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On the spectrum of the price of stocks related to the Barrier option

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ได้รับทุนอุดหนุนการวิจัยจาก กองทุนวิจัยมหาวิทยาลัยนเรศวร

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บทคัดย่อ(ภาษาไทย)

งานวิจัยนี้ ทำการศึกษาถึงการมีจริงและการมีเพียงผลเฉลยเดียวสำหรับปัญหาค่าขอบของสมการเชิงอนุพันธ์เชิงเศษส่วนของฮิลเฟอร์ไม่เชิงเส้น โดยประยุกต์ใช้ทฤษฎีบทจุดตรึง กล่าวคือสำหรับปัญหาการมีจริงจะประยุกต์ใช้ทฤษฎีบทจุดตรึงชาวเฟอร์ และสำหรับปัญหาการมีผลเฉลยเดียวจะประยุกต์ใช้ทฤษฎีบทการหดตัวบานาชและทฤษฎีบทจุดตรึงบอยวอง พร้อมทั้งยกตัวอย่างการนำทฤษฎีบทดังกล่าวไปใช้

ซึ่งงานวิจัยนี้มีประโยชน์ในการประยุกต์ใช้ในวิทยาศาสตร์หลายๆ สาขา เช่น ฟิสิกส์ เคมี หรือแม้กระทั่งคณิตศาสตร์ทางการเงิน

บทคัดย่อ(ภาษาอังกฤษ)

In this research, we study existence and uniqueness of solutions for boundary value problem of nonlinear implicit Hilfer fractional differential equations, via standard fixed point theorems. The existence is proved by using Schaufer's fixed point theorem while the existence and uniqueness by Banach contraction mapping principle and Boyd–Wong fixed point theorem. Illustrative examples are also discussed.

LIST OF CONTENT

CHAPTER 1

INTRODUCTION

Fractional calculus is a generalization of ordinary differentiation and integration to positive integer order. See, for example, the books such as [1-4].

In recent years, the interest in the study of fractional differential equations has been growing rapidly. Fractional differential equations have arisen in mathematical models of systems and processes in various fields such as aerodynamics, acoustics, mechanics, electromagnetism, signal processing, control theory, robotics, population dynamics, finance, etc.

In the past, many mathematicians have paid much attention to the study of fractional differential equations, see for example, the article such as [7-26]. Because it can be applied to answer many scientific problems.

Recently, Furati et al. (2012) [9] studied the existence and stablity for Hilfer fractional differential equations with Hilfer fractional derivatives with fractional integral initial condition given by

$$D_{a^+}^{\alpha,\beta}x(t) = f(t,x(t)), \quad \forall t \in (a,b],$$
$$I_{a^+}^{1-\gamma}x(a) = x_a, \quad \gamma = \alpha + \beta - \alpha\beta,$$

where $D_{a+}^{\alpha,\beta}$ is Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$, $I_{a+}^{1-\gamma}$ is the Riemann-Liouville fractional integral of order $1-\gamma$ and $f:(a,b]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a function. We prove existence and uniqueness of global solutions in the space of weighted continuous functions.The stability of the solution for a weighted Cauchy-type problem is also analyzed.

Abbas et al. (2017) [23] discussed the existence and Ulam-Hyers-Rassias stability results for a class of functional differential equations involving the Hilfer-Hadamard fractional derivative with Hadamard fractional integral initial condition given by

$$
\begin{aligned}
{}^{H}D_{1}^{\alpha,\beta}x(t) &= f(t,x(t)), \ \ \forall t \in J,\\
{}^{H}I_{1}^{1-\gamma}x(1) &= \phi, \ \ \gamma = \alpha+\beta-\alpha\beta,
\end{aligned}
$$

where ${}^{H}D_{1}^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order $\alpha \in (1,2]$ and type $\beta \in [0,1]$, $\phi \in \mathbb{R}$, ${}^{H}I_{1}^{1-\gamma}$ is the Hadamard fractional integral of order $1-\gamma$ and $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given function where $J := [1,T], T > 1$.

Ahmad et al. (2018) [24] discussed the existence and uniqueness result of solutions for boundary value problems for Hilfer-Hadamard fractional differential equations of the form

$$
\begin{aligned}
{}^{H}D^{\alpha,\beta}x(t) + f(t,x(t)) = 0, \ \ \forall t \in J,\\
x(1+\varepsilon) = 0, \ \ {}^{H}D^{1,1}x(e) = \nu\ {}^{H}D^{1,1}x(\zeta),
\end{aligned}
$$

where ${}^{H}D_{1}^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order $\alpha \in (1,2]$ and type $\beta \in [0,1]$, $\nu \in [0,1)$, $\zeta \in (1,e)$, $\varepsilon \in (0,1)$, ${}^{H}D^{1,1} = t\frac{d}{dt}$ and $f : J \times \mathbb{R} \longrightarrow \mathbb{R}^{+}$ is a given function where $J := (1,e]$. Our analysis relies on the Banach's fixed point theorem and Leray- Schauder alternative.

Dhaigude et al. (2018) [25] investigated the existence and uniqueness for Hilfer fractional differential equations with Hilfer fractional derivatives with fractional integral initial condition given by

$$
\begin{aligned}
D_{a^+}^{\alpha,\beta}x(t) &= f(t,x(t)), \ \ \forall t \in (a,b],\\
I_{a^+}^{n-\gamma}x(a) &= b_k, \ \ k \in \{1,2,...,n\}, \ \ n = -[-\alpha],
\end{aligned}
$$

where $D_{a^+}^{\alpha,\beta}$ is Hilfer fractional derivative of order α and type $\beta \in [0,1]$, $I_{a^+}^{1-\gamma}$ is the Riemann-Liouville fractional integral of order $1-\gamma$ where $\gamma = \alpha+\beta-\alpha\beta$ and $f : (a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function.

Harikrishnan et al. (2018) [26] studied the existence and stablity for Hilfer fractional differential equations with two Hilfer fractional derivatives with fractional integral initial conditions

$$D_{a^+}^{\alpha_1,\beta}\left(D_{a^+}^{\alpha_2,\beta}+\lambda\right)x(t)=f(t,x(t)),\ \ \forall t\in(a,b],$$
$$I_{a^+}^{1-\gamma}x(a)=x,\ \ \gamma=\alpha+\beta-\alpha\beta,$$

where $D_{a^+}^{\alpha_1,\beta}$, $D_{a^+}^{\alpha_2,\beta}$ are two Hilfer fractional derivative of order $\alpha_1,\alpha_2\in(0,1)$ and type $\beta\in[0,1]$, $I_{a^+}^{1-\gamma}$ is the Riemann-Liouville fractional integral of order $1-\gamma$ and $f:(a,b]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a function. Our analysis relies on the Krasnoselskii fixed point theorem and the Banach's fixed point theorem for existence and Ulam type stability for stablity result.

In this research, we study the existence and uniqueness of nonlinear implicit Hilfer fractional differential equations

$$D_{a^+}^{\alpha_1,\beta}D_{a^+}^{\alpha_2,\beta}x(t)=f(t,x(t),D_{a^+}^{\alpha_i,\beta}x(t),D_{a^+}^{\alpha_1,\beta}D_{a^+}^{\alpha_2,\beta}x(t)),\quad t\in(a,b] \tag{1.1}$$

with Hilfer fractional differential boundary conditions

$$x(a)=\rho_1 D_{a^+}^{\alpha_1,\beta}x(\theta_1),\qquad x(b)=\rho_2 D_{a^+}^{\alpha_2,\beta}x(\theta_2), \tag{1.2}$$

where $D_{a^+}^{\alpha_i,\beta}$ is the Hilfer fractional derivative of order $\alpha_i\in(0,1)$ and type $\beta\in[0,1]$ and $\gamma_i=\alpha_i+\beta-\alpha_i\beta$, $\nu_j\in\mathbb{R}_{++}:=(0,\infty)$, $\rho_j\in\mathbb{R}_+:=[0,\infty)$, $\theta_j\in(a,b)$ for $i,j\in\{1,2\}$ and $f:(a,b]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is a given function space, $I_{a^+}^{(\cdot)}$ is the Riemann-Liouville fractional integral of positive order. We give some examples illustrate the existence and stablity of problem by applying over main results.

1.1 Preliminaries

In this section, we suggest some basic definitions, lemmas and theorems are using in this research.

Definition 1.1 (The Riemann-Liouville Fractional Integral [4]). The Riemann-Liouville fractional integral of order $\alpha\in\mathbb{R}_{++}$ of a function $h:(a,\infty)\to\mathbb{R}$ is defined by

$$I_{a^+}^{\alpha}h(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}h(s)ds,$$

provided the right-hand side is point-wise defined on (a,∞), where Γ is the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt.$$

From Definition 1.1, if $\alpha = n \in \mathbb{N}$, the definition coincide with the n-th integrals of the form

$$I_{a+}^n h(t) = \int_a^t \int_a^{s_1} \cdots \int_a^{s_{n-1}} h(s_n)ds_n \ \ldots \ ds_2 \ ds_1 = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}h(s)ds.$$

Definition 1.2 (The Riemann-Liouville Fractional Derivative [4]). The Riemann-Liouville fractional derivative of order $\alpha \in \mathbb{R}_+$ of a function $h : (a,\infty) \to \mathbb{R}$ is defined by

$$D_{a+}^\alpha h(t) := D^n I_{a+}^{n-\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1}h(s)ds,$$

where $D^n \equiv \frac{d^n}{dt^n}$ and $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of a real number α, provided the right-hand side is point-wise defined on (a,∞).

In particular, when $\alpha = n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, then

$$D_{a+}^0 h(t) = h(t) \text{ and } D_{a+}^n h(t) = h^{(n)}(t),$$

where $h^{(n)}(t)$ is the usual derivative of $h(t)$ of order n.

Definition 1.3 (The Caputo Fractional Derivative [4]). The Caputo fractional derivative of order $\alpha \in \mathbb{R}_+$ of a function $h : (a,\infty) \to \mathbb{R}$ is defined by

$${}^cD_{a+}^\alpha h(t) := I_{a+}^{n-\alpha} D^n h(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1}h^{(n)}(s)ds.$$

provided the right-hand side is point-wise defined on (a,∞).

In particular, when $\alpha = n \in \mathbb{N}_0$, then ${}^cD_{a+}^n h(t) = h^{(n)}(t)$.

Theorem 1.4 ([4]). If $\alpha \notin \mathbb{N}_0$ and $n = [\alpha] + 1$, then the Caputo fractional derivatives coincide with the Riemann-Liouville fractional derivatives in the following cases:

$$ {}^cD^{\alpha}_{a^+}h(t) = D^{\alpha}_{a^+}h(t), $$

if $h(a) = h'(a) = \cdots = h^{(n-1)}(a) = 0.$

In particular, when $\alpha \in (0,1)$, we have

$$ {}^cD^{\alpha}_{a^+}h(t) = D^{\alpha}_{a^+}h(t), \quad \text{when} \quad h(a) = 0. $$

Next, we guide space of a continuous functions [16]. Let $[a,b]$ is a finite interval and $0 \leq \gamma < 1$ as the following:

(1) The space of continuous function f from $(a,b]$ to $\mathbb{R}$ is defined by

$$ C_{\gamma}([a,b],\mathbb{R}) := \Big\{ f : (a,b] \longrightarrow \mathbb{R} \ : \ (x-a)^{\gamma} f(x) \in C([a,b],\mathbb{R}) \Big\}, $$

where $C([a,b],\mathbb{R})$ is a space of continuous function f from $[a,b]$ to $\mathbb{R}$. Inparticular, if $\gamma = 0$ then $C_0([a,b],\mathbb{R}) = C([a,b],\mathbb{R})$.

(2) The space of n-time continuously differentiable function f from $(a,b]$ to $\mathbb{R}$ is defined by

$$ C^n_{\gamma}([a,b],\mathbb{R}) := \Big\{ f \in C^{n-1}([a,b],\mathbb{R}) : \ f^{(n)} \in C_{\gamma}([a,b],\mathbb{R}) \Big\}, $$

where $C^n([a,b],\mathbb{R})$ is a space of n-time continuously differentiable function f from $[a,b]$ to $\mathbb{R}$.

We recommend some properties of Riemann-Liouville fractional integration and fractional derivative as follows:

Lemma 1.5 ([4]). If $\alpha, \beta \in \mathbb{R}_{++}$ and $n = [\alpha] + 1$, then

$$ I^{\alpha}_{a^+}(t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\beta+\alpha-1} \tag{1.3} $$

and

$$D^{\alpha}_{a^+}(t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}.$$

On the other hand, for $k \in \{1, 2, ..., n\}$,

$$D^{\alpha}_{a^+}(t-a)^{\alpha-k} = 0.$$

Theorem 1.6. The equation (1.3) by using Definition 1.1 can be written as

$$\int_a^t (t-s)^{\alpha-1}(s-a)^{\beta-1}ds = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}(t-a)^{\alpha+\beta-1}.$$

Lemma 1.7 ([4]). Let $\alpha, \beta \in \mathbb{R}_{++}$ and $h \in C([a,b], \mathbb{R})$. Then

$$I^{\alpha}_{a^+} I^{\beta}_{a^+} h(t) = I^{\alpha+\beta}_{a^+} h(t) \tag{1.4}$$

for any $t \in [a, b]$.

Lemma 1.8 ([4]). Let $\alpha, \beta \in \mathbb{R}_{++}$ and $h \in C([a,b], \mathbb{R})$. Then

$$D^{\beta}_{a^+} I^{\alpha}_{a^+} h(t) = I^{\alpha-\beta}_{a^+} h(t)$$

for any $t \in [a, b]$.

Lemma 1.9 ([4]). Let $\alpha \in \mathbb{R}_{++}$ and $h \in C([a,b], \mathbb{R})$. Then

$$D^{\alpha}_{a^+} I^{\alpha}_{a^+} h(t) = h(t) \tag{1.5}$$

for any $t \in [a, b]$.

Lemma 1.10 ([4]). If $\alpha \in \mathbb{R}_{++}, n = [\alpha] + 1$ and $h \in C([a,b], \mathbb{R})$, $I^{n-\alpha}_{a^+} h \in C^n([a,b], \mathbb{R})$, then

$$I^{\alpha}_{a^+} D^{\alpha}_{a^+} h(t) = h(t) - \sum_{k=1}^{n} \frac{D^{n-k} I^{n-\alpha}_{a^+} h(a)}{\Gamma(\alpha-k+1)}(t-a)^{\alpha-k} \tag{1.6}$$

for any $t \in [a, b]$.

In particular, if $\alpha \in (0, 1)$, then

$$I^{\alpha}_{a^+} D^{\alpha}_{a^+} h(t) = h(t) - \frac{I^{1-\alpha}_{a^+} h(a)}{\Gamma(\alpha)}(t-a)^{\alpha-1}.$$

Lemma 1.11 ([9]). Let $\alpha \in \mathbb{R}_{++}$ and $h \in C([a,b],\mathbb{R})$. Then

$$I_{a^+}^{\alpha}h(a) = \lim_{x\longrightarrow a^+} I_{a^+}^{\alpha}h(t) = 0. \tag{1.7}$$

Lemma 1.12 ([4,9]). Let $\alpha,\beta \in \mathbb{R}_{++}, \gamma \in [0,1)$ and $n = [\alpha]+1$. The following assertions are then true:

(a) If $h \in C_\gamma([a,b],\mathbb{R})$, then the relation in (1.4) holds at any point $t \in (a,b]$.

(b) If $h \in C_\gamma([a,b],\mathbb{R})$, then the equality in (1.5) holds at any point $t \in (a,b]$.

(c) If $h \in C_\gamma([a,b],\mathbb{R})$ and $I_{a^+}^{n-\alpha}h \in C_\gamma^n([a,b],\mathbb{R})$, then the relation in (1.6) holds at any point $t \in (a,b]$.

(d) If $h \in C_\gamma([a,b],\mathbb{R})$, then the relation in (1.7) holds.

We recommend some properties of Riemann-Liouville fractional integration and Caputo fractional derivative as follows:

Lemma 1.13 ([4]). If $\alpha,\beta \in \mathbb{R}_{++}$ and $n = [\alpha]+1$, then

$${}^cD_{a^+}^{\alpha}(t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}$$

where $\beta > n$ and

$${}^cD_{a^+}^{\alpha}(t-a)^{k-1} = 0.$$

for all $k \in \{1,2,...,n\}$.

Lemma 1.14 ([4]). If $\alpha \in \mathbb{R}_{++}$ and $h \in C([a,b],\mathbb{R})$, then

$${}^cD_{a^+}^{\alpha}I_{a^+}^{\alpha}h(t) = h(t)$$

for any $t \in [a,b]$.

Lemma 1.15 ([4]). Let $\alpha \in \mathbb{R}_{++}$ and $n = [\alpha]+1$. If $h \in C^n([a,b],\mathbb{R})$, then

$$I_{a^+}^{\alpha}\ {}^cD_{a^+}^{\alpha}h(t) = h(t) - \sum_{k=0}^{n-1}\frac{h^{(k)}(a)}{k!}(t-a)^k$$

for any $t \in [a,b]$.

In particular, if $\alpha \in (0,1)$, then

$$I_{a^+}^{\alpha}\ {}^{c}D_{a^+}^{\alpha}h(t) = h(t) - h(a).$$

Definition 1.16 (The Hilfer Fractional Derivative [16]). Let $\alpha \in \mathbb{R}_{++}$ and $\beta \in [0,1]$. The Hilfer fractional derivative of order α and type β of a function $h : (a,\infty) \to \mathbb{R}$ is defined by

$$D_{a^+}^{\alpha,\beta}h(t) = I_{a^+}^{\beta(n-\alpha)}D^{n}I_{a^+}^{(n-\alpha)(1-\beta)}h(t),$$

where $n = [\alpha] + 1$ and $D^n = \dfrac{d^n}{dt^n}$. If $\gamma = \alpha + n\beta - \alpha\beta$, then can be written as

$$D_{a^+}^{\alpha,\beta}h(t) = I_{a^+}^{\beta(n-\alpha)}D_{a^+}^{\gamma}h(t). \tag{1.8}$$

Note that, the generalization for definition of Hilfer fractional derivatives, if $\beta = 0$ coincides with the Riemann-Liouville derivative and for $\beta = 1$ with the Caputo derivative, that is

$$D_{a^+}^{\alpha,\beta}h(t) = \begin{cases} D_{a^+}^{\alpha}h(t), & \beta = 0, \\ {}^{c}D_{a^+}^{\alpha}h(t), & \beta = 1. \end{cases}$$

In particular, when $\alpha \in (0,1)$, we have $\gamma = \alpha + \beta - \alpha\beta$. From (1.8) one has

$$D_{a^+}^{\alpha,\beta}h(t) = I_{a^+}^{\beta(1-\alpha)}D_{a^+}^{\gamma}h(t).$$

Moreover, the parameter γ satisfies $\gamma \in (0,1]$, $\gamma \leq \alpha$, $\gamma < \beta$ and $1-\gamma < 1-\beta(1-\alpha)$.

In this section, we introduce definition of the normed space and the equipments used on the normed space.

Definition 1.17 (Normed Space). Let X be vector space over the field $\mathbb{R}$, X is a map $\|\cdot\| : X \longrightarrow \mathbb{R}_+$ such that for any $x, y \in X$

$$(1) \quad \|x\| = 0 \iff x = 0$$

$$(2) \quad \|\alpha x\| = |\alpha| \, \|x\| \quad \text{for all } \alpha \in \mathbb{R}$$

$$(3) \quad \|x+y\| \leq \|x\| + \|y\|.$$

The pair $(X, \|\cdot\|)$ is called a normed space.

Definition 1.18. Let $X := (X, \|\cdot\|)$ be a normed space and $M \subset X$.

(1) M is said to be compact if every sequence of point in M has a convergent subsequence.

(2) A sequence $\{f_n\}$ of function $f_n : M \longrightarrow X$ is uniformly bounded if there is $K > 0$ such that $\|f_n\| \leq K$ for $n \in \mathbb{N}$.

(3) A family $F = \{f_n\}$ of functions from M to X is equicontinuous if for every $\varepsilon > 0$ and for every $x_0 \in M$ there exists $\delta(x_0, \varepsilon) > 0$ such that $|f_n(x) - f_n(x_0)| < \varepsilon$ whenever for every $x \in M$, $|x - x_0| < \delta$ and for every $f_n \in F$, that is

$$\lim_{x \longrightarrow x_0} \left| f_n(x) - f_n(x_0) \right| = 0.$$

Next, we instruct definition of the Banach space and the property used operate on the Banach space.

Definition 1.19 (Banach Space). A Banach space is a normed space X that is complete, i.e., every Cauchy sequence in X is convergent.

Theorem 1.20 (Arzelà–Ascoli Theorem). Let U be compact set in Banach space X. If $\{f_n\}$ is a uniformly bounded and equicontinuous sequence of function, $f_n : U \longrightarrow X$, then there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which converges uniformly on U to some limit function $f : U \longrightarrow X$.

Theorem 1.21 (Subspace of a Banach space). A subspace Y of a Banach space X is complete if and only if the set Y is closed in X.

Definition 1.22 (Contraction Mapping). Let X be a Banach space. A mapping $T : X \longrightarrow X$ is called a contraction if there exists a constant $L \in (0, 1)$, such that

$$\left\|Tu - Tv\right\| \leq L\left\|u - v\right\|$$

for all $u, v \in X$.

Definition 1.23 (Nonlinear contraction). Let X be a Banach space and $T : X \longrightarrow X$ be a mapping. T is said to be a nonlinear contraction if there exists a continuous increasing function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\psi(0) = 0$ and $\psi(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$ with the property

$$\left\|Tu - Tv\right\| \leq \psi\left(\|u - v\|\right)$$

for all $u, v \in X$.

In this section, we will use fixed point theory above Banach space for prove that the existence and uniqueness of Hilfer fractional differential equations the following:

Theorem 1.24 (Schaufer's Fixed Point Theorem [3]). Let X be a Banach space, $T : X \longrightarrow X$ continuous and compact mapping. Such that the set

$$B := \left\{x \in X : x = \lambda Tx,\ \exists\lambda \in [0, 1]\right\}$$

is bounded. Then T has fixed point in X.

Theorem 1.25 (Banach's contraction principle [3]). Let T be a contraction on a Banach space X. Then T has a unique fixed point in X.

Theorem 1.26 (Boyd-Wong Fixed Point Theorem [22]). Let X be a Banach space and let $T : X \longrightarrow X$ be a nonlinear contraction. Then T has a unique fixed point in X.

CHAPTER 2

MAIN RESULTS

2.1 Existence and uniqueness results for the boundary value problem

In this section, we introduce prove existence and uniqueness a boundary value problem of nonlinear implicit Hilfer fractional differential equations with Hilfe fractional differential boundary conditions as follows (1.1) and (1.2).

