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โครงการ : วิธีการฟอร์เวิร์ดแบคเวิร์ดเพนนัลตี้สำหรับการแก้ปัญหาอินคลูชันแบบทางเดียว

A new forward-backward penalty scheme for solving monotone Inclusion problems

ผู้วิจัย

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ชื่อโครงการ : วิธีการฟอร์เวิร์ดแบคเวิร์ดเพนนัลตี้สำหรับการแก้ปัญหาอินคลูชันแบบทางเดียว

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โครงการวิจัยนี้เราได้สร้างวิธีการแบบมานน์ชนิดใหม่ที่ผนวกทั้งพจน์ที่เป็นอินเนอร์เชียลและพจน์ที่เป็นค่าความคลาดเคลื่อนเพื่อที่จะค้นหาจุดตรึงของการส่งแบบไม่ขยายในปริภูมิฮิลเบิร์ต เราได้แสดงให้เห็นถึงการลู่เข้าแบบเข้มของวิธีทำซ้ำภายใต้สมมติฐานที่เหมาะสมบางประการเพื่อที่จะหาผลเฉลยสู่ปัญหาจุดตรึงที่เราสนใจ ประโยชน์ของทฤษฎีบทหลักสามารถนำไปประยุกต์ใช้ในการประมาณศูนย์ของผลรวมของตัวดำเนินการทางเดียวสามตัวดำเนินการได้ เราได้เปรียบเทียบประสิทธิภาพการลู่เข้าสำหรับสามวิธีการ คือ วิธีการแบบใหม่ที่เราได้นำเสนอ อัลกอริธึมที่เป็นแบบของมานน์ที่ปราศจากพจน์ที่เป็นอินเนอร์เชียลและพจน์ที่เป็นค่าความคลาดเคลื่อน และอัลกอริธึมที่เป็นแบบของฮาลเพิร์น ในรูปแบบของปัญหาค่าต่ำสุดเชิงคอนเวกซ์ด้วยเงื่อนไขบังคับของการแปลงเชิงเส้นแบบอสมมาตรที่ไม่ใช่ศูนย์ ในตอนท้าย เราได้แสดงให้เห็นภาพสำหรับการนำไปใช้ประโยชน์ได้จริงของอัลกอริธึมที่เราได้สร้างมาผ่านการทดสอบเชิงตัวเลขซึ่งเกี่ยวข้องสัมพันธ์กับปัญหาการกู้คืนรูปภาพ

คำสำคัญ: วิธีอินเนอร์เชียล อัลกอริธึมที่เป็นแบบของมานน์ ฟอร์เวิร์ดแบคเวิร์ดอัลกอริธึม ปัญหาอินคลูชันแบบทางเดียว การส่งแบบไม่ขยาย

ABSTRACT

Project Code: R2563C019

Project Title: A new forward-backward penalty scheme for solving monotone inclusion problems

Researcher: Associate Professor Dr. Kasamsuk Ungchittrakool

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In this project, we establish a new Mann-type method combining both inertial terms and errors to find a fixed point of a nonexpansive mapping in a Hilbert space. We show strong convergence of the iterate under some appropriate assumptions in order to find a solution to an investigative fixed point problem. For the virtue of the main theorem, it can be applied to an approximately zero point of the sum of three monotone operators. We compare the convergent performance of our proposed method, the Mann-type algorithm without both inertial terms and errors, and the Halpern-type algorithm in convex minimization problem with the constraint of a non-zero asymmetric linear transformation. Finally, we illustrate the functionality of the algorithm through numerical experiments addressing image restoration problems.

Keywords: inertial method, Mann-type algorithm, forward-backward algorithm, monotone inclusion problem, nonexpansive mapping

CHAPTER I

EXECUTIVE SUMMARY

In this project, we proposed a new Mann-type method combining both inertial terms and errors to solve the fixed point problem for a nonexpansive mapping. We also prove the strong convergence of the proposed algorithm under some sufficient conditions of involved parameters.

Let a nonexpansive mapping T from $\mathcal{H}$ into itself be such that $\mathbf{Fix}(T) \neq \emptyset$. We propose the following algorithm.

$$\textbf{(Algorithm 1)} \quad \begin{cases} x_0, x_1 \in \mathcal{C}, \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = \delta_n y_n + \alpha_n(T\delta_n y_n - \delta_n y_n) + \varepsilon_n, \end{cases} \tag{1}$$

for all $n \geq 1$, where $(\theta_n)_{n\geq 0} \subseteq [0, \theta]$ with $\theta \in [0, 1)$, $(\alpha_n)_{n\geq 0}$ and $(\delta_n)_{n\geq 0}$ are sequences in $(0, 1]$ and $(\varepsilon_n)_{n\geq 0}$ is a sequence in $\mathcal{H}$.

