สัญญาเลขที่ R2563C006

รายงานวิจัยฉบับสมบูรณ์

โครงการ: บนข้อคาดเดาเกี่ยวกับตัวกำหนดของมาร์คัสและเดอโอลิไวรา

On the Marcus–de Oliveira determinantal conjecture

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สนับสนุนโดย

งบประมาณรายได้มหาวิทยาลัยนเรศวร

ปีงบประมาณ 2563

ชื่อโครงการ บนข้อคาดเดาเกี่ยวกับตัวกำหนดของมาร์คัสและเดอโอลิไวรา
On the Marcus-de Oliveira determinantal conjecture

ชื่อผู้วิจัย รองศาสตราจารย์ ดร.กิจติ รอดเทศ
หน่วยงานที่สังกัด ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยนเรศวร
ได้รับทุนอุดหนุนการวิจัยจาก งบประมาณรายได้มหาวิทยาลัยนเรศวร ปีงบประมาณ 2563
จำนวนเงิน 220,000 บาท
ระยะเวลาการทำวิจัย 1 ปี

บทคัดย่อ(ภาษาไทย)

ในงานวิจัยนี้ ผู้วิจัยได้ค้นพบตลาสของเมทริกซ์ปรกติซึ่งสร้างจากเมทริกซ์จัตุรัสใด ๆ ที่สอดคล้องกับข้อคาดเดาเกี่ยวกับตัวกำหนดของมาร์คัสและเดอโอลิไวรา และได้ค้นพบการดำเนินการบนเมทริกซ์ปรกติที่ยังคงรักษาความสอดคล้องกับข้อคาดเดาดังกล่าวไว้ด้วย

บทคัดย่อ(ภาษาอังกฤษ)

In this project, we discover a class of normal matrices constructed from any square matrix that affirms the Marcus-de Oliveira determinantal conjecture. We also found some operations on normal matrices preserving the conjecture.

กิตติกรรมประกาศ (Acknowledgement)

รายงานการวิจัยฉบับนี้สำเร็จลุล่วงได้ ข้าพเจ้าขอขอบพระคุณทาง มหาวิทยาลัยนเรศวร ที่ให้ทุนอุดหนุนการวิจัยจากงบประมาณรายได้มหาวิทยาลัยนเรศวร ประจำปีงบประมาณ 2563 เป็นจำนวนเงินทั้งสิ้น 220,000 บาท

กิจติ รอดเทศ

สารบัญ

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Chapter 1

Introduction

Throughout, n will denote a positive integer. The determinant conjecture of Marcus and de Oliveira states that the determinant of the sum of two n by n normal matrices A and B belongs to the convex hull of the $n!$ σ-points, $z_\sigma := \prod_{i=1}^{n}(a_i + b_{\sigma(i)})$, indexed by $\sigma \in S_n$, where a_i's and b_j's are eigenvalues of A and B, respectively (see [9],[3],[11]). We briefly write as $(A, B) \in MOC$ if the pair of normal matrices A, B affirms the Marcus and de Oliveira conjecture, i.e.,

$$\det(A + B) \in co(\{z_\sigma | \sigma \in S_n\}).$$

In [8], Fiedler showed that, for two hermitian matrices A, B

$$\Delta(A, B) := \{\det(A + UBU^*) | U \in U_n(\mathbb{C})\}$$

is a line segment with σ-points as endpoints, where $U_n(\mathbb{C})$ denotes the set of all unitary matrices of dimension $n \times n$. This result, in fact, motivates the conjecture. As a consequence of Fiedler's result, $(A, B) \in MOC$ for any pair of skew-hermitian matrices A, B.

In [1], N. Bebiano, A. Kovacec, and J.da Providencia provided that if A is positive definite and B a non-real scalar multiple of a hermitian matrix, then $(A, B) \in MOC$. They also obtained that if eigenvalues of A are pairwise distinct complex numbers lying on a line l and all eigenvalues of B lie on a parallel to l, then $(A, B) \in MOC$. S.W. Drury showed that $(A, B) \in MOC$ for the case that A is hermitian and B is non-real scalar multiple of a hermitian matrix (essentially hermitian matrix) in [5] and the case that $A = sU$ and $B = tV$ for $s, t \in \mathbb{C}$ and $U, V \in U_n(\mathbb{C})$ in [6].

It is also known that, for normal matrices $A, B \in M_n(\mathbb{C})$ (the set of all $n \times n$ matrices over $\mathbb{C}$), $(A, B) \in MOC$: if $\det(A + B) = 0$ ([7]); if the point z_σ lie all on a straight line ([10]); if $n = 2, 3$ ([3, 2]); if A or B has only two distinct eigenvalues, one of them simple, ([3]). However, it seems that there is no new affirmative class of normal matrices to this conjecture after the year 2007.

Chapter 2

Operations preserving the conjecture

2.1 Direct sum

It is well known that if two n by n matrices A and B are normal, then so is their direct sum $A \oplus B$. Also, the spectrum of the sum is clearly given by $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$.