Lemma 2.1. Let $\alpha_j \in (0,1), \beta \in [0,1], \gamma_j = \alpha_j + \beta - \alpha_j\beta$ for $j = 1,2$ and $h \in \mathcal{C}$ where $J := [a,b]$ and $\gamma_1 < \gamma_2,\ \alpha_1 < \gamma_2,\ \alpha_1 < \alpha_2 + \gamma_1,\ \eta = b - a$. Then the BVP

$$D_{a^+}^{\alpha_1,\beta} D_{a^+}^{\alpha_2,\beta} x(t) = h(t), \quad t \in (a,b], \tag{2.1}$$

$$x(a) = \rho_1 D_{a^+}^{\alpha_1,\beta} x(\theta_1), \quad x(b) = \rho_2 D_{a^+}^{\alpha_2,\beta} x(\theta_2) \tag{2.2}$$

with $\rho_j \in \mathbb{R}_+$ and $\theta_j \in (a,b)$ is equivalent to the integral equation

$$x(t) = \frac{1}{\mathcal{A}}\left[\Big(\Delta_1(h) - \mathcal{A}_1(h)\Big)(t-a)^{\gamma_2-1} + \Big(\Delta_2(h) - \mathcal{A}_2(h)\Big)(t-a)^{\gamma_1+\alpha_2-1}\right] + I_{a^+}^{\alpha_1+\alpha_2} h(t),$$

where $\xi_j = \theta_j - a,\ \mathcal{A} := \Delta - \mathcal{W} \neq 0,\ \mathcal{W} := |W|,\ \mathcal{A}_j(h) := |A_j(h)|,$

$$\Delta := \frac{\rho_1\rho_2\Gamma(\gamma_2)\Gamma(\gamma_1+\alpha_2)\xi_1^{\gamma_2-\alpha_1-1}\xi_2^{\gamma_1-1}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\gamma_1)},$$

$$W := \begin{bmatrix} \eta^{\alpha_2+\gamma_1-1} & \frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)\Gamma(\gamma_1)} \\ \eta^{\gamma_2-1} & \frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)} \end{bmatrix}$$

and

$$\Delta_1(h) := \frac{\rho_1 \eta^{\gamma_1+\alpha_2-1}}{\Gamma(\alpha_2)} \int_a^{\theta_1} (\theta_1 - s)^{\alpha_2-1} h(s)ds - \frac{\rho_1 \Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)\Gamma(\alpha_1+\alpha_2)} \int_a^b (b-s)^{\alpha_1+\alpha_2-1} h(s)ds,$$

$$\Delta_2(h) := \frac{\rho_1 \Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2)} \int_a^b (b-s)^{\alpha_1+\alpha_2-1} h(s)ds,$$

$$A_1(h) := \begin{bmatrix} \frac{\rho_1}{\Gamma(\alpha_2)} \int_a^{\theta_1} (\theta_1 - s)^{\alpha_2-1} h(s)ds & \frac{\rho_1 \Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)} \\ \frac{\rho_2}{\Gamma(\alpha_1)} \int_a^{\theta_2} (\theta_2 - s)^{\alpha_1-1} h(s)ds & \frac{\rho_2 \Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\gamma_1)} \end{bmatrix},$$

$$A_2(h) := \begin{bmatrix} \frac{\rho_1 \Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)} & \frac{\rho_1}{\Gamma(\alpha_2)} \int_a^{\theta_1} (\theta_1 - s)^{\alpha_2-1} h(s)ds \\ -\eta^{\gamma_2-1} & \frac{\rho_2}{\Gamma(\alpha_1)} \int_a^{\theta_2} (\theta_2 - s)^{\alpha_1-1} h(s)ds \end{bmatrix}.$$

Proof. From (2.1), taking the Riemann-Liouville fractional integral of order α_1, we get

$$I_{a^+}^{\alpha_1} D_{a^+}^{\alpha_1,\beta} D_{a^+}^{\alpha_2,\beta} x(t) = I_{a^+}^{\alpha_1} h(t).$$

Since $\alpha_1 \in (0,1)$, $n = 1$. By Definition 1.16 and Lemma 1.7 with $\gamma_1 = \alpha_1 + \beta - \alpha_1\beta$ so that

$$I_{a^+}^{\gamma_1} D_{a^+}^{\gamma_1} \left(D_{a^+}^{\alpha_2,\beta} x(t) \right) = I_{a^+}^{\alpha_1} h(t).$$

Since $D_{a^+}^{\alpha_2,\beta} x \in C$, by Proposition ?? we obtain $D_{a^+}^{\alpha_2,\beta} x \in C_{\gamma_1}(J,\mathbb{R})$ and $I_{a^+}^{1-\gamma_1} D_{a^+}^{\alpha_2,\beta} x \in C_{\gamma_1}^1(J,\mathbb{R})$, by Lemma 1.12(c) so that

$$D_{a^+}^{\alpha_2,\beta} x(t) - \frac{I_{a^+}^{1-\gamma_1}(D_{a^+}^{\alpha_2,\beta} x(a))}{\Gamma(\gamma_1)} (t-a)^{\gamma_1-1} = I_{a^+}^{\alpha_1} h(t).$$

That is

$$D_{a^+}^{\alpha_2,\beta} x(t) - c_0 (t-a)^{\gamma_1-1} = I_{a^+}^{\alpha_1} h(t).$$

Taking the Riemann-Liouville fractional integral of order α_2, we have

$$I_{a^+}^{\alpha_2} D_{a^+}^{\alpha_2,\beta} x(t) - c_0 I_{a^+}^{\alpha_2} (t-a)^{\gamma_1-1} = I_{a^+}^{\alpha_2} I_{a^+}^{\alpha_1} h(t).$$

Since $\alpha_2 \in (0,1)$, $n = 1$. Similary, by Definition 1.16 and Lemma 1.7 with $\gamma_2 = \alpha_2 + \beta - \alpha_2\beta$ so that

$$I_{a^+}^{\gamma_2} D_{a^+}^{\gamma_2} x(t) - c_0 I_{a^+}^{\alpha_2} (t-a)^{\gamma_1-1} = I_{a^+}^{\alpha_1+\alpha_2} h(t).$$

Since $x \in \mathcal{C}$, by Proposition ?? we obtain $x \in C_{\gamma_2}(J,\mathbb{R})$ and $I_{a^+}^{1-\gamma_2}x \in C_{\gamma_2}^1(J,\mathbb{R})$, by Lemma 1.12(c) we obtain

$$x(t) - \frac{I_{a^+}^{1-\gamma_2}x(a)}{\Gamma(\gamma_2)}(t-a)^{\gamma_2-1} - \frac{c_0\Gamma(\gamma_1)}{\Gamma(\gamma_1+\alpha_2)}(t-a)^{\gamma_1+\alpha_2-1} = I_{a^+}^{\alpha_1+\alpha_2}h(t).$$

So that

$$x(t) = c_1(t-a)^{\gamma_2-1} + c_2(t-a)^{\gamma_1+\alpha_2-1} + I_{a^+}^{\alpha_1+\alpha_2}h(t). \tag{2.3}$$

From (2.3), taking the Hilfer fractional derivative of order α_1 and type β, we consider

$$\begin{aligned}
D_{a^+}^{\alpha_1,\beta}x(t) &= c_1 D_{a^+}^{\alpha_1,\beta}(t-a)^{\gamma_2-1} + c_2 D_{a^+}^{\alpha_1,\beta}(t-a)^{\gamma_1+\alpha_2-1} + D_{a^+}^{\alpha_1,\beta}I_{a^+}^{\alpha_1+\alpha_2}h(t)\\
&= c_1 I_{a^+}^{\beta-\alpha_1\beta}D_{a^+}^{\gamma_1}(t-a)^{\gamma_2-1} + c_2 I_{a^+}^{\beta-\alpha_1\beta}D_{a^+}^{\gamma_1}(t-a)^{\gamma_1+\alpha_2-1} + I_{a^+}^{\beta-\alpha_1\beta}D_{a^+}^{\gamma_1}I_{a^+}^{\alpha_1+\alpha_2}h(t)\\
&= c_1 I_{a^+}^{\beta-\alpha_1\beta}\left[\frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2-\gamma_1)}(t-a)^{\gamma_2-\gamma_1-1}\right] + c_2 I_{a^+}^{\beta-\alpha_1\beta}\left[\frac{\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1+\alpha_2-\gamma_1)}(t-a)^{\gamma_1+\alpha_2-\gamma_1-1}\right]\\
&\quad + I_{a^+}^{\beta-\alpha_1\beta}I_{a^+}^{\alpha_1+\alpha_2-\gamma_1}h(t)\\
&= \frac{c_1\Gamma(\gamma_2)}{\Gamma(\gamma_2-\gamma_1)}I_{a^+}^{\beta-\alpha_1\beta}(t-a)^{\gamma_2-\gamma_1-1} + \frac{c_2\Gamma(\gamma_1+\alpha_2)}{\Gamma(\alpha_2)}I_{a^+}^{\beta-\alpha_1\beta}(t-a)^{\alpha_2-1} + I_{a^+}^{\beta-\alpha_1\beta+\alpha_1+\alpha_2-\gamma_1}h(t)\\
&= \frac{c_1\Gamma(\gamma_2)}{\Gamma(\gamma_2-\gamma_1)}\left[\frac{\Gamma(\gamma_2-\gamma_1)}{\Gamma(\gamma_2-\gamma_1+\beta-\alpha_1\beta)}(t-a)^{\gamma_2-\gamma_1+\beta-\alpha_1\beta-1}\right]\\
&\quad + \frac{c_2\Gamma(\gamma_1+\alpha_2)}{\Gamma(\alpha_2)}\left[\frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2+\beta-\alpha_1\beta)}(t-a)^{\alpha_2+\beta-\alpha_1\beta-1}\right] + I_{a^+}^{\gamma_1+\alpha_2-\gamma_1}h(t)\\
&= \frac{c_1\Gamma(\gamma_2)}{\Gamma(\gamma_2-\gamma_1+\gamma_1-\alpha_1)}(t-a)^{\gamma_2-\gamma_1+\gamma_1-\alpha_1-1} + \frac{c_2\Gamma(\gamma_1+\alpha_2)}{\Gamma(\alpha_2+\gamma_1-\alpha_1)}(t-a)^{\alpha_2+\gamma_1-\alpha_1-1} + I_{a^+}^{\alpha_2}h(t).
\end{aligned}$$

Thus

$$D_{a^+}^{\alpha_1,\beta}x(t) = \frac{c_1\Gamma(\gamma_2)}{\Gamma(\gamma_2-\alpha_1)}(t-a)^{\gamma_2-\alpha_1-1} + \frac{c_2\Gamma(\gamma_1+\alpha_2)}{\Gamma(\alpha_2-\alpha_1+\gamma_1)}(t-a)^{\alpha_2-\alpha_1+\gamma_1-1} + I_{a^+}^{\alpha_2}h(t). \tag{2.4}$$

Similary, taking the Hilfer fractional derivative of order α_2 and type β of (2.3), we obtain

$$\begin{aligned}
D_{a^+}^{\alpha_2,\beta}x(t) &= c_1 D_{a^+}^{\alpha_2,\beta}(t-a)^{\gamma_2-1} + c_2 D_{a^+}^{\alpha_2,\beta}(t-a)^{\gamma_1+\alpha_2-1} + D_{a^+}^{\alpha_2,\beta} I_{a^+}^{\alpha_1+\alpha_2} h(t) \\
&= c_1 I_{a^+}^{\beta-\alpha_2\beta} D_{a^+}^{\gamma_2}(t-a)^{\gamma_2-1} + c_2 I_{a^+}^{\beta-\alpha_2\beta} D_{a^+}^{\gamma_2}(t-a)^{\gamma_1+\alpha_2-1} + I_{a^+}^{\beta-\alpha_2\beta} D_{a^+}^{\gamma_2} I_{a^+}^{\alpha_1+\alpha_2} h(t) \\
&= c_1 I_{a^+}^{\beta-\alpha_1\beta}(0) + c_2 I_{a^+}^{\beta-\alpha_2\beta}\left[\frac{\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1+\alpha_2-\gamma_2)}(t-a)^{\gamma_1+\alpha_2-\gamma_2-1}\right] \\
&\quad + I_{a^+}^{\beta-\alpha_2\beta} I_{a^+}^{\alpha_1+\alpha_2-\gamma_2} h(t) \\
&= \frac{c_2\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1-\beta+\alpha_2\beta)} I_{a^+}^{\beta-\alpha_2\beta}(t-a)^{\gamma_1-\beta+\alpha_2\beta-1} + I_{a^+}^{\beta-\alpha_2\beta+\alpha_1+\alpha_2-\gamma_2} h(t) \\
&= \frac{c_2\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1-\beta+\alpha_2\beta)}\left[\frac{\Gamma(\gamma_1-\beta+\alpha_2\beta)}{\Gamma(\gamma_1-\beta+\alpha_2\beta+\beta-\alpha_2\beta)}(t-a)^{\gamma_1-\beta+\alpha_2\beta+\beta-\alpha_2\beta-1}\right] \\
&\quad + I_{a^+}^{\gamma_2+\alpha_1-\gamma_2} h(t).
\end{aligned}$$

So that

$$D_{a^+}^{\alpha_2,\beta}x(t) = \frac{c_2\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1)}(t-a)^{\gamma_1-1} + I_{a^+}^{\alpha_1}h(t). \tag{2.5}$$

From the first condition (2.2) and (2.4), one gets

$$\rho_1\left[\frac{c_1\Gamma(\gamma_2)}{\Gamma(\gamma_2-\alpha_1)}(\theta_1-a)^{\gamma_2-\alpha_1-1} + \frac{c_2\Gamma(\gamma_1+\alpha_2)}{\Gamma(\alpha_2-\alpha_1+\gamma_1)}(\theta_1-a)^{\alpha_2-\alpha_1+\gamma_1-1} + I_{a^+}^{\alpha_2}h(\theta_1)\right] = 0,$$

or

$$c_1\left[\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)}\right] + c_2\left[\frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)}\right] = -\frac{\rho_1}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}h(s)ds. \tag{2.6}$$

From the second condition (2.2) and (2.4), we obtain

$$c_1(b-a)^{\gamma_2-1} + c_2(b-a)^{\gamma_1+\alpha_2-1} + I_{a^+}^{\alpha_1+\alpha_2}h(b) = \rho_2\left[\frac{c_2\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1)}(\theta_2-a)^{\gamma_1-1} + I_{a^+}^{\alpha_1}h(\theta_2)\right]$$

or

$$c_1\eta^{\gamma_2-1}+c_2\eta^{\gamma_1+\alpha_2-1}+\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^b(b-s)^{\alpha_1+\alpha_2-1}h(s)ds$$
$$=c_2\left[\frac{\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\gamma_1)}\right]+\frac{\rho_2}{\Gamma(\alpha_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}h(s)ds.$$

That is

$$c_1\eta^{\gamma_2-1}+c_2\left[\eta^{\gamma_1+\alpha_2-1}-\frac{\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\gamma_1)}\right]$$
$$=\frac{\rho_2}{\Gamma(\alpha_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}h(s)ds-\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^b(b-s)^{\alpha_1+\alpha_2-1}h(s)ds \quad (2.7)$$

From (2.6) and (2.7) we find c_1 and c_2 by solving linear system of equation,

$$\begin{bmatrix}\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)} & \frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)}\\ \eta^{\gamma_2-1} & \eta^{\gamma_1+\alpha_2-1}-\frac{\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\gamma_1)}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$$
$$=\begin{bmatrix}-\frac{\rho_1}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}h(s)ds\\ \frac{\rho_2}{\Gamma(\alpha_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}h(s)ds-\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^b(b-s)^{\alpha_1+\alpha_2-1}h(s)ds\end{bmatrix}.$$

Let us consider

$$|A|:=\begin{vmatrix}\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)} & \frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)}\\ \eta^{\gamma_2-1} & \eta^{\gamma_1+\alpha_2-1}-\frac{\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\gamma_1)}\end{vmatrix}$$
$$=\left(\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)}\right)\left(\eta^{\gamma_1+\alpha_2-1}-\frac{\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\gamma_1)}\right)-\left(\frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)}\right)\eta^{\gamma_2-1}$$
$$=\left(\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)}\right)\eta^{\gamma_1+\alpha_2-1}-\left(\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)}\right)\left(\frac{\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\gamma_1)}\right)$$
$$-\left(\frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)}\right)\eta^{\gamma_2-1}$$

$$= -\frac{\rho_1\rho_2\Gamma(\gamma_2)\Gamma(\gamma_1+\alpha_2)\xi_1^{\gamma_2-\alpha_1-1}\xi_2^{\gamma_1-1}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\gamma_1)} + \left[\eta^{\gamma_1+\alpha_2-1}\left(\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)}\right) - \eta^{\gamma_2-1}\right.$$

$$\left.\times\left(\frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)}\right)\right]$$

$$= -\frac{\rho_1\rho_2\Gamma(\gamma_2)\Gamma(\gamma_1+\alpha_2)\xi_1^{\gamma_2-\alpha_1-1}\xi_2^{\gamma_1-1}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\gamma_1)} + \begin{vmatrix} \eta^{\alpha_2+\gamma_1-1} & \frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)\Gamma(\gamma_1)} \\ \eta^{\gamma_2-1} & \frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)} \end{vmatrix}$$

$$= -\Delta + |W| = -\Big(\Delta - \mathcal{W}\Big) = -\mathcal{A}.$$

Since $|A| \neq 0$, by Cramer's rule, we have c_1 as

$$c_1 = \frac{1}{|A|}\begin{vmatrix} -\frac{\rho_1}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}h(s)ds & \frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)} \\ \frac{\rho_2}{\Gamma(\alpha_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}h(s)ds - \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^b(b-s)^{\alpha_1+\alpha_2-1}h(s)ds & \eta^{\gamma_1+\alpha_2-1} - \frac{\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\gamma_1)} \end{vmatrix}$$

$$= \frac{1}{|A|}\left[\left(-\frac{\rho_1}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}h(s)ds\right)\left(\eta^{\gamma_1+\alpha_2-1} - \frac{\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\gamma_1)}\right)\right.$$

$$-\left(\frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)}\right)\left(\frac{\rho_2}{\Gamma(\alpha_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}h(s)ds - \frac{1}{\Gamma(\alpha_1+\alpha_2)}\right.$$

$$\left.\left.\times\int_a^b(b-s)^{\alpha_1+\alpha_2-1}h(s)ds\right)\right]$$

$$= \frac{1}{|A|}\left[-\eta^{\gamma_1+\alpha_2-1}\left(\frac{\rho_1}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}h(s)ds\right) + \left(\frac{\rho_1}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}h(s)ds\right)\right.$$

$$\times\left(\frac{\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\gamma_1)}\right) - \left(\frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)}\right)\left(\frac{\rho_2}{\Gamma(\alpha_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}h(s)ds\right)$$

$$\left.+\left(\frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)}\right)\left(\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^b(b-s)^{\alpha_1+\alpha_2-1}h(s)ds\right)\right]$$

$$= \frac{1}{|A|}\left[-\frac{\rho_1\eta^{\gamma_1+\alpha_2-1}}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}h(s)ds + \frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)\Gamma(\alpha_1+\alpha_2)}\right.$$

$$\times\int_a^b(b-s)^{\alpha_1+\alpha_2-1}h(s)ds + \left\{\left(\frac{\rho_1}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}h(s)ds\right)\left(\frac{\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\gamma_1)}\right)\right.$$

$$\left.\left.-\left(\frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)}\right)\left(\frac{\rho_2}{\Gamma(\alpha_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}h(s)ds\right)\right\}\right]$$

$$= \frac{1}{|A|}\Bigg[-\Bigg(\frac{\rho_1\eta^{\gamma_1+\alpha_2-1}}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}h(s)ds - \frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)\Gamma(\alpha_1+\alpha_2)}$$

$$\times\int_a^b(b-s)^{\alpha_1+\alpha_2-1}h(s)ds + \begin{vmatrix}\frac{\rho_1}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}h(s)ds & \frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)} \\ \frac{\rho_2}{\Gamma(\alpha_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}h(s)ds & \frac{\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\gamma_1)}\end{vmatrix}\Bigg]$$

$$= \frac{1}{|A|}\Big(-\Delta_1(h)+|A_1(h)|\Big) = \frac{1}{-\mathcal{A}}\Big[-\Big(\Delta_1(h)-\mathcal{A}_1(h)\Big)\Big]$$

$$= \frac{1}{\mathcal{A}}\Big(\Delta_1(h)-\mathcal{A}_1(h)\Big).$$

Next, we obtain c_2 as

$$c_2 = \frac{1}{|A|}\begin{vmatrix}\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)} & -\frac{\rho_1}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}h(s)ds \\ \eta^{\gamma_2-1} & \frac{\rho_2}{\Gamma(\alpha_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}h(s)ds - \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^b(b-s)^{\alpha_1+\alpha_2-1}h(s)ds\end{vmatrix}$$

$$= \frac{1}{|A|}\Bigg[\Bigg(\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)}\Bigg)\Bigg(\frac{\rho_2}{\Gamma(\alpha_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}h(s)ds - \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^b(b-s)^{\alpha_1+\alpha_2-1}h(s)ds\Bigg)$$

$$-\Bigg(-\frac{\rho_1}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}h(s)ds\Bigg)\eta^{\gamma_2-1}\Bigg]$$

$$= \frac{1}{|A|}\Bigg[-\Bigg(\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)}\Bigg)\Bigg(\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^b(b-s)^{\alpha_1+\alpha_2-1}h(s)ds\Bigg) + \Bigg\{\Bigg(\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)}\Bigg)$$

$$\times\Bigg(\frac{\rho_2}{\Gamma(\alpha_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}h(s)ds\Bigg) + \eta^{\gamma_2-1}\Bigg(\frac{\rho_1}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}h(s)ds\Bigg)\Bigg\}\Bigg]$$

$$= \frac{1}{|A|}\Bigg[-\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2)}\int_a^b(b-s)^{\alpha_1+\alpha_2-1}h(s)ds + \Bigg\{\Bigg(\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)}\Bigg)$$

$$\times\Bigg(\frac{\rho_2}{\Gamma(\alpha_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}h(s)ds\Bigg) - \Bigg(\frac{\rho_1}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}h(s)ds\Bigg)\Big(-\eta^{\gamma_2-1}\Big)\Bigg\}\Bigg]$$

$$= \frac{1}{|A|}\Bigg[-\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2)}\int_a^b(b-s)^{\alpha_1+\alpha_2-1}h(s)ds + \begin{vmatrix}\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)} & \frac{\rho_1}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}h(s)ds \\ -\eta^{\gamma_2-1} & \frac{\rho_2}{\Gamma(\alpha_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}h(s)ds\end{vmatrix}\Bigg]$$

$$= \frac{1}{|A|}\Big(-\Delta_2(h)+|A_2(h)|\Big) = \frac{1}{-\mathcal{A}}\Big[-\Big(\Delta_2(h)-\mathcal{A}_2(h)\Big)\Big]$$

$$= \frac{1}{\mathcal{A}}\Big(\Delta_2(h)-\mathcal{A}_2(h)\Big).$$

Substituting the values of c_1 and c_2 in (2.3), we obtain

$$x(t) = \frac{1}{\mathcal{A}}\left[\Big(\Delta_1(h) - \mathcal{A}_1(h)\Big)(t-a)^{\gamma_2-1} + \Big(\Delta_2(h) - \mathcal{A}_2(h)\Big)(t-a)^{\gamma_1+\alpha_2-1}\right] + I_{a^+}^{\alpha_1+\alpha_2}h(t).$$

□

In this the boundary value problem (1.1) and (1.2), we investigate the existence and stability by the space

$$C_{\alpha_1,\alpha_2,\beta}(J,\mathbb{R}) := \left\{x \in \mathcal{C} : D_{a^+}^{\alpha_i,\beta}x \in \mathcal{C} \text{ for } i = 1,2\right\} \tag{2.8}$$

equipped with the norm defined by

$$\left\|x\right\|_{C_{\alpha_1,\alpha_2,\beta}} := \left\|x\right\| + \left\|D_{a^+}^{\alpha_1,\beta}x\right\| + \left\|D_{a^+}^{\alpha_2,\beta}x\right\| \tag{2.9}$$

where $\|x\| = \max_{t\in J}|x(t)|$ for any $x \in \mathcal{C}$.