Assumption 1 *Let $(\alpha_n)_{n\geq 0}$ and $(\delta_n)_{n\geq 0}$ be sequences in $(0, 1]$ and let $(\varepsilon_n)_{n\geq 0}$ be a sequence in $\mathcal{H}$. Assume the conditions are verifiable, as follows.*

1. $\liminf\limits_{n\to+\infty} \alpha_n > 0$ *and* $\sum_{n\geq 1} |\alpha_n - \alpha_{n-1}| < +\infty$,
2. $\lim\limits_{n\to+\infty} \delta_n = 1$, $\sum_{n\geq 0}(1 - \delta_n) = +\infty$ *and* $\sum_{n\geq 1} |\delta_n - \delta_{n-1}| < +\infty$,
3. $\sum_{n\geq 0} \|\varepsilon_n\| < +\infty$.

Theorem 2 *Let $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping such that $\mathbf{Fix}(T) \neq \emptyset$ and let $(x_n)_{n\geq 0}$ be generated by* **Algorithm 1**. *Let $(\theta_n)_{n\geq 0}$ be a sequence in $[0, \theta]$ with $\theta \in [0, 1)$ such that $\sum_{n\geq 1} \theta_n \|x_n - x_{n-1}\| < +\infty$. Suppose Assumption 1 holds. Then, the sequence $(x_n)_{n\geq 0}$ strongly converges to $x^* := \mathbf{proj}_{\mathbf{Fix}(T)}(0)$.*

We also apply results to approximate the solutions of the monotone inclusion problems.

Let C be a nonempty closed convex subset of $\mathcal{H}$. The *indicator function* is defined by

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise}, \end{cases}$$

for all $x \in \mathcal{H}$. We consider the monotone inclusion problem as follows:

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in Ax + Bx + Cx, \tag{2}$$

where $A : \mathcal{H} \to 2^{\mathcal{H}}$ and $B : \mathcal{H} \to 2^{\mathcal{H}}$ are maximal monotone operators and $C : \mathcal{H} \to \mathcal{H}$ is a δ-cocoercive operator with $\delta > 0$. We assume that $\mathbf{zer}(A + B + C) \neq \emptyset$. We propose the following algorithm for solving the problem (2).

$$\textbf{(Algorithm 2)} \quad \begin{cases} a_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = J_{\mu B}(\delta_n a_n), \\ z_n = J_{\mu A}(2y_n - \delta_n a_n - \mu C y_n), \\ x_{n+1} = \delta_n a_n + \alpha_n(z_n - y_n) + \varepsilon_n, \end{cases} \tag{3}$$

for all $n \geq 1$, where $x_0, x_1 \in \mathcal{H}$, $\mu \in (0, 2\delta)$, $(\theta_n)_{n\geq 1} \subseteq [0, \theta]$ with $\theta \in [0, 1)$, and $(\alpha_n)_{n\geq 0}$ and $(\delta_n)_{n\geq 0}$ are sequences in $(0, 1]$ and $(\varepsilon_n)_{n\geq 0}$ is a sequence in $\mathcal{H}$.

Furthermore, we also provided a numerical example to compare the proposed algorithm with other algorithms in the convex minimization problem. Finally, we use our method to solve image restoration problems.

For the suggestion on this research area in the future, it might be possible to put (and/or add) the new position of the inertial term(s) in the considered algorithm together with changing the controlling scalars in order to improve the numerical performance.

CHAPTER II

CONTENTS OF RESEARCH

In this project, we obtain one publication that published in the international journal as the following:

1

Natthaphon Artsawang and **Kasamsuk Ungchittrakool** (2020). Inertial Mann-type algorithm for a nonexpansive mapping to solve monotone inclusion and image restoration problems. Symmetry. 12, 750: 17 pages (ISI Impact Factor 2020 : 2.645)

Let $\mathcal{H}$ be a real Hilbert space with an inner product and corresponding norm which is denoted by the notations $\langle \cdot, \cdot \rangle$ and $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$, respectively. Let $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping, that is, $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in \mathcal{H}$. Given C a nonempty closed convex subset of $\mathcal{H}$. The set of all fixed points of the operator T is denoted by $\mathbf{Fix}(T) := \{x \in \mathcal{H} : Tx = x\}$. The metric projection of $\mathcal{H}$ onto C, $\mathbf{proj}_C : \mathcal{H} \to C$ is defined by $\mathbf{proj}_C(x) = \arg\min_{c \in C} \|x - c\|$ for all $x \in \mathcal{H}$, see more detail in [1] and the references therein.

