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Linear functions on the orthogonal group

ฟังก์ชันเชิงเส้นบนกรุปเชิงตั้งฉาก

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สนับสนุนโดยมหาวิทยาลัยนเรศวร

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Abstract

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Abstract: The Lucas sequence is an integer sequence defined similarly to the Fibonacci sequence and the Pell sequence. There has been studied continuously about the reciprocal sums derived from the Fibonacci sequence, the Pell sequence, and the Lucas sequence. These results are beautiful in the involving fields. Recently, there has been found the interesting results for the partial finite sum of the reciprocals of the even and odd terms in the Fibonacci sequence and the Pell sequence. In this research, we aim to find the exact formula for the partial finite sum of the reciprocals of the even and odd terms in the Lucas sequence.

keywords: Lucas numbers, reciprocal, finite sum, even and odd terms

บทคัดย่อ: ลำดับลูคัสเป็นลำดับของจำนวนเต็มซึ่งนิยามคล้ายกับลำดับฟิโบนัคซีและลำดับเพลล์ ได้มีการศึกษาอย่างต่อเนื่องเกี่ยวกับผลรวมส่วนกลับซึ่งได้มาจากลำดับฟิโบนัคซี ลำดับเพลล์ และลำดับลูคัส ผลลัพธ์เหล่านี้สวยงามในสาขาวิชาที่เกี่ยวข้อง เมื่อเร็วๆนี้ได้มีการค้นพบผลลัพธ์ที่น่าสนใจสำหรับผลรวมจำกัดของส่วนกลับของพจน์คู่และคี่ในลำดับฟิโบนัคซีและลำดับเพลล์ ในงานวิจัยนี้ เรามีจุดประสงค์เพื่อหาสูตรแน่นอนสำหรับผลรวมจำกัดของส่วนกลับของพจน์คู่และคี่ในลำดับลูคัส

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1. Abstract

The Lucas sequence is an integer sequence defined similarly to the Fibonacci sequence and the Pell sequence. There has been studied continuously about the reciprocal sums derived from the Fibonacci sequence, the Pell sequence, and the Lucas sequence. These results are beautiful in the involving fields. Recently, there has been found the interesting results for the partial finite sum of the reciprocals of the even and odd terms in the Fibonacci sequence and the Pell sequence. In this research, we aim to find the exact formula for the partial finite sum of the reciprocals of the even and odd terms in the Lucas sequence.

2. Executive summary

2.1 Introduction to research

The Fibonacci numbers are named after Italian mathematician, Leonardo of Pisa, later known as Fibonacci. They are the numbers derived from an integer sequence called the Fibonacci sequence such that begin with 0 and 1, and the number after that is the sum of the two previous terms. That is,

$$F_0 = 0,\ F_1 = 1, \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

The Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$. Fibonacci numbers are strongly related to the golden ratio and appear often in mathematics.

The Pell numbers are defined by

$$P_0 = 0,\ P_1 = 1, \text{ and } P_n = 2P_{n-1} + P_{n-2} \text{ for } n \geq 2.$$

They are $0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, \ldots$.

The Lucas numbers are named after the mathematician Francois Edouard Anatole Lucas. Each Lucas number is defined to be the sum of its two previous terms, as follows.

$$L_0 = 2,\ L_1 = 1, \text{ and } L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 2.$$

This is like the Fibonacci numbers, except the starting value. The Lucas numbers are $2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots$.

2.2 Literature review

The reciprocal sums of the Fibonacci numbers are studied in 2009 by Ohtsuka and Nakamura [5].

Theorem 1. *For all positive integers $n \geq 2$,*

$$\left\lfloor \left(\sum_{k=1}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_n - F_{n-1}, & \text{if } n \text{ is even}, \\ F_n - F_{n-1} - 1, & \text{if } n \text{ is odd}, \end{cases} \tag{1}$$

where $\lfloor \bullet \rfloor$ denotes the floor function.

Later, in 2015, Wang and Wen [6] extend Theorem 1 to the finite partial sums.

Theorem 2. *(i) For all positive integers $n \geq 4$,*

$$\left\lfloor \left(\sum_{k=n}^{2n} \frac{1}{F_k} \right)^{-1} \right\rfloor = F_n - F_{n-1}. \tag{2}$$

(ii) For all positive integers $m \geq 3$ and $n \geq 2$,

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_n - F_{n-1}, & \text{if } n \text{ is even}, \\ F_n - F_{n-1} - 1, & \text{if } n \text{ is odd}. \end{cases} \tag{3}$$

Wang and Zhang [7] investigated more about the reciprocal sums of the Fibonacci numbers with even and odd indexes.