Proposition 2.2. The space $\left(C_{\alpha_1,\alpha_2,\beta}(J,\mathbb{R}), \|\cdot\|_{C_{\alpha_1,\alpha_2,\beta}}\right)$ is a Banach space.

Proof. First, we must show that $(C_{\alpha_1,\alpha_2,\beta}(J,\mathbb{R}), \|\cdot\|_{C_{\alpha_1,\alpha_2,\beta}})$ is a normed space. Let $f,g \in C_{\alpha_1,\alpha_2,\beta}(J,\mathbb{R})$, we divide the proof into three steps.
Step I : To show that $\|f\|_{C_{\alpha_1,\alpha_2,\beta}} = 0$ if and only if $f = 0$.
($\Longrightarrow$) Suppose that $\|f\|_{C_{\alpha_1,\alpha_2,\beta}} = 0$. Thus

$$\left\|f\right\| + \left\|D_{a^+}^{\alpha_1,\beta}f\right\| + \left\|D_{a^+}^{\alpha_2,\beta}f\right\| = 0.$$

Thus $\|f\| = 0$ and $\|D_{a^+}^{\alpha_j,\beta}f\| = 0$ for $j = 1,2$. If $\|f\| = 0$, by $\mathcal{C}$ is a normed space, we have $f = 0$. Similary, if $\|D_{a^+}^{\alpha_j,\beta}f\| = 0$ then $D_{a^+}^{\alpha_j,\beta}f = 0$ so that for any $t \in J$, we have

$$D_{a^+}^{\alpha_j,\beta}f(t) = 0$$

Taking the Riemann-Liouville fractional integral of order $\alpha_j \in (0,1)$, one gets

$$I_{a^+}^{\alpha_j}D_{a^+}^{\alpha_j,\beta}f(t) = 0.$$

By Definition 1.16 and Lemma 1.7 with $\gamma_j = \alpha_j + \beta - \alpha_j\beta$ so that

$$I_{a^+}^{\gamma_j} D_{a^+}^{\gamma_j} f(t) = 0.$$

By Lemma 1.12(c), we have $f(t) = c(t-a)^{\gamma_1 - 1} = c(t-a)^{\gamma_2 - 1}$ for all $t \in J$. Then $c = 0$, thus $f(t) = 0$ for any $t \in J$. Hence $f = 0$.

($\Longleftarrow$) Clearly, by assumption.

Step II : To show that $\|\mu f\|_{C_{\alpha_1,\alpha_2,\beta}} = |\mu| \|f\|_{C_{\alpha_1,\alpha_2,\beta}}$ for any $\mu \in \mathbb{R}$.

Let $\mu \in \mathbb{R}$. Since C is a normed space, we have

$$\begin{aligned}
\left\|\mu f\right\|_{C_{\alpha_1,\alpha_2,\beta}} &= \left\|\mu f\right\| + \left\|D_{a^+}^{\alpha_1,\beta}(\mu f)\right\| + \left\|D_{a^+}^{\alpha_2,\beta}(\mu f)\right\| \\
&= \left\|\mu f\right\| + \left\|\mu D_{a^+}^{\alpha_1,\beta} f\right\| + \left\|\mu D_{a^+}^{\alpha_2,\beta} f\right\| \\
&= |\mu|\left\|f\right\| + |\mu|\left\|D_{a^+}^{\alpha_1,\beta} f\right\| + |\mu|\left\|D_{a^+}^{\alpha_2,\beta} f\right\| \\
&= |\mu|\left(\left\|f\right\| + \left\|D_{a^+}^{\alpha_1,\beta} f\right\| + \left\|D_{a^+}^{\alpha_2,\beta} f\right\|\right) \\
&= |\mu|\left\|f\right\|_{C_{\alpha_1,\alpha_2,\beta}}.
\end{aligned}$$

Step III : To show that $\|f+g\|_{C_{\alpha_1,\alpha_2,\beta}} \leq \|f\|_{C_{\alpha_1,\alpha_2,\beta}} + \|g\|_{C_{\alpha_1,\alpha_2,\beta}}$.

Since C is a normed space, we have

$$\begin{aligned}
\left\|f+g\right\|_{C_{\alpha_1,\alpha_2,\beta}} &= \left\|f+g\right\| + \left\|D_{a^+}^{\alpha_1,\beta}(f+g)\right\| + \left\|D_{a^+}^{\alpha_2,\beta}(f+g)\right\| \\
&= \left\|f+g\right\| + \left\|D_{a^+}^{\alpha_1,\beta} f + D_{a^+}^{\alpha_1,\beta} g\right\| + \left\|D_{a^+}^{\alpha_2,\beta} f + D_{a^+}^{\alpha_2,\beta} g\right\| \\
&\leq \left(\left\|f\right\| + \left\|g\right\|\right) + \left(\left\|D_{a^+}^{\alpha_1,\beta} f\right\| + \left\|D_{a^+}^{\alpha_1,\beta} g\right\|\right) + \left(\left\|D_{a^+}^{\alpha_2,\beta} f\right\| + \left\|D_{a^+}^{\alpha_2,\beta} g\right\|\right) \\
&= \left(\left\|f\right\| + \left\|D_{a^+}^{\alpha_1,\beta} f\right\| + \left\|D_{a^+}^{\alpha_2,\beta} f\right\|\right) + \left(\left\|g\right\| + \left\|D_{a^+}^{\alpha_1,\beta} g\right\| + \left\|D_{a^+}^{\alpha_2,\beta} g\right\|\right) \\
&= \left\|f\right\|_{C_{\alpha_1,\alpha_2,\beta}} + \left\|g\right\|_{C_{\alpha_1,\alpha_2,\beta}}.
\end{aligned}$$

From step I, step II and step III, we have $(C_{\alpha_1,\alpha_2,\beta}(J,\mathbb{R}), \|\cdot\|_{C_{\alpha_1,\alpha_2,\beta}})$ is a normed space.

Next, we must show that $C_{\alpha_1,\alpha_2,\beta}(J,\mathbb{R})$ is a closed in $\mathcal{C}$. Let $\{f_n\}$ be a sequence in $C_{\alpha_1,\alpha_2,\beta}(J,\mathbb{R})$ such that $f_n \longrightarrow f$. That is, for each $t \in J$

$$\lim_{n\longrightarrow\infty} f_n(t) = f(t). \tag{2.10}$$

Next, we shall show that for $j = 1,2$,

$$\lim_{n\longrightarrow\infty} D_{a^+}^{\alpha_j,\beta} f_n(t) = D_{a^+}^{\alpha_j,\beta} f(t) \text{ for all } t \in J. \tag{2.11}$$

Let $t \in J$, by (2.10) we obtain for $j = 1,2$,

$$\begin{aligned}
\lim_{n\longrightarrow\infty} D_{a^+}^{\alpha_j,\beta} f_n(t) &= \lim_{n\longrightarrow\infty} I_{a^+}^{\beta-\alpha_j\beta} D_{a^+}^{\gamma_j} f_n(t) \\
&= \lim_{n\longrightarrow\infty} \frac{1}{\Gamma(\beta-\alpha_j\beta)} \int_a^t (t-s)^{\beta-\alpha_j\beta-1}\Big(D_{a^+}^{\gamma_j} f_n(s)\Big)ds \\
&= \frac{1}{\Gamma(\beta-\alpha_j\beta)} \int_a^t (t-s)^{(\beta-\alpha_j\beta)-1}\Big(\lim_{n\longrightarrow\infty} D_{a^+}^{\gamma_j} f_n(s)\Big)ds \\
&= I_{a^+}^{\beta-\alpha_j\beta}\Big(\lim_{n\longrightarrow\infty} D_{a^+}^{\gamma_j} f_n(t)\Big) \\
&= I_{a^+}^{\beta-\alpha_j\beta}\left[\lim_{n\longrightarrow\infty} \frac{1}{\Gamma(1-\gamma_j)} \frac{d}{dt} \int_a^t (t-s)^{1-\gamma_j-1} f_n(s)ds\right] \\
&= I_{a^+}^{\beta-\alpha_j\beta}\left[\frac{1}{\Gamma(1-\gamma_j)} \frac{d}{dt} \int_a^t (t-s)^{1-\gamma_j-1}\Big(\lim_{n\longrightarrow\infty} f_n(s)\Big)ds\right] \\
&= I_{a^+}^{\beta-\alpha_j\beta}\left[\frac{1}{\Gamma(1-\gamma_j)} \frac{d}{dt} \int_a^t (t-s)^{1-\gamma_j-1} f(s)ds\right] \\
&= I_{a^+}^{\beta-\alpha_j\beta} D_{a^+}^{\gamma_j} f(t) \\
&= D_{a^+}^{\alpha_j,\beta} f(t).
\end{aligned}$$

Hence (2.11) holds. Since $f_n \in C_{\alpha_1,\alpha_2,\beta}(J,\mathbb{R})$, thus $D_{a^+}^{\alpha_j,\beta} f_n \in \mathcal{C}$ for $j = 1,2$. So that, for any $t_0 \in J$

$$\lim_{t\longrightarrow t_0} D_{a^+}^{\alpha_j,\beta} f_n(t) = D_{a^+}^{\alpha_j,\beta} f_n(t_0). \tag{2.12}$$

We will show that $f \in C_{\alpha_1,\alpha_2,\beta}(J,\mathbb{R})$, i.e. $D_{a^+}^{\alpha_j,\beta} f \in \mathcal{C}$ for $j = 1,2$. That is

$$\lim_{t\longrightarrow t_0} D_{a^+}^{\alpha_j,\beta} f(t) = D_{a^+}^{\alpha_j,\beta} f(t_0) \text{ for any } t_0 \in J.$$

Let $t, t_0 \in J$, by (2.11) and (2.12) so that for $j = 1, 2$,

$$\begin{aligned}
\lim_{t \longrightarrow t_0} D_{a^+}^{\alpha_j,\beta} f(t) &= \lim_{t \longrightarrow t_0} \left(\lim_{n \longrightarrow \infty} D_{a^+}^{\alpha_j,\beta} f_n(t) \right) \\
&= \lim_{n \longrightarrow \infty} \left(\lim_{t \longrightarrow t_0} D_{a^+}^{\alpha_j,\beta} f_n(t) \right) \\
&= \lim_{n \longrightarrow \infty} D_{a^+}^{\alpha_j,\beta} f_n(t_0) \\
&= D_{a^+}^{\alpha_j,\beta} f(t_0).
\end{aligned}$$

Hence $D_{a^+}^{\alpha_j,\beta} f \in \mathcal{C}$, so that $f \in C_{\alpha_1,\alpha_2,\beta}(J, \mathbb{R})$. Thus $C_{\alpha_1,\alpha_2,\beta}(J, \mathbb{R})$ is closed in $\mathcal{C}$. Since $C_{\alpha_1,\alpha_2,\beta}(J, \mathbb{R})$ is a subspace of $\mathcal{C}$, by Theorem 1.21 so that $C_{\alpha_1,\alpha_2,\beta}(J, \mathbb{R})$ is a complete. Therefore $(C_{\alpha_1,\alpha_2,\beta}(J, \mathbb{R}), \|\cdot\|_{C_{\alpha_1,\alpha_2,\beta}})$ is a Banach space. □

Let $E := C_{\alpha_1,\alpha_2,\beta}(J, \mathbb{R})$ denoted by (2.8) and endowed with the norm defined by (2.9). We transform the BVP (1.1) and (1.2) into a fixed point problem. In view of Lemma 2.1, we consider the operator $\mathcal{G} : E \to E$ defined by

$$\begin{aligned}
\mathcal{G}x(t) :=& \frac{1}{\mathcal{A}} \left[\left(\Delta_1(\hat{f}_x) - \mathcal{A}_1(\hat{f}_x) \right) (t-a)^{\gamma_2 - 1} + \left(\Delta_2(\hat{f}_x) - \mathcal{A}_2(\hat{f}_x) \right) (t-a)^{\gamma_1 + \alpha_2 - 1} \right] \\
&+ I_{a^+}^{\alpha_1+\alpha_2} \hat{f}_x(t) \qquad (2.13)
\end{aligned}$$

for $x \in E$ and $t \in J$, where $\hat{f}_x(t) := f(t, x(t), D_{a^+}^{\alpha_i,\beta} x(t), D_{a^+}^{\alpha_1,\beta} D_{a^+}^{\alpha_2,\beta} x(t))$, $i = 1, 2$.

In the following, for the sake of convenience, set

$$\begin{aligned}
\sigma_0 :=& \frac{1}{|\mathcal{A}|} \left[\left(\frac{\rho_1 \eta^{\gamma_1+\alpha_2-1} \xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{\rho_1 \Gamma(\gamma_1+\alpha_2) \xi_1^{\alpha_2-\alpha_1+\gamma_1-1} \eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)\Gamma(\alpha_2-\alpha_1+\gamma_1)} + \frac{\rho_1\rho_2 \Gamma(\gamma_1+\alpha_2) \xi_2^{\gamma_1-1} \xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)\Gamma(\gamma_1)} \right.\right. \\
&\left. + \frac{\rho_1\rho_2 \Gamma(\gamma_1+\alpha_2) \xi_1^{\alpha_2-\alpha_1+\gamma_1-1} \xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2-\alpha_1+\gamma_1)} \right) \eta^{\gamma_2-1} + \left(\frac{\rho_1 \Gamma(\gamma_2) \xi_1^{\gamma_2-\alpha_1-1} \eta^{\alpha_1+\alpha_2}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2+1)} + \frac{\rho_1 \eta^{\gamma_2-1} \xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)} \right. \\
&\left.\left. + \frac{\rho_1\rho_2 \Gamma(\gamma_2) \xi_1^{\gamma_2-\alpha_1-1} \xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\gamma_2-\alpha_1)} \right) \eta^{\gamma_1+\alpha_2-1} \right] + \frac{\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)},
\end{aligned}$$

$$
\begin{aligned}
\sigma_1 := \frac{1}{|\mathcal{A}|}\Bigg[\Bigg(&\frac{\rho_1\eta^{\gamma_1+\alpha_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)}+\frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}+\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)\Gamma(\gamma_1)}\\
&+\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\Bigg)\frac{\Gamma(\gamma_2)\eta^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)}+\Bigg(\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2+1)}\\
&+\frac{\rho_1\rho_2\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\gamma_2-\alpha_1)}+\frac{\rho_1\eta^{\gamma_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)}\Bigg)\frac{\Gamma(\gamma_1+\alpha_2)\eta^{\gamma_1+\alpha_2-\alpha_1-1}}{\Gamma(\gamma_1+\alpha_2-\alpha_1)}\Bigg]+\frac{\eta^{\alpha_2}}{\Gamma(\alpha_2+1)},\\
\sigma_2 := \frac{1}{|\mathcal{A}|}\Bigg[\Bigg(&\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2+1)}+\frac{\rho_1\rho_2\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\gamma_2-\alpha_1)}+\frac{\rho_1\eta^{\gamma_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)}\Bigg)\\
&\times\frac{\Gamma(\gamma_1+\alpha_2)\eta^{\gamma_1-1}}{\Gamma(\gamma_1)}\Bigg]+\frac{\eta^{\alpha_1}}{\Gamma(\alpha_1+1)},\\
\zeta_1 := &\frac{\rho_1\eta^{\gamma_1+\alpha_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)}+\frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)\Gamma(\alpha_1+\alpha_2+1)}+\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\\
&+\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2}\xi_2^{\gamma_1-1}}{\Gamma(\gamma_1)\Gamma(\alpha_2+1)},\\
\zeta_2 := &\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2+1)}+\frac{\rho_1\rho_2\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\gamma_2-\alpha_1)}+\frac{\rho_1\xi_1^{\alpha_2}\eta^{\gamma_2-1}}{\Gamma(\alpha_2+1)}.
\end{aligned}
$$

Next, we prove existence and uniqueness results based on Schaefer's fixed point theorem, Banach's contraction principle and Boyd-Wong fixed point theorem, respectively.

2.2 Existence result via Schaefer's fixed point theorem

Theorem 2.3. Assume that (A_1) and (A_2) hold. Then the BVP (1.1) and (1.2) has at least one solution on J.

Proof. We shall use Schaefer's fixed point theorem to prove that $\mathcal{G}$, defined as (2.13) has a fixed point.

First, we will show that $\mathcal{G}$ is a continuous. Let $\{x_n\}$ be a sequence such that $x_n \longrightarrow x$ in E. Then for each $t \in J$, consider

$$
\begin{aligned}
&\Big|\mathcal{G}x_n(t)-\mathcal{G}x(t)\Big| \\
&= \Bigg|\Bigg[\frac{1}{\mathcal{A}}\Bigg\{\Big(\Delta_1(\hat{f}_{x_n})-\mathcal{A}_1(\hat{f}_{x_n})\Big)(t-a)^{\gamma_2-1}+\Big(\Delta_2(\hat{f}_{x_n})-\mathcal{A}_2(\hat{f}_{x_n})\Big)(t-a)^{\gamma_1+\alpha_2-1}\Bigg\} \\
&\quad +I_{a^+}^{\alpha_1+\alpha_2}\hat{f}_{x_n}(t)\Bigg]-\Bigg[\frac{1}{\mathcal{A}}\Bigg\{\Big(\Delta_1(\hat{f}_x)-\mathcal{A}_1(\hat{f}_x)\Big)(t-a)^{\gamma_2-1}+\Big(\Delta_2(\hat{f}_x)-\mathcal{A}_2(\hat{f}_x)\Big) \\
&\quad \times(t-a)^{\gamma_1+\alpha_2-1}\Bigg\}+I_{a^+}^{\alpha_1+\alpha_2}\hat{f}_x(t)\Bigg]\Bigg| \\
&= \Bigg|\frac{1}{\mathcal{A}}\Bigg[\Bigg\{\Big(\Delta_1(\hat{f}_{x_n})-\mathcal{A}_1(\hat{f}_{x_n})\Big)-\Big(\Delta_1(\hat{f}_x)-\mathcal{A}_1(\hat{f}_x)\Big)\Bigg\}(t-a)^{\gamma_2-1} \\
&\quad +\Bigg\{\Big(\Delta_2(\hat{f}_{x_n})-\mathcal{A}_2(\hat{f}_{x_n})\Big)-\Big(\Delta_2(\hat{f}_x)-\mathcal{A}_2(\hat{f}_x)\Big)\Bigg\}(t-a)^{\gamma_1+\alpha_2-1}\Bigg] \\
&\quad +\Bigg(I_{a^+}^{\alpha_1+\alpha_2}\hat{f}_{x_n}(t)-I_{a^+}^{\alpha_1+\alpha_2}\hat{f}_x(t)\Bigg)\Bigg| \\
&\le \frac{1}{|\mathcal{A}|}\Bigg[\Bigg|\Big(\Delta_1(\hat{f}_{x_n})-\mathcal{A}_1(\hat{f}_{x_n})\Big)-\Big(\Delta_1(\hat{f}_x)-\mathcal{A}_1(\hat{f}_x)\Big)\Bigg|(t-a)^{\gamma_2-1} \\
&\quad +\Bigg|\Big(\Delta_2(\hat{f}_{x_n})-\mathcal{A}_2(\hat{f}_{x_n})\Big)-\Big(\Delta_2(\hat{f}_x)-\mathcal{A}_2(\hat{f}_x)\Big)\Bigg|(t-a)^{\gamma_1+\alpha_2-1}\Bigg] \\
&\quad +\Bigg|I_{a^+}^{\alpha_1+\alpha_2}\hat{f}_{x_n}(t)-I_{a^+}^{\alpha_1+\alpha_2}\hat{f}_x(t)\Bigg| \\
&\le \frac{1}{|\mathcal{A}|}\Bigg[\Bigg|\Big(\Delta_1(\hat{f}_{x_n})-\Delta_1(\hat{f}_x)\Big)-\Big(\mathcal{A}_1(\hat{f}_{x_n})-\mathcal{A}_1(\hat{f}_x)\Big)\Bigg|(b-a)^{\gamma_2-1} \\
&\quad +\Bigg|\Big(\Delta_2(\hat{f}_{x_n})-\Delta_2(\hat{f}_x)\Big)-\Big(\mathcal{A}_2(\hat{f}_{x_n})-\mathcal{A}_2(\hat{f}_x)\Big)\Bigg|(b-a)^{\gamma_1+\alpha_2-1}\Bigg] \\
&\quad +\Bigg|\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t(t-s)^{\alpha_1+\alpha_2-1}\hat{f}_{x_n}(s)ds-\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t(t-s)^{\alpha_1+\alpha_2-1}\hat{f}_x(s)ds\Bigg| \\
&\le \frac{1}{|\mathcal{A}|}\Bigg[\Bigg\{\Big|\Delta_1(\hat{f}_{x_n})-\Delta_1(\hat{f}_x)\Big|+\Big|\mathcal{A}_1(\hat{f}_{x_n})-\mathcal{A}_1(\hat{f}_x)\Big|\Bigg\}(b-a)^{\gamma_2-1} \\
&\quad +\Bigg\{\Big|\Delta_2(\hat{f}_{x_n})-\Delta_2(\hat{f}_x)\Big|+\Big|\mathcal{A}_2(\hat{f}_{x_n})-\mathcal{A}_2(\hat{f}_x)\Big|\Bigg\}(b-a)^{\gamma_1+\alpha_2-1}\Bigg] \\
&\quad +\frac{1}{\Gamma(\alpha_1+\alpha_2)}\Bigg|\int_a^t(t-s)^{\alpha_1+\alpha_2-1}\Big(\hat{f}_{x_n}(s)-\hat{f}_x(s)\Big)ds\Bigg|
\end{aligned}
$$

$$\leq \frac{1}{|\mathcal{A}|}\Bigg[\Bigg(\Big|\Delta_1(\hat{f}_{x_n}) - \Delta_1(\hat{f}_x)\Big| + \Big|\mathcal{A}_1(\hat{f}_{x_n}) - \mathcal{A}_1(\hat{f}_x)\Big|\Bigg)\eta^{\gamma_2-1}$$

$$+\Bigg(\Big|\Delta_2(\hat{f}_{x_n}) - \Delta_2(\hat{f}_x)\Big| + \Big|\mathcal{A}_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_x)\Big|\Bigg)\eta^{\gamma_1+\alpha_2-1}\Bigg]$$

$$+\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t (t-s)^{\alpha_1+\alpha_2-1}\Big|\hat{f}_{x_n}(s) - \hat{f}_x(s)\Big|ds. \tag{2.14}$$