Proposition 2.1. *Let $A, B \in M_n(\mathbb{C})$ and $C, D \in M_m(\mathbb{C})$ be normal. If $(A, B) \in MOC$ and $(C, D) \in MOC$, then $(A \oplus C, B \oplus D) \in MOC$.*

Proof. Suppose that $\{a_i \mid 1 \leq i \leq n\}$, $\{b_i \mid 1 \leq i \leq n\}$, $\{c_i \mid 1 \leq i \leq m\}$ and $\{d_i \mid 1 \leq i \leq m\}$ are ordered set of the eigenvalues of A, B, C and D, respectively. Denote $e_i := a_i$, $f_i := b_i$ for $i = 1, \ldots, n$ and $e_{n+j} = c_j$, $f_{n+j} = d_j$ for $j = 1, \ldots, m$. Then, $\{e_i \mid 1 \leq i \leq n+m\}$ and $\{f_i \mid 1 \leq i \leq n+m\}$ are ordered set of the eigenvalues of $A \oplus C$ and $B \oplus D$, respectively. For each $\sigma \in S_n, \pi \in S_m$ and $\theta \in S_{n+m}$, denote z_σ, v_π and w_θ the product $\prod_{i=1}^{n}(a_i - b_{\sigma(i)})$, $\prod_{i=1}^{m}(c_i - d_{\pi(i)})$ and $\prod_{i=1}^{n+m}(e_i - f_{\theta(i)})$, respectively. Suppose that $(A, B) \in MOC$ and $(C, D) \in MOC$, then

$$\det(A - B) = \sum_{\sigma \in S_n} t_\sigma z_\sigma \text{ and } \det(C - D) = \sum_{\pi \in S_m} s_\pi v_\pi,$$

where $t_\sigma, s_\pi \in [0, 1]$ such that $\sum_{\sigma \in S_n} t_\sigma = 1$ and $\sum_{\sigma \in S_m} s_\pi = 1$. Note that

$$\begin{aligned} \det(A \oplus C - B \oplus D) &= \det((A - B) \oplus (C - D)) \\ &= \det(A - B) \cdot \det(C - D) \\ &= (\sum_{\sigma \in S_n} t_\sigma z_\sigma)(\sum_{\pi \in S_m} s_\pi v_\pi) \\ &= \sum_{\sigma \in S_n, \pi \in S_m} (t_\sigma s_\pi)(z_\sigma v_\pi). \end{aligned}$$

For each $\sigma \in S_n$ and $\pi \in S_m$, we define a permutation $\theta(\sigma, \pi) \in S_{n+m}$ by

$$\theta(\sigma, \pi) := \begin{pmatrix} 1 & \cdots & n & n+1 & \cdots & n+m \\ \sigma(1) & \cdots & \sigma(n) & n+\pi(1) & \cdots & n+\pi(m) \end{pmatrix},$$

then $w_{\theta(\sigma,\pi)} = z_\sigma v_\pi$. Since, for each $\sigma \in S_n$ and $\pi \in S_m$, $t_\sigma s_\pi \in [0, 1]$ and

$$\sum_{\sigma \in S_n, \pi \in S_m} (t_\sigma s_\pi) = (\sum_{\sigma \in S_n} t_\sigma)(\sum_{\sigma \in S_m} s_\pi) = (1)(1) = 1,$$

we conclude that

$$\det(A \oplus C - B \oplus D) \in co\{w_{\theta(\sigma,\pi)} \mid \sigma \in S_n, \pi \in S_m\} \subseteq co\{w_\theta \mid \theta \in S_{n+m}\}.$$

Hence $(A \oplus C, B \oplus D) \in MOC$. □

2.2 Kronecker product and Kronecker sum

It is well known that if $A \in M_n(\mathbb{C})$ and $B \in M_m(\mathbb{C})$, then so is their Kronecker product $A \otimes B$ and $\det(A \otimes B) = (\det(A))^m(\det(B))^n$. Also, if $\{a_i \mid 1 \leq i \leq n\}$ and $\{b_j \mid 1 \leq j \leq m\}$ are the ordered set of eigenvalues for A and B, respectively, then $\{a_i b_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ is the ordered set of eigenvalues for $A \otimes B$.

Proposition 3.1. *Let $A, B \in M_n(\mathbb{C})$ and $C \in M_m(\mathbb{C})$ be normal matrices. If $(A, B) \in MOC$, then $(A \otimes C, B \otimes C) \in MOC$.*

Proof. Let $\{a_i \mid 1 \leq i \leq n\}$, $\{b_i \mid 1 \leq i \leq n\}$ and $\{c_j \mid 1 \leq j \leq m\}$ be the ordered spectrums of A, B and C respectively. Now, for each $1 \leq i \leq n$ and $1 \leq j \leq m$, we define

$$x_{(j-1)n+i} := a_i c_j \quad \text{and} \quad y_{(j-1)n+i} := b_i c_j.$$

Then, $\{x_k \mid 1 \leq k \leq mn\}$ and $\{y_k \mid 1 \leq k \leq mn\}$ form an ordered spectrums of $A \otimes C$ and $B \otimes C$, respectively. For each $\sigma \in S_n$ and $\pi \in S_{mn}$, we denote $z_\sigma := \prod_{i=1}^{n}(a_i - b_{\sigma(i)})$ and $v_\pi := \prod_{k=1}^{mn}(x_k - y_{\pi(k)})$. Suppose that $(A, B) \in MOC$, then we can write $\det(A - B) = \sum_{\sigma \in S_n} t_\sigma z_\sigma$, where $t_\sigma \in [0, 1]$ and $\sum_{\sigma \in S_n} t_\sigma = 1$. Now, by Multinomial theorem, we have