Theorem 3. *For any positive integers m and n,*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{2k}} \right) \right\rfloor = \begin{cases} F_{2n} - F_{2n-2}, & \text{if } m = 2 \text{ and } n \geq 3, \\ F_{2n} - F_{2n-1} - 1, & \text{if } m \geq 3 \text{ and } n \geq 1. \end{cases} \tag{4}$$

Theorem 4. *For all integers $n \geq 1$ and $m \geq 2$,*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{2k-1}} \right)^{-1} \right\rfloor = F_{2n-1} - F_{2n-3}. \tag{5}$$

Recently, in 2017, Choo [1, 2] has found some interesting results about the reciprocal sum of the Pell numbers and the Lucas numbers.

Theorem 5. *For all positive integers m and n. We have*

$$\left\lfloor \left(\sum_{k=n}^{m} \frac{1}{P_k} \right)^{-1} \right\rfloor = \begin{cases} P_n - P_{n-1}, & \text{if } n \geq 2 \text{ with } n \text{ is even and } m \geq 2n, \\ P_n - P_{n-1} - 1, & \text{if } n \geq 1 \text{ with } n \text{ is odd and } m \geq 3n. \end{cases} \tag{6}$$

Theorem 6. *For all positive integers m and n. We have*

$$\left\lfloor \left(\sum_{k=n}^{m} \frac{1}{L_k} \right)^{-1} \right\rfloor = \begin{cases} L_n - L_{n-1}, & \text{if } n \geq 3 \text{ with } n \text{ is odd and } m \geq 2n, \\ L_n - L_{n-1} - 1, & \text{if } n \geq 4 \text{ with } n \text{ is even and } m \geq 3n. \end{cases} \tag{7}$$

It can be seen that the results (3), (6), and (7) are similar and in the same direction. So, it is natural to ask whether there are the same phenomenal for other sums? This motivated Janphaisaeng and Sookcharoenpinyo to investigate the partial finite sums for the reciprocal of the Pell numbers with even and odd indexes.

Theorem 7. *For any positive integers m and n,*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{P_{2k}} \right) \right\rfloor = \begin{cases} P_{2n} - P_{2n-2}, & \text{if } m = 2 \text{ and } n \geq 3, \\ P_{2n} - P_{2n-2} - 1, & \text{if } m \geq 3 \text{ and } n \geq 1. \end{cases} \tag{8}$$

Theorem 8. *For all integers* $n \geq 2$ *and* $m \geq 2$,

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{P_{2k-1}} \right)^{-1} \right\rfloor = P_{2n-1} - P_{2n-3}. \qquad (9)$$

2.3 Objectives

The results (8) and (9) are good and closed to those of the Fibonacci numbers as we expected. These results show the beauty in mathematics. So, we are interested in finding exact formula for the partial finite sum of the even-indexed and the odd-indexed reciprocal Lucas numbers. We hope to see results that are similar to those of the Fibonacci and the Pell numbers. As a consequence, we can guess formulas for the reciprocal sums of the Pell numbers, the Lucas numbers, and the other related numbers from those of the Fibonacci numbers.

3. Results and discussion

We begin this section with some identities involving the Lucas numbers that will be used in the proof of our main theorems. They are similar to identities of the Fibonacci numbers and the Pell numbers.

Lemma 9. *For any positive integer* $n \geq 1$, *we have*

$$L_n^2 - L_{n-1}L_{n+1} = 5(-1)^n. \qquad (10)$$

Proof. We proceed by induction on n. It is clearly true for $n = 1$. Assuming the result holds for any positive integer k. We have

$$
\begin{aligned}
L_{k+1}^2 - L_k L_{k+2} &= L_{k+1}\left(L_k + L_{k-1}\right) - L_k\left(L_{k+1} + L_k\right) \\
&= L_{k+1}L_k + L_{k+1}L_{k-1} - L_k L_{k+1} - L_k^2 \\
&= L_{k+1}L_{k-1} - L_k^2 \\
&= -5(-1)^{k-1} \\
&= 5(-1)^k .
\end{aligned}
$$

This completes the induction on n. ■

Lemma 10. *For any positive integer $n \geq 2$, we have*

$$L_n^2 - L_{n-2}L_{n+2} = 5(-1)^{n-1}. \tag{11}$$

Proof. By the definition of the Lucas numbers and Lemma 9, we obtain that

$$
\begin{aligned}
L_n^2 - L_{n-2}L_{n+2} &= \left(L_{n-1} + L_{n-2}\right)^2 - L_{n-2}\left(L_{n+1} + L_n\right) \\
&= L_{n-1}^2 + 2L_{n-1}L_{n-2} + L_{n-2}^2 - L_{n-2}L_{n+1} - L_{n-2}L_n \\
&= L_{n-1}^2 + 2L_{n-1}L_{n-2} + L_{n-2}^2 - L_{n-2}\left(L_n + L_{n-1}\right) - L_{n-2}\left(L_{n-1} + L_{n-2}\right) \\
&= L_{n-1}^2 - L_{n-2}L_n \\
&= 5(-1)^{n-1}.
\end{aligned}
$$