We see that

$$\Big|\Delta_1(\hat{f}_{x_n}) - \Delta_1(\hat{f}_x)\Big|$$

$$= \Bigg|\Bigg(\frac{\rho_1\eta^{\gamma_1+\alpha_2-1}}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\hat{f}_{x_n}(s)ds - \frac{\rho_1\eta^{\gamma_1+\alpha_2-1}}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\hat{f}_x(s)ds\Bigg)$$

$$-\Bigg(\frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_1+\alpha_2)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\int_a^b (b-s)^{\alpha_1+\alpha_2-1}\hat{f}_{x_n}(s)ds$$

$$-\frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_1+\alpha_2)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\int_a^b (b-s)^{\alpha_1+\alpha_2-1}\hat{f}_x(s)ds\Bigg)\Bigg|$$

$$= \Bigg|\frac{\rho_1\eta^{\gamma_1+\alpha_2-1}}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\Big(\hat{f}_{x_n}(s) - \hat{f}_x(s)\Big)ds - \frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_1+\alpha_2)\Gamma(\alpha_2-\alpha_1+\gamma_1)}$$

$$\times\int_a^b (b-s)^{\alpha_1+\alpha_2-1}\Big(\hat{f}_{x_n}(s) - \hat{f}_x(s)\Big)ds\Bigg|$$

$$\leq \frac{\rho_1\eta^{\gamma_1+\alpha_2-1}}{\Gamma(\alpha_2)}\Bigg|\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\Big(\hat{f}_{x_n}(s) - \hat{f}_x(s)\Big)ds\Bigg|$$

$$+\frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_1+\alpha_2)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\Bigg|\int_a^b (b-s)^{\alpha_1+\alpha_2-1}\Big(\hat{f}_{x_n}(s) - \hat{f}_x(s)\Big)ds\Bigg|$$

$$\leq \frac{\rho_1\eta^{\gamma_1+\alpha_2-1}}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\Big|\hat{f}_{x_n}(s) - \hat{f}_x(s)\Big|ds$$

$$+\frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_1+\alpha_2)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\int_a^b (b-s)^{\alpha_1+\alpha_2-1}\Big|\hat{f}_{x_n}(s) - \hat{f}_x(s)\Big|ds, \tag{2.15}$$

$$
\begin{aligned}
&\left|\Delta_2(\hat{f}_{x_n}) - \Delta_2(\hat{f}_x)\right| \\
&\quad = \left|\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2)}\int_a^b (b-s)^{\alpha_1+\alpha_2-1}\hat{f}_{x_n}(s)ds \right.\\
&\qquad \left. -\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2)}\int_a^b (b-s)^{\alpha_1+\alpha_2-1}\hat{f}_{x}(s)ds\right| \\
&\quad = \left|\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2)}\int_a^b (b-s)^{\alpha_1+\alpha_2-1}\Big(\hat{f}_{x_n}(s)-\hat{f}_x(s)\Big)ds\right| \\
&\quad \le \frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2)}\left|\int_a^b (b-s)^{\alpha_1+\alpha_2-1}\Big(\hat{f}_{x_n}(s)-\hat{f}_x(s)\Big)ds\right| \\
&\quad \le \frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2)}\int_a^b (b-s)^{\alpha_1+\alpha_2-1}\Big|\hat{f}_{x_n}(s)-\hat{f}_x(s)\Big|ds, \qquad (2.16)
\end{aligned}
$$

$$
\begin{aligned}
&\left|\mathcal{A}_1(\hat{f}_{x_n}) - \mathcal{A}_1(\hat{f}_x)\right| \\
&\quad = \left|\left\{\left(\frac{\rho_1}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\hat{f}_{x_n}(s)ds\right)\left(\frac{\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\gamma_1)}\right)\right.\right. \\
&\qquad \left. -\left(\frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)}\right)\left(\frac{\rho_2}{\Gamma(\alpha_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}\hat{f}_{x_n}(s)ds\right)\right\} \\
&\qquad -\left\{\left(\frac{\rho_1}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\hat{f}_{x}(s)ds\right)\left(\frac{\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\gamma_1)}\right)\right. \\
&\qquad \left.\left. -\left(\frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)}\right)\left(\frac{\rho_2}{\Gamma(\alpha_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}\hat{f}_{x}(s)ds\right)\right\}\right| \\
&\quad = \left|\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\alpha_2)\Gamma(\gamma_1)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\Big(\hat{f}_{x_n}(s)-\hat{f}_x(s)\Big)ds\right. \\
&\qquad \left. -\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}\Big(\hat{f}_{x_n}(s)-\hat{f}_x(s)\Big)ds\right|
\end{aligned}
$$

$$\leq \frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\alpha_2)\Gamma(\gamma_1)}\left|\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\Big(\hat{f}_{x_n}(s)-\hat{f}_x(s)\Big)ds\right|$$

$$+\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\left|\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}\Big(\hat{f}_{x_n}(s)-\hat{f}_x(s)\Big)ds\right|$$

$$\leq \frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\alpha_2)\Gamma(\gamma_1)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\Big|\hat{f}_{x_n}(s)-\hat{f}_x(s)\Big|ds$$

$$+\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}\Big|\hat{f}_{x_n}(s)-\hat{f}_x(s)\Big|ds \tag{2.17}$$

and

$$\Big|\mathcal{A}_2(\hat{f}_{x_n})-\mathcal{A}_2(\hat{f}_x)\Big|$$

$$= \Bigg|\Bigg\{\left(\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)}\right)\left(\frac{\rho_2}{\Gamma(\alpha_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}\hat{f}_{x_n}(s)ds\right)$$

$$-\left(\frac{\rho_1}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\hat{f}_{x_n}(s)ds\right)\Big(-\eta^{\gamma_2-1}\Big)\Bigg\}$$

$$-\Bigg\{\left(\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)}\right)\left(\frac{\rho_2}{\Gamma(\alpha_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}\hat{f}_x(s)ds\right)$$

$$-\left(\frac{\rho_1}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\hat{f}_x(s)ds\right)\Big(-\eta^{\gamma_2-1}\Big)\Bigg\}\Bigg|$$

$$= \Bigg|\frac{\rho_1\rho_2\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\alpha_1)\Gamma(\gamma_2-\alpha_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}\Big(\hat{f}_{x_n}(s)-\hat{f}_x(s)\Big)ds$$

$$+\frac{\rho_1\eta^{\gamma_2-1}}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\Big(\hat{f}_{x_n}(s)-\hat{f}_x(s)\Big)ds\Bigg|$$

$$\leq \frac{\rho_1\rho_2\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\alpha_1)\Gamma(\gamma_2-\alpha_1)}\left|\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}\Big(\hat{f}_{x_n}(s)-\hat{f}_x(s)\Big)ds\right|$$

$$+\frac{\rho_1\eta^{\gamma_2-1}}{\Gamma(\alpha_2)}\left|\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\Big(\hat{f}_{x_n}(s)-\hat{f}_x(s)\Big)ds\right|$$

$$\leq \frac{\rho_1\rho_2\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\alpha_1)\Gamma(\gamma_2-\alpha_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}\Big|\hat{f}_{x_n}(s)-\hat{f}_x(s)\Big|ds$$
$$+\frac{\rho_1\eta^{\gamma_2-1}}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\Big|\hat{f}_{x_n}(s)-\hat{f}_x(s)\Big|ds. \tag{2.18}$$

Since f is continuous, we have $\Big|\hat{f}_{x_n}(s)-\hat{f}_x(s)\Big| \longrightarrow 0$ as $x_n \longrightarrow x$. From (2.15), (2.16), (2.17) and (2.18), so that

$$\Big\|\mathcal{G}x_n-\mathcal{G}x\Big\| \longrightarrow 0 \text{ as } x_n \longrightarrow x. \tag{2.19}$$

From (2.13), taking the Hilfer fractional derivative of order α_1 and type β, one get

$$D_{a^+}^{\alpha_1,\beta}\mathcal{G}x(t)=\frac{1}{\mathcal{A}}\Bigg[\frac{(\Delta_1(\hat{f}_x)-\mathcal{A}_1(\hat{f}_x))\Gamma(\gamma_2)}{\Gamma(\gamma_2-\alpha_1)}(t-a)^{\gamma_2-\alpha_1-1}+\frac{(\Delta_2(\hat{f}_x)-\mathcal{A}_2(\hat{f}_x))\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1+\alpha_2-\alpha_1)}$$
$$\times(t-a)^{\alpha_2-\alpha_1+\gamma_1-1}\Bigg]+I_{a^+}^{\alpha_2}\hat{f}_x(t).$$

Similary, taking the Hilfer fractional derivative of order α_2 and type β of (4.11), we obtain

$$D_{a^+}^{\alpha_2,\beta}\mathcal{G}x(t)=\frac{1}{\mathcal{A}}\Bigg[\frac{(\Delta_2(\hat{f}_x)-\mathcal{A}_2(\hat{f}_x))\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1)}(t-a)^{\gamma_1-1}\Bigg]+I_{a^+}^{\alpha_1}\hat{f}_x(t).$$

Also, for each $t\in J$ we consider

$$\Big|D_{a^+}^{\alpha_1,\beta}\mathcal{G}x_n(t)-D_{a^+}^{\alpha_1,\beta}\mathcal{G}x(t)\Big|$$
$$=\Bigg|\Bigg[\frac{1}{\mathcal{A}}\Bigg\{\frac{(\Delta_1(\hat{f}_{x_n})-\mathcal{A}_1(\hat{f}_{x_n}))\Gamma(\gamma_2)}{\Gamma(\gamma_2-\alpha_1)}(t-a)^{\gamma_2-\alpha_1-1}+\frac{(\Delta_2(\hat{f}_{x_n})-\mathcal{A}_2(\hat{f}_{x_n}))\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1+\alpha_2-\alpha_1)}$$
$$\times(t-a)^{\alpha_2-\alpha_1+\gamma_1-1}\Bigg\}+I_{a^+}^{\alpha_2}\hat{f}_{x_n}(t)\Bigg]-\Bigg[\frac{1}{\mathcal{A}}\Bigg\{\frac{(\Delta_1(\hat{f}_x)-\mathcal{A}_1(\hat{f}_x))\Gamma(\gamma_2)}{\Gamma(\gamma_2-\alpha_1)}(t-a)^{\gamma_2-\alpha_1-1}$$
$$+\frac{(\Delta_2(\hat{f}_x)-\mathcal{A}_2(\hat{f}_x))\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1+\alpha_2-\alpha_1)}(t-a)^{\alpha_2-\alpha_1+\gamma_1-1}\Bigg\}+I_{a^+}^{\alpha_2}\hat{f}_x(t)\Bigg]\Bigg|$$

$$
\begin{aligned}
&= \left|\frac{1}{\mathcal{A}}\left[\frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2-\alpha_1)}\left\{\left(\Delta_1(\hat{f}_{x_n})-\mathcal{A}_1(\hat{f}_{x_n})\right)-\left(\Delta_1(\hat{f}_x)-\mathcal{A}_1(\hat{f}_x)\right)\right\}(t-a)^{\gamma_2-\alpha_1-1}\right.\right.\\
&\quad\left.+\frac{\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1+\alpha_2-\alpha_1)}\left\{\left(\Delta_2(\hat{f}_{x_n})-\mathcal{A}_2(\hat{f}_{x_n})\right)-\left(\Delta_2(\hat{f}_x)-\mathcal{A}_2(\hat{f}_x)\right)\right\}(t-a)^{\alpha_2-\alpha_1+\gamma_1-1}\right]\\
&\quad\left.+\left(I_{a^+}^{\alpha_2}\hat{f}_{x_n}(t)-I_{a^+}^{\alpha_2}\hat{f}_x(t)\right)\right|\\
&\le \frac{1}{|\mathcal{A}|}\left[\frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2-\alpha_1)}\left|\left(\Delta_1(\hat{f}_{x_n})-\Delta_1(\hat{f}_x)\right)-\left(\mathcal{A}_1(\hat{f}_{x_n})-\mathcal{A}_1(\hat{f}_x)\right)\right|(b-a)^{\gamma_2-\alpha_1-1}\right.\\
&\quad\left.+\frac{\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1+\alpha_2-\alpha_1)}\left|\left(\Delta_2(\hat{f}_{x_n})-\Delta_2(\hat{f}_x)\right)-\left(\mathcal{A}_2(\hat{f}_{x_n})-\mathcal{A}_2(\hat{f}_x)\right)\right|(b-a)^{\alpha_2-\alpha_1+\gamma_1-1}\right]\\
&\quad+\left|\frac{1}{\Gamma(\alpha_2)}\int_a^t(t-s)^{\alpha_2-1}\hat{f}_{x_n}(s)ds-\frac{1}{\Gamma(\alpha_2)}\int_a^t(t-s)^{\alpha_2-1}\hat{f}_x(s)ds\right|\\
&\le \frac{1}{|\mathcal{A}|}\left[\frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2-\alpha_1)}\left\{\left|\Delta_1(\hat{f}_{x_n})-\Delta_1(\hat{f}_x)\right|+\left|\mathcal{A}_1(\hat{f}_{x_n})-\mathcal{A}_1(\hat{f}_x)\right|\right\}(b-a)^{\gamma_2-\alpha_1-1}\right.\\
&\quad\left.+\frac{\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1+\alpha_2-\alpha_1)}\left\{\left|\Delta_2(\hat{f}_{x_n})-\Delta_2(\hat{f}_x)\right|+\left|\mathcal{A}_2(\hat{f}_{x_n})-\mathcal{A}_2(\hat{f}_x)\right|\right\}(b-a)^{\alpha_2-\alpha_1+\gamma_1-1}\right]\\
&\quad+\frac{1}{\Gamma(\alpha_2)}\left|\int_a^t(t-s)^{\alpha_2-1}\left(\hat{f}_{x_n}(s)-\hat{f}_x(s)\right)ds\right|\\
&\le \frac{1}{|\mathcal{A}|}\left[\frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2-\alpha_1)}\left(\left|\Delta_1(\hat{f}_{x_n})-\Delta_1(\hat{f}_x)\right|+\left|\mathcal{A}_1(\hat{f}_{x_n})-\mathcal{A}_1(\hat{f}_x)\right|\right)\eta^{\gamma_2-\alpha_1-1}\right.\\
&\quad\left.+\frac{\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1+\alpha_2-\alpha_1)}\left(\left|\Delta_2(\hat{f}_{x_n})-\Delta_2(\hat{f}_x)\right|+\left|\mathcal{A}_2(x_n)-\mathcal{A}_2(\hat{f}_x)\right|\right)\eta^{\alpha_2-\alpha_1+\gamma_1-1}\right]\\
&\quad+\frac{1}{\Gamma(\alpha_2)}\int_a^t(t-s)^{\alpha_2-1}\left|\hat{f}_{x_n}(s)-\hat{f}_x(s)\right|ds.
\end{aligned}
$$

Since f is continuous and from (2.15), (2.16), (2.17) and (2.18), we obtain

$$\left\|D_{a^+}^{\alpha_1,\beta}\mathcal{G}x_n-D_{a^+}^{\alpha_1,\beta}\mathcal{G}x\right\|\longrightarrow 0 \text{ as } x_n\longrightarrow x. \tag{2.20}$$

For each $t\in J$, we consider

$$\begin{aligned}
&\left|D_{a^+}^{\alpha_2,\beta}\mathcal{G}x_n(t) - D_{a^+}^{\alpha_2,\beta}\mathcal{G}x(t)\right| \\
&= \Bigg|\left[\frac{1}{\mathcal{A}}\left\{\frac{(\Delta_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x_n}))\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1)}(t-a)^{\gamma_1-1}\right\} + I_{a^+}^{\alpha_1}\hat{f}_{x_n}(t)\right] \\
&\quad -\left[\frac{1}{\mathcal{A}}\left\{\frac{(\Delta_2(\hat{f}_{x}) - \mathcal{A}_2(\hat{f}_{x}))\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1)}(t-a)^{\gamma_1-1}\right\} + I_{a^+}^{\alpha_1}\hat{f}_{x}(t)\right]\Bigg| \\
&= \Bigg|\frac{1}{\mathcal{A}}\left[\frac{\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1)}\left\{\Big(\Delta_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x_n})\Big) - \Big(\Delta_2(\hat{f}_{x}) - \mathcal{A}_2(\hat{f}_{x})\Big)\right\}(t-a)^{\gamma_1-1}\right] \\
&\quad + \left(I_{a^+}^{\alpha_1}\hat{f}_{x_n}(t) - I_{a^+}^{\alpha_1}\hat{f}_{x}(t)\right)\Bigg| \\
&\le \frac{1}{|\mathcal{A}|}\left[\frac{\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1)}\left|\Big(\Delta_2(\hat{f}_{x_n}) - \Delta_2(\hat{f}_{x})\Big) - \Big(\mathcal{A}_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x})\Big)\right|(b-a)^{\gamma_1-1}\right] \\
&\quad + \left|\frac{1}{\Gamma(\alpha_1)}\int_a^t (t-s)^{\alpha_1-1}\hat{f}_{x_n}(s)ds - \frac{1}{\Gamma(\alpha_1)}\int_a^t (t-s)^{\alpha_1-1}\hat{f}_{x}(s)ds\right| \\
&\le \frac{1}{|\mathcal{A}|}\left[\frac{\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1)}\left\{\left|\Delta_2(\hat{f}_{x_n}) - \Delta_2(\hat{f}_{x})\right| + \left|\mathcal{A}_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x})\right|\right\}(b-a)^{\gamma_1-1}\right] \\
&\quad + \frac{1}{\Gamma(\alpha_1)}\left|\int_a^t (t-s)^{\alpha_1-1}\Big(\hat{f}_{x_n}(s) - \hat{f}_{x}(s)\Big)ds\right| \\
&\le \frac{1}{|\mathcal{A}|}\left[\frac{\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1)}\left(\left|\Delta_2(\hat{f}_{x_n}) - \Delta_2(\hat{f}_{x})\right| + \left|\mathcal{A}_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x})\right|\right)\eta^{\gamma_1-1}\right] \\
&\quad + \frac{1}{\Gamma(\alpha_1)}\int_a^t (t-s)^{\alpha_1-1}\left|\hat{f}_{x_n}(s) - \hat{f}_{x}(s)\right|ds.
\end{aligned}$$

Since f is continuous and from (2.15), (2.16), (2.17) and (2.18), we obtain

$$\left\|D_{a^+}^{\alpha_2,\beta}\mathcal{G}x_n - D_{a^+}^{\alpha_2,\beta}\mathcal{G}x\right\| \longrightarrow 0 \text{ as } x_n \longrightarrow x. \tag{2.21}$$

We see that

$$\left\|\mathcal{G}x_n - \mathcal{G}x\right\|_E = \left\|\mathcal{G}x_n - \mathcal{G}x\right\| + \left\|D_{a^+}^{\alpha_1,\beta}\mathcal{G}x_n - D_{a^+}^{\alpha_1,\beta}\mathcal{G}x\right\| + \left\|D_{a^+}^{\alpha_2,\beta}\mathcal{G}x_n - D_{a^+}^{\alpha_2,\beta}\mathcal{G}x\right\|.$$

From (2.19), (2.20) and (2.21), we have

$$\left\|\mathcal{G}x_n - \mathcal{G}x\right\|_E \longrightarrow 0 \text{ as } x_n \longrightarrow x.$$

Hence $\mathcal{F}$ is continuous.

Next, we must show that $\mathcal{G}$ is compact on $\bar{B}_R := \{x \in E : \|x\|_E \leq R\}$. That is $\mathcal{G}(\bar{B}_R)$ is compact set. Let $\{g_n\}$ be a sequence in $\mathcal{G}(\bar{B}_R)$. Define by $g_n(t) := \mathcal{G}x_n(t)$ for any $n \in \mathbb{N}$. We divide the proof into two steps.