Problem: the fixed point problem for the mapping T generally denote as,

$$\text{find } x \in \mathcal{H} \text{ such that } x = Tx.$$

Recently, Bot et al. [2] proposed a new Mann-type algorithm (**MTA**) to solve the fixed point problem for a nonexpansive mapping and proved strong convergence of the iterate without using viscosity and projection method under some control conditions of parameters sequences. Their algorithm was defined by

$$(\mathbf{MTA}) \quad x_{n+1} = (1 - \alpha_n)\delta_n x_n + \alpha_n T \delta_n x_n, \quad \forall n \geq 1, \tag{1.1}$$

where $x_1 \in \mathcal{H}$ and $(\alpha_n)_{n \geq 0}$, $(\delta_n)_{n \geq 0}$ are sequences in $(0, 1]$.

Polyak [3] firstly proposed an inertial extrapolation as an acceleration process to solve the smooth convex minimization problem. An inertial algorithm is a two-step iterative method and the next iterate is defined by making use of the previous two iterates. It is well known that combining an inertial term in an algorithm can accelerate the speed of convergence of the sequence generated by the algorithm. Subsequently, there are many authors who are interested in studying the inertial-type algorithm. By using the concept of the inertial method, the technique of Halpern method and error terms, Shehu et al. [5] introduced an algorithm for solving a fixed point of a nonexpansive mapping which was defined as follows:

$$\begin{cases} x_0, x_1 \in \mathcal{H}, \\ y_n = x_n + \theta_n (x_n - x_{n-1}), \\ x_{n+1} = \alpha_n x_0 + \delta_n y_n + \gamma_n T y_n + e_n, \end{cases} \tag{1.2}$$

for all $n \geq 1$, where $(\theta_n)_{n \geq 0} \subseteq [0, \theta]$ with $\theta \in [0, 1)$, $(\alpha_n)_{n \geq 0}$, $(\delta_n)_{n \geq 0}$ and $(\gamma_n)_{n \geq 0}$ are sequences in $(0, 1]$ and $(e_n)_{n \geq 0}$ is a sequence in $\mathcal{H}$.

Being motivated by the above facts, we intend to accelerate the speed of convergence by avoiding the viscosity concept, hence, we propose a Mann-type method combining both inertial terms and errors for finding a fixed point of a nonexpansive mapping in a Hilbert space.

Let a nonexpansive mapping T from $\mathcal{H}$ into itself be such that $\mathbf{Fix}(T) \neq \emptyset$. We propose the following algorithm.

$$
\textbf{(Algorithm 1)} \quad \begin{cases} x_0, x_1 \in \mathcal{C}, \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = \delta_n y_n + \alpha_n(T\delta_n y_n - \delta_n y_n) + \varepsilon_n, \end{cases} \tag{1.3}
$$

for all $n \geq 1$, where $(\theta_n)_{n\geq 0} \subseteq [0, \theta]$ with $\theta \in [0, 1)$, $(\alpha_n)_{n\geq 0}$ and $(\delta_n)_{n\geq 0}$ are sequences in $(0, 1]$ and $(\varepsilon_n)_{n\geq 0}$ is a sequence in $\mathcal{H}$.

On the other hand, for the set of all zeros of the sum of three monotone operators A, B, C as the following

$$
\text{find } x \in \mathcal{H} \text{ such that } 0 \in Ax + Bx + Cx, \tag{1.4}
$$

where A, B, C are maximal monotone operators on a Hilbert space $\mathcal{H}$ and C is δ-cocoercive with parameter δ. The problem (1.4) was considered by Davis and Yin [6] and it can be reformulated to the fixed point problem for nonexpansive mappings. Therefore, it is interesting to study the fixed point problem in order to apply for solving the zeros problem of maximal monotone operators.

For the applications, we can formulate the main problem, that is, the fixed point problem in order to apply in the case of finding a zero point of the sum of three maximal monotone operators. Furthermore, the convergence behavior between the algorithms that obtained from the **Algorithm 1** are illustrated by some numerical experiment.

Assumption 1.1. *Let $(\alpha_n)_{n\geq 0}$ and $(\delta_n)_{n\geq 0}$ be sequences in $(0, 1]$ and let $(\varepsilon_n)_{n\geq 0}$ be a sequence in $\mathcal{H}$. Assume the conditions are verifiable, as follows.*

1. $\liminf_{n\to+\infty} \alpha_n > 0$ *and* $\sum_{n\geq 1} |\alpha_n - \alpha_{n-1}| < +\infty$,
2. $\lim_{n\to+\infty} \delta_n = 1$, $\sum_{n\geq 0}(1 - \delta_n) = +\infty$ *and* $\sum_{n\geq 1} |\delta_n - \delta_{n-1}| < +\infty$,
3. $\sum_{n\geq 0} \|\varepsilon_n\| < +\infty$.