■

Lemma 11. *For any positive integers $a \geq 1$ and $b \geq 1$, we have*

$$L_a L_b + L_{a+1}L_{b+1} = L_{a+b} + L_{a+b+2}. \tag{12}$$

Proof. Let $a \geq 1$ be a positive integer. We proceed by induction on $b \geq 1$. For $b = 1$, we have

$$
\begin{aligned}
L_a L_1 + L_{a+1}L_2 = L_a + 3L_{a+1} &= \left(L_a + L_{a+1}\right) + L_{a+1} + L_{a+1} \\
&= \left(L_{a+2} + L_{a+1}\right) + L_{a+1} \\
&= L_{a+3} + L_{a+1}
\end{aligned}
$$

Let k be any positive integer. Assume that the equation (12) holds for any positive integer $b \leq k$. We get

$$L_aL_{k+1}+L_{a+1}L_{k+2}=L_a\left(L_k+L_{k-1}\right)+L_{a+1}\left(L_{k+1}+L_k\right)$$
$$=L_aL_k+L_aL_{k-1}+L_{a+1}L_{k+1}+L_{a+1}L_k$$
$$=\left(L_aL_k+L_{a+1}L_{k+1}\right)+\left(L_aL_{k-1}+L_{a+1}L_k\right)$$
$$=\left(L_{a+k}+L_{a+k+2}\right)+\left(L_{a+k-1}+L_{a+k+1}\right)$$
$$=\left(L_{a+k-1}+L_{a+k}\right)+\left(L_{a+k+1}+L_{a+k+2}\right)$$
$$=L_{a+k}+L_{a+k+3}.$$

This completes the mathematical induction on b. ■

Now, we are ready to prove the case of even term.

Theorem 12. *For all positive integers* $m\geq 2$ *and* $n\geq 2$, *we have*

$$\left\lfloor\left(\sum_{k=n}^{mn}\frac{1}{L_{2k}}\right)^{-1}\right\rfloor=L_{2n}-L_{2n-2}. \tag{13}$$

Proof. Equation (13) is equivalent to $L_{2n}-L_{2n-2}\leq\left(\sum_{k=n}^{mn}\frac{1}{L_{2k}}\right)^{-1}<L_{2n}-L_{2n-2}+1$, or

$$\frac{1}{L_{2n}-L_{2n-2}+1}<\sum_{k=n}^{mn}\frac{1}{L_{2k}}\leq\frac{1}{L_{2n}-L_{2n-2}}. \tag{14}$$

By elementary calculation we derive that, for $k\geq 1$,

$$\frac{1}{L_{2k}-L_{2k-2}+1}-\frac{1}{L_{2k}}-\frac{1}{L_{2k+2}-L_{2k}+1}$$
$$=\frac{L_{2k}\left(L_{2k+2}-L_{2k}+1\right)-\left(L_{2k}-L_{2k-2}+1\right)\left(L_{2k+2}-L_{2k}+1\right)-L_{2k}\left(L_{2k}-L_{2k-2}+1\right)}{L_{2k}\left(L_{2k}-L_{2k-2}+1\right)\left(L_{2k+2}-L_{2k}+1\right)}$$
$$=\frac{-L_{2k}^2+L_{2k-2}L_{2k+2}+L_{2k-2}-L_{2k+2}-1}{L_{2k}\left(L_{2k}-L_{2k-2}+1\right)\left(L_{2k+2}-L_{2k}+1\right)}.$$

By Lemma 10, we get

$$\frac{1}{L_{2k}-L_{2k-2}+1}-\frac{1}{L_{2k}}-\frac{1}{L_{2k+2}-L_{2k}+1}=\frac{-5(-1)^{2k-1}+L_{2k-2}-L_{2k+2}-1}{L_{2k}\left(L_{2k}-L_{2k-2}+1\right)\left(L_{2k+2}-L_{2k}+1\right)}$$
$$=\frac{4+L_{2k-2}-L_{2k+2}}{L_{2k}\left(L_{2k}-L_{2k-2}+1\right)\left(L_{2k+2}-L_{2k}+1\right)}.$$