Step I : To show that $\{g_n\}$ is uniformly bounded. We will find K, such that $\|g_n\|_E \leq K$, for any $n \in \mathbb{N}$ and $x \in \bar{B}_R$. Consider,

$$
\begin{aligned}
&\left|g_n(t)\right| \\
&= \left|\frac{1}{\mathcal{A}}\left[\left(\Delta_1(\hat{f}_{x_n}) - \mathcal{A}_1(\hat{f}_{x_n})\right)(t-a)^{\gamma_2-1} + \left(\Delta_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x_n})\right)(t-a)^{\gamma_1+\alpha_2-1}\right]\right. \\
&\quad \left. + I_{a^+}^{\alpha_1+\alpha_2}\hat{f}_{x_n}(t)\right| \\
&\leq \frac{1}{|\mathcal{A}|}\left|\left(\Delta_1(\hat{f}_{x_n}) - \mathcal{A}_1(\hat{f}_{x_n})\right)(t-a)^{\gamma_2-1} + \left(\Delta_2(x_n) - \mathcal{A}_2(x_n)\right)(t-a)^{\gamma_1+\alpha_2-1}\right| \\
&\quad + \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t (t-s)^{\alpha_1+\alpha_2-1}\left|\hat{f}_{x_n}(s)\right|ds \\
&\leq \frac{1}{|\mathcal{A}|}\left[\left|\Delta_1(\hat{f}_{x_n}) - \mathcal{A}_1(\hat{f}_{x_n})\right|(b-a)^{\gamma_2-1} + \left|\Delta_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x_n})\right|(b-a)^{\gamma_1+\alpha_2-1}\right] \\
&\quad + \frac{D}{\Gamma(\alpha_1+\alpha_2)}\int_a^t (t-s)^{\alpha_1+\alpha_2-1}ds \\
&= \frac{1}{|\mathcal{A}|}\left[\left|\Delta_1(\hat{f}_{x_n}) - \mathcal{A}_1(\hat{f}_{x_n})\right|\eta^{\gamma_2-1} + \left|\Delta_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x_n})\right|\eta^{\gamma_1+\alpha_2-1}\right] \\
&\quad + \frac{D}{\Gamma(\alpha_1+\alpha_2)}\left(\frac{(t-a)^{\alpha_1+\alpha_2}}{\alpha_1+\alpha_2}\right) \\
&\leq \frac{1}{|\mathcal{A}|}\left[\left|\Delta_1(\hat{f}_{x_n}) - \mathcal{A}_1(\hat{f}_{x_n})\right|\eta^{\gamma_2-1} + \left|\Delta_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x_n})\right|\eta^{\gamma_1+\alpha_2-1}\right] + \frac{D\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}.
\end{aligned}
\tag{2.22}
$$

By (A_2), for each $t \in J$ we get that

$$
\begin{aligned}
&\left|\Delta_1(\hat{f}_{x_n}) - \mathcal{A}_1(\hat{f}_{x_n})\right| \\
&= \Bigg|\Bigg(\frac{\rho_1\eta^{\gamma_1+\alpha_2-1}}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\hat{f}_{x_n}(s)ds - \frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)\Gamma(\alpha_1+\alpha_2)} \\
&\quad \times\int_a^b(b-s)^{\alpha_1+\alpha_2-1}\hat{f}_{x_n}(s)ds\Bigg) - \Bigg\{\Bigg(\frac{\rho_1}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\hat{f}_{x_n}(s)ds\Bigg) \\
&\quad \times\Bigg(\frac{\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\gamma_1)}\Bigg) - \Bigg(\frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)}\Bigg) \\
&\quad \times\Bigg(\frac{\rho_2}{\Gamma(\alpha_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}\hat{f}_{x_n}(s)ds\Bigg)\Bigg\}\Bigg| \\
&= \Bigg|\frac{\rho_1\eta^{\gamma_1+\alpha_2-1}}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\hat{f}_{x_n}(s)ds - \frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)\Gamma(\alpha_1+\alpha_2)} \\
&\quad \times\int_a^b(b-s)^{\alpha_1+\alpha_2-1}\hat{f}_{x_n}(s)ds - \frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\gamma_1)\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\hat{f}_{x_n}(s)ds \\
&\quad -\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}\hat{f}_{x_n}(s)ds\Bigg| \\
&\le \frac{\rho_1\eta^{\gamma_1+\alpha_2-1}}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\left|\hat{f}_{x_n}(s)\right|ds + \frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)\Gamma(\alpha_1+\alpha_2)} \\
&\quad \times\int_a^b(b-s)^{\alpha_1+\alpha_2-1}\left|\hat{f}_{x_n}(s)\right|ds + \frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\gamma_1)\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\left|\hat{f}_{x_n}(s)\right|ds \\
&\quad +\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}\left|\hat{f}_{x_n}(s)\right|ds \\
&\le \frac{\rho_1\eta^{\gamma_1+\alpha_2-1}D}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}ds + \frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}D}{\Gamma(\alpha_2-\alpha_1+\gamma_1)\Gamma(\alpha_1+\alpha_2)}\int_a^b(b-s)^{\alpha_1+\alpha_2-1}ds \\
&\quad +\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}D}{\Gamma(\gamma_1)\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}ds + \frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}D}{\Gamma(\alpha_1)\Gamma(\alpha_2-\alpha_1+\gamma_1)} \\
&\quad \times\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}ds \\
&= D\Bigg[\frac{\rho_1\eta^{\gamma_1+\alpha_2-1}}{\Gamma(\alpha_2)}\Bigg(\frac{(\theta_1-a)^{\alpha_2}}{\alpha_2}\Bigg) + \frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)\Gamma(\alpha_1+\alpha_2)}\Bigg(\frac{(b-a)^{\alpha_1+\alpha_2}}{\alpha_1+\alpha_2}\Bigg) \\
&\quad +\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\gamma_1)\Gamma(\alpha_2)}\Bigg(\frac{(\theta_1-a)^{\alpha_2}}{\alpha_2}\Bigg) + \frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\Bigg(\frac{(\theta_2-a)^{\alpha_1}}{\alpha_1}\Bigg)\Bigg] \\
&= D\Bigg[\frac{\rho_1\eta^{\gamma_1+\alpha_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_2-\alpha_1+\gamma_1)\Gamma(\alpha_1+\alpha_2+1)} + \frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2}\xi_2^{\gamma_1-1}}{\Gamma(\gamma_1)\Gamma(\alpha_2+1)} \\
&\quad +\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\Delta_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x_n})\Big| \\
&= \Bigg|\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2)}\int_a^b (b-s)^{\alpha_1+\alpha_2-1}\hat{f}_{x_n}(s)ds - \Bigg\{\Bigg(\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)}\Bigg) \\
&\quad \times\Bigg(\frac{\rho_2}{\Gamma(\alpha_1)}\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}\hat{f}_{x_n}(s)ds\Bigg) - \Bigg(\frac{\rho_1}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\hat{f}_{x_n}(s)ds\Bigg) \\
&\quad \times\Big(-\eta^{\gamma_2-1}\Big)\Bigg\}\Bigg| \\
&= \Bigg|\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2)}\int_a^b (b-s)^{\alpha_1+\alpha_2-1}\hat{f}_{x_n}(s)ds - \frac{\rho_1\rho_2\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\alpha_1)\Gamma(\gamma_2-\alpha_1)} \\
&\quad \times\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}\hat{f}_{x_n}(s)ds + \frac{\rho_1\eta^{\gamma_2-1}}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\hat{f}_{x_n}(s)ds\Bigg| \\
&\le \frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2)}\int_a^b (b-s)^{\alpha_1+\alpha_2-1}\Big|\hat{f}_{x_n}(s)\Big|ds + \frac{\rho_1\rho_2\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\alpha_1)\Gamma(\gamma_2-\alpha_1)} \\
&\quad \times\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}\Big|\hat{f}_{x_n}(s)\Big|ds + \frac{\rho_1\eta^{\gamma_2-1}}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}\Big|\hat{f}_{x_n}(s)\Big|ds \\
&\le \frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}D}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2)}\int_a^b (b-s)^{\alpha_1+\alpha_2-1}ds + \frac{\rho_1\rho_2\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}D}{\Gamma(\alpha_1)\Gamma(\gamma_2-\alpha_1)} \\
&\quad \times\int_a^{\theta_2}(\theta_2-s)^{\alpha_1-1}ds + \frac{\rho_1\eta^{\gamma_2-1}D}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1-s)^{\alpha_2-1}ds \\
&= D\Bigg[\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2)}\Bigg(\frac{(b-a)^{\alpha_1+\alpha_2}}{\alpha_1+\alpha_2}\Bigg) + \frac{\rho_1\rho_2\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\alpha_1)\Gamma(\gamma_2-\alpha_1)}\Bigg(\frac{(\theta_2-a)^{\alpha_1}}{\alpha_1}\Bigg) \\
&\quad +\frac{\rho_1\eta^{\gamma_2-1}}{\Gamma(\alpha_2)}\Bigg(\frac{(\theta_1-a)^{\alpha_2}}{\alpha_2}\Bigg)\Bigg] \\
&= D\Bigg[\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2+1)} + \frac{\rho_1\rho_2\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\gamma_2-\alpha_1)} + \frac{\rho_1\xi_1^{\alpha_2}\eta^{\gamma_2-1}}{\Gamma(\alpha_2+1)}\Bigg] \\
&= D\zeta_2. \qquad (2.24)
\end{aligned}
$$

From (2.22), (2.23) and (2.24) we obtain

$$
\Big|g_n(t)\Big| \le D\Bigg[\frac{1}{|\mathcal{A}|}\Big(\zeta_1\eta^{\gamma_2-1}+\zeta_2\eta^{\gamma_1+\alpha_2-1}\Big) + \frac{\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}\Bigg] = D\delta_0 \qquad (2.25)
$$

where

$$\delta_0 := \frac{1}{|\mathcal{A}|}\left(\zeta_1\eta^{\gamma_2-1} + \zeta_2\eta^{\gamma_1+\alpha_2-1}\right) + \frac{\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} > 0.$$

Hence $\left\|g_n\right\| = \max_{t\in J}\left|g_n(t)\right| \leq D\delta_0$. From (2.23) and (2.24), we consider

$$\begin{aligned}
&\left|D_{a^+}^{\alpha_1,\beta} g_n(t)\right| \\
&= \left|\frac{1}{\mathcal{A}}\left[\frac{(\Delta_1(\hat{f}_{x_n}) - \mathcal{A}_1(\hat{f}_{x_n}))\Gamma(\gamma_2)}{\Gamma(\gamma_2-\alpha_1)}(t-a)^{\gamma_2-\alpha_1-1} + \frac{(\Delta_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x_n}))\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1+\alpha_2-\alpha_1)}\right.\right. \\
&\qquad \left.\left.\times(t-a)^{\alpha_2-\alpha_1+\gamma_1-1}\right] + I_{a^+}^{\alpha_2}\hat{f}_{x_n}(t)\right| \\
&\leq \frac{1}{|\mathcal{A}|}\left|\frac{(\Delta_1(\hat{f}_{x_n}) - \mathcal{A}_1(\hat{f}_{x_n}))\Gamma(\gamma_2)}{\Gamma(\gamma_2-\alpha_1)}(t-a)^{\gamma_2-\alpha_1-1} + \frac{(\Delta_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x_n}))\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1+\alpha_2-\alpha_1)}\right. \\
&\qquad \left.\times(t-a)^{\alpha_2-\alpha_1+\gamma_1-1}\right| + \frac{1}{\Gamma(\alpha_2)}\int_a^t (t-s)^{\alpha_2-1}\left|\hat{f}_{x_n}(s)\right|ds \\
&\leq \frac{1}{|\mathcal{A}|}\left[\frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2-\alpha_1)}\left|\Delta_1(\hat{f}_{x_n}) - \mathcal{A}_1(\hat{f}_{x_n})\right|(b-a)^{\gamma_2-\alpha_1-1} + \frac{\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1+\alpha_2-\alpha_1)}\right. \\
&\qquad \left.\times\left|\Delta_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x_n})\right|(b-a)^{\alpha_2-\alpha_1+\gamma_1-1}\right] + \frac{D}{\Gamma(\alpha_2)}\int_a^t (t-s)^{\alpha_2-1}ds \\
&= \frac{1}{|\mathcal{A}|}\left[\frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2-\alpha_1)}\left|\Delta_1(\hat{f}_{x_n}) - \mathcal{A}_1(\hat{f}_{x_n})\right|\eta^{\gamma_2-\alpha_1-1} + \frac{\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1+\alpha_2-\alpha_1)}\right. \\
&\qquad \left.\times\left|\Delta_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x_n})\right|\eta^{\alpha_2-\alpha_1+\gamma_1-1}\right] + \frac{D}{\Gamma(\alpha_2)}\left(\frac{(t-a)^{\alpha_2}}{\alpha_2}\right) \\
&\leq D\frac{1}{|\mathcal{A}|}\left[\frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2-\alpha_1)}\left|\Delta_1(\hat{f}_{x_n}) - \mathcal{A}_1(\hat{f}_{x_n})\right|\eta^{\gamma_2-\alpha_1-1} + \frac{\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1+\alpha_2-\alpha_1)}\right. \\
&\qquad \left.\times\left|\Delta_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x_n})\right|\eta^{\alpha_2-\alpha_1+\gamma_1-1}\right] + \frac{D\eta^{\alpha_2}}{\Gamma(\alpha_2+1)} \\
&\leq D\left[\frac{1}{|\mathcal{A}|}\left(\frac{\Gamma(\gamma_2)\zeta_1\eta^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)} + \frac{\Gamma(\gamma_1+\alpha_2)\zeta_2\eta^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\gamma_1+\alpha_2-\alpha_1)}\right) + \frac{\eta^{\alpha_2}}{\Gamma(\alpha_2+1)}\right] \\
&= D\delta_1 \qquad (2.26)
\end{aligned}$$

Step II : To show that $\{g_n\}$ is equicontinuous. Let $a \le t_0 \le t \le b$, and $x \in \bar{B}_R$. Consider,

$$\begin{aligned}
&\left|g_n(t) - g_n(t_0)\right| \\
&= \Bigg|\Bigg[\frac{1}{\mathcal{A}}\Big\{\Big(\Delta_1(\hat{f}_{x_n}) - \mathcal{A}_1(\hat{f}_{x_n})\Big)(t-a)^{\gamma_2-1} + \Big(\Delta_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x_n})\Big)(t-a)^{\gamma_1+\alpha_2-1}\Big\} \\
&\quad + I_{a^+}^{\alpha_1+\alpha_2}\hat{f}_{x_n}(t)\Bigg] \\
&\quad - \Bigg[\frac{1}{\mathcal{A}}\Big\{\Big(\Delta_1(\hat{f}_{x_n}) - \mathcal{A}_1(\hat{f}_{x_n})\Big)(t_0-a)^{\gamma_2-1} + \Big(\Delta_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x_n})\Big)(t_0-a)^{\gamma_1+\alpha_2-1}\Big\} \\
&\quad + I_{a^+}^{\alpha_1+\alpha_2}\hat{f}_{x_n}(t_0)\Bigg]\Bigg| \\
&= \Bigg|\frac{1}{\mathcal{A}}\Bigg[\Big(\Delta_1(\hat{f}_{x_n}) - \mathcal{A}_1(\hat{f}_{x_n})\Big)\Big\{(t-a)^{\gamma_2-1} - (t_0-a)^{\gamma_2-1}\Big\} + \Big(\Delta_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x_n})\Big) \\
&\quad \Big\{(t-a)^{\gamma_1+\alpha_2-1} - (t_0-a)^{\gamma_1+\alpha_2-1}\Big\}\Bigg] + \Big(I_{a^+}^{\alpha_1+\alpha_2}\hat{f}_{x_n}(t) - I_{a^+}^{\alpha_1+\alpha_2}\hat{f}_{x_n}(t_0)\Big)\Bigg| \\
&\le \frac{1}{|\mathcal{A}|}\Bigg[\left|\Delta_1(\hat{f}_{x_n}) - \mathcal{A}_1(\hat{f}_{x_n})\right|\left|(t-a)^{\gamma_2-1} - (t_0-a)^{\gamma_2-1}\right| + \left|\Delta_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x_n})\right| \\
&\quad \times\left|(t-a)^{\gamma_1+\alpha_2-1} - (t_0-a)^{\gamma_1+\alpha_2-1}\right|\Bigg] + \left|I_{a^+}^{\alpha_1+\alpha_2}\hat{f}_{x_n}(t) - I_{a^+}^{\alpha_1+\alpha_2}\hat{f}_{x_n}(t_0)\right|. \qquad (2.28)
\end{aligned}$$

By (A_2), for each $t, t_0 \in J$ with $t_0 \le t$ we consider

$$\begin{aligned}
&\left|I_{a^+}^{\alpha_1+\alpha_2}\hat{f}_{x_n}(t) - I_{a^+}^{\alpha_1+\alpha_2}\hat{f}_{x_n}(t_0)\right| \\
&\qquad = \frac{1}{\Gamma(\alpha_1+\alpha_2)}\left|\int_a^t (t-s)^{\alpha_1+\alpha_2-1}\hat{f}_{x_n}(s)ds - \int_a^t (t_0-s)^{\alpha_1+\alpha_2-1}\hat{f}_{x_n}(s)ds\right| \\
&\qquad = \frac{1}{\Gamma(\alpha_1+\alpha_2)}\Bigg|\int_a^{t_0}\Big((t-s)^{\alpha_1+\alpha_2-1} - (t_0-s)^{\alpha_1+\alpha_2-1}\Big)\hat{f}_{x_n}(s)ds \\
&\qquad\quad + \int_{t_0}^t (t-s)^{\alpha_1+\alpha_2-1}\hat{f}_{x_n}(s)ds\Bigg|
\end{aligned}$$

$$\leq \frac{1}{\Gamma(\alpha_1+\alpha_2)}\left[\int_a^{t_0}\left|(t-s)^{\alpha_1+\alpha_2-1}-(t_0-s)^{\alpha_1+\alpha_2-1}\right|\left|\hat{f}_{x_n}(s)\right|ds\right.$$

$$\left.+\int_{t_0}^{t}(t-s)^{\alpha_1+\alpha_2-1}\left|\hat{f}_{x_n}(s)\right|ds\right]$$

$$\leq \frac{D}{\Gamma(\alpha_1+\alpha_2)}\left[\int_a^{t_0}\left|(t-s)^{\alpha_1+\alpha_2-1}-(t_0-s)^{\alpha_1+\alpha_2-1}\right|ds+\int_{t_0}^{t}(t-s)^{\alpha_1+\alpha_2-1}ds\right].$$

Taking $t\longrightarrow t_0$, thus

$$\left|I_{a^+}^{\alpha_1+\alpha_2}\hat{f}_{x_n}(t)-I_{a^+}^{\alpha_1+\alpha_2}\hat{f}_{x_n}(t_0)\right|\longrightarrow 0.$$

From (2.28), we see that, as $t\longrightarrow t_0$, so that

$$\left|g_n(t)-g_n(t_0)\right|\longrightarrow 0.$$

Next, we consider

$$\left|D_{a^+}^{\alpha_1,\beta}g_n(t)-D_{a^+}^{\alpha_1,\beta}g_n(t_0)\right|$$

$$=\left|\left[\frac{1}{\mathcal{A}}\left\{\frac{(\Delta_1(\hat{f}_{x_n})-\mathcal{A}_1(\hat{f}_{x_n}))\Gamma(\gamma_2)}{\Gamma(\gamma_2-\alpha_1)}(t-a)^{\gamma_2-\alpha_1-1}+\frac{(\Delta_2(\hat{f}_{x_n})-\mathcal{A}_2(\hat{f}_{x_n}))\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1+\alpha_2-\alpha_1)}\right.\right.\right.$$

$$\left.\left.\times(t-a)^{\alpha_2-\alpha_1+\gamma_1-1}\right\}+I_{a^+}^{\alpha_2}\hat{f}_{x_n}(t)\right]-\left[\frac{1}{\mathcal{A}}\left\{\frac{(\Delta_1(\hat{f}_{x_n})-\mathcal{A}_1(\hat{f}_{x_n}))\Gamma(\gamma_2)}{\Gamma(\gamma_2-\alpha_1)}(t_0-a)^{\gamma_2-\alpha_1-1}\right.\right.$$

$$\left.\left.\left.+\frac{(\Delta_2(\hat{f}_{x_n})-\mathcal{A}_2(\hat{f}_{x_n}))\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1+\alpha_2-\alpha_1)}(t_0-a)^{\alpha_2-\alpha_1+\gamma_1-1}\right\}+I_{a^+}^{\alpha_2}\hat{f}_{x_n}(t_0)\right]\right|$$

$$= \left| \frac{1}{\mathcal{A}} \left[\frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2 - \alpha_1)} \left(\Delta_1(\hat{f}_{x_n}) - \mathcal{A}_1(\hat{f}_{x_n}) \right) \left\{ (t-a)^{\gamma_2 - \alpha_1 - 1} - (t_0 - a)^{\gamma_2 - \alpha_1 - 1} \right\} \right. \right.$$

$$\left. + \frac{\Gamma(\gamma_1 + \alpha_2)}{\Gamma(\gamma_1 + \alpha_2 - \alpha_1)} \left(\Delta_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x_n}) \right) \left\{ (t-a)^{\alpha_2 - \alpha_1 + \gamma_1 - 1} - (t_0 - a)^{\alpha_2 - \alpha_1 + \gamma_1 - 1} \right\} \right]$$

$$\left. + \left(I_{a^+}^{\alpha_2} \hat{f}_{x_n}(t) - I_{a^+}^{\alpha_2} \hat{f}_{x_n}(t_0) \right) \right|$$

$$\leq \frac{1}{|\mathcal{A}|} \left[\frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2 - \alpha_1)} \left| \Delta_1(\hat{f}_{x_n}) - \mathcal{A}_1(\hat{f}_{x_n}) \right| \left| (t-a)^{\gamma_2 - \alpha_1 - 1} - (t_0 - a)^{\gamma_2 - \alpha_1 - 1} \right| \right.$$

$$\left. + \frac{\Gamma(\gamma_1 + \alpha_2)}{\Gamma(\gamma_1 + \alpha_2 - \alpha_1)} \left| \Delta_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x_n}) \right| \left| (t-a)^{\alpha_2 - \alpha_1 + \gamma_1 - 1} - (t_0 - a)^{\alpha_2 - \alpha_1 + \gamma_1 - 1} \right| \right]$$

$$+ \left| I_{a^+}^{\alpha_2} \hat{f}_{x_n}(t) - I_{a^+}^{\alpha_2} \hat{f}_{x_n}(t_0) \right|. \tag{2.29}$$

Next, we consider

$$\left| D_{a^+}^{\alpha_2, \beta} g_n(t) - D_{a^+}^{\alpha_2, \beta} g_n(t_0) \right|$$

$$= \left| \left[\frac{1}{\mathcal{A}} \left\{ \frac{(\Delta_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x_n}))\Gamma(\gamma_1 + \alpha_2)}{\Gamma(\gamma_1)} (t-a)^{\gamma_1 - 1} \right\} + I_{a^+}^{\alpha_1} \hat{f}_{x_n}(t) \right] \right.$$

$$\left. - \left[\frac{1}{\mathcal{A}} \left\{ \frac{(\Delta_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x_n}))\Gamma(\gamma_1 + \alpha_2)}{\Gamma(\gamma_1)} (t_0-a)^{\gamma_1 - 1} \right\} + I_{a^+}^{\alpha_1} \hat{f}_{x_n}(t_0) \right] \right|$$

$$= \left| \frac{1}{\mathcal{A}} \left[\frac{\Gamma(\gamma_1 + \alpha_2)}{\Gamma(\gamma_1)} \left(\Delta_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x_n}) \right) \left\{ (t-a)^{\gamma_1 - 1} - (t_0 - a)^{\gamma_1 - 1} \right\} \right] \right.$$

$$\left. + \left(I_{a^+}^{\alpha_1} \hat{f}_{x_n}(t) - I_{a^+}^{\alpha_1} \hat{f}_{x_n}(t_0) \right) \right|$$

$$\leq \frac{1}{|\mathcal{A}|} \left[\frac{\Gamma(\gamma_1 + \alpha_2)}{\Gamma(\gamma_1)} \left| \Delta_2(\hat{f}_{x_n}) - \mathcal{A}_2(\hat{f}_{x_n}) \right| \left| (t-a)^{\gamma_1 - 1} - (t_0 - a)^{\gamma_1 - 1} \right| \right]$$

$$+ \left| I_{a^+}^{\alpha_1} \hat{f}_{x_n}(t) - I_{a^+}^{\alpha_1} \hat{f}_{x_n}(t_0) \right|. \tag{2.30}$$

Clearly, for each $t, t_0 \in J$ with $t_0 \leq t$ then for $j = 1,2$ we get that

$$\left| I_{a^+}^{\alpha_j} \hat{f}_{x_n}(t) - I_{a^+}^{\alpha_j} \hat{f}_{x_n}(t_0) \right| \longrightarrow 0 \text{ as } t \longrightarrow t_0.$$

From (2.29) and (2.30), thking $t \longrightarrow t_0$, so that for $j = 1,2$

$$\left| D_{a^+}^{\alpha_j,\beta} g_n(t) - D_{a^+}^{\alpha_j,\beta} g_n(t_0) \right| \longrightarrow 0.$$

Hence $\{g_n\}$ is equicontinuous. From Step I and Step II, by Arzelà–Ascoli theorem, we get that $\mathcal{G}(\bar{B}_R)$ is compact set, so we have $\mathcal{G}$ is compact on $\bar{B}_R$.

