together with (3.15), (3.16) and $\lim_{n\to\infty} \alpha_n = 0$, we obtain that

$$\lim_{n\to\infty} D_f(z_{n+1}, y_n) = 0,$$

again using Lemma 2.11, we have

$$\lim_{n\to\infty} \|z_{n+1} - y_n\| = 0. \tag{3.17}$$

Taking into account

$$\|x_n - y_n\| \leq \|x_n - z_{n+1}\| + \|z_{n+1} - y_n\|,$$

and using (3.11) and (3.17), we get that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.18}$$

Since f is uniformly Frêchet differentiable, it follows from Lemma 2.10 that ∇f is uniformly continuous on bounded subsets of E. This, together with (3.17) and (3.18), we obtain that

$$\lim_{n\to\infty} \|\nabla f(z_{n+1}) - \nabla f(y_n)\| = 0 \tag{3.19}$$

and

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(y_n)\| = 0. \tag{3.20}$$

Furthermore, in view of the property of the Legendre function f, we obtain that

$$\begin{aligned}\|\nabla f(x_n) - \nabla f(y_n)\| &= \|\nabla f(x_n) - \nabla f(\nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(u_n)))\| \\ &= \|\nabla f(x_n) - \alpha_n \nabla f(z_n) - (1-\alpha_n)\nabla f(u_n)\| \\ &= \|\alpha_n(\nabla f(x_n) - \nabla f(z_n)) + (1-\alpha_n)(\nabla f(x_n) - \nabla f(u_n))\| \\ &= \|\alpha_n(\nabla f(x_n) - \nabla f(z_n)) \\ &\quad + (1-\alpha_n)(\nabla f(\nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(Tx_n))))\| \\ &= \|\alpha_n(\nabla f(x_n) - \nabla f(z_n)) \\ &\quad + (1-\alpha_n)(1-\beta_n)(\nabla f(x_n) - \nabla f(Tx_n))\| \\ &\geq (1-\alpha_n)(1-\beta_n)\|\nabla f(x_n) - \nabla f(Tx_n)\| \\ &\quad - \alpha_n\|\nabla f(x_n) - \nabla f(z_n)\|, \end{aligned} \tag{3.21}$$

which implies that

$$\begin{aligned}(1-\alpha_n)(1-\beta_n)\|\nabla f(x_n) - \nabla f(Tx_n)\| &\leq \alpha_n\|\nabla f(x_n) - \nabla f(z_n)\| \\ &\quad + \|\nabla f(x_n) - \nabla f(y_n)\|. \end{aligned} \tag{3.22}$$

Since $\liminf_{n\to\infty}(1-\alpha_n)(1-\beta_n) > 0$, there exists $a > 0$ such that

$$\begin{aligned} a\|\nabla f(x_n) - \nabla f(Tx_n)\| &\leq (1-\alpha_n)(1-\beta_n)\|\nabla f(x_n) - \nabla f(Tx_n)\| \\ &\leq \alpha_n\|\nabla f(x_n) - \nabla f(z_n)\| + \|\nabla f(x_n) - \nabla f(y_n)\|. \end{aligned} \tag{3.23}$$

This, together with (3.20), $\lim_{n\to\infty} \alpha_n = 0$ and the boundedness of $\{\nabla f(x_n)\}$ and $\{\nabla f(z_n)\}$, we obtain that

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(Tx_n)\| = 0. \tag{3.24}$$

Since $\{x_n\}$ is a bounded sequence in C, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^* \in C$ as $k \to \infty$. It follows from (3.11) and (3.17), there exist subsequences $\{y_{n_k}\}$ of $\{y_n\}$ and $\{z_{n_k}\}$ of $\{z_n\}$ such that $y_{n_k} \rightharpoonup x^*$ and $z_{n_k} \rightharpoonup x^*$ as $k \longrightarrow \infty$, respectively. In view of f is the Legendre function and f^* is uniformly Frêchet differentiable on bounded subsets of E^* together with (3.24), it implies that

$$\lim_{n\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0. \tag{3.25}$$

Since the asymptotic fixed point with $x_{n_k} \rightharpoonup x^*$ and (3.25), we conclude that x^* is a asymptotic fixed point of T. Since T is a Bregman relatively nonexpansive mapping, x^* is a fixed point of T. Further, in the light of $Res^f_{G,\varphi,\Psi}(y_n) = z_{n+1}$ and Definition 2.20, it follows that, for each $y \in C$,