By summing both sides from n to mn, we get

$$\frac{1}{L_{2n}-L_{2n-2}+1}-\sum_{k=n}^{mn}\frac{1}{L_{2k}}-\frac{1}{L_{2mn+2}-L_{2mn}+1}=\sum_{k=n}^{mn}\frac{4+L_{2k-2}-L_{2k+2}}{L_{2k}\left(L_{2k}-L_{2k-2}+1\right)\left(L_{2k+2}-L_{2k}+1\right)}.$$

Then

$$\begin{aligned}\sum_{k=n}^{mn}\frac{1}{L_{2k}}&=\frac{1}{L_{2n}-L_{2n-2}+1}-\frac{1}{L_{2mn+2}-L_{2mn}+1}+\sum_{k=n}^{mn}\frac{L_{2k+2}-L_{2k-2}-4}{L_{2k}\left(L_{2k}-L_{2k-2}+1\right)\left(L_{2k+2}-L_{2k}+1\right)}\\&>\frac{1}{L_{2n}-L_{2n-2}+1}-\frac{1}{L_{2mn+2}-L_{2mn}+1}+\frac{L_{2n+2}-L_{2n-2}-4}{L_{2n}\left(L_{2n}-L_{2n-2}+1\right)\left(L_{2n+2}-L_{2n}+1\right)}.\end{aligned}$$

Since $n\geq 2$, $L_{2n-1}\geq L_3=4$. Therefore, $L_{2n}=L_{2n-1}+L_{2n-2}\geq L_{2n-2}+4$. It follows that $L_{2n+2}-L_{2n-2}-4\geq L_{2n+2}-L_{2n}=L_{2n+1}$. Then

$$\begin{aligned}\left(L_{2n+2}-L_{2n-2}-4\right)\left(L_{2mn+2}-L_{2mn}+1\right)&\geq L_{2n+1}\left(L_{2mn+2}-L_{2mn}+1\right)\\&=L_{2n+1}\left(L_{2mn+1}+1\right)\\&\geq L_{2n+1}\left(L_{4n+1}+1\right)\\&>L_{2n+1}L_{4n+1}\\&=L_{2n+1}\left(L_{4n}+L_{4n-1}\right)\\&>L_{2n+1}\left(L_{4n}+L_{4n-2}\right).\end{aligned}$$

By Lemma 11, we get that $L_{4n}+L_{4n-2}\geq L_{2n}L_{2n}$. Since $L_{2n-2}\geq L_2=3$, we get

$$\begin{aligned}\left(L_{2n+2}-L_{2n-2}-4\right)\left(L_{2mn+2}-L_{2mn}+1\right)&>L_{2n+1}L_{2n}L_{2n}\\&=L_{2n+1}L_{2n}\left(L_{2n-1}+L_{2n-2}\right)\\&\geq L_{2n+1}L_{2n}\left(L_{2n-1}+3\right)\\&=L_{2n+1}L_{2n}L_{2n-1}+3L_{2n+1}L_{2n}\\&>L_{2n+1}L_{2n}L_{2n-1}+L_{2n}L_{2n-1}+L_{2n+1}L_{2n}+L_{2n}\\&=L_{2n}\left(L_{2n+1}L_{2n-1}+L_{2n-1}+L_{2n+1}+1\right)\\&=L_{2n}\left(L_{2n-1}+1\right)\left(L_{2n+1}+1\right)\\&=L_{2n}\left(L_{2n}-L_{2n-2}+1\right)\left(L_{2n+2}-L_{2n}+1\right).\end{aligned}$$

It follows that $\dfrac{L_{2n+2}-L_{2n-2}-4}{L_{2n}\left(L_{2n}-L_{2n-2}+1\right)\left(L_{2n+2}-L_{2n}+1\right)}>\dfrac{1}{L_{2mn+2}-L_{2mn}+1}$. Hence,

$$\sum_{k=n}^{mn}\frac{1}{L_{2k}} > \frac{1}{L_{2n}-L_{2n-2}+1} \tag{15}$$

On the other hand, by elementary calculation and Lemma 10, we derive that, for $k \geq 1$,

$$\begin{aligned}\frac{1}{L_{2k}-L_{2k-2}}-\frac{1}{L_{2k}}-\frac{1}{L_{2k+2}-L_{2k}} &= \frac{-L_{2k}^2+L_{2k-2}L_{2k+2}}{L_{2k}\left(L_{2k}-L_{2k-2}\right)\left(L_{2k+2}-L_{2k}\right)} \\ &= \frac{5}{L_{2k}\left(L_{2k}-L_{2k-2}\right)\left(L_{2k+2}-L_{2k}\right)} \\ &> 0.\end{aligned}$$

Summing both sides from n to mn, we get

$$\frac{1}{L_{2n}-L_{2n-2}}-\sum_{k=n}^{mn}\frac{1}{L_{2k}}-\frac{1}{L_{2mn+2}-L_{2mn}} > 0,$$

which implies that

$$\sum_{k=n}^{mn}\frac{1}{L_{2k}} < \frac{1}{L_{2n}-L_{2n-2}}-\frac{1}{L_{2mn+2}-L_{2mn}} < \frac{1}{L_{2n}-L_{2n-2}}. \tag{16}$$

Finally, combining (16) and (15), we obtain (13) as desired. ■

Before proving the case of odd term, we need some help from the following lemmas.