Finally, we will show that $B = \{x \in E : x = \lambda \mathcal{G}x, \ \exists \lambda \in [0,1]\}$ is bounded. Let $t \in J$ and $x \in B$, then $x(t) = \lambda \mathcal{G}x(t)$ for some $\lambda \in [0,1]$. By (2.23) and (2.24), we obtain

$$\left| \lambda \mathcal{G}x(t) \right| \leq D\delta_0 \text{ and } \left| D_{a^+}^{\alpha_j,\beta} \lambda \mathcal{G}x(t) \right| \leq D\delta_j$$

for $j = 1,2$. We see that

$$\begin{aligned} \left\| x \right\|_E &= \left\| \lambda \mathcal{G}x \right\|_E \\ &= \left\| \lambda \mathcal{G}x \right\| + \left\| D_{a^+}^{\alpha_1,\beta} \lambda \mathcal{G}x \right\| + \left\| D_{a^+}^{\alpha_2,\beta} \lambda \mathcal{G}x \right\| \\ &= \max_{t \in J} \left| \lambda \mathcal{G}x(t) \right| + \max_{t \in J} \left| D_{a^+}^{\alpha_1,\beta} \lambda \mathcal{G}x(t) \right| + \max_{t \in J} \left| D_{a^+}^{\alpha_2,\beta} \lambda \mathcal{G}x(t) \right| \\ &\leq D(\delta_0 + \delta_1 + \delta_2). \end{aligned}$$

Hence B is bounded. By Schaufer's fixed point theorem, we get that $\mathcal{G}$ has a fixed point. Therefore, BVP (1.1) and (1.2) has at least one solution on J. □

2.3 Existence result via Banach's contraction

Theorem 2.4. Assume that (A_1) holds. In addition, (A_6) There exist constants $K, L, M \in \mathbb{R}_{++}$ such that

$$\left| f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2) \right| \leq K \left| u_1 - u_2 \right| + L \left| v_1 - v_2 \right| + M \left| w_1 - w_2 \right|$$

for any $u_1, v_1, w_1, u_2, v_2, w_2 \in \mathbb{R}$ and $t \in J$. If $\mathcal{P} := \sigma\Lambda_i < 1$ for $i = 1, 2$ where

$$\Lambda_1 := \max\left\{K, L, \frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\right\}, \tag{2.31}$$

$$\Lambda_2 := \max\left\{1, K, L + \frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\right\} \tag{2.32}$$

and $\sigma := \sigma_0 + \sigma_1 + \sigma_2$, then the BVP (1.1) and (1.2) has a unique solution on J.

Proof. We shall use Banach's contraction principle to prove that $\mathcal{G}$, defined as (2.13) has a unique fixed point. We will show that $\mathcal{G}$ is a contraction on E. Let $x_1, x_2 \in E$, for each $t \in J$. We consider

$$\begin{aligned}
&\left|D_{a^+}^{\alpha_1,\beta} D_{a^+}^{\alpha_2,\beta} x_1(t) - D_{a^+}^{\alpha_1,\beta} D_{a^+}^{\alpha_2,\beta} x_2(t)\right| \\
&= \left|I_{a^+}^{\beta-\alpha_1\beta} D_{a^+}^{\gamma_1}\left(D_{a^+}^{\alpha_2,\beta} x_1(t)\right) - I_{a^+}^{\beta-\alpha_1\beta} D_{a^+}^{\gamma_1}\left(D_{a^+}^{\alpha_2,\beta} x_2(t)\right)\right| \\
&= \left|\frac{1}{\Gamma(\beta-\alpha_1\beta)}\left[\int_a^t (t-s)^{\beta-\alpha_1\beta-1} D_{a^+}^{\gamma_1}\left(D_{a^+}^{\alpha_2,\beta} x_1(s)\right) ds - \int_a^t (t-s)^{\beta-\alpha\beta_1-1} D_{a^+}^{\gamma_1}\left(D_{a^+}^{\alpha_2,\beta} x_2(s)\right) ds\right]\right| \\
&= \frac{1}{\Gamma(\beta-\alpha_1\beta)}\left|\int_a^t (t-s)^{\beta-\alpha_1\beta-1}\left[D_{a^+}^{\gamma_1}\left(D_{a^+}^{\alpha_2,\beta} x_1(s)\right) - D_{a^+}^{\gamma_1}\left(D_{a^+}^{\alpha_2,\beta} x_2(s)\right)\right] ds\right| \\
&\leq \frac{1}{\Gamma(\beta-\alpha_1\beta)}\left[\int_a^t (t-s)^{\beta-\alpha_1\beta-1}\left|D_{a^+}^{\gamma_1}\left(D_{a^+}^{\alpha_2,\beta} x_1(s)\right) - D_{a^+}^{\gamma_1}\left(D_{a^+}^{\alpha_2,\beta} x_2(s)\right)\right| ds\right] \\
&= \frac{1}{\Gamma(\beta-\alpha_1\beta)}\left[\int_a^t (t-s)^{\beta-\alpha_1\beta-1}\left|\frac{1}{\Gamma(1-\gamma_1)}\left\{\frac{d}{ds}\int_a^s (s-\tau)^{1-\gamma_1-1} D_{a^+}^{\alpha_2,\beta} x_1(\tau) d\tau\right.\right.\right. \\
&\quad \left.\left.\left. - \frac{d}{ds}\int_a^s (s-\tau)^{1-\gamma_1-1} D_{a^+}^{\alpha_2,\beta} x_2(\tau) d\tau\right\}\right| ds\right] \\
&= \frac{1}{\Gamma(\beta-\alpha_1\beta)\Gamma(1-\gamma_1)}\left[\int_a^t (t-s)^{\beta-\alpha_1\beta-1}\left|\frac{d}{ds}\int_a^s (s-\tau)^{-\gamma_1}\left(D_{a^+}^{\alpha_2,\beta} x_1(\tau) - D_{a^+}^{\alpha_2,\beta} x_2(\tau)\right) d\tau\right| ds\right] \\
&= \frac{1}{\Gamma(\beta-\alpha_1\beta)\Gamma(1-\gamma_1)}\left[\int_a^t (t-s)^{\beta-\alpha_1\beta-1}\left\{\frac{d}{ds}\left|\int_a^s (s-\tau)^{-\gamma_1}\left(D_{a^+}^{\alpha_2,\beta} x_1(\tau) - D_{a^+}^{\alpha_2,\beta} x_2(\tau)\right) d\tau\right|\right\} ds\right] \\
&\leq \frac{1}{\Gamma(\beta-\alpha_1\beta)\Gamma(1-\gamma_1)}\left[\int_a^t (t-s)^{\beta-\alpha_1\beta-1}\left\{\frac{d}{ds}\int_a^s (s-\tau)^{-\gamma_1}\left|D_{a^+}^{\alpha_2,\beta} x_1(\tau) - D_{a^+}^{\alpha_2,\beta} x_2(\tau)\right| d\tau\right\} ds\right] \\
&\leq \frac{\|D_{a^+}^{\alpha_2,\beta} x_1 - D_{a^+}^{\alpha_2,\beta} x_2\|}{\Gamma(\beta-\alpha_1\beta)\Gamma(1-\gamma_1)}\int_a^t (t-s)^{\beta-\alpha_1\beta-1}\left[\frac{d}{ds}\int_a^s (s-\tau)^{-\gamma_1} d\tau\right] ds
\end{aligned}$$

$$= \frac{\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\|}{\Gamma(\beta-\alpha_1\beta)\Gamma(1-\gamma_1)} \int_a^t (t-s)^{\beta-\alpha_1\beta-1}\left[\frac{d}{ds}\left(\frac{(s-a)^{1-\gamma_1}}{1-\gamma_1}\right)\right]ds$$

$$= \frac{\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\|}{\Gamma(\beta-\alpha_1\beta)\Gamma(1-\gamma_1)} \int_a^t (t-s)^{\beta-\alpha_1\beta-1}(s-a)^{-\gamma_1}ds$$

$$= \frac{\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\|}{\Gamma(1-\gamma_1)} I_{a^+}^{\beta-\alpha_1\beta}(t-a)^{-\gamma_1}$$

$$= \frac{\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\|}{\Gamma(1-\gamma_1)} I_{a^+}^{\beta-\alpha_1\beta}(t-a)^{(1-\gamma_1)-1}$$

$$= \frac{\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\|}{\Gamma(1-\gamma_1)}\left[\frac{\Gamma(1-\gamma_1)}{\Gamma[(\beta-\alpha_1\beta)+(1-\gamma_1)]}(t-a)^{(\beta-\alpha_1\beta)+(1-\gamma_1)-1}\right]$$

$$= \frac{(t-a)^{\beta-\alpha_1\beta-\gamma_1}}{\Gamma(\beta-\alpha_1\beta-\gamma_1+1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|$$

$$= \frac{(t-a)^{\alpha_1+\beta-\alpha_1\beta-\gamma_1-\alpha_1}}{\Gamma(\alpha_1+\beta-\alpha_1\beta-\gamma_1+1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|$$

$$= \frac{(t-a)^{-\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|$$

$$\le \frac{(t-a)^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|$$

$$\le \frac{\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|.$$

Thus

$$\left|D_{a^+}^{\alpha_1,\beta}D_{a^+}^{\alpha_2,\beta}x_1(t) - D_{a^+}^{\alpha_1,\beta}D_{a^+}^{\alpha_2,\beta}x_2(t)\right| \le \frac{\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|. \tag{2.33}$$

By (A_6) and (2.33), for $i = 1,2$ we obtain

$$\left|\hat{f}_{x_1}(t) - \hat{f}_{x_2}(t)\right|$$

$$= \left|f(t, x_1(t), D_{a^+}^{\alpha_i,\beta}x_1(t), D_{a^+}^{\alpha_1,\beta}D_{a^+}^{\alpha_2,\beta}x_1(t)) - f(t, x_2(t), D_{a^+}^{\alpha_i,\beta}x_2(t), D_{a^+}^{\alpha_1,\beta}D_{a^+}^{\alpha_2,\beta}x_2(t))\right|$$

$$\le K\left|x_1(t) - x_2(t)\right| + L\left|D_{a^+}^{\alpha_i,\beta}x_1(t) - D_{a^+}^{\alpha_i,\beta}x_2(t)\right| + M\left|D_{a^+}^{\alpha_1,\beta}D_{a^+}^{\alpha_2,\beta}x_1(t) - D_{a^+}^{\alpha_1,\beta}D_{a^+}^{\alpha_2,\beta}x_2(t)\right|$$

$$\le K\left\|x_1 - x_2\right\| + L\left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|. \tag{2.34}$$

From (2.15), (2.16), (2.17), (2.18), and (2.34), we have

$$\left|\Delta_1(\hat{f}_{x_1}) - \Delta_1(\hat{f}_{x_2})\right|$$

$$\leq \left(K\left\|x_1 - x_2\right\| + L\left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right)$$

$$\times\left[\frac{\rho_1\eta^{\gamma_1+\alpha_2-1}}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1 - s)^{\alpha_2-1}ds + \frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_1+\alpha_2)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\int_a^b(b-s)^{\alpha_1+\alpha_2-1}ds\right]$$

$$= \left(K\left\|x_1 - x_2\right\| + L\left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right)$$

$$\times\left[\frac{\rho_1\eta^{\gamma_1+\alpha_2-1}}{\Gamma(\alpha_2)}\left(\frac{(\theta_1-a)^{\alpha_2}}{\alpha_2}\right) + \frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_1+\alpha_2)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\left(\frac{(b-a)^{\alpha_1+\alpha_2}}{\alpha_1+\alpha_2}\right)\right]$$

$$= \left(K\left\|x_1 - x_2\right\| + L\left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right)$$

$$\times\left[\frac{\rho_1\eta^{\gamma_1+\alpha_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\right], \tag{2.35}$$

$$\left|\Delta_2(\hat{f}_{x_1}) - \Delta_2(\hat{f}_{x_2})\right|$$

$$\leq \left(K\left\|x_1 - x_2\right\| + L\left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right)$$

$$\times\left[\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2)}\int_a^b(b-s)^{\alpha_1+\alpha_2-1}ds\right]$$

$$= \left(K\left\|x_1 - x_2\right\| + L\left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right)$$

$$\times\left[\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2)}\left(\frac{(b-a)^{\alpha_1+\alpha_2}}{\alpha_1+\alpha_2}\right)\right]$$

$$= \left(K\left\|x_1 - x_2\right\| + L\left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right)$$

$$\times\left[\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2+1)}\right], \tag{2.36}$$

$$
\begin{aligned}
&\left|\mathcal{A}_1(\hat{f}_{x_1}) - \mathcal{A}_1(\hat{f}_{x_2})\right| \\
&\leq \left(K\left\|x_1 - x_2\right\| + L\left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right) \\
&\quad \times\left[\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\alpha_2)\Gamma(\gamma_1)}\int_a^{\theta_1}(\theta_1 - s)^{\alpha_2-1}ds + \frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\right. \\
&\quad \left.\times\int_a^{\theta_2}(\theta_2 - s)^{\alpha_1-1}ds\right] \\
&= \left(K\left\|x_1 - x_2\right\| + L\left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right) \\
&\quad \times\left[\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}}{\Gamma(\alpha_2)\Gamma(\gamma_1)}\left(\frac{(\theta_1-a)^{\alpha_2}}{\alpha_2}\right) + \frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\left(\frac{(\theta_2-a)^{\alpha_1}}{\alpha_1}\right)\right] \\
&= \left(K\left\|x_1 - x_2\right\| + L\left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right) \\
&\quad \times\left[\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)\Gamma(\gamma_1)} + \frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\right] \qquad (2.37)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\mathcal{A}_2(\hat{f}_{x_1}) - \mathcal{A}_2(\hat{f}_{x_2})\right| \\
&\leq \left(K\left\|x_1 - x_2\right\| + L\left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right) \\
&\quad\times\left[\frac{\rho_1\rho_2\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\alpha_1)\Gamma(\gamma_2-\alpha_1)}\int_a^{\theta_2}(\theta_2 - s)^{\alpha_1-1}ds + \frac{\rho_1\eta^{\gamma_2-1}}{\Gamma(\alpha_2)}\int_a^{\theta_1}(\theta_1 - s)^{\alpha_2-1}ds\right] \\
&= \left(K\left\|x_1 - x_2\right\| + L\left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right) \\
&\quad\times\left[\frac{\rho_1\rho_2\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}}{\Gamma(\alpha_1)\Gamma(\gamma_2-\alpha_1)}\left(\frac{(\theta_2 - a)^{\alpha_1}}{\alpha_1}\right) + \frac{\rho_1\eta^{\gamma_2-1}}{\Gamma(\alpha_2)}\left(\frac{(\theta_1 - a)^{\alpha_2}}{\alpha_2}\right)\right] \\
&= \left(K\left\|x_1 - x_2\right\| + L\left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right) \\
&\quad\times\left[\frac{\rho_1\rho_2\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\gamma_2-\alpha_1)} + \frac{\rho_1\eta^{\gamma_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)}\right]. \qquad (2.38)
\end{aligned}
$$

From (2.35), (2.36), (2.37) and (2.38), for any $x_1, x_2 \in E$ we have

$$
\begin{aligned}
&\Big|\mathcal{G}x_1(t)-\mathcal{G}x_2(t)\Big| \\
&\le \frac{1}{|\mathcal{A}|}\Bigg[\bigg(\Big(\Big|\Delta_1(\hat{f}_{x_1})-\Delta_1(\hat{f}_{x_2})\Big|+\Big|\mathcal{A}_1(\hat{f}_{x_1})-\mathcal{A}_1(\hat{f}_{x_2})\Big|\Big)\eta^{\gamma_2-1}+\Big(\Big|\Delta_2(\hat{f}_{x_1})-\Delta_2(\hat{f}_{x_2})\Big| \\
&\quad +\Big|\mathcal{A}_2(\hat{f}_{x_1})-\mathcal{A}_2(\hat{f}_{x_2})\Big|\Big)\eta^{\gamma_1+\alpha_2-1}\bigg]+\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t (t-s)^{\alpha_1+\alpha_2-1}\Big|\hat{f}_{x_1}(s)-\hat{f}_{x_2}(s)\Big|ds \\
&\le \left(K\Big\|x_1-x_2\Big\|+L\Big\|D_{a^+}^{\alpha_1,\beta}x_1-D_{a^+}^{\alpha_1,\beta}x_2\Big\|+\frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\Big\|D_{a^+}^{\alpha_2,\beta}x_1-D_{a^+}^{\alpha_2,\beta}x_2\Big\|\right) \\
&\quad\times\Bigg[\frac{1}{|\mathcal{A}|}\Bigg[\Bigg\{\left(\frac{\rho_1\eta^{\gamma_1+\alpha_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)}+\frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\right) \\
&\quad+\left(\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)\Gamma(\gamma_1)}+\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\right)\Bigg\}\eta^{\gamma_2-1} \\
&\quad+\Bigg\{\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2+1)}+\left(\frac{\rho_1\rho_2\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\gamma_2-\alpha_1)}+\frac{\rho_1\eta^{\gamma_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)}\right)\Bigg\}\eta^{\gamma_1+\alpha_2-1}\Bigg] \\
&\quad+\frac{1}{\Gamma(\alpha_1+\alpha_2)}\left(\frac{(t-a)^{\alpha_1+\alpha_2}}{\alpha_1+\alpha_2}\right)\Bigg] \\
&\le \left(K\Big\|x_1-x_2\Big\|+L\Big\|D_{a^+}^{\alpha_1,\beta}x_1-D_{a^+}^{\alpha_1,\beta}x_2\Big\|+\frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\Big\|D_{a^+}^{\alpha_2,\beta}x_1-D_{a^+}^{\alpha_2,\beta}x_2\Big\|\right) \\
&\quad\times\Bigg[\frac{1}{|\mathcal{A}|}\Bigg\{\Bigg(\frac{\rho_1\eta^{\gamma_1+\alpha_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)}+\frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}+\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)\Gamma(\gamma_1)} \\
&\quad+\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\Bigg)\eta^{\gamma_2-1}+\Bigg(\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2+1)}+\frac{\rho_1\eta^{\gamma_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)} \\
&\quad+\frac{\rho_1\rho_2\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\gamma_2-\alpha_1)}\Bigg)\eta^{\gamma_1+\alpha_2-1}\Bigg\}+\frac{\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}\Bigg] \\
&= \sigma_0\left(K\Big\|x_1-x_2\Big\|+L\Big\|D_{a^+}^{\alpha_1,\beta}x_1-D_{a^+}^{\alpha_1,\beta}x_2\Big\|+\frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\Big\|D_{a^+}^{\alpha_2,\beta}x_1-D_{a^+}^{\alpha_2,\beta}x_2\Big\|\right), \qquad (2.39)
\end{aligned}
$$

$$\left|D_{a^+}^{\alpha_1,\beta}\mathcal{G}x_1(t)-D_{a^+}^{\alpha_1,\beta}\mathcal{G}x_2(t)\right|$$

$$\leq \frac{1}{|\mathcal{A}|}\left[\frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2-\alpha_1)}\left\{\left|\Delta_1(\hat{f}_{x_1})-\Delta_1(\hat{f}_{x_2})\right|+\left|\mathcal{A}_1(\hat{f}_{x_1})-\mathcal{A}_1(\hat{f}_{x_2})\right|\right\}\eta^{\gamma_2-\alpha_1-1}\right.$$

$$\left.+\frac{\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1+\alpha_2-\alpha_1)}\left\{\left|\Delta_2(\hat{f}_{x_1})-\Delta_2(\hat{f}_{x_2})\right|+\left|\mathcal{A}_2(\hat{f}_{x_1})-\mathcal{A}_2(\hat{f}_{x_2})\right|\right\}\eta^{\alpha_2-\alpha_1+\gamma_1-1}\right]$$

$$+\frac{1}{\Gamma(\alpha_2)}\int_a^t(t-s)^{\alpha_2-1}\left|\hat{f}_{x_1}(s)-\hat{f}_{x_2}(s)\right|ds$$

$$\leq \left(K\left\|x_1-x_2\right\|+L\left\|D_{a^+}^{\alpha_i,\beta}x_1-D_{a^+}^{\alpha_i,\beta}x_2\right\|+\frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1-D_{a^+}^{\alpha_2,\beta}x_2\right\|\right)$$

$$\times\left[\frac{1}{|\mathcal{A}|}\left[\frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2-\alpha_1)}\left\{\left(\frac{\rho_1\eta^{\gamma_1+\alpha_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)}+\frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\right)\right.\right.\right.$$

$$\left.+\left(\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)\Gamma(\gamma_1)}+\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\right)\right\}\eta^{\gamma_2-\alpha_1-1}$$

$$+\frac{\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1+\alpha_2-\alpha_1)}\left\{\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2+1)}+\left(\frac{\rho_1\rho_2\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\gamma_2-\alpha_1)}\right.\right.$$

$$\left.\left.\left.+\frac{\rho_1\eta^{\gamma_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)}\right)\right\}\eta^{\alpha_2-\alpha_1+\gamma_1-1}\right]+\frac{1}{\Gamma(\alpha_2)}\left(\frac{(t-a)^{\alpha_2-1}}{\alpha_2}\right)\right]$$

$$\leq \left(K\left\|x_1-x_2\right\|+L\left\|D_{a^+}^{\alpha_i,\beta}x_1-D_{a^+}^{\alpha_i,\beta}x_2\right\|+\frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1-D_{a^+}^{\alpha_2,\beta}x_2\right\|\right)$$

$$\times\left[\frac{1}{|\mathcal{A}|}\left[\left\{\frac{\rho_1\eta^{\gamma_1+\alpha_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)}+\frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}+\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)\Gamma(\gamma_1)}\right.\right.\right.$$

$$\left.+\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\right\}\frac{\Gamma(\gamma_2)\eta^{\gamma_2-\alpha_1-1}}{\Gamma(\gamma_2-\alpha_1)}+\left\{\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2+1)}\right.$$

$$\left.\left.+\frac{\rho_1\rho_2\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\gamma_2-\alpha_1)}+\frac{\rho_1\eta^{\gamma_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)}\right\}\frac{\Gamma(\gamma_1+\alpha_2)\eta^{\gamma_1+\alpha_2-\alpha_1-1}}{\Gamma(\gamma_1+\alpha_2-\alpha_1)}\right]+\frac{\eta^{\alpha_2}}{\Gamma(\alpha_2+1)}\right]$$

$$= \sigma_1\left(K\left\|x_1-x_2\right\|+L\left\|D_{a^+}^{\alpha_i,\beta}x_1-D_{a^+}^{\alpha_i,\beta}x_2\right\|+\frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1-D_{a^+}^{\alpha_2,\beta}x_2\right\|\right) \qquad (2.40)$$

and

$$\left|D_{a^+}^{\alpha_2,\beta}\mathcal{G}x_1(t)-D_{a^+}^{\alpha_2,\beta}\mathcal{G}x_2(t)\right|$$

$$\leq \frac{1}{|\mathcal{A}|}\left[\frac{\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1)}\left(\left|\Delta_2(\hat{f}_{x_1})-\Delta_2(\hat{f}_{x_2})\right|+\left|\mathcal{A}_2(\hat{f}_{x_1})-\mathcal{A}_2(\hat{f}_{x_2})\right|\right)\eta^{\gamma_1-1}\right]+\frac{1}{\Gamma(\alpha_1)}$$

$$\times\int_a^t(t-s)^{\alpha_1-1}\left|\hat{f}_{x_1}(s)-\hat{f}_{x_2}(s)\right|ds$$

$$\leq \left(K\left\|x_1-x_2\right\|+L\left\|D_{a^+}^{\alpha_i,\beta}x_1-D_{a^+}^{\alpha_i,\beta}x_2\right\|+\frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1-D_{a^+}^{\alpha_2,\beta}x_2\right\|\right)$$

$$\times\left[\frac{1}{|\mathcal{A}|}\left[\frac{\Gamma(\gamma_1+\alpha_2)}{\Gamma(\gamma_1)}\left\{\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2+1)}+\left(\frac{\rho_1\rho_2\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\gamma_2-\alpha_1)}\right.\right.\right.\right.$$

$$\left.\left.\left.\left.+\frac{\rho_1\eta^{\gamma_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)}\right)\right\}\eta^{\gamma_1-1}\right]+\frac{1}{\Gamma(\alpha_1)}\left(\frac{(t-a)^{\alpha_1}}{\alpha_1}\right)\right]$$

$$\leq \left(K\left\|x_1-x_2\right\|+L\left\|D_{a^+}^{\alpha_i,\beta}x_1-D_{a^+}^{\alpha_i,\beta}x_2\right\|+\frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1-D_{a^+}^{\alpha_2,\beta}x_2\right\|\right)$$

$$\times\left[\frac{1}{|\mathcal{A}|}\left[\left\{\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2+1)}+\frac{\rho_1\rho_2\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\gamma_2-\alpha_1)}+\frac{\rho_1\eta^{\gamma_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)}\right\}\right.\right.$$

$$\left.\left.\times\frac{\Gamma(\gamma_1+\alpha_2)\eta^{\gamma_1-1}}{\Gamma(\gamma_1)}\right]+\frac{\eta^{\alpha_1}}{\Gamma(\alpha_1+1)}\right]$$

$$=\sigma_2\left(K\left\|x_1-x_2\right\|+L\left\|D_{a^+}^{\alpha_i,\beta}x_1-D_{a^+}^{\alpha_i,\beta}x_2\right\|+\frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1-D_{a^+}^{\alpha_2,\beta}x_2\right\|\right). \quad (2.41)$$