We have verified Assumption 1.1 as shown in the following remark.

Remark 1.2. *Let $z \in \mathcal{H}$. We set $\delta_n = 1 - \frac{1}{n+2}$, $\alpha_n = \frac{1}{4} - \frac{1}{(n+3)^2}$ and $\varepsilon_n = \frac{z}{(n+1)^3}$ for all $n \geq 0$. It's easy to see that the Assumption 1.1 is satisfied.*

2 Main results

Lemma 2.1. *Let $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping such that* $\text{Fix}(T) \neq \emptyset$ *and let $(x_n)_{n\geq 0}$ be generated by* Algorithm 1. *Let $(\theta_n)_{n\geq 0}$ be a sequence in $[0, \theta]$ with $\theta \in [0, 1)$ such that $\sum_{n\geq 1} \theta_n \|x_n - x_{n-1}\| < +\infty$. Suppose Assumption 1.1 holds. Then $(x_n)_{n\geq 0}$ is bounded.*

Theorem 2.2. *Let $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping such that* $\mathbf{Fix}(T) \neq \emptyset$ *and let $(x_n)_{n\geq 0}$ be generated by* **Algorithm 1**. *Let $(\theta_n)_{n\geq 0}$ be a sequence in $[0, \theta]$ with $\theta \in [0, 1)$ such that $\sum_{n\geq 1} \theta_n \|x_n - x_{n-1}\| < +\infty$. Suppose Assumption 1.1 holds. Then, the sequence $(x_n)_{n\geq 0}$ strongly converges to $x^* := \mathbf{proj}_{\mathbf{Fix}(T)}(0)$.*

3 Applications

Let $\mathcal{C}$ be a nonempty closed convex subset of $\mathcal{H}$. The *indicator function* is defined by

$$
\delta_{\mathcal{C}}(x) = \begin{cases} 0, & \text{if } x \in \mathcal{C}, \\ +\infty, & \text{otherwise,} \end{cases}
$$

for all $x \in \mathcal{H}$. We consider the monotone inclusion problem as follows:

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in Ax + Bx + Cx, \tag{3.1}$$

where $A : \mathcal{H} \to 2^{\mathcal{H}}$ and $B : \mathcal{H} \to 2^{\mathcal{H}}$ are maximal monotone operators and $C : \mathcal{H} \to \mathcal{H}$ is a δ-cocoercive operator with $\delta > 0$. We assume that $\mathbf{zer}(A + B + C) \neq \emptyset$. We propose the following algorithm for solving the problem (3.1).

$$\textbf{(Algorithm 2)} \quad \begin{cases} a_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = J_{\mu B}(\delta_n a_n), \\ z_n = J_{\mu A}(2y_n - \delta_n a_n - \mu C y_n), \\ x_{n+1} = \delta_n a_n + \alpha_n(z_n - y_n) + \varepsilon_n, \end{cases} \tag{3.2}$$

for all $n \geq 1$, where $x_0, x_1 \in \mathcal{H}$, $\mu \in (0, 2\delta)$, $(\theta_n)_{n\geq 1} \subseteq [0, \theta]$ with $\theta \in [0, 1)$, and $(\alpha_n)_{n\geq 0}$ and $(\delta_n)_{n\geq 0}$ are sequences in $(0, 1]$ and $(\varepsilon_n)_{n\geq 0}$ is a sequence in $\mathcal{H}$.

Theorem 3.1. *Let $A, B : \mathcal{H} \to 2^{\mathcal{H}}$ be two maximal monotone operators and $C : \mathcal{H} \to \mathcal{H}$ be δ-cocoercive with $\delta > 0$. Suppose that $\mathrm{zer}(A + B + C) \neq \emptyset$. Let $(\theta_n)_{n\geq 1}$ be a sequence in $[0, \theta]$ with $\theta \in [0, 1)$ and $\mu \in (0, 2\delta)$. Let $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 1}$ and $(z_n)_{n\geq 1}$ be generated by* Algorithm **2**. *Assume that the Assumption 1.1 holds and $\sum_{n\geq 1} \theta_n \|x_n - x_{n-1}\| < +\infty$. Then the following statements are true:*

1. *$(x_n)_{n\geq 0}$ strongly converges to $x^* := \mathbf{proj}_{\mathbf{Fix}(T)}(0)$, where $T := J_{\mu A} \circ (2J_{\mu B} - Id - \mu C \circ J_{\mu B}) + Id - J_{\mu B}$ for some $\mu > 0$.*
2. *$(y_n)_{n\geq 1}$ and $(z_n)_{n\geq 1}$ strongly converge to $J_{\mu B}(x^*) \in \mathrm{zer}(A + B + C)$.*

Using similar arguments as in Theorem 3.1 and set $Bx = 0$ for all $x \in \mathcal{H}$, it yields the following results.