Lemma 13. *For any positive integer* $n \geq 1$, *we have*

$$L_{2n}+L_{2n+2} = L_{n+1}L_{n+2}-L_{n-1}L_{n} \tag{17}$$

Proof. Setting $a = n-1$ and $b = n+1$ in Lemma 11, we get

$$\begin{aligned} L_{2n}+L_{2n+2} &= L_{n-1}L_{n+1}+L_{n}L_{n+2} \\ &= L_{n-1}\left(L_{n+2}-L_{n}\right)+L_{n}L_{n+2} \\ &= L_{n-1}L_{n+2}-L_{n-1}L_{n}+L_{n}L_{n+2} \\ &= \left(L_{n-1}+L_{n}\right)L_{n+2}-L_{n-1}L_{n} \\ &= L_{n+1}L_{n+2}-L_{n-1}L_{n}. \end{aligned}$$

∎

Lemma 14. *Let a and b be two integers with $a \geq b \geq 0$. If $n > a$, then*

$$L_{n+a}L_{n-a-1} - L_{n+b}L_{n-b-1} = (-1)^{n-a} L_{a+b+1}L_{a-b} + 2(-1)^{n-a+1} L_{2a+1}. \quad (18)$$

Proof. We proceed by induction on n. For $n = a+1$, we have

$$L_{2a+1}L_0 - L_{a+b+1}L_{a-b} = 2L_{2a+1} - L_{a+b+1}L_{a-b} = (-1)^1 L_{a+b+1}L_{a-b} + 2(-1)^2 L_{2a+1}.$$

Assume that the result holds for any integer $n > a$. Applying Lemma 11 twice and using the induction hypothesis, we get

$$\begin{aligned} L_{n+a+1}L_{n-a} - L_{n+b+1}L_{n-b} &= (L_{n+a+1}L_{n-a} + L_{n+a}L_{n-a-1}) - L_{n+b+1}L_{n-b} - L_{n+a}L_{n-a-1} \\ &= L_{2n+1} + L_{2n-1} - L_{n+b+1}L_{n-b} - L_{n+a}L_{n-a-1} \\ &= (L_{n+b+1}L_{n-b} + L_{n+b}L_{n-b-1}) - L_{n+b+1}L_{n-b} - L_{n+a}L_{n-a-1} \\ &= L_{n+b}L_{n-b-1} - L_{n+a}L_{n-a-1} \\ &= (-1)^{n-a+1} L_{a+b+1}L_{a-b} + 2(-1)^{n-a+2} L_{2a+1}, \end{aligned}$$

which completes the mathematical induction on n. ∎

Theorem 15. *For all positive integers $n \geq 6$, we have*

$$\left\lfloor \left(\sum_{k=n}^{2n} \frac{1}{L_{2k-1}} \right)^{-1} \right\rfloor = L_{2n-1} - L_{2n-3}. \quad (19)$$

Proof. We must show that

$$\frac{1}{L_{2n-1} - L_{2n-3} + 1} < \sum_{k=n}^{mn} \frac{1}{L_{2k-1}} \leq \frac{1}{L_{2n-1} - L_{2n-3}}. \quad (20)$$

By elementary calculation and Lemma 10, we derive that, for $k \geq 2$,

$$\frac{1}{L_{2k-1}-L_{2k-3}+1}-\frac{1}{L_{2k-1}}-\frac{1}{L_{2k+1}-L_{2k-1}+1}$$
$$=\frac{L_{2k-3}L_{2k+1}-L_{2k-1}^2+L_{2k-3}-L_{2k+1}-1}{L_{2k-1}\left(L_{2k-1}-L_{2k-3}+1\right)\left(L_{2k+1}-L_{2k-1}+1\right)}$$
$$=\frac{-5(-1)^{2k-2}+L_{2k-3}-L_{2k+1}-1}{L_{2k-1}\left(L_{2k-1}-L_{2k-3}+1\right)\left(L_{2k+1}-L_{2k-1}+1\right)}$$
$$=\frac{L_{2k-3}-L_{2k+1}-6}{L_{2k-1}\left(L_{2k-1}-L_{2k-3}+1\right)\left(L_{2k+1}-L_{2k-1}+1\right)}.$$