$$\begin{aligned}
\det(A \otimes C - B \otimes C) &= (\det(A - B))^m(\det(C))^n \\
&= \Big(\sum_{\sigma \in S_n} t_\sigma z_\sigma\Big)^m\Big(\prod_{j=1}^{m} c_j\Big)^n \\
&= \sum_{k_1 + \cdots + k_{n!} = m} \binom{m}{k_1, \ldots, k_{n!}} (t_{\sigma_1} z_{\sigma_1})^{k_1} \cdots (t_{\sigma_{n!}} z_{\sigma_{n!}})^{k_{n!}} \Big[\prod_{j=1}^{m} c_j^n\Big] \\
&= \sum_{k_1 + \cdots + k_{n!} = m} \binom{m}{k_1, \ldots, k_{n!}} (t_{\sigma_1}^{k_1} \cdots t_{\sigma_{n!}}^{k_{n!}})\Big(\Big[z_{\sigma_1}^{k_1} \prod_{j=1}^{k_1} c_j^n\Big] \cdots \Big[z_{\sigma_{n!}}^{k_{n!}} \prod_{j=s}^{m} c_j^n\Big]\Big),
\end{aligned}$$

where $s = k_1 + \cdots + k_{n!-1} + 1$. For each $1 \leq l \leq n!$, we have

$$\begin{aligned}
z_{\sigma_l}^{k_l} \prod_{j=\hat{k}_{l-1}+1}^{\hat{k}_l} c_j^n &= \Big[\prod_{i=1}^{n}(a_i - b_{\sigma_l(i)}) \cdot c_{\hat{k}_{l-1}+1}^n\Big] \cdots \Big[\prod_{i=1}^{n}(a_i - b_{\sigma_l(i)}) \cdot c_{\hat{k}_l}^n\Big] \\
&= \Big[\prod_{i=1}^{n}(a_i c_{\hat{k}_{l-1}+1} - b_{\sigma_l(i)} c_{\hat{k}_{l-1}+1})\Big] \cdots \Big[\prod_{i=1}^{n}(a_i c_{\hat{k}_l} - b_{\sigma_l(i)} c_{\hat{k}_l})\Big] \\
&= \Big[\prod_{i=1}^{n}(x_{\hat{k}_{l-1}n+i} - y_{\hat{k}_{l-1}n+\sigma_l(i)})\Big] \cdots \Big[\prod_{i=1}^{n}(x_{(\hat{k}_l-1)n+i} - y_{(\hat{k}_l-1)n+\sigma_l(i)})\Big],
\end{aligned}$$

where $\hat{k}_{l-1} = k_1 + \cdots + k_{l-1}$ and $\hat{k}_l = \hat{k}_{l-1} + k_l$. Denote $\tilde{k} = (k_1, \ldots, k_{n!})$ and denote $[n + j] = \{j + 1, \ldots, j + n\}$, for each integer j. Define a permutation $\sigma_{\tilde{k}} \in S_{mn}$ by

$$\begin{pmatrix} [n] & \cdots & [n + (k_1 - 1)n] & \cdots & [n + (\hat{k}_{n!-1})n] & \cdots & [n + (\hat{k}_{n!} - 1)n] \\ \sigma_1([n]) & \cdots & k_1 + \sigma_1([n]) & \cdots & (\hat{k}_{n!-1})n + \sigma_{k_{n!}}([n]) & \cdots & (\hat{k}_{n!} - 1)n + \sigma_{k_{n!}}([n]) \end{pmatrix},$$

i.e., permute each block $[n]$ in the first k_1 blocks by σ_1 and so on. Then,

$$\det(A \otimes C - B \otimes C) = \sum_{\tilde{k} \in K_m} \binom{m}{\tilde{k}} t_{\sigma_1}^{k_1} \cdots t_{\sigma_{n!}}^{k_{n!}} \cdot v_{\sigma_{\tilde{k}}},$$

where $K_m = \{\tilde{k} := (k_1, \ldots, k_{n!}) \mid k_1 + \cdots + k_{n!} = m\}$. Since

$$\sum_{\tilde{k} \in K_m} \binom{m}{\tilde{k}} t_{\sigma_1}^{k_1} \cdots t_{\sigma_{n!}}^{k_{n!}} = (\sum_{\sigma \in S_n} t_\sigma)^m = 1^m = 1$$

and each term in this sum belong to $[0, 1]$, we conclude that

$$\det(A \otimes C - B \otimes C) \in co\{v_{\sigma_{\tilde{k}}} \mid \tilde{k} \in K_m\} \subseteq co\{v_\pi \mid \pi \in S_{mn}\}.$$

Therefore, $(A \otimes C, B \otimes C) \in MOC$. □

Remark 3.2. *For normal matrices $A, B \in M_n(\mathbb{C})$ and $C, D \in M_n(\mathbb{C})$, if we can show that $(A \otimes C, B \otimes D) \in MOC$ whenever $(A, B) \in MOC$ and $(C, D) \in MOC$, then the followings are immediate consequences:*

(1) *The results in* [1] *and* [2].

(2) *If $(A, B) \in MOC$, then $(A^{\otimes m}, B^{\otimes m}) \in MOC$.*

Recall that the Kronecker sum of two matrices $A \in M_n(\mathbb{C})$ and $B \in M_m(\mathbb{C})$ denoted by $A \boxplus B$ is defined to be

$$A \boxplus B := I_m \otimes A + B \otimes I_n.$$

Note that, in general $(A \boxplus B) \otimes C \neq (A \otimes C) \boxplus (B \otimes C)$. Moreover, if A, B are normal, so is their Kronecker sum. The spectrum of the Kronecker sum of A, B can be obtained from

$$\sigma(A \boxplus B) = \sigma(A) + \sigma(B) = \{a + b \mid a \in \sigma(A), b \in \sigma(B)\}.$$

By using similar arguments as in the proof of Proposition 3.1, we have:

Proposition 3.3. *Let $A, B \in M_n(\mathbb{C})$ and $C \in M_m(\mathbb{C})$ be normal matrices. If $(A, B) \in MOC$, then $(A \boxplus C, B \boxplus C) \in MOC$.*

Chapter 3

Main results

Let X be a square $n \times n$ complex matrix and s be a complex number. It is a direct calculation to see that

$$N(X,s) := \begin{pmatrix} X & (X-sI)^* \\ (X-sI)^* & X \end{pmatrix}$$

is a normal matrix of size $2n \times 2n$ and thus it is a normal dilation of X. We will see (in the proof of the main result) that the eigenvalues of $N(X,s)$ lie on both real and imaginary axis and thus this matrix need not be essentially hermitian or a scalar multiple of a unitary matrix. In this short note, we show that:

Theorem 0.1. *Let $X, Y \in M_n(\mathbb{C})$ and $s, t \in \mathbb{C}$. Then $(N(X,s), N(Y,t)) \in MOC$.*

Note that if $A \in M_n(\mathbb{C})$ is normal then UAU^* is also normal for any $U \in U_n(\mathbb{C})$. Then $VN(X,s)V^*$ is also a normal dilation of X for any $V \in U_{2n}(\mathbb{C})$. Moreover, since the conjecture is invariant under simultaneous unitary similarity, we also deduce from Theorem 0.1 that $(VN(X,s)V^*, VN(Y,t)V^*) \in MOC$ for any $V \in U_{2n}(\mathbb{C})$.

To prove the main result, we will use the following lemmas.

Lemma 0.2. *Let $A, B \in M_n(\mathbb{C})$ and $C, D \in M_m(\mathbb{C})$ be normal. If $(A,B) \in MOC$ and $(C,D) \in MOC$, then $(A \oplus C, B \oplus D) \in MOC$.*

Proof. Suppose that $\{a_i \,|\, 1 \le i \le n\}$, $\{b_i \,|\, 1 \le i \le n\}$, $\{c_i \,|\, 1 \le i \le m\}$ and $\{d_i \,|\, 1 \le i \le m\}$ are ordered set of the eigenvalues of A, B, C and D, respectively. Denote $e_i := a_i$, $f_i := b_i$ for $i = 1, \ldots, n$ and $e_{n+j} = c_j$, $f_{n+j} = d_j$ for $j = 1, \ldots, m$. Then, $\{e_i \,|\, 1 \le i \le n+m\}$ and $\{f_i \,|\, 1 \le i \le n+m\}$ are ordered set of the eigenvalues of $A \oplus C$ and $B \oplus D$, respectively. For each $\sigma \in S_n$, $\pi \in S_m$ and $\theta \in S_{n+m}$, denote z_σ, v_π and w_θ the product $\prod_{i=1}^{n}(a_i + b_{\sigma(i)})$, $\prod_{i=1}^{m}(c_i + d_{\pi(i)})$ and $\prod_{i=1}^{n+m}(e_i + f_{\theta(i)})$, respectively. Suppose that $(A,B) \in MOC$ and $(C,D) \in MOC$, then

$$\det(A+B) = \sum_{\sigma \in S_n} t_\sigma z_\sigma \text{ and } \det(C+D) = \sum_{\pi \in S_m} s_\pi v_\pi,$$

where $t_\sigma, s_\pi \in [0,1]$ such that $\sum_{\sigma \in S_n} t_\sigma = 1$ and $\sum_{\sigma \in S_m} s_\pi = 1$. Note that

$$\begin{aligned} \det(A \oplus C + B \oplus D) &= \det((A+B) \oplus (C+D)) \\ &= \det(A+B) \cdot \det(C+D) \\ &= (\sum_{\sigma \in S_n} t_\sigma z_\sigma)(\sum_{\pi \in S_m} s_\pi v_\pi) \\ &= \sum_{\sigma \in S_n, \pi \in S_m} (t_\sigma s_\pi)(z_\sigma v_\pi). \end{aligned}$$

For each $\sigma \in S_n$ and $\pi \in S_m$, define a permutation $\theta(\sigma,\pi) \in S_{n+m}$ by

$$\theta(\sigma,\pi) := \begin{pmatrix} 1 & \cdots & n & n+1 & \cdots & n+m \\ \sigma(1) & \cdots & \sigma(n) & n+\pi(1) & \cdots & n+\pi(m) \end{pmatrix}$$

Then $w_{\theta(\sigma,\pi)} = z_\sigma v_\pi$. Since, for each $\sigma \in S_n$ and $\pi \in S_m$, $t_\sigma s_\pi \in [0,1]$ and

$$\sum_{\sigma \in S_n, \pi \in S_m} (t_\sigma s_\pi) = (\sum_{\sigma \in S_n} t_\sigma)(\sum_{\sigma \in S_m} s_\pi) = (1)(1) = 1,$$

we conclude that

$$\det(A \oplus C + B \oplus D) \in co\{w_{\theta(\sigma,\pi)} \,|\, \sigma \in S_n, \pi \in S_m\} \subseteq co\{w_\theta \,|\, \theta \in S_{n+m}\}.$$

Hence $(A \oplus C, B \oplus D) \in MOC$. □

To be a self contained material, we record a result of S.W. Drury.

Theorem 0.3. [4] *Let A and B be hermitian matrices with the given eigenvalues $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ respectively. Let $(t_1, \ldots, t_n)$ be the eigenvalues of $A + B$. Then*

$$\prod_{j=1}^{n}(\lambda + t_j) \in co\{\prod_{j=1}^{n}(\lambda + a_j + b_{\sigma(j)})|\sigma \in S_n\},$$

where co denotes the convex hull in the space of polynomials and λ is an indeterminate.