Corollary 3.2. *Let $A : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator and $C : \mathcal{H} \to \mathcal{H}$ a δ-cocoercive operator with $\delta > 0$ and $\mathrm{zer}(A + C) \neq \emptyset$. Let $\mu \in (0, 2\delta)$ and $(x_n)_{n\geq 0}$ be generated by the following iterative scheme*

$$\begin{cases} x_0, x_1 \in \mathcal{H}, \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (1 - \alpha_n)\delta_n y_n + \alpha_n J_{\mu A}(\delta_n y_n - \mu C \delta_n y_n) + \varepsilon_n, \end{cases} \tag{3.3}$$

for all $n \geq 1$, where $(\theta_n)_{n\geq 1} \subseteq [0, \theta]$ with $\theta \in [0, 1)$, and $(\alpha_n)_{n\geq 0}$ and $(\delta_n)_{n\geq 0}$ are sequences in $(0, 1]$ and $(\varepsilon_n)_{n\geq 0}$ is a sequence in $\mathcal{H}$. Assume that the Assumption 1.1 holds and $\sum_{n\geq 1} \theta_n \|x_n - x_{n-1}\| < +\infty$.

Then, the sequence $(x_n)_{n\geq 0}$ strongly converges to a point $\mathbf{proj}_{\mathbf{zer}(A+C)}(0)$.

Using similar arguments as in Theorem 3.1 and set $Bx = 0$ for all $x \in \mathcal{H}$, we can prove the following results.

Corollary 3.3. *Let $A : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator and $C : \mathcal{H} \to \mathcal{H}$ a δ-cocoercive operator with $\delta > 0$ and $\mathbf{zer}(A + C) \neq \emptyset$. Let $\mu \in (0, 2\delta)$ and $(x_n)_{n\geq 0}$ be generated by the following iterative scheme*

$$\begin{cases} x_0, x_1 \in \mathcal{H}, \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (1 - \alpha_n)\delta_n y_n + \alpha_n J_{\mu A}(\delta_n y_n - \mu C \delta_n y_n) + \varepsilon_n, \end{cases} \tag{3.4}$$

for all $n \geq 1$, where $(\theta_n)_{n\geq 1} \subseteq [0, \theta]$ with $\theta \in [0, 1)$, and $(\alpha_n)_{n\geq 0}$ and $(\delta_n)_{n\geq 0}$ are sequences in $(0, 1]$ and $(\varepsilon_n)_{n\geq 0}$ is a sequence in $\mathcal{H}$. Assume that the Assumption 1.1 holds and $\sum_{n\geq 1} \theta_n \|x_n - x_{n-1}\| < +\infty$.

Then, the sequence $(x_n)_{n\geq 0}$ strongly converges to a point $\mathbf{proj}_{\mathbf{zer}(A+C)}(0)$.

4 Numerical experiments

To illustrate the behavior of the proposed iterative method, we provide a numerical example in a convex minimization problem and compare the convergence performance of the proposed algorithm with some algorithms in the literature. Moreover, we also employ our algorithm in the context of image restoration problems. All the experiments are implemented in MATLAB R2016b running on a MacBook Air 13-inch, Early 2017 with a 1.8 GHz Intel Core i5 processor and 8 GB 1600 MHz DDR3 memory.

4.1 Convex Minimization Problems

In this subsection, we present some comparisons among **Algorithm 2**, **MTA**, and Shehu et al. algorithm (1.2) ([5, Algorothm 3.1]) in convex minimization problem.

Example 4.1. Let $f : \mathbb{R}^s \to \mathbb{R}$ be defied by $f(x) = \|x\|_1$ for all $x \in \mathbb{R}^s$, $g : \mathbb{R}^s \to \mathbb{R}$ be defined by indicator function $g(x) = \delta_W(x)$ with $W := \{x : \mathbf{A}x = \mathrm{b}\}$ for all $x \in \mathbb{R}^s$, where $\mathbf{A} : \mathbb{R}^s \to \mathbb{R}^l$ is a non-zero linear transformation, $\mathrm{b} \in \mathbb{R}^l$ and $s > l$ and $h : \mathbb{R}^s \to \mathbb{R}$ be defied by $h(x) = \frac{1}{2}\|x\|_2^2$ for all $x \in \mathbb{R}^s$. Since $s > l$, we get that $\mathbf{A}$ is an asymmetric transformation. Finding the solution of the following problem:

$$\begin{aligned} &\text{minimize } \|x\|_1 + \delta_W(x) + \frac{1}{2}\|x\|_2^2 \\ &\text{subject to } x \in \mathbb{R}^s. \end{aligned} \tag{4.1}$$

The problem (4.1) can be written in the form of the problem (3.1) as:

$$\text{find } x \in \mathbb{R}^s \text{ such that } 0 \in \partial\|x\|_1 + \partial\delta_W(x) + \nabla h(x), \tag{4.2}$$

where $A = \partial\|\cdot\|_1$, $B = \partial\delta_W(\cdot)$ and $C = \nabla h(\cdot)$.