Summing both sides from n to $2n$ gives

$$\frac{1}{L_{2n-1}-L_{2n-3}+1}-\sum_{k=n}^{2n}\frac{1}{L_{2k-1}}-\frac{1}{L_{4n+1}-L_{4n-1}+1}$$
$$=\sum_{k=n}^{2n}\frac{L_{2k-3}-L_{2k+1}-6}{L_{2k-1}\left(L_{2k-1}-L_{2k-3}+1\right)\left(L_{2k+1}-L_{2k-1}+1\right)}$$

Then

$$\sum_{k=n}^{2n}\frac{1}{L_{2k-1}}=\frac{1}{L_{2n-1}-L_{2n-3}+1}-\frac{1}{L_{4n+1}-L_{4n-1}+1}$$
$$+\sum_{k=n}^{2n}\frac{L_{2k+1}-L_{2k-3}+6}{L_{2k-1}\left(L_{2k-1}-L_{2k-3}+1\right)\left(L_{2k+1}-L_{2k-1}+1\right)}$$
$$>\frac{1}{L_{2n-1}-L_{2n-3}+1}-\frac{1}{L_{4n+1}-L_{4n-1}+1}+\frac{1}{L_{2n-1}\left(L_{2n+1}-L_{2n-1}+1\right)}.$$

By Lemma 11, we get

$$L_{4n+1}-L_{4n-1}+1=L_{4n}+1=L_{4n-1}+L_{4n-3}+1$$
$$=L_{2n}L_{2n-1}+L_{2n-1}L_{2n-2}+1$$
$$=L_{2n-1}\left(L_{2n}+L_{2n-2}\right)+1$$
$$=L_{2n-1}\left(L_{2n+1}-L_{2n-1}+L_{2n-2}\right)+1$$
$$>L_{2n-1}\left(L_{2n+1}-L_{2n-1}+1\right).$$

Therefore,

$$\sum_{k=n}^{2n}\frac{1}{L_{2k-1}}>\frac{1}{L_{2n-1}-L_{2n-3}+1}. \tag{21}$$

On the other hand, by elementary calculation and Lemma 9, we derive that, for $k \geq 2$,

$$\frac{1}{L_{2k-1}-L_{2k-3}}-\frac{1}{L_{2k-1}}-\frac{1}{L_{2k+1}-L_{2k-1}}=\frac{L_{2k-3}L_{2k+1}-L_{2k-1}^2}{L_{2k-1}\left(L_{2k-1}-L_{2k-3}\right)\left(L_{2k+1}-L_{2k-1}\right)}$$
$$=\frac{-5(-1)^{2k-2}}{L_{2k-1}\left(L_{2k-1}-L_{2k-3}\right)\left(L_{2k+1}-L_{2k-1}\right)}$$
$$=\frac{-5}{L_{2k-1}\left(L_{2k-1}-L_{2k-3}\right)\left(L_{2k+1}-L_{2k-1}\right)}. \quad (22)$$

Summing both sides from n to $2n$ gives

$$\frac{1}{L_{2n-1}-L_{2n-3}}-\sum_{k=n}^{2n}\frac{1}{L_{2k-1}}-\frac{1}{L_{4n+1}-L_{4n-1}}=\sum_{k=n}^{2n}\frac{-5}{L_{2k-1}\left(L_{2k-1}-L_{2k-3}\right)\left(L_{2k+1}-L_{2k-1}\right)},$$

which implies that

$$\sum_{k=n}^{2n}\frac{1}{L_{2k-1}}=\frac{1}{L_{2n-1}-L_{2n-3}}-\frac{1}{L_{4n+1}-L_{4n-1}}+\sum_{k=n}^{2n}\frac{5}{L_{2k-1}\left(L_{2k-1}-L_{2k-3}\right)\left(L_{2k+1}-L_{2k-1}\right)}.$$

We know that $L_{2n-4}=L_{2n-5}+L_{2n-6}=2L_{2n-6}+L_{2n-7}=3L_{2n-7}+2L_{2n-8}\geq 5L_{2n-8}$. By replacing n by $2n-1$ in Lemma 13, we get

$$5\left(L_{4n+1}-L_{4n-1}\right)L_{2n-8}=5L_{4n}L_{2n-8}\leq L_{4n}L_{2n-4}$$
$$=\left(L_{2n}L_{2n+1}-L_{2n-2}L_{2n-1}-L_{4n-2}\right)L_{2n-4}$$
$$\leq L_{2n}L_{2n+1}L_{2n-4}-L_{2n-2}L_{2n-1}L_{2n-4}.$$