$$\begin{aligned} G(z_{n_k+1}, y) &+ \varphi(y) - \varphi(z_{n_k+1}) + \langle \Psi(y_{n_k}), y - z_{n_k+1}\rangle \\ &+ \langle \nabla f(z_{n_k+1}) - \nabla f(y_{n_k}), y - z_{n_k+1}\rangle \geq 0, \ \forall y \in C, \end{aligned}$$

and hence, combining this with the Assumption 2.19 (ii), we obtain that

$$\begin{aligned}\varphi(y)-\varphi(z_{n_k+1})+\langle\Psi(y_{n_k}),y-z_{n_k+1}\rangle+\langle\nabla f(z_{n_k+1})-\nabla f(y_{n_k}),y-z_{n_k+1}\rangle \\ \geq -G(z_{n_k+1},y) \\ \geq G(y,z_{n_k+1}),\ \forall y\in C.\end{aligned}$$

For any $y\in C$ and $t\in(0,1]$, we let $y_t=ty+(1-t)x^*\in C$. This implies that

$$\begin{aligned}\varphi(y_t)-\varphi(z_{n_k+1})+\langle\Psi(y_{n_k}),y_t-z_{n_k+1}\rangle+\langle\nabla f(z_{n_k+1})-\nabla f(y_{n_k}),y_t-z_{n_k+1}\rangle \\ \geq G(y_t,z_{n_k+1}).\end{aligned}$$

Since $G(x,y)$ in the second variable y and φ are lower semicontinuous,

$$\begin{aligned}\liminf_{k\to\infty}(G(y_t,z_{n_k+1})-\varphi(y_t)+\varphi(z_{n_k+1})+\langle\Psi(y_{n_k}),z_{n_k+1}-y_t\rangle) \\ \leq \liminf_{k\to\infty}\langle\nabla f(z_{n_k+1})-\nabla f(y_{n_k}),y_t-z_{n_k+1}\rangle,\end{aligned}$$

it follows that

$$G(y_t,x^*)-\varphi(y_t)+\varphi(x^*)+\langle\Psi(x^*),x^*-y_t\rangle\leq 0. \tag{3.26}$$

Furthermore, we next consider the following inequality,

$$\begin{aligned}0&=G(y_t,y_t)+\varphi(y_t)-\varphi(y_t)+\langle\Psi(x^*),y_t-y_t\rangle\\&=G(y_t,ty+(1-t)x^*)+\varphi(ty+(1-t)x^*)-t\varphi(y_t)-(1-t)\varphi(y_t)\\&\quad+\langle\Psi(x^*),ty+(1-t)x^*-t(y_t)-(1-t)y_t\rangle\\&\leq tG(y_t,y)+(1-t)G(y_t,x^*)+t\varphi(y)+(1-t)\varphi(x^*)-t\varphi(y_t)\\&\quad-(1-t)\varphi(y_t)+t\langle\Psi(x^*),y-y_t\rangle+(1-t)\langle\Psi(x^*),x^*-y_t\rangle\\&=t(G(y_y,y)+\varphi(y)-\varphi(y_t)+\langle\Psi(x^*),y-y_t\rangle)\\&\quad+(1-t)(G(y_t,x^*)+\varphi(x^*)-\varphi(y_t)+\langle\Psi(x^*),x^*-y_t\rangle)\\&\leq t(G(y_t,y)+\varphi(y)-\varphi(y_t)+\langle\Psi(x^*),y-y_t\rangle),\end{aligned} \tag{3.27}$$

which implies that

$$G(y_t,y)+\varphi(y)-\varphi(y_t)+\langle\Psi(x^*),y-y_t\rangle\geq 0. \tag{3.28}$$

Moreover, it follows from the Assumption 2.19 (iii), we conclude that

$$\begin{aligned}0&\leq\limsup_{t\to0^+}(G(y_t,y)+\varphi(y)-\varphi(y_t)+\langle\Psi(x^*),y-y_t\rangle)\\&=\limsup_{t\to0^+}(G(ty+(1-t)x^*,y)+\varphi(y)-\varphi(ty+(1-t)x^*)\\&\qquad+\langle\Psi(x^*),y-ty-(1-t)x^*\rangle)\\&\leq G(x^*,y)+\varphi(y)-\varphi(x^*)+\langle\Psi(x^*),y-x^*\rangle.\end{aligned} \tag{3.29}$$