In this setting, we have $J_{\mu\partial\delta_W}(x) = x + \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}(\mathrm{b} - \mathbf{A}x)$,

$$J_{\mu\partial\|\cdot\|_1}(x) = (\max\{0, 1 - \frac{\mu}{|x^1|}\}x_1, \max\{0, 1 - \frac{\mu}{|x^2|}\}x_2, ..., \max\{0, 1 - \frac{\mu}{|x^s|}\}x_s),$$

and $\nabla h(x) = x$, where $x = (x^1, x^2, ..., x^s) \in \mathbb{R}^s$.

We begin with the problem by random vectors $z, x_0, x_1 \in \mathbb{R}^s$ and $\mathrm{b} \in \mathbb{R}^l$ and matrix $\mathbf{A} \in \mathbb{R}^{l\times s}$. Next, we compare the **Algorithm 2** performance with two remained performance. The parameters that are used in our algorithm are chosen as follows: $\alpha_n = 1 - \frac{1}{(n+2)^2}$, $\delta_n = 1 - \frac{1}{n+2}$, $\varepsilon_n = \frac{z}{(100n)^2}$, and

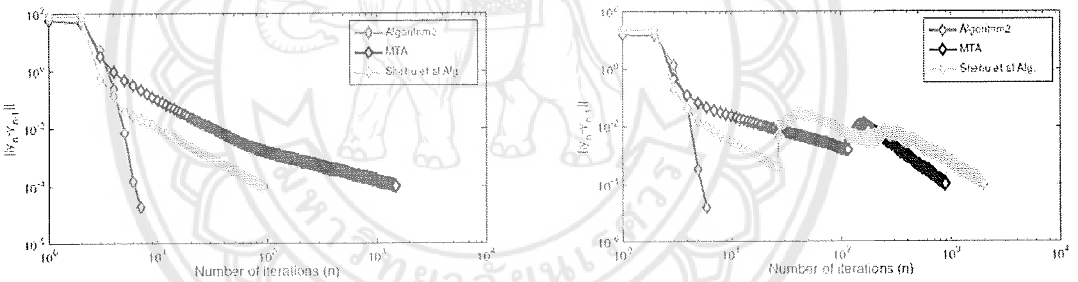
$$\theta_n = \begin{cases} \min\left\{\frac{1}{2}, \frac{1}{(n+1)^2\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \frac{1}{2}, & \text{otherwise.} \end{cases} \tag{4.3}$$

We choose $\alpha_n = \frac{1}{n+1}$, $\delta_n = \gamma_n = \frac{1}{2(n+1)}$ and $e_n = \varepsilon_n$ for the algorithm of Shehu et al. (1.2) in [5]. We obtain the CPU times (seconds) and the number of iterations by using the stopping criteria : $\|y_n - y_{n-1}\| \leq 10^{-4}$.

Table 1: Comparison: **Algorithm 2**, MTA and Shehu et al. Alg. (1.2)

(l, s)	**Algorithm 2**		MTA		**Shehu et al. Alg.** (1.2)	
	CPU Time (s)	Iterations	CPU Time (s)	Iterations	CPU Time (s)	Iterations
(20,700)	0.0218	7	0.0428	278	0.0756	626
(20,800)	0.0189	7	0.0914	350	0.1745	796
(20,7000)	0.0302	7	1.7751	1273	0.0977	53
(20,8000)	0.0308	6	1.2419	1290	0.0671	54
(200,7000)	0.0365	8	1.9452	858	4.6538	2028
(200,8000)	0.0406	7	2.5115	977	0.1425	53
(500,7000)	0.0403	7	4.1647	892	8.3620	1956
(500,8000)	0.0548	8	4.3239	813	9.0929	1835
(1000,7000)	0.0703	7	6.7954	786	14.1693	1751
(1000,8000)	0.0728	7	7.8302	825	16.3752	1784
(3000,7000)	0.1597	7	18.0559	779	44.8129	1940
(3000,8000)	0.1763	7	22.3514	841	49.6872	1891
(100,80000)	0.1376	8	26.6863	1489	1.5926	94
(1000,80000)	0.6949	8	344.7048	3289	9.4181	93

In table 1 we present a comparison among the numerical results of **Algorithm 2**, **MTA**, and Shehu et al. Algorithm (1.2) in different sizes of matrix **A**. The smallest number of iterations is generated by **Algorithm 2** for all different sizes of matrix **A**. Moreover, **Algorithm 2** requires the least CPU computation time to reach the optimality tolerance for all cases.