As a corollary of the above theorem, we have that:

Lemma 0.4. *Let $X, Y \in M_n(\mathbb{C})$ and $\alpha, \beta \in \mathbb{C}$. Then $(X - X^* + \alpha I_n, Y - Y^* + \beta I_n) \in MOC$ and $(X + X^* + \alpha I_n, Y + Y^* + \beta I_n) \in MOC$.*

Proof. Since $X + X^*$ and $Y + Y^*$ are hermitian, by Theorem 0.3, we deduce directly that $(X + X^* + \alpha I_n, Y + Y^* + \beta I_n) \in MOC$. Since $X - X^*$ and $Y - Y^*$ are skew-hermitian, $i(X - X^*)$ and $i(Y - Y^*)$ are hermitian. Again, by Theorem 0.3, $(X - X^* + \alpha I_n, Y - Y^* + \beta I_n) \in MOC$. □

Proof. (Theorem 0.1) Let U be the block matrix in $M_{2n}(\mathbb{C})$ defined by

$$U := \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & I_n \\ -I_n & I_n \end{pmatrix}.$$

It is a direct computation to see that U is a unitary matrix and

$$U^* \begin{pmatrix} M & N \\ N & M \end{pmatrix} U = (M - N) \oplus (M + N),$$

for any $M, N \in M_n(\mathbb{C})$. Let $A := X - X^* + (\overline{s})I_n$, $B := Y - Y^* + \overline{t}I_n$, $C := X + X^* - (\overline{s})I_n$, and $D := Y + Y^* - \overline{t}I_n$. By Lemma 0.4, the pair of normal matrices (A, B) and (C, D) satisfy the conjecture. Hence, by Lemma 0.2, $(A \oplus C, B \oplus D) \in MOC$. Therefore,

$$(N(X, s), N(Y, t)) = (U(A \oplus C)U^*, U(B \oplus D)U^*) \in MOC,$$

which completes the proof. □

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Appendix

ตารางเปรียบเทียบ

วัตถุประสงค์ที่วางไว้ และ สิ่งที่ได้

วัตถุประสงค์ที่วางไว้	สิ่งที่ได้
ศึกษาข้อคาดเดาของมาร์คัส และ เดอโอลิไวรา	เข้าใจข้อคาดเดาดังกล่าวได้ในภาพรวม
หาการดำเนินการบนเซตของเมทริกซ์ปรกติที่รักษาข้อคาดเดาของมาร์คัส และ เดอโอลิไวรา	ค้นพบการดำเนินการบนเซตของเมทริกซ์ปรกติที่รักษาข้อคาดเดาของมาร์คัส และ เดอโอลิไวรา เช่น ผลบวกตรง ผลคูณเทนเซอร์ของเมทริกซ์
หากลุ่มของเมทริกซ์ปรกติที่สอดคล้องกับข้อคาดเดาของมาร์คัส และ เดอโอลิไวรา ในวงกว้างมากกว่าข้อสรุปที่มีอยู่ในปัจจุบัน	ค้นพบกลุ่มของเมทริกซ์ปรกติที่สอดคล้องกับข้อคาดเดาของมาร์คัส และ เดอโอลิไวรา ในวงกว้างมากกว่าข้อสรุปที่มีอยู่ในปัจจุบัน

จดหมายตอบรับการพิจารณาตีพิมพ์

OaM-2619, decision

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Sat 18/07/2020 09:12
To:

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Dear Professor Kijti Rodtes,

It is my pleasure to inform you that your paper OaM-2619 "*A class of normal dilation matrices affirming the Marcus-de Oliveira Conjecture*" has been accepted for publication in Operators and Matrices. Please send to us a TeX file of your paper by replying to this e-mail.

Thank you for submitting your paper to our journal.

On behalf of the chief-editor,
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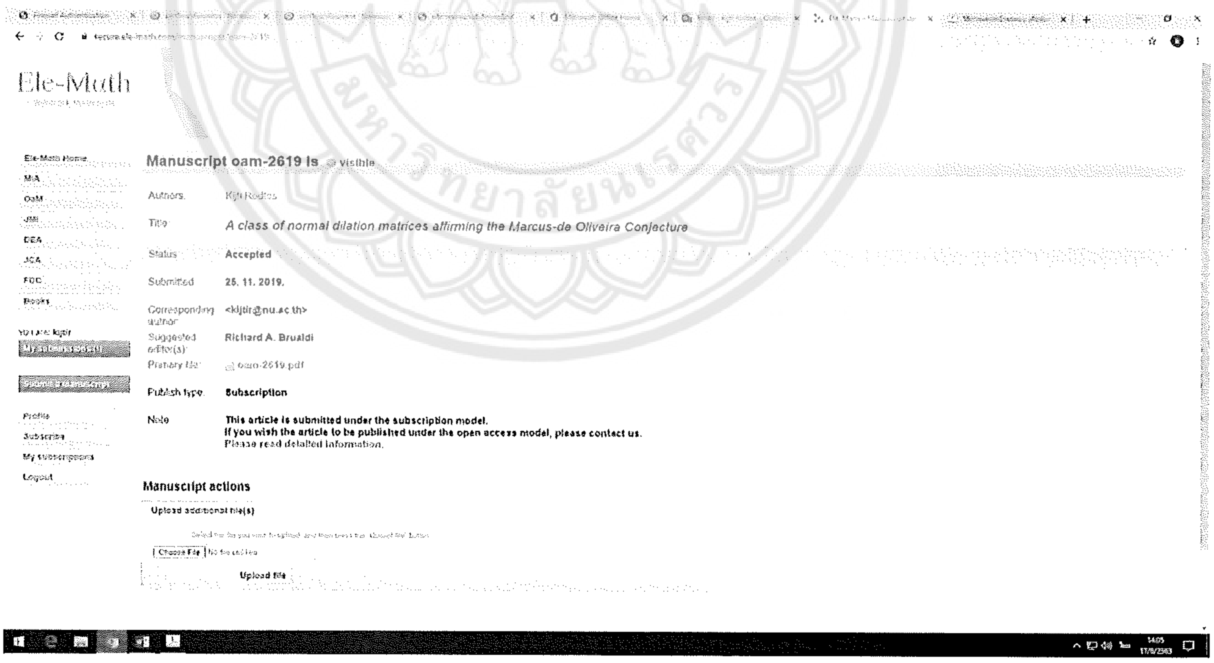