From (2.39), (2.40) and (2.41), we see that

$$
\begin{aligned}
&\left\|\mathcal{G}x_1-\mathcal{G}x_2\right\|_E \\
&= \left\|\mathcal{G}x_1-\mathcal{G}x_2\right\| + \left\|D_{a^+}^{\alpha_i,\beta}\mathcal{G}x_1 - D_{a^+}^{\alpha_i,\beta}\mathcal{G}x_2\right\| + \left\|D_{a^+}^{\alpha_2,\beta}\mathcal{G}x_1 - D_{a^+}^{\alpha_1,\beta}\mathcal{G}x_2\right\| \\
&= \max_{t\in J}\left|\mathcal{G}x_1(t)-\mathcal{G}x_2(t)\right| + \max_{t\in J}\left|D_{a^+}^{\alpha_i,\beta}\mathcal{G}x_1(t) - D_{a^+}^{\alpha_i,\beta}\mathcal{G}x_2(t)\right| + \max_{t\in J}\left|D_{a^+}^{\alpha_2,\beta}\mathcal{G}x_1(t) - D_{a^+}^{\alpha_2,\beta}\mathcal{G}x_2(t)\right| \\
&\leq \left(\sigma_0+\sigma_1+\sigma_2\right)\left[K\left\|x_1-x_2\right\| + L\left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right] \\
&= \sigma\left[K\left\|x_1-x_2\right\| + L\left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right].
\end{aligned}
$$

Case i=1 :

$$
\begin{aligned}
&\left\|\mathcal{G}x_1-\mathcal{G}x_2\right\|_E \\
&\leq \sigma\left[K\left\|x_1-x_2\right\| + L\left\|D_{a^+}^{\alpha_1,\beta}x_1 - D_{a^+}^{\alpha_1,\beta}x_2\right\| + \frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right] \\
&\leq \sigma\max\left\{K, L, \frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\right\}\left[\left\|x_1-x_2\right\| + \left\|D_{a^+}^{\alpha_1,\beta}x_1 - D_{a^+}^{\alpha_1,\beta}x_2\right\| + \left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right] \\
&= \sigma\Lambda_1\left\|x_1-x_2\right\|_E.
\end{aligned}
$$

Case i=2:

$$
\begin{aligned}
&\left\|\mathcal{G}x_1-\mathcal{G}x_2\right\|_E \\
&\leq \sigma\left[K\left\|x_1-x_2\right\| + L\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\| + \frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right] \\
&\leq \sigma\left[K\left\|x_1-x_2\right\| + \left\|D_{a^+}^{\alpha_1,\beta}x_1 - D_{a^+}^{\alpha_1,\beta}x_2\right\| + \left(L+\frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\right)\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right] \\
&\leq \sigma\max\left\{1, K, L+\frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\right\}\left[\left\|x_1-x_2\right\| + \left\|D_{a^+}^{\alpha_1,\beta}x_1 - D_{a^+}^{\alpha_1,\beta}x_2\right\| + \left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right] \\
&= \sigma\Lambda_2\left\|x_1-x_2\right\|_E.
\end{aligned}
$$

From case i=1 and case i=2, we get that

$$\left\|\mathcal{G}x_1-\mathcal{G}x_2\right\|_E \leq \mathcal{P}\left\|x_1-x_2\right\|_E.$$

Since $\mathcal{P} < 1$, so that $\mathcal{G}$ is a contraction on E. By Banach's contraction principle, we have $\mathcal{G}$ has a unique fixed pointn in E. Therefore, BVP (1.1) and (1.2) has a unique solution on J. □

2.4 Existence result via Boyd-Wong fixed point theorem

Theorem 2.5. Assume that (A_1) and (A_4) hold. If $\Omega := \sigma\phi(b)\Theta_i \leq 1$ for $i = 1,2$ where

$$\Theta_1 := \max\left\{1, \frac{\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\right\}, \tag{2.42}$$

$$\Theta_2 := 1 + \frac{\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}. \tag{2.43}$$

Then the BVP (1.1) and (1.2) has a unique solution on J.

Proof. We will use Boyd-Wong fixed point theorem to prove that $\mathcal{G}$, defined as (2.13) has a unique fixed point. We must show that $\mathcal{G}$ is a nonlinear contraction on E. Define an increasing continuous function $\psi_\omega : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ for $\omega \in (0,1)$ by $\psi_\omega(\varepsilon) = \omega\varepsilon$ for all $\varepsilon \geq 0$. Let $x_1, x_2 \in E$, for each $t \in J$. By (A_4) and (2.33), for $i = 1,2$ we consider

$$\begin{aligned}
&\left|\hat{f}_{x_1}(t) - \hat{f}_{x_2}(t)\right| \\
&= \left|f(t, x_1(t), D_{a^+}^{\alpha_i,\beta}x_1(t), D_{a^+}^{\alpha_1,\beta}D_{a^+}^{\alpha_2,\beta}x_1(t)) - f(t, x_2(t), D_{a^+}^{\alpha_i,\beta}x_2(t), D_{a^+}^{\alpha_1,\beta}D_{a^+}^{\alpha_2,\beta}x_2(t))\right| \\
&\leq \omega\phi(t)\left[\left|x_1(t) - x_2(t)\right| + \left|D_{a^+}^{\alpha_i,\beta}x_1(t) - D_{a^+}^{\alpha_i,\beta}x_2(t)\right| + \left|D_{a^+}^{\alpha_1,\beta}D_{a^+}^{\alpha_2,\beta}x_1(t) - D_{a^+}^{\alpha_1,\beta}D_{a^+}^{\alpha_2,\beta}x_2(t)\right|\right] \\
&\leq \omega\phi(t)\left[\left\|x_1 - x_2\right\| + \left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right].
\end{aligned} \tag{2.44}$$

Since $\phi \in C(J, \mathbb{R}_+)$ is increasing and by (2.15), (2.16), (2.17), (2.18) and (2.44), we have

$$\left|\Delta_1(\hat{f}_{x_1}) - \Delta_1(\hat{f}_{x_2})\right|$$
$$\leq \omega\phi(b)\left[\left\|x_1 - x_2\right\| + \left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right]$$
$$\times\left(\frac{\rho_1\eta^{\gamma_1+\alpha_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{\rho_1\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\right), \tag{2.45}$$

$$\left|\Delta_2(\hat{f}_{x_1}) - \Delta_2(\hat{f}_{x_2})\right|$$
$$\leq \omega\phi(b)\left[\left\|x_1 - x_2\right\| + \left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right]$$
$$\times\left(\frac{\rho_1\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\gamma_2-\alpha_1)\Gamma(\alpha_1+\alpha_2+1)}\right), \tag{2.46}$$

$$\left|\mathcal{A}_1(\hat{f}_{x_1}) - \mathcal{A}_1(\hat{f}_{x_2})\right|$$
$$\leq \omega\phi(b)\left[\left\|x_1 - x_2\right\| + \left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right]$$
$$\times\left(\frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_2^{\gamma_1-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)\Gamma(\gamma_1)} + \frac{\rho_1\rho_2\Gamma(\gamma_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1+\gamma_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2-\alpha_1+\gamma_1)}\right) \tag{2.47}$$

and

$$\left|\mathcal{A}_2(\hat{f}_{x_1}) - \mathcal{A}_2(\hat{f}_{x_2})\right|$$
$$\leq \omega\phi(b)\left[\left\|x_1 - x_2\right\| + \left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right]$$
$$\times\left(\frac{\rho_1\rho_2\Gamma(\gamma_2)\xi_1^{\gamma_2-\alpha_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\gamma_2-\alpha_1)} + \frac{\rho_1\eta^{\gamma_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)}\right). \tag{2.48}$$

From (2.45), (2.46), (2.47) and (2.48), for any $x_1, x_2 \in E$ we have

$$\left\|\mathcal{G}x_1 - \mathcal{G}x_2\right\|$$
$$= \max_{t\in J}\left|\mathcal{G}x_1(t) - \mathcal{G}x_2(t)\right|$$
$$\leq \sigma_0\omega\phi(b)\left[\left\|x_1 - x_2\right\| + \left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right],$$

$$\left\|D_{a^+}^{\alpha_1,\beta}\mathcal{G}x_1 - D_{a^+}^{\alpha_1,\beta}\mathcal{G}x_2\right\|$$
$$= \max_{t\in J}\left|D_{a^+}^{\alpha_1,\beta}\mathcal{G}x_1(t) - D_{a^+}^{\alpha_1,\beta}\mathcal{G}x_2(t)\right|$$
$$\leq \sigma_1\omega\phi(b)\left[\left\|x_1 - x_2\right\| + \left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right],$$

and

$$\left\|D_{a^+}^{\alpha_2,\beta}\mathcal{G}x_1 - D_{a^+}^{\alpha_2,\beta}\mathcal{G}x_2\right\|$$
$$= \max_{t\in J}\left|D_{a^+}^{\alpha_2,\beta}\mathcal{G}x_1(t) - D_{a^+}^{\alpha_2,\beta}\mathcal{G}x_2(t)\right|$$
$$\leq \sigma_2\omega\phi(b)\left[\left\|x_1 - x_2\right\| + \left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right].$$

We see that

$$\left\|\mathcal{G}x_1 - \mathcal{G}x_2\right\|_E$$
$$= \left\|\mathcal{G}x_1 - \mathcal{G}x_2\right\| + \left\|D_{a^+}^{\alpha_i,\beta}\mathcal{G}x_1 - D_{a^+}^{\alpha_i,\beta}\mathcal{G}x_2\right\| + \left\|D_{a^+}^{\alpha_2,\beta}\mathcal{G}x_1 - D_{a^+}^{\alpha_1,\beta}\mathcal{G}x_2\right\|$$
$$\leq \left(\sigma_0 + \sigma_1 + \sigma_2\right)\omega\phi(b)\left[\left\|x_1 - x_2\right\| + \left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right]$$
$$= \sigma\omega\phi(b)\left[\left\|x_1 - x_2\right\| + \left\|D_{a^+}^{\alpha_i,\beta}x_1 - D_{a^+}^{\alpha_i,\beta}x_2\right\| + \frac{\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right].$$

Case i=1 :

$$\left\|\mathcal{G}x_1 - \mathcal{G}x_2\right\|_E$$
$$\leq \sigma\omega\phi(b)\left[\left\|x_1 - x_2\right\| + \left\|D_{a^+}^{\alpha_1,\beta}x_1 - D_{a^+}^{\alpha_1,\beta}x_2\right\| + \frac{\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right]$$
$$\leq \sigma\omega\phi(b)\max\left\{1, \frac{\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\right\}\left[\left\|x_1 - x_2\right\| + \left\|D_{a^+}^{\alpha_1,\beta}x_1 - D_{a^+}^{\alpha_1,\beta}x_2\right\| + \left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right]$$
$$= \sigma\omega\phi(b)\Theta_1\left\|x_1 - x_2\right\|_E.$$

Case i=2:

$$
\begin{aligned}
&\left\|\mathcal{G}x_1 - \mathcal{G}x_2\right\|_E \\
&\leq \sigma\omega\phi(b)\left[\left\|x_1 - x_2\right\| + \left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\| + \frac{\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right] \\
&\leq \sigma\omega\phi(b)\left[\left\|x_1 - x_2\right\| + \left\|D_{a^+}^{\alpha_1,\beta}x_1 - D_{a^+}^{\alpha_1,\beta}x_2\right\| + \left(1 + \frac{\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\right)\left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right] \\
&\leq \sigma\omega\phi(b)\left(1 + \frac{\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\right)\left[\left\|x_1 - x_2\right\| + \left\|D_{a^+}^{\alpha_1,\beta}x_1 - D_{a^+}^{\alpha_1,\beta}x_2\right\| + \left\|D_{a^+}^{\alpha_2,\beta}x_1 - D_{a^+}^{\alpha_2,\beta}x_2\right\|\right] \\
&= \sigma\omega\phi(b)\Theta_2\left\|x_1 - x_2\right\|_E.
\end{aligned}
$$

Since $\Omega \leq 1$, by case i=1 and case i=2 so that

$$
\begin{aligned}
\left\|\mathcal{G}x_1 - \mathcal{G}x_2\right\|_E &\leq \sigma\omega\phi(b)\Theta_i\left\|x_1 - x_2\right\|_E \\
&= \omega\Omega\left\|x_1 - x_2\right\|_E \\
&\leq \omega\left\|x_1 - x_2\right\|_E = \psi_\omega\left(\|x_1 - x_2\|_E\right).
\end{aligned}
$$

Hence $\mathcal{G}$ is a nonlinear contraction on E, by Boyd-Wong fixed point theorem, we have $\mathcal{G}$ has a unique fixed point in E. Therefore, BVP (1.1) and (1.2) has a unique solution on J.

□

2.5 Special cases for the boundary value problem

In this section, we recommend for some special cases for a boundary value problem of nonlinear implicit Hilfer fractional differential equations (1.1) with Hilfer fractional differential boundary conditions (1.2).

2.5.1 Boundary value problem of nonlinear implicit Riemann-Liouville fractional differential equations (Case $\beta = 0$)

By generalization of Hilfer fractional derivatives, for $\beta = 0$ we obtain that boundary value problem of nonlinear implicit Riemann-Liouville fractional differential equations with Riemann-Liouville fractional differential boundary conditions as the following:

$$D_{a^+}^{\alpha_1} D_{a^+}^{\alpha_2} x(t) = f(t, x(t), D_{a^+}^{\alpha_i} x(t), D_{a^+}^{\alpha_1} D_{a^+}^{\alpha_2} x(t)), \quad t \in (a, b] \tag{2.49}$$

$$x(a) = \rho_1 D_{a^+}^{\alpha_1} x(\theta_1), \quad x(b) = \rho_2 D_{a^+}^{\alpha_2} x(\theta_2) \tag{2.50}$$

where $\alpha_1 < \alpha_2$. Therefore, we have the following existence and stability results. We will set positive constants $\sigma = \sigma_0 + \sigma_1 + \sigma_2$ where

$$\mathcal{A} = \frac{\rho_1\rho_2\Gamma(\alpha_2)\Gamma(\alpha_1+\alpha_2)\xi_1^{\alpha_2-\alpha_1-1}\xi_2^{\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)\Gamma(\alpha_1)} - \frac{\rho_1\Gamma(\alpha_2)\xi_1^{\alpha_2-\alpha_1-1}\eta^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_2-\alpha_1)} + \frac{\rho_1\Gamma(\alpha_1+\alpha_2)\xi_1^{\alpha_2-1}\eta^{\alpha_2-1}}{\Gamma(\alpha_2)},$$

$$\sigma_0 = \frac{1}{|\mathcal{A}|}\Bigg[\Bigg(\frac{\rho_1\eta^{\alpha_1+\alpha_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{\rho_1\Gamma(\alpha_1+\alpha_2)\xi_1^{\alpha_2-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)\Gamma(\alpha_2)} + \frac{\rho_1\rho_2\Gamma(\alpha_1+\alpha_2)\xi_2^{\alpha_1-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)\Gamma(\alpha_1)}$$
$$+\frac{\rho_1\rho_2\Gamma(\alpha_1+\alpha_2)\xi_1^{\alpha_2-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2)}\Bigg)\eta^{\alpha_2-1} + \Bigg(\frac{\rho_1\Gamma(\alpha_2)\xi_1^{\alpha_2-\alpha_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_2-\alpha_1)\Gamma(\alpha_1+\alpha_2+1)} + \frac{\rho_1\rho_2\Gamma(\alpha_2)\xi_1^{\alpha_2-\alpha_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2-\alpha_1)}$$
$$+\frac{\rho_1\eta^{\alpha_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)}\Bigg)\eta^{\alpha_1+\alpha_2-1}\Bigg] + \frac{\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)},$$

$$\sigma_1 = \frac{1}{|\mathcal{A}|}\Bigg[\Bigg(\frac{\rho_1\eta^{\alpha_1+\alpha_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{\rho_1\Gamma(\alpha_1+\alpha_2)\xi_1^{\alpha_2-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)\Gamma(\alpha_2)} + \frac{\rho_1\rho_2\Gamma(\alpha_1+\alpha_2)\xi_1^{\alpha_2-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2)}$$
$$+\frac{\rho_1\rho_2\Gamma(\alpha_1+\alpha_2)\xi_2^{\alpha_1-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)\Gamma(\alpha_1)}\Bigg)\frac{\Gamma(\alpha_2)\eta^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)} + \Bigg(\frac{\rho_1\Gamma(\alpha_2)\xi_1^{\alpha_2-\alpha_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_2-\alpha_1)\Gamma(\alpha_1+\alpha_2+1)} + \frac{\rho_1\eta^{\alpha_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)}$$
$$+\frac{\rho_1\rho_2\Gamma(\alpha_2)\xi_1^{\alpha_2-\alpha_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2-\alpha_1)}\Bigg)\frac{\Gamma(\alpha_1+\alpha_2)\eta^{+\alpha_2-1}}{\Gamma(\alpha_2)}\Bigg] + \frac{\eta^{\alpha_2}}{\Gamma(\alpha_2+1)},$$

$$\sigma_2 = \frac{1}{|\mathcal{A}|}\Bigg[\Bigg(\frac{\rho_1\Gamma(\alpha_2)\xi_1^{\alpha_2-\alpha_1-1}\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_2-\alpha_1)\Gamma(\alpha_1+\alpha_2+1)} + \frac{\rho_1\rho_2\Gamma(\alpha_2)\xi_1^{\alpha_2-\alpha_1-1}\xi_2^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2-\alpha_1)} + \frac{\rho_1\eta^{\alpha_2-1}\xi_1^{\alpha_2}}{\Gamma(\alpha_2+1)}\Bigg)\frac{\Gamma(\alpha_1+\alpha_2)\eta^{\alpha_1-1}}{\Gamma(\alpha_1)}\Bigg]$$
$$+\frac{\eta^{\alpha_1}}{\Gamma(\alpha_1+1)}.$$

Corollary 2.6. Assume that (A_1) and (A_2) hold. Then the BVP (2.49) and (2.50) has at least one solution on J.

Corollary 2.7. Assume that (A_1) and (A_6) hold. If $\mathcal{P} = \sigma\Lambda_i < 1$ for $i = 1,2$ where Λ_1 and Λ_2 are defined as (2.31) and (2.32) respectively, then the BVP (2.49) and (2.50) has a unique solution on J.

Corollary 2.8. Assume that (A_1) and (A_4) hold. If $\Omega = \sigma\phi(b)\Theta_i \leq 1$ for $i = 1,2$ where Θ_1 and Θ_2 are defined as (2.42) and (2.43) respectively, then the BVP (2.49) and (2.50) has a unique solution on J.

Corollary 2.9. Assume that (A_1) and (A_6) hold. If $\mathcal{P} = \sigma\Lambda_i < 1$ for $i = 1,2$, then the problem (2.49) and (2.50) is Ulam-Hyers stable.

Corollary 2.10. Assume that $(A_1), (A_5), (A_6)$ and (A_7) hold. If $\mathcal{P} = \sigma\Lambda_i < 1$ for $i = 1,2$, then the problem (2.49) and (2.50) is Ulam-Hyers-Rassias stable with respect to φ.