(a) Case: $(l, s) = (100, 80, 000)$ (b) Case: $(l, s) = (500, 7000)$

Figure 1: Illustration the behavior of $\|y_n - y_{n-1}\|$ for **Algorithm 2**, MTA, and Shehu et al. Alg. (1.2)

Figure 1 shows the behavior of $\|y_k - y_{k-1}\|$ for **Algorithm 2**, MTA, and Shehu et al. Algorithm (1.2) in two different choices of (l, s). We can observe that by using our algorithm the behavior of the red line **Algorithm 2** is the best performance.

4.2 Image Restoration Problems

In this subsection, we apply the proposed algorithm, image restoration problems, which involves deblurring and denoising images. We consider the degradation model that represents an actual image restoration problems or through the least useful mathematical abstractions thereof.

$$y = Hx + w, \tag{4.4}$$

where y, H, x and w represent the degraded image, degradation operator or blurring operator, original image and noise operator, respectively.

The reconstructed image is obtained by solving the following regularized least-squares problem

$$\min_x \left\{ \frac{1}{2}\|Hx - y\|_2^2 + \mu\phi(x) \right\}, \tag{4.5}$$

where $\mu > 0$ is the regularization parameter and $\phi(\cdot)$ is the regularization functional. Well-known regularization functional that is used to remove noise in the restoration problem is the l_1 norm, which is called Tikhonov regularization [7]. The problem (4.5) can be written in the form of the following problem as:

$$\text{find } x \in \arg\min_{x\in\mathbb{R}^k} \left\{ \frac{1}{2}\|Hx - y\|_2^2 + \mu\|x\|_1 \right\}, \tag{4.6}$$

where y is the degraded image and H is a bounded linear operator. Note that problem (4.6) is a spacial case of problem (1.4) by setting $A = \partial f(\cdot)$, $B = 0$, and $C = \nabla L(\cdot)$ where $f(x) = \|x\|_1$ and $L(x) = \frac{1}{2}\|Hx - y\|_2^2$. This setting we have that $C(x) = \nabla L(x) = H^*(Hx - y)$, where H^* is a transpose of H. We begin the problem by choosing images and degrade them by random noise and different types of blurring. The random noise in this study is provided by Gaussian white noise of zero mean and 0.001 variance. We solve the problem (4.6) by using our algorithm in Corollary 3.3. We set $\alpha_n = 1 - \frac{1}{(n+1)^2}$, $\delta_n = 1 - \frac{1}{100n+1}$, $\mu = 0.001$, $\varepsilon_n = 0$ and θ_n is defined as (4.3).

We compare our proposed algorithm with the inertial Mann-type algorithm that was introduced by Kitkuan et al. [4]. In Kitkuan et al. Algorithm ([4, Algorithm in Theorem 3.1]), we choose $\varsigma_n = \theta_n$, $\alpha_n = \frac{1}{n+1}$, $\lambda_n = 0.001$ and $h(x) = \frac{1}{12}\|x\|_2^2$. We assess the quality of the reconstructed image by using the signal to noise ratio (SNR) for monochrome images which is defined by

$$\mathrm{SNR}(n) = 20\log_{10}\frac{\|x\|_2^2}{\|x - x_n\|_2^2},$$

where x and x_n denote the original and the restored image at iteration n, respectively.

For colour images, we estimate the quality of the reconstructed image by using the normalized colour difference (NCD) [8] which is defined by

$$\mathrm{NCD}(n) = \frac{\sum_{i=1}^{N}\sum_{j=1}^{M}\sqrt{(L_{i,j}^o - L_{i,j}(n))^2 + (u_{i,j}^o - u_{i,j}(n))^2 + (v_{i,j}^o - v_{i,j}(n))^2}}{\sum_{i=1}^{N}\sum_{j=1}^{M}\sqrt{(L_{i,j}^o)^2 + (u_{i,j}^o)^2 + (v_{i,j}^o)^2}},$$

where i, j are indices of the sample position, N, M characterize an image size and $L_{i,j}^o$, $u_{i,j}^o$, $v_{i,j}^o$ and $L_{i,j}(n)$, $u_{i,j}(n)$, $v_{i,j}(n)$ are values of the perceived lightness and two representatives of chrominance related to the original and the restored image at iteration n, respectively.