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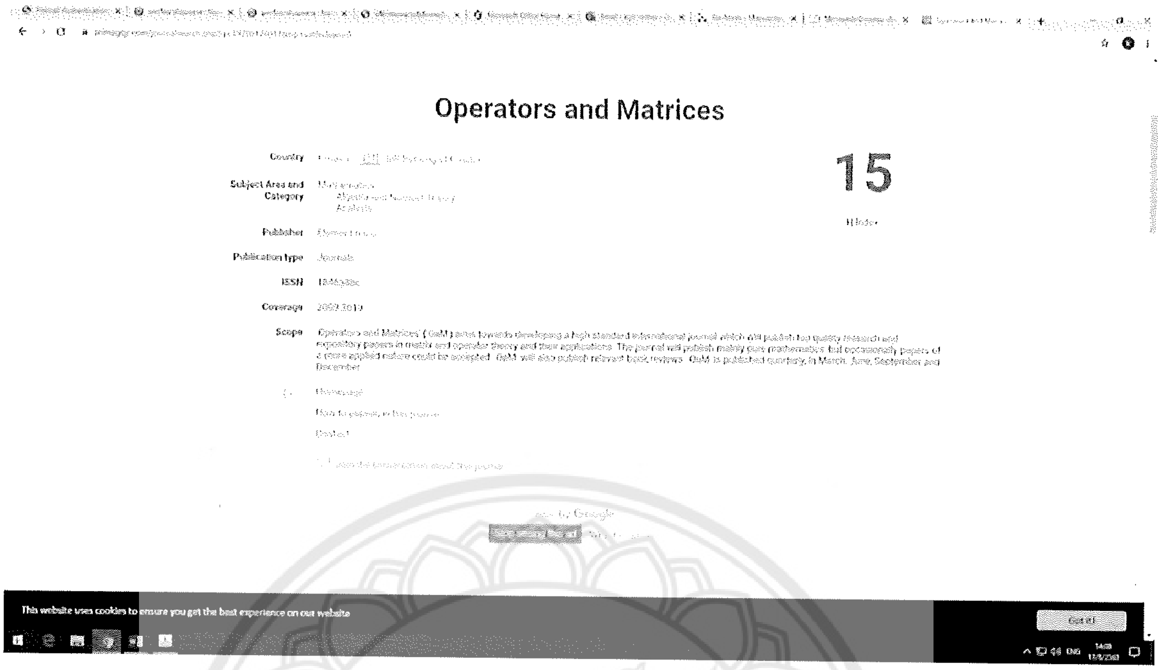