2.5.2 Boundary value problem of nonlinear implicit Caputo fractional differential equations (Case $\beta = 1$)

By generalization of Hilfer fractional derivatives, for $\beta = 1$ we get that boundary value problem of nonlinear implicit Caputo fractional differential equations with Caputo fractional differential boundary conditions as the following:

$$ {}^{c}D_{a^+}^{\alpha_1}\, {}^{c}D_{a^+}^{\alpha_2} x(t) = f(t, x(t), {}^{c}D_{a^+}^{\alpha_i} x(t), {}^{c}D_{a^+}^{\alpha_1}\, {}^{c}D_{a^+}^{\alpha_2} x(t)), \quad t \in (a,b] \tag{2.51} $$

$$ x(a) = \rho_1 {}^{c}D_{a^+}^{\alpha_1} x(\theta_1), \qquad x(b) = \rho_2 {}^{c}D_{a^+}^{\alpha_2} x(\theta_2). \tag{2.52} $$

Therefore, we obtain the following existence and stability results. We will set positive constants $\sigma = \sigma_0 + \sigma_1 + \sigma_2$ with

$$\mathcal{A} = \eta^{\alpha_2} - \rho_2\Gamma(\alpha_2+1) + \frac{\rho_1\Gamma(\alpha_2+1)\xi_1^{\alpha_2-\alpha_1}}{\Gamma(\alpha_2-\alpha_2+1)},$$

$$\sigma_0 = \frac{1}{|\mathcal{A}|}\Bigg[\left(\frac{\rho_1\eta^{\alpha_2}\xi_1^{\alpha_2-\alpha_1}}{\Gamma(\alpha_2+1)} + \frac{\rho_1\Gamma(\alpha_2+1)\eta^{\alpha_1+\alpha_2}\xi_1^{\alpha_2-\alpha_1}}{\Gamma(\alpha_1+\alpha_2+1)\Gamma(\alpha_2-\alpha_1+1)} + \frac{\rho_1\rho_2\Gamma(\alpha_2+1)\xi_1^{\alpha_2-\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2-\alpha_1+1)} + \rho_1\rho_2\xi_1^{\alpha_2-\alpha_1}\right)$$
$$+\left(\frac{\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{\rho_2}{\Gamma(\alpha_1+1)} + \frac{\rho_1\xi_1^{\alpha_2-\alpha_1}}{\Gamma(\alpha_2+1)}\right)\eta^{\alpha_2}\Bigg] + \frac{\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)},$$

$$\sigma_1 = \frac{1}{|\mathcal{A}|}\left(\frac{\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{\rho_2}{\Gamma(\alpha_1+1)} + \frac{\rho_1\xi_1^{\alpha_2-\alpha_1}}{\Gamma(\alpha_2+1)}\right)\frac{\Gamma(\alpha_2+1)\eta^{\alpha_2}}{\Gamma(\alpha_2-\alpha_1+1)} + \frac{\eta^{\alpha_2}}{\Gamma(\alpha_2+1)},$$

$$\sigma_2 = \frac{1}{|\mathcal{A}|}\left(\frac{\eta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{\rho_2}{\Gamma(\alpha_1+1)} + \frac{\rho_1\xi_1^{\alpha_2-\alpha_1}}{\Gamma(\alpha_2+1)}\right)\Gamma(\alpha_2+1)\eta^{\alpha_2} + \frac{\eta^{\alpha_1}}{\Gamma(\alpha_1+1)}.$$

Corollary 2.11. Assume that (A_1) and (A_2) hold. Then the BVP (2.51) and (2.52) has at least one solution on J.

Corollary 2.12. Assume that (A_1) and (A_6) hold. If $\mathcal{P} = \sigma\Lambda < 1$ where $\Lambda := \max\{K, L, M\}$, then the BVP (2.51) and (2.52) has a unique solution on J.

Corollary 2.13. Assume that (A_1) and (A_4) hold. If $\Omega = \sigma\phi(b) \leq 1$, then the BVP (2.51) and (2.52) has a unique solution on J.

Corollary 2.14. Assume that (A_1) and (A_6) hold. If $\mathcal{P} = \sigma\Lambda < 1$, then the problem (2.51) and (2.52) is Ulam-Hyers stable.

Corollary 2.15. Assume that $(A_1), (A_5), (A_6)$ and (A_7) hold. If $\mathcal{P} = \sigma\Lambda < 1$, then the problem (2.51) and (2.52) is Ulam-Hyers-Rassias stable with respect to φ.

2.6 Examples

To validation our result, we give examples for the boundary value problem of nonlinear implicit Hilfer fractional differential equations with Hilfer fractional derivative boundary conditions to illustration as following.

Example 2.1. Consider the following boundary value problem

$$D^{\frac{7}{10},\frac{1}{4}}_{\frac{\pi}{6}^+} D^{\frac{9}{10},\frac{1}{4}}_{\frac{\pi}{6}^+} x(t) = \frac{(1+t^2)\cos x(t)}{\left|D^{\frac{9}{10},\frac{1}{4}}_{\frac{\pi}{6}^+} x(t)\right| + 2} + \frac{3}{e^{6t}\left(\left|D^{\frac{7}{10},\frac{1}{4}}_{\frac{\pi}{6}^+} D^{\frac{9}{10},\frac{1}{4}}_{\frac{\pi}{6}^+} x(t)\right| + 1\right)}, \quad \forall t \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right] \tag{2.53}$$

$$x\left(\frac{\pi}{6}\right) = \ln 3 D^{\frac{7}{10},\frac{1}{4}}_{\frac{\pi}{6}^+} x\left(\frac{\pi}{4}\right), \qquad x\left(\frac{\pi}{4}\right) = \ln 7 D^{\frac{9}{10},\frac{1}{4}}_{\frac{\pi}{6}^+} x\left(\frac{\pi}{3}\right). \tag{2.54}$$

Since the BVP (2.53) and (2.54), we know that

$$\alpha_1 = \frac{7}{10}, \ \alpha_2 = \frac{9}{10}, \ \beta = \frac{1}{4}, \ a = \frac{\pi}{6}, \ b = \frac{\pi}{2}, \ \eta = b - a = \frac{\pi}{3}, \ \gamma_1 = \frac{31}{40}, \ \gamma_2 = \frac{37}{40},$$
$$\rho_1 = \ln 3, \ \ \rho_2 = \ln 7, \ \ \theta_1 = \frac{\pi}{4}, \ \ \theta_2 = \frac{\pi}{3}, \ \ \xi_1 =, \frac{\pi}{12}, \ \ \xi_2 = \frac{\pi}{6}$$

such that $\gamma_1 < \gamma_2$, $\alpha_1 < \gamma_2$ and $\alpha_1 < \alpha_2 + \gamma_1$. We see that (A_1) holds, because

$$f(t,u,v,w) = \frac{(1+t^2)\cos u}{|v| + 2} + \frac{3}{e^{6t}(|w|+1)}$$

is a continuous on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Let $t \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$. Next, we will show that (A_2) holds.

That is to find $D \in \mathbb{R}_{++}$ such that $|f(t,u,v,w)| \le D$. Let $u, v, w \in \mathbb{R}$, consider

$$\begin{aligned}
\left|f(t,u,v,w)\right| &= \left|\frac{(1+t^2)\cos u}{|v|+2} + \frac{3}{e^{6t}(|w|+1)}\right| \\
&\le \frac{(1+t^2)|\cos u|}{|v|+2} + \frac{3}{e^{6t}(|w|+1)} \\
&\le \frac{1}{0+2}\left(1 + \frac{\pi^2}{4}\right) + \frac{3}{e^{\pi}(0+1)} \\
&= \frac{4+\pi^2}{8} + \frac{3}{e^{\pi}}.
\end{aligned}$$

That is there exists

$$D := \frac{4+\pi^2}{8} + \frac{3}{e^{\pi}} \in \mathbb{R}_{++}$$

such that $|f(t,u,v,w)| \leq D$. Hence (A_2) holds. By Theorem 2.3, the BVP (2.53) and (2.54) has at least one solution on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

Example 2.2. Consider the following boundary value problem

$$D_{1^+}^{\frac{2}{3},\frac{1}{7}} D_{1^+}^{\frac{3}{4},\frac{1}{7}} x(t) = \frac{e^{-3t}\left(\left|x(t)\right| + \left|D_{1^+}^{\frac{2}{3},\frac{1}{7}} x(t)\right| + \left|D_{1^+}^{\frac{2}{3},\frac{1}{7}} D_{1^+}^{\frac{3}{4},\frac{1}{7}} x(t)\right|\right)}{(1+2e^{3t})\left(1 + \left|x(t)\right| + \left|D_{1^+}^{\frac{2}{3},\frac{1}{7}} x(t)\right| + \left|D_{1^+}^{\frac{2}{3},\frac{1}{7}} D_{1^+}^{\frac{3}{4},\frac{1}{7}} x(t)\right|\right)}, \ \forall t \in \left(1, \frac{4}{3}\right], \tag{2.55}$$

$$x(1) = \frac{4}{7} D_{1^+}^{\frac{2}{3},\frac{1}{7}} x\left(\frac{10}{9}\right), \qquad x\left(\frac{4}{3}\right) = \frac{8}{7} D_{1^+}^{\frac{3}{4},\frac{1}{7}} x\left(\frac{10}{9}\right). \tag{2.56}$$

Since the BVP (2.55) and (2.56), we know that

$$\alpha_1 = \frac{2}{3}, \ \alpha_2 = \frac{3}{4}, \ \beta = \frac{1}{7}, \ a = 1, \ b = \frac{4}{3}, \ \eta = b - a = \frac{1}{3}, \ \gamma_1 = \frac{15}{21}, \ \gamma_2 = \frac{11}{14},$$

$$\rho_1 = \frac{2}{3}, \ \rho_2 = \frac{8}{7}, \ \theta_1 = \theta_2 = \frac{10}{9}, \ \xi_1 = \xi_2 = \frac{1}{9}$$

such that $\gamma_1 < \gamma_2$, $\alpha_1 < \gamma_2$ and $\alpha_1 < \alpha_2 + \gamma_1$. In this case, consider for $i = 1$. We see that (A_1) holds, because

$$f(t,u,v,w) = \frac{e^{-3t}(|u| + |v| + |w|)}{(1+2e^{3t})(1 + |u| + |v| + |w|)}$$

is a continuous on $[1,3] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Let $t \in [1,3]$. Next, we will show that (A_6) holds. Let $u_1, v_1, w_1, u_2, v_2, w_2 \in \mathbb{R}$, consider

$$\left|f(t,u_1,v_1,w_1)-f(t,u_2,v_2,w_2)\right|$$

$$= \left|\frac{e^{-3t}(|u_1|+|v_1|+|w_1|)}{(1+2e^{3t})(1+|u_1|+|v_1|+|w_1|)} - \frac{e^{-3t}(|u_2|+|v_2|+|w_2|)}{(1+2e^{3t})(1+|u_2|+|v_2|+|w_2|)}\right|$$

$$= \frac{e^{-3t}}{(1+2e^{3t})}\left|\frac{|u_1|+|v_1|+|w_1|}{1+|u_1|+|v_1|+|w_1|} - \frac{|u_2|+|v_2|+|w_2|}{1+|u_2|+|v_2|+|w_2|}\right|$$

$$= \frac{e^{-3t}}{(1+2e^{3t})}\cdot\frac{\left|(|u_1|-|u_2|)+(|v_1|-|v_2|)+(|w_1|-|w_2|)\right|}{(1+|u_1|+|v_1|+|w_1|)(1+|u_2|+|v_2|+|w_2|)}$$

$$\le \frac{e^{-3t}}{(1+2e^{3t})}\left[\left||u_1|-|u_2|\right|+\left||v_1|-|v_2|\right|+\left||w_1|-|w_2|\right|\right]$$

$$\le \frac{e^{-3}}{(1+2e^{3})}\left[\left|u_1-u_2\right|+\left|v_1-v_2\right|+\left|w_1-w_2\right|\right].$$

We have,

$$\left|f(t,u_1,v_1,w_1)-f(t,u_2,v_2,w_2)\right| \le \frac{e^{-3}}{1+2e^3}\left|u_1-u_2\right|+\frac{e^{-3}}{1+2e^3}\left|v_1-v_2\right|+\frac{e^{-3}}{1+2e^3}\left|w_1-w_2\right|.$$

Hence (A_6) holds which $K=L=M=\dfrac{e^{-3}}{1+2e^3}$. Through a simple calculation, such that

$$\sigma_0 \approx 0.9104, \quad \sigma_1 \approx 0.9554, \quad \text{and} \quad \sigma_2 \approx 0.8419,$$

we get,

$$\sigma = \sigma_0+\sigma_1+\sigma_2 \approx 0.9104+0.9554+0.8419 \approx 2.7077.$$

From in case $i=1$, by (2.31) we obtain

$$\Lambda_1 = \max\left\{K, L, \frac{M\eta^{\alpha_1}}{\Gamma(1-\alpha_1)}\right\} \approx 0.0012.$$

We see that

$$\mathcal{P} = \sigma\Lambda_1 \approx (2.7077)(0.0012) \approx 0.0033.$$

Hence $\mathcal{P} < 1$. By Theorem 2.4, the BVP (2.55) and (2.56) has a unique solution on $\left[1, \frac{4}{3}\right]$.

Next, we give examples for the boundary value problem of nonlinear implicit Caputo fractional differential equations with Caputo fractional derivative boundary conditions to illustration as following.

Example 2.3. Consider the following boundary value problem

$$ {}^cD_{1^+}^{\frac{1}{4}}\, {}^cD_{1^+}^{\frac{3}{4}}x(t) = \frac{1}{5}t^2x(t) + t\,{}^cD_{1^+}^{\frac{1}{4}}x(t) - \frac{1}{2}\left|{}^cD_{1^+}^{\frac{1}{4}}\, {}^cD_{1^+}^{\frac{3}{4}}x(t)\right|, \quad \forall t \in (1,2] \tag{2.57} $$

$$ x(1) = \frac{5}{4}\,{}^cD_{1^+}^{\frac{1}{4}}x\left(\frac{3}{2}\right), \qquad x(2) = \ln 5\,{}^cD_{1^+}^{\frac{3}{4}}x\left(\frac{3}{2}\right). \tag{2.58} $$

Since the BVP (2.57) and (2.58), we know that

$$ \alpha_1 = \frac{1}{4},\ \alpha_2 = \frac{3}{4},\ a = 1,\ b = 2,\ \eta = 1,\ \rho_1 = \frac{5}{4},\ \rho_2 = \ln 5,\ \theta_1 = \theta_2 = \frac{3}{2},\ \xi_1 = \xi_2 = \frac{1}{2}. $$

We see that (A_1) holds, because

$$ f(t,u,v,w) = \frac{1}{5}t^2u + tv - \frac{1}{2}\left|w\right| $$

is a continuous on $[1,2] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Let $t \in [1,2]$. Next, we will show that (A_2) holds. That is to find $D \in \mathbb{R}_{++}$ such that $|f(t,u,v,w)| \leq D$. Let $u, v, w \in \mathbb{R}$, consider

$$ \begin{aligned} \left|f(t,u,v,w)\right| &= \left|\frac{1}{5}t^2u + tv - \frac{1}{2}\left|w\right|\right| \\ &\leq \frac{1}{5}t^2\left|u\right| + t\left|u\right| + \frac{1}{2}\left|w\right| \\ &\leq \frac{1}{5}(4)\left|u\right| + 2\left|u\right| + \frac{1}{2}\left|w\right|. \end{aligned} $$

That is there exists

$$ D := \frac{4}{5}\left|u\right| + 2\left|u\right| + \frac{1}{2}\left|w\right| \in \mathbb{R}_{++} $$

such that $|f(t,u,v,w)| \le D$. Hence (A_2) holds. By Corollary 2.11, the BVP (2.57) and (2.58) has at least one solution on $[1,2]$.

Example 2.4. Consider the following boundary value problem

$$ {}^cD_{1^+}^{\frac{2}{3}}\,{}^cD_{1^+}^{\frac{3}{4}}x(t) = \frac{\left|x(t)\right| + \left|{}^cD_{1^+}^{\frac{3}{4}}x(t)\right| + \left|{}^cD_{1^+}^{\frac{2}{3}}\,{}^cD_{1^+}^{\frac{3}{4}}x(t)\right|}{e^{6t}\left(4 + \left|x(t)\right| + \left|{}^cD_{1^+}^{\frac{3}{4}}x(t)\right| + \left|{}^cD_{1^+}^{\frac{2}{3}}\,{}^cD_{1^+}^{\frac{3}{4}}x(t)\right|\right)}, \quad \forall t \in \left(1, \frac{4}{3}\right] \tag{2.59} $$

$$ x(1) = \frac{1}{6}\,{}^cD_{1^+}^{\frac{2}{3}}x\left(\frac{9}{8}\right), \qquad x\left(\frac{4}{3}\right) = \sqrt{7}\,{}^cD_{1^+}^{\frac{3}{4}}x\left(\frac{9}{8}\right). \tag{2.60} $$

Since the BVP (2.59) and (2.60), we know that

$$ \alpha_1 = \frac{2}{3},\ \alpha_2 = \frac{3}{4},\ a = 1,\ b = \frac{4}{3},\ \eta = \frac{1}{3},\ \rho_1 = \frac{1}{6},\ \rho_2 = \sqrt{7},\ \theta_1 = \theta_2 = \frac{9}{8},\ \xi_1 = \xi_2 = \frac{1}{8}. $$

We see that (A_1) holds, because

$$ f(t,u,v,w) = \frac{|u| + |v| + |w|}{e^{6t}\left(4 + |u| + |v| + |w|\right)} $$

is a continuous on $\left[1, \frac{4}{3}\right] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Let $t \in \left[1, \frac{4}{3}\right]$. Next, we will show that (A_6) holds. Let $u_1, v_1, w_1, u_2, v_2, w_2 \in \mathbb{R}$, so that

$$ \begin{aligned} \left|f(t,u_1,v_1,w_1) - f(t,u_2,v_2,w_2)\right| &= \left|\frac{|u_1| + |v_1| + |w_1|}{e^{6t}(4 + |u_1| + |v_1| + |w_1|)} - \frac{|u_2| + |v_2| + |w_2|}{e^{6t}(4 + |u_2| + |v_2| + |w_2|)}\right| \\ &= \frac{1}{e^{6t}}\left|\frac{|u_1| + |v_1| + |w_1|}{4 + |u_1| + |v_1| + |w_1|} - \frac{|u_2| + |v_2| + |w_2|}{4 + |u_2| + |v_2| + |w_2|}\right| \\ &\le \frac{1}{e^{6t}} \cdot \frac{\left||u_1| - |u_2|\right| + \left||v_1| - |v_2|\right| + \left||w_1| - |w_2|\right|}{(4 + |u_1| + |v_1| + |w_1|)(4 + |u_2| + |v_2| + |w_2|)} \\ &\le \frac{1}{16e^6}\left[\left|u_1 - u_2\right| + \left|v_1 - v_2\right| + \left|w_1 - w_2\right|\right]. \end{aligned} \tag{2.61} $$

Hence (A_6) holds which $K = L = M = \frac{1}{16e^6}$. Through a simple calculation, such that

$$ \sigma_0 \approx 1.3416, \quad \sigma_1 \approx 1.2027, \quad \text{and} \quad \sigma_2 \approx 1.2276, $$

one has,

$$\sigma = \sigma_0 + \sigma_1 + \sigma_2 \approx 1.3416 + 1.2027 + 1.2276 \approx 3.7719. \tag{2.62}$$

Thus $\Lambda = \max\{K, L, M\} = \frac{1}{16e^6}$. We see that

$$\mathcal{P} = \sigma\Lambda \approx (3.7719)\left(\frac{1}{16e^6}\right) \approx 0.0006.$$

Hence $\mathcal{P} < 1$. By Corollary 2.12, the BVP (2.59) and (2.60) has a unique solution on $\left[1, \frac{4}{3}\right]$.

Next, we will check condition (A_4) holds. Let $t \in \left[1, \frac{4}{3}\right]$ and $u_1, v_1, w_1, u_2, v_2, w_2 \in \mathbb{R}$, choose $\phi(t) = \frac{1}{64}t \in C\left(\left[1, \frac{4}{3}\right], \mathbb{R}_+\right)$ is increasing. From (2.61), we get

$$\begin{aligned} \left|f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2)\right| &\leq \frac{4}{e^6}\left(\frac{1}{64}t\right)\left[\left|u_1 - u_2\right| + \left|v_1 - v_2\right| + \left|w_1 - w_2\right|\right] \\ &= \omega\phi(t)\left[\left|u_1 - u_2\right| + \left|v_1 - v_2\right| + \left|w_1 - w_2\right|\right]. \end{aligned}$$

Hence (A_4) holds which $\omega = 4e^{-6} \approx 0.0099$. From (2.62) and $\phi\left(\frac{4}{3}\right) = \frac{1}{48}$, one has

$$\Omega = \sigma\phi\left(\frac{4}{3}\right) \approx (3.7719)\left(\frac{1}{48}\right) \approx 0.0786.$$

Hence $\Omega \leq 1$. By Corollary 2.13, the BVP (2.59) and (2.60) has a unique solution on $\left[1, \frac{4}{3}\right]$.

CHAPTER 3

CONCLUSION

In this research, we studied existence and uniqueness of solutions for the boundary value problem (1.1) and (1.2) that is Theorem 2.3, Theorem 2.4, Theorem 2.5. We prove these five theories with examples.

Finally, we suggest for some special cases for a boundary value problem of nonlinear implicit Hilfer fractional differential equations (1.1) with Hilfer fractional differential boundary conditions (1.2), in this case $\beta = 0$ we get that boundary value problem of nonlinear implicit Riemann-Liouville fractional differential equations with Riemann-Liouville fractional differential boundary conditions as the following:

$$D_{a^+}^{\alpha_1} D_{a^+}^{\alpha_2} x(t) = f(t, x(t), D_{a^+}^{\alpha_i} x(t), D_{a^+}^{\alpha_1} D_{a^+}^{\alpha_2} x(t)), \quad t \in (a, b],$$

$$x(a) = \rho_1 D_{a^+}^{\alpha_1} x(\theta_1), \qquad x(b) = \rho_2 D_{a^+}^{\alpha_2} x(\theta_2),$$

we have three main theories for existence and uniqueness of the boundary value problem (2.49) and (2.50) that is Corollary 2.6, Corollary 2.7, Corollary 2.8. An examples is presented to illustrate the theory.

In this case $\beta = 1$ we get that boundary value problem of nonlinear implicit Caputo fractional differential equations with Caputo fractional differential boundary conditions as the following:

$${}^cD_{a^+}^{\alpha_1}\, {}^cD_{a^+}^{\alpha_2} x(t) = f(t, x(t), {}^cD_{a^+}^{\alpha_i} x(t), {}^cD_{a^+}^{\alpha_1}\, {}^cD_{a^+}^{\alpha_2} x(t)), \quad t \in (a, b],$$

$$x(a) = \rho_1 {}^cD_{a^+}^{\alpha_1} x(\theta_1), \qquad x(b) = \rho_2 {}^cD_{a^+}^{\alpha_2} x(\theta_2),$$

we have three main theories for existence and uniqueness of the boundary value problem (2.51) and (2.52) that is Corollary 2.11, Corollary 2.12, Corollary 2.13 and two main theories for stablity of the problem (2.51) and (2.52) that is Corollary 2.14, Corollary 2.15. Finally, examples are provided to illustrate the results.

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