This implies that x^* is a solution of the generalized mixed equilibrium problem, and hence $x^*\in GMEP(G)$. To sum up, we have $x^*\in\Omega:=F(T)\cap GMEP(G)$. Finally, we now prove that $\{x_n\}$ converges strongly to $\bar{x}=proj_{\Omega}^{f}(x_1)$. It follows from the definition of the Bregman projection together with Ω is a nonempty closed and convex subset of E, we obtain that $proj^f_\Omega(x_1)$ is well-defined. Let $\bar{x} = proj^f_\Omega(x_1)$. In view of $x_{n+1} = proj^f_{C_n \cap Q_n}(x_1)$ and $proj^f_\Omega(x_1) \in \Omega \subseteq C_n \cap Q_n$, we obtain that

$$D_f(x_{n+1}, x_1) \le D_f(proj^f_\Omega(x_1), x_1). \tag{3.30}$$

It follows from Lemma 2.24, we conclude that $\{x_n\}$ converges strongly to $\bar{x} = proj^f_\Omega(x_1)$. This completes the proof. □

If we assume that Ψ is a zero mapping in Algorithm (3.1), then we get the following corollary.

Corollary 3.3. *Let E be a reflexive Banach space with dual E^* and C be a nonempty closed convex subset of E such that $C \subset int(domf)$. Let $f : E \longrightarrow (-\infty, +\infty]$ be a strongly coercive Legendre function which is bounded uniformly Fréchet differentiable and totally convex on bounded subsets of E, $G : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying the Assumption 2.19 and $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex. Let $T : C \longrightarrow C$ be a Bregman relatively nonexpansive mapping. Assume that $F(T) \cap MEP(G) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by the iterative scheme:*

$$\left\{\begin{array}{l} x_1, z_1 \in C; \\ u_n = \nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(Tx_n)); \\ y_n = \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(u_n)); \\ z_{n+1} = Res^f_{G,\varphi}(u_n); \\ C_n = \{z \in C : D_f(z, z_{n+1}) \le \alpha_n D_f(z, z_n) \\ \qquad\qquad +(1-\alpha_n)D_f(z, x_n)\}; \\ Q_n = \{z \in C : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n \rangle \le 0\}; \\ x_{n+1} = proj^f_{C_n \cap Q_n} x_1, \quad \forall n \ge 1, \end{array}\right. \tag{3.31}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ such that

$$\lim_{n\to\infty} \alpha_n = 0, \quad \liminf_{n\to\infty} (1-\alpha_n)(1-\beta_n) > 0.$$

Then, $\{x_n\}$ converges strongly to $proj^f_{\overline{F(T) \cap MEP(G)}}(x_1)$.

4. Numerical Examples

In this section, we will present a numerical example in the N-dimensional space of real numbers to show that our algorithm is efficient and illustrate the numerical example for comparing the values of sequences generated by iteration (1.4) and (3.31).

If E is a uniformly smooth and uniformly convex Banach space, then an important and interesting Legendre function is $f(x) = \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$ satisfying all assumptions in Theorem 3.2. In this case the gradient ∇f of f coincides with the generalized duality mapping of E, i.e., $\nabla f = J_p$ $(1 < p < \infty)$. In particular in Hilbert spaces, $\nabla f = I$, the identity mapping (see [31])

Let $\Pi^N_{i=1}[a,b]$ be the set of vectors $x \in \mathbb{R}^N$ where each component of x contained in the interval $[a,b]$.

Example 4.1. Let $E = \mathbb{R}^N$ with the Euclidean norm and $C = \Pi_{i=1}^{N}[-\frac{\pi}{2}, \frac{\pi}{2}]$. Let $f : \mathbb{R}^N \to (-\infty, +\infty]$ defined by $f(x) = \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$. The mapping $T : C \to C$ is given by $T(x) = \frac{1}{3}x$, we have T is a Bregman relatively nonexpansive mapping and 0 is the unique fixed point of T. Let $G : C \times C \to \mathbb{R}$ be defined by $G(x, y) = y(x - y)$, $\varphi : C \to \mathbb{R}$ be defined by $\varphi(x) = x^2$, $\Psi : C \to \mathbb{R}^N$ be defined as $\Psi(x) = \sin(x)$ where $x, y \in C$. We get that $0 \in GMEP(G)$ and hence $0 \in \Omega = F(T) \cap GMEP(G)$. We illustrate the results for three cases $p = 2,\ 2.5,\ 3$. For experiment, we randomly generated starting points $x_1, z_1 \in \prod_{i=1}^{N}[-\frac{\pi}{2}, \frac{\pi}{2}]$ with the following control parameter $\alpha_n = \frac{1}{(n+1)^p}$. The following two cases of the control parameter β_n are considered:

Case 1. $\beta_n = 10^{-10} + \frac{1}{n+2}$.