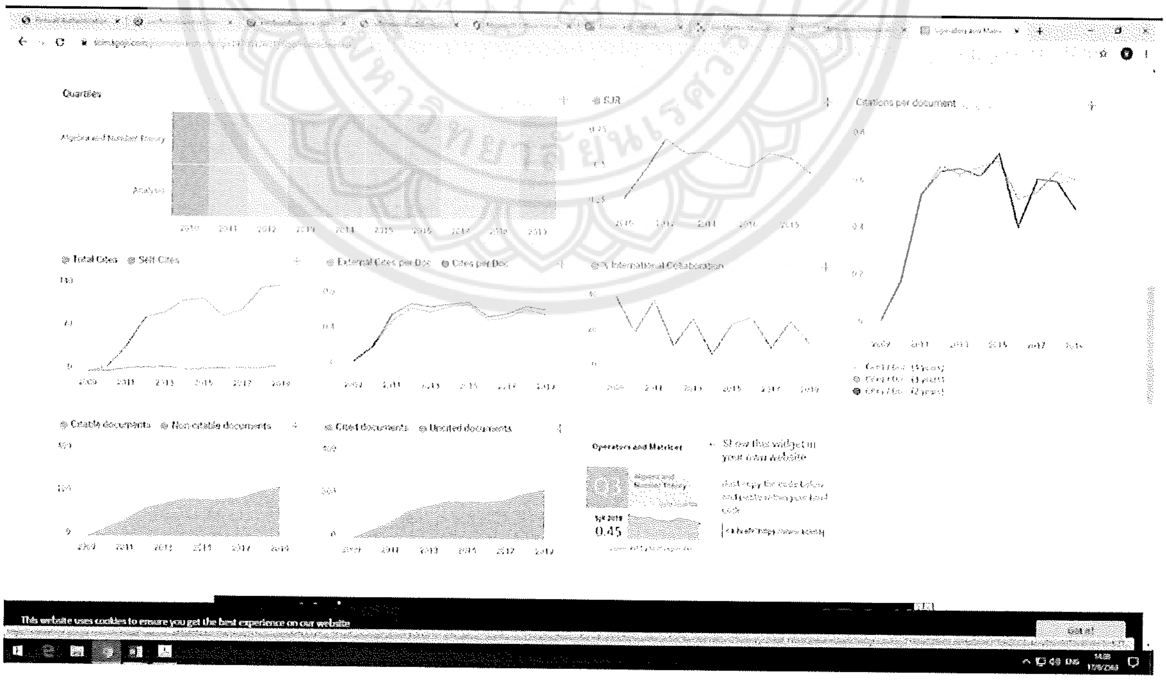

Revised Manuscript

A CLASS OF NORMAL DILATION MATRICES AFFIRMING THE MARCUS-DE OLIVEIRA CONJECTURE

KIJTI RODTES

ABSTRACT. In this article, we prove a class of normal dilation matrices affirming the Marcus-de Oliveira conjecture.

Throughout, n will denote a positive integer. The determinant conjecture of Marcus and de Oliveira states that the determinant of the sum of two n by n normal matrices A and B belongs to the convex hull of the $n!$ σ-points, $z_\sigma := \prod_{i=1}^{n}(a_i + b_{\sigma(i)})$, indexed by $\sigma \in S_n$, where a_i's and b_j's are eigenvalues of A and B, respectively (see [9],[3],[11]). We briefly write as $(A, B) \in MOC$ if the pair of normal matrices A, B affirms the Marcus and de Oliveira conjecture, i.e.,

$$\det(A + B) \in co(\{z_\sigma | \sigma \in S_n\}).$$

In [8], Fiedler showed that, for two hermitian matrices A, B

$$\Delta(A, B) := \{\det(A + UBU^*) | U \in U_n(\mathbb{C})\}$$

is a line segment with σ-points as endpoints, where $U_n(\mathbb{C})$ denotes the set of all unitary matrices of dimension $n \times n$. This result, in fact, motivates the conjecture. As a consequence of Fiedler's result, $(A, B) \in MOC$ for any pair of skew-hermitian matrices A, B.

In [1], N. Bebiano, A. Kovacec, and J.da Providencia provided that if A is positive definite and B a non-real scalar multiple of a hermitian matrix, then $(A, B) \in MOC$. They also obtained that if eigenvalues of A are pairwise distinct complex numbers lying on a line l and all eigenvalues of B lie on a parallel to l, then $(A, B) \in MOC$. S.W. Drury showed that $(A, B) \in MOC$ for the case that A is hermitian and B is non-real scalar multiple of a hermitian matrix (essentially hermitian matrix) in [5] and the case that $A = sU$ and $B = tV$ for $s, t \in \mathbb{C}$ and $U, V \in U_n(\mathbb{C})$ in [6].

It is also known that, for normal matrices $A, B \in M_n(\mathbb{C})$ (the set of all $n \times n$ matrices over $\mathbb{C}$), $(A, B) \in MOC$: if $\det(A + B) = 0$ ([7]); if the point z_σ lie all on a straight line ([10]); if $n = 2, 3$ ([3, 2]); if A or B has only two distinct eigenvalues, one of them simple,

Keywords: Normal dilation, Normal matrices, Marcus de Oliveira Conjecture
MSC(2010): 15A15; 15A60; 15A86.

([3]). However, it seems that there is no new affirmative class of normal matrices to this conjecture after the year 2007.

Let X be a square $n \times n$ complex matrix and s be a complex number. It is a direct calculation to see that

$$N(X,s) := \begin{pmatrix} X & (X-sI)^* \\ (X-sI)^* & X \end{pmatrix}$$

is a normal matrix of size $2n \times 2n$ and thus it is a normal dilation of X. We will see (in the proof of the main result) that the eigenvalues of $N(X,s)$ lie on both real and imaginary axis and thus this matrix need not be essentially hermitian or a scalar multiple of a unitary matrix. In this short note, we show that:

Theorem 0.1. *Let $X, Y \in M_n(\mathbb{C})$ and $s, t \in \mathbb{C}$. Then $(N(X,s), N(Y,t)) \in MOC$.*

Note that if $A \in M_n(\mathbb{C})$ is normal then UAU^* is also normal for any $U \in U_n(\mathbb{C})$. Then $VN(X,s)V^*$ is also a normal dilation of X for any $V \in U_{2n}(\mathbb{C})$. Moreover, since the conjecture is invariant under simultaneous unitary similarity, we also deduce from Theorem 0.1 that $(VN(X,s)V^*, VN(Y,t)V^*) \in MOC$ for any $V \in U_{2n}(\mathbb{C})$.

To prove the main result, we will use the following lemmas.