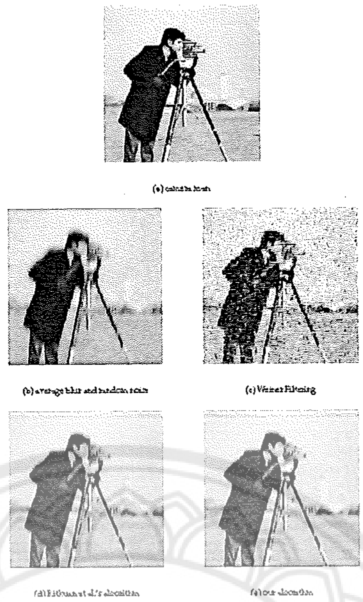


Figure 2: Figure (a) shows the original image 'camera man', figure (b) shows the images degraded by average blur and random noise (Gaussian noise) and figure (c), (d), (e) show the reconstructed image by using Weiner filter, Kitkuan et al. algorithm, and our algorithm (3.4)., respectively.

Figure 3: Figure (a) shows the original image 'Artsawang', figure (b) shows the images degraded by Gaussian blur and random noise (Gaussian noise) and figure (c), (d), (e) show the reconstructed image by using Weiner filter, Kitkuan et al. algorithm, and our algorithm (3.4)., respectively.

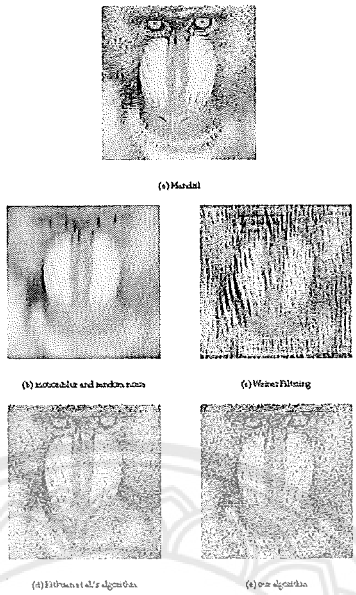
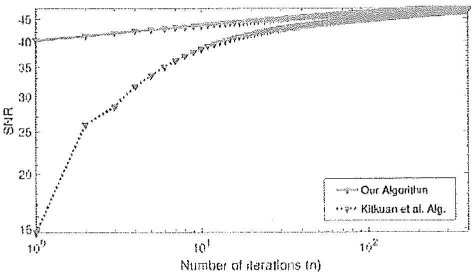
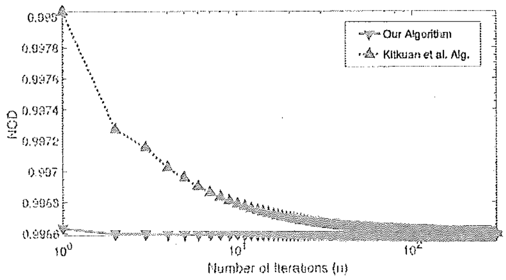


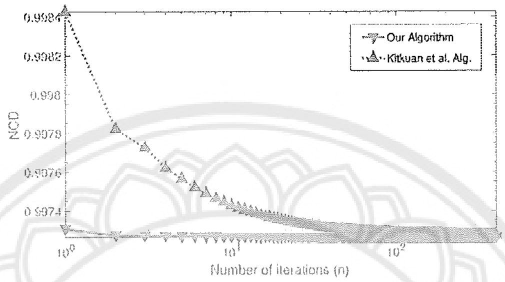
Figure 4: Figure (a) shows the original image 'Mandril', figure (b) shows the images degraded by motion blur and random noise (Gaussian noise) and figure (c), (d), (e) show the reconstructed image by using Weiner filter, Kitkuan et al. algorithm, and our algorithm (3.4)., respectively.



(a) camera man

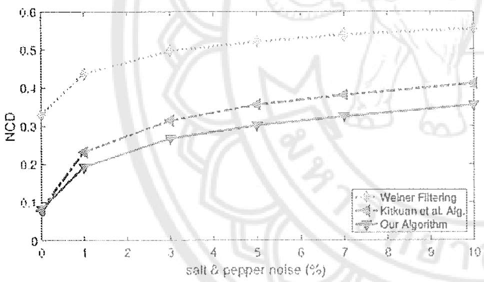


(b) Artsawang

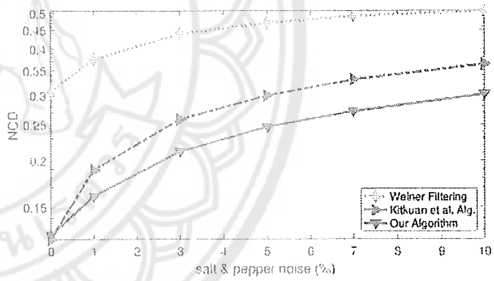


(c) Mandril

Figure 5. (a) The behavior of SNR for two algorithms in Figure 2d