Setting $a=2$ and $b=0$, and replacing n by $2n-1$ in Lemma 14, we get

$$L_{2n+1}L_{2n-4}-L_{2n-1}L_{2n-2}=(-1)^{2n-3}L_3L_2+2(-1)^{2n-2}L_5=-(4)(3)+2(11)=10. \quad (23)$$

Setting $a=1$ and $b=0$, and replacing n by $2n-1$ in Lemma 14, we get

$$L_{2n}L_{2n-3} - L_{2n-1}L_{2n-2} = (-1)^{2n-2} L_2L_1 + 2(-1)^{2n-1} L_3 = 3 - 2(4) = 3 - 8 = -5. \tag{24}$$

Then from (23) and (24), we obtain

$$\begin{aligned}5(L_{4n+1} - L_{4n-1})L_{2n-8} &\le L_{2n}(L_{2n-1}L_{2n-2} + 10) - (L_{2n}L_{2n-3} + 5)L_{2n-4} \\ &= L_{2n}L_{2n-1}L_{2n-2} + 10L_{2n} - L_{2n}L_{2n-3}L_{2n-4} - 5L_{2n-4}.\end{aligned}$$

Since $n \ge 6$, $L_{2n-3} \ge L_9 = 76$ and $L_{2n-4} \ge L_8 = 47$. Therefore,

$$\begin{aligned}5(L_{4n+1} - L_{4n-1})L_{2n-8} &\le L_{2n}L_{2n-1}L_{2n-2} + 10L_{2n} - 10L_{2n} - 5L_{2n-4} \\ &< L_{2n}L_{2n-1}L_{2n-2} \\ &= (L_{2n+1} - L_{2n-1})L_{2n-1}(L_{n-1} - L_{n-3}).\end{aligned}$$

It follows that $\dfrac{5(L_{4n+1} - L_{4n-1})}{L_{2n-1}(L_{2n-1} - L_{2n-3})(L_{2n+1} - L_{2n-1})} < \dfrac{1}{L_{2n-8}}$. Therefore,

$$\begin{aligned}\sum_{k=n}^{2n} \frac{5(L_{4n+1} - L_{4n-1})}{L_{2k-1}(L_{2k-1} - L_{2k-3})(L_{2k+1} - L_{2k-1})} &\le \sum_{k=n}^{2n} \frac{5(L_{4n+1} - L_{4n-1})}{L_{2n-1}(L_{2n-1} - L_{2n-3})(L_{2n+1} - L_{2n-1})} \\ &< \sum_{k=n}^{2n} \frac{1}{L_{2n-8}} \\ &= \frac{n+1}{L_{2n-8}} \\ &\le 1.\end{aligned}$$

This means $\displaystyle\sum_{k=n}^{2n} \frac{5}{L_{2k-1}(L_{2k-1} - L_{2k-3})(L_{2k+1} - L_{2k-1})} \le \frac{1}{L_{4n+1} - L_{4n-1}}$. Then we get

$$\sum_{k=n}^{2n} \frac{1}{L_{2k-1}} < \frac{1}{L_{2n-1} - L_{2n-3}}. \tag{25}$$

Finally, combining (25) and (11), we obtain (20) as desired. ■

Theorem 16. *For all positive integers $m \ge 3$ and $n \ge 2$, we have*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{L_{2k-1}} \right)^{-1} \right\rfloor = L_{2n-1} - L_{2n-3} - 1. \tag{26}$$

Proof. We have to show that

$$\frac{1}{L_{2n-1}-L_{2n-3}} < \sum_{k=n}^{mn} \frac{1}{L_{2k-1}} \le \frac{1}{L_{2n-1}-L_{2n-3}-1}. \tag{27}$$

From equation (22), summing both sides from n to mn yields

$$\frac{1}{L_{2n-1}-L_{2n-3}} - \sum_{k=n}^{mn} \frac{1}{L_{2k-1}} - \frac{1}{L_{2mn+1}-L_{2mn-1}} = \sum_{k=n}^{mn} \frac{-5}{L_{2k-1}\left(L_{2k-1}-L_{2k-3}\right)\left(L_{2k+1}-L_{2k-1}\right)},$$

which implies that

$$\begin{aligned}\sum_{k=n}^{mn} \frac{1}{L_{2k-1}} &= \frac{1}{L_{2n-1}-L_{2n-3}} - \frac{1}{L_{2mn+1}-L_{2mn-1}} + \sum_{k=n}^{mn} \frac{5}{L_{2k-1}\left(L_{2k-1}-L_{2k-3}\right)\left(L_{2k+1}-L_{2k-1}\right)} \\ &> \frac{1}{L_{2n-1}-L_{2n-3}} - \frac{1}{L_{2mn+1}-L_{2mn-1}} + \frac{5}{L_{2n-1}\left(L_{2n-1}-L_{2n-3}\right)\left(L_{2n+1}-L_{2n-1}\right)}.\end{aligned}$$