Lemma 0.2. *Let $A, B \in M_n(\mathbb{C})$ and $C, D \in M_m(\mathbb{C})$ be normal. If $(A,B) \in MOC$ and $(C,D) \in MOC$, then $(A \oplus C, B \oplus D) \in MOC$.*

Proof. Suppose that $\{a_i \mid 1 \le i \le n\}$, $\{b_i \mid 1 \le i \le n\}$, $\{c_i \mid 1 \le i \le m\}$ and $\{d_i \mid 1 \le i \le m\}$ are ordered set of the eigenvalues of A, B, C and D, respectively. Denote $e_i := a_i$, $f_i := b_i$ for $i = 1, \ldots, n$ and $e_{n+j} = c_j$, $f_{n+j} = d_j$ for $j = 1, \ldots, m$. Then, $\{e_i \mid 1 \le i \le n+m\}$ and $\{f_i \mid 1 \le i \le n+m\}$ are ordered set of the eigenvalues of $A \oplus C$ and $B \oplus D$, respectively. For each $\sigma \in S_n, \pi \in S_m$ and $\theta \in S_{n+m}$, denote z_σ, v_π and w_θ the product $\prod_{i=1}^n (a_i + b_{\sigma(i)})$, $\prod_{i=1}^m (c_i + d_{\pi(i)})$ and $\prod_{i=1}^{n+m} (e_i + f_{\theta(i)})$, respectively. Suppose that $(A,B) \in MOC$ and $(C,D) \in MOC$, then

$$\det(A+B) = \sum_{\sigma \in S_n} t_\sigma z_\sigma \text{ and } \det(C+D) = \sum_{\pi \in S_m} s_\pi v_\pi,$$

where $t_\sigma, s_\pi \in [0,1]$ such that $\sum_{\sigma \in S_n} t_\sigma = 1$ and $\sum_{\sigma \in S_m} s_\pi = 1$. Note that

$$\begin{aligned} \det(A \oplus C + B \oplus D) &= \det((A+B) \oplus (C+D)) \\ &= \det(A+B) \cdot \det(C+D) \\ &= (\sum_{\sigma \in S_n} t_\sigma z_\sigma)(\sum_{\pi \in S_m} s_\pi v_\pi) \\ &= \sum_{\sigma \in S_n, \pi \in S_m} (t_\sigma s_\pi)(z_\sigma v_\pi). \end{aligned}$$

For each $\sigma \in S_n$ and $\pi \in S_m$, define a permutation $\theta(\sigma,\pi) \in S_{n+m}$ by

$$\theta(\sigma,\pi) := \begin{pmatrix} 1 & \cdots & n & n+1 & \cdots & n+m \\ \sigma(1) & \cdots & \sigma(n) & n+\pi(1) & \cdots & n+\pi(m) \end{pmatrix}$$

Then $w_{\theta(\sigma,\pi)} = z_\sigma v_\pi$. Since, for each $\sigma \in S_n$ and $\pi \in S_m$, $t_\sigma s_\pi \in [0,1]$ and

$$\sum_{\sigma \in S_n, \pi \in S_m} (t_\sigma s_\pi) = (\sum_{\sigma \in S_n} t_\sigma)(\sum_{\sigma \in S_m} s_\pi) = (1)(1) = 1,$$

we conclude that

$$\det(A \oplus C + B \oplus D) \in co\{w_{\theta(\sigma,\pi)} \mid \sigma \in S_n, \pi \in S_m\} \subseteq co\{w_\theta \mid \theta \in S_{n+m}\}.$$

Hence $(A \oplus C, B \oplus D) \in MOC$. □

To be a self contained material, we record a result of S.W. Drury.

Theorem 0.3. [4] *Let A and B be hermitian matrices with the given eigenvalues $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ respectively. Let $(t_1, \ldots, t_n)$ be the eigenvalues of $A + B$. Then*

$$\prod_{j=1}^{n}(\lambda + t_j) \in co\{\prod_{j=1}^{n}(\lambda + a_j + b_{\sigma(j)}) \mid \sigma \in S_n\},$$

where co denotes the convex hull in the space of polynomials and λ is an indeterminate.

As a corollary of the above theorem, we have that:

Lemma 0.4. *Let $X, Y \in M_n(\mathbb{C})$ and $\alpha, \beta \in \mathbb{C}$. Then $(X - X^* + \alpha I_n, Y - Y^* + \beta I_n) \in MOC$ and $(X + X^* + \alpha I_n, Y + Y^* + \beta I_n) \in MOC$.*

Proof. Since $X + X^*$ and $Y + Y^*$ are hermitian, by Theorem 0.3, we deduce directly that $(X + X^* + \alpha I_n, Y + Y^* + \beta I_n) \in MOC$. Since $X - X^*$ and $Y - Y^*$ are skew-hermitian, $i(X - X^*)$ and $i(Y - Y^*)$ are hermitian. Again, by Theorem 0.3, $(X - X^* + \alpha I_n, Y - Y^* + \beta I_n) \in MOC$.

□

Proof. *(Theorem 0.1)* Let U be the block matrix in $M_{2n}(\mathbb{C})$ defined by

$$U := \frac{1}{\sqrt{2}}\begin{pmatrix} I_n & I_n \\ -I_n & I_n \end{pmatrix}.$$

It is a direct computation to see that U is a unitary matrix and

$$U^*\begin{pmatrix} M & N \\ N & M \end{pmatrix}U = (M - N) \oplus (M + N),$$

for any $M, N \in M_n(\mathbb{C})$. Let $A := X - X^* + (\overline{s})I_n$, $B := Y - Y^* + \overline{t}I_n$, $C := X + X^* - (\overline{s})I_n$, and $D := Y + Y^* - \overline{t}I_n$. By Lemma 0.4, the pair of normal matrices (A, B) and (C, D) satisfy the conjecture. Hence, by Lemma 0.2, $(A \oplus C, B \oplus D) \in MOC$. Therefore,

$$(N(X, s), N(Y, t)) = (U(A \oplus C)U^*, U(B \oplus D)U^*) \in MOC,$$

which completes the proof. □

Acknowledgments

The author would like to thank Prof Tin Yau Tam for bringing this topic to the author. He would like to thank the referee(s) for valuable comments to improve the paper. He also would like to thank Naresuan University for the financial support on the project number R2563C006.

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