Case 2. $\beta_n = 0.99 - \frac{1}{n+2}$.

Using Algorithm (3.1) with the initial points x_1, z_1, generated by randomly 10 starting points and presented results are on average. We get the following observation for different iterations using the stopping criterion $\|x_{n+1} - x_n\| < 10^{-3}$, we have the numerical result for supporting our main result in Table 1.

TABLE 1. The numerical results for different parameters β_n

Size		Average Iteration		Average Times (sec)	
N	p	β_n defined as in Case 1	β_n defined as in Case 2	β_n defined as in Case 1	β_n defined as in Case 2
5	2	430	450	0.070	0.100
	2.5	821	911	0.156	0.218
	3	852	1000	0.210	0.22
10	2	559	560	0.12	0.234
	2.5	963	1094	0.128	0.238
	3	1159	1227	0.262	0.238

From Table 1, we may suggest that the smallest size of parameter β_n defined as $\beta_n = 10^{-10} + \frac{1}{n+2}$ provides less computational times and iterations than other cases. As Example 4.1, we consider $p = 2$ for comparing numerically in Algorithm (3.31) with Algoritm (1.4).

Example 4.2. Let $E = \mathbb{R}^N$ and $C = \prod_{i=1}^{N}[-\pi, \pi]$. Let $f : \mathbb{R}^N \to (-\infty, +\infty]$ defined by $f(x) = \frac{1}{2}\|x\|^2$ and $T : C \to C$ be defined by $Tx = \frac{3}{4}x$. Let $G : C \times C \to \mathbb{R}$ be defined by $G(x, y) = x(y - x)$, $\varphi : C \to \mathbb{R}$ be defined by $\varphi(x) = \frac{x}{2}$ and $\phi : C \times C \to \mathbb{R}$ be defined by $\phi(x, y) = y - x$, for all $x, y \in C$. Let the sequence $\{x_n\}$ generated by Algorithm (1.4) and Algorithm (3.31) with randomly the initial point $x_1, z_1 \in \prod_{i=1}^{N}[-\pi, \pi]$. We consider the problem setting and the control parameters as in Example 4.1, using $\alpha_n = \frac{1}{n+1}$ with only the case of parameter $\beta_n = 10^{-10} + \frac{1}{n+2}$. We compare Algorithm (1.4) with Algorithm (3.31) using the stopping criterion $\|x_{n+1} - x_n\| < 10^{-3}$, we have the numerical result in Table 2 with randomly 10 starting points x_1, z_1.

TABLE 2. The numerical results for comparing Algorithm (1.4) and Algorithm (3.31)

Size	Average Iterations		Average Times (sec)	
$\mathbb{R}^N$	Algorithm (3.31)	Algorithm (1.4)	Algorithm (3.31)	Algorithm (1.4)
5	204	217	0.0470	0.0610
10	269	283	0.0530	0.0590
50	454	472	0.0870	0.0886
100	569	600	0.1110	0.1310
500	996	1045	0.1550	0.2380

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References

[1] Alber, Y. I., Metric and generalized projection operators in Banach spaces: Properties and applications, In: Kartsatos, A.G. (ed.) Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Lecture Notes in Pure and Appl. Math., Marcel Dekker, New York, **178** (1996), 15–50

[2] Agarwal, R. P., Chen, J. W., Cho, Y. J., Strong convergence theorems for equilibrium problems and weak Bregman relatively nonexpansive mappings in Banach spaces. J. Inequal. Appl., **119** (2013)

[3] Bauschke, H. H., Borwein, J. M., Combettes, P. L., Essential smoothness, essential strict convexity, and Legendre function in Banach spaces. Comm. Contemp. Math., **3** (2001), 615–647

[4] Bauschke, H. H., Borwein, J. M., Combettes, P. L., Bregman monotone optimization algorithms. SIAM J. Control Optim., **42** (2003), 596–636

[5] Blum, E., Oettli, W., From optimization and variational inequalities to equilibrium problems. Math. Stud., **63** (1994), 123–145

[6] Bregman, L. M., The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming. USSR. Comput. Math. Math. Phys., **7** (1967), 200–217

[7] Bruck, R. E., Reich, S., Nonexpansive projections and resolvents of accretive operators in Banach spaces. Houst. J. Math., **3** (1977), 459–470