Since $m \ge 3$, applying Lemma 11 twice we get

$$\begin{aligned}5\left(L_{2mn+1}-L_{2mn-1}\right) = 5L_{2mn} &> 4L_{6n} \\ &\ge 2L_{4n}L_{2n-1} \\ &\ge L_{2n}L_{2n-1}L_{2n-1} \\ &> L_{2n}L_{2n-1}L_{2n-2} = \left(L_{2n+1}-L_{2n-1}\right)L_{2n-1}\left(L_{2n-1}-L_{2n-3}\right).\end{aligned}$$

Then we get $\dfrac{5}{L_{2n-1}\left(L_{2n-1}-L_{2n-3}\right)\left(L_{2n+1}-L_{2n-1}\right)} \ge \dfrac{1}{L_{2mn+1}-L_{2mn-1}}$. Therefore,

$$\sum_{k=n}^{mn} \frac{1}{L_{2k-1}} > \frac{1}{L_{2n-1}-L_{2n-3}}. \tag{28}$$

On the other hand, by elementary calculation and Lemma 10, we derive

that, for $k \ge 2$,

$$\frac{1}{L_{2k-1}-L_{2k-3}-1}-\frac{1}{L_{2k-1}}-\frac{1}{L_{2k+1}-L_{2k-1}-1}$$
$$=\frac{L_{2k-3}L_{2k+1}-L_{2k-1}^2-L_{2k-3}+L_{2k+1}-1}{L_{2k-1}(L_{2k-1}-L_{2k-3}-1)(L_{2k+1}-L_{2k-1}-1)}$$
$$=\frac{-5(-1)^{2k-2}-L_{2k-3}+L_{2k+1}-1}{L_{2k-1}(L_{2k-1}-L_{2k-3}-1)(L_{2k+1}-L_{2k-1}-1)}$$
$$=\frac{L_{2k+1}-L_{2k-3}-6}{L_{2k-1}(L_{2k-1}-L_{2k-3}-1)(L_{2k+1}-L_{2k-1}-1)}.$$

Since

$$L_{2k+1}-L_{2k-3}=L_{2k}+L_{2k-1}+L_{2k-2}-L_{2k-1}=L_{2k}+L_{2k-2}\geq L_4+L_2=7+3=10>6,$$

$$\frac{1}{L_{2k-1}-L_{2k-3}-1}-\frac{1}{L_{2k-1}}-\frac{1}{L_{2k+1}-L_{2k-1}-1}>0.$$

Summing both sides from n to mn yields

$$\frac{1}{L_{2n-1}-L_{2n-3}-1}-\sum_{k=n}^{mn}\frac{1}{L_{2k-1}}-\frac{1}{L_{2mn+1}-L_{2mn-1}-1}>0,$$

which implies that

$$\sum_{k=n}^{mn}\frac{1}{L_{2k-1}}<\frac{1}{L_{2n-1}-L_{2n-3}-1}-\frac{1}{L_{2mn+1}-L_{2mn-1}-1}<\frac{1}{L_{2n-1}-L_{2n-3}-1}. \quad (29)$$

Finally, combining (29) and (28), we obtain (27) as desired. ■

Furthermore, letting m tends to infinity in Theorem 12 and Theorem 16, we obtain the infinite sums as followings.

Corollary 17. *For any positive integer $n\geq 2$, we have*

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{L_{2k}}\right)^{-1}\right\rfloor=L_{2n}-L_{2n-2}. \quad (30)$$

Corollary 18. *For any positive integer* $n \geq 2$, *we have*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{L_{2k-1}} \right)^{-1} \right\rfloor = L_{2n-1} - L_{2n-3} - 1. \tag{31}$$

5. Conclusion

We can see that our results are slightly different from those of the Fibonacci numbers and the Pell numbers. Our formula for the even term, equation (13), looks like the formula for the odd term of the Fibonacci numbers, equation (5), and the Pell numbers, equation (9), whereas our formula for the odd term, equations (19) and (26), looks like the formula for the even term of the Fibonacci numbers (4), and the Pell numbers, equation (7). The reason may be the starting value. The Fibonacci numbers and the Pell numbers begin with the same value 0 and 1, but the Lucas numbers begin with 2 and 1. Therefore, we obtain the different results.

6. References

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