[8] Butnariu, D., Iusem, A. N., Totally convex functions for fixed points computation and infinite dimensional optimization, Applied Optimization. Kluwer Academic, Dordrecht **40** (2000)

[9] Butnariu, D., Iusem, A. N., Zalineacu, C., On uniform convexity, totally convexity and convergence of the proximal point and outer Bregman projection algorithms in Banach spaces. J. Convex Anal., **10** (2003), 35–61

[10] Butnariu, D., Reich, S., Zaslavski, A. J., Asymptotic behavior of relatively nonexpansive operators in Banach spaces. J. Appl. Anal., **7** (2001), 151–174

[11] Butnariu, D., Resmerita, E., Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces. Abstr. Appl. Anal., **2006**, 14 pages

[12] Chen, J. W., Wan, Z. P., Yuan, L. Y., Zheng, Y., Approximation of fixed points of weak Bregman relatively nonexpansive mappings in Banach spaces. Inter. J. Math. Math. Sci., **23** (2011), Article ID 420192

[13] Ceng, L. C., Yao, J. C., A hybrid iterative scheme for mixed equilibrium problems and fixed point problems. J. Comput. Appl. Math., **214** (2008), 186–201

[14] Combettes, P. L., Hirstoaga, S. A., Equilibrium programming in Hilbert spaces. J. Nonlinear Convex Anal. **6** (2005) 117–136

[15] Darvish, V., A new algorithm for mixed equilibrium problem and Bregman strongly nonexpansive mapping in Banach spaces. Math. FA., **1** (2015)

[16] Darvish, V., Strong convergence theorem for generalized mixed equilibrium problems and Bregman nonexpansive mapping in Banach space. Mathematica Moravica, Vol. **20** (2016), 69-87

[17] Kazmi, K. R., Ali, R., Yousuf, S., Generalized equilibrium and fixed point problems for Bregman relatively nonexpansive mappings in Banach spaces. J. Fixed Point Theory Appl., **20** (2018)

[18] Moudafi, A., Viscosity approximation methods for fixed-point problems. J. Math. Anal. Appl. **241** (2000) 46–55

[19] Phelps, R. P., Convex Functions, Monotone Operators, and Differentiability. Springer-Verlag: Berlin, Germany, (1993)

[20] Reich, S., A weak convergence theorem for the alternating method with Bregman distances. In: Theory and applications of nonlinear operators. Marcel Dekker, New York, (1996), 313–318

[21] Reich, S., Sabach, S., A strong convergence theorem for a proximal-type algorithm in reflexive Banach space. J. Nonlinear Convex Anal., **10** (2009), 471–485

[22] Reich, S., Sabach, S., Two strong convergence theorems for a proximal method in reflexive Banach spaces. Numer. Funct. Anal. Optim., **31** (2010), 22–44

[23] Reich, S., Sabach, S., A projection method for solving nonlinear problems in reflexive Banach spaces. J. Fixed Point Theory Appl., **9** (2011), 101–116

[24] Reich, S., Sabach, S., Existence and Approximation of Fixed Points of Bregman Firmly Nonexpansive Mappings in Reflexive Banach Spaces. Springer: New York, NY, USA, (2011), 301–316

[25] Sabach, S., Products of finitely many resolvents of maximal monotone mappings in reflexive Banach spaces. SIAM J. Optim., **21** (2011), 1289–1308

[26] Shahzad, N., Zegeye, H., Convergence theorem for common fixed points of a finite family of multi-valued Bregman relatively nonexpansive mappings. Fixed Point Theory Appl., **2014**, 152

[27] Tada, A., Takahashi, W., Strong convergence theorem for an equilibrium problem and a nonexpansive mapping, in: W. Takahashi, T. Tanaka(Eds.), Nonlinear Analysis and Convex Analysis, Yokohama Publishers, Yokohama, (2006)

[28] Takahashi, S., Takahashi, W., Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces. J. Math. Anal. Appl. **331** (2007), 506–515

[29] Ugwunnadi, G.C., Ali, B., Idris, I., Minjibir, M.S., Strong convergence theorem for quasi-Bregman strictly pseudocontractive mappings and equilibrium problems in Banach spaces. Fixed Point Theory Appl., **2014**, 231

[30] Zalinescu, C., Convex Analysis in General Vector spaces. World Scientific Publishing Co., Inc.: River Edge, NJ, USA, (2002)

[31] Zegeye, H., Shahzad, N., Strong convergence theorems for a common fixed point of a finite family of Bregman weak relativity nonexpansive mappings in reflexive Banach spaces. The Scientific World, (2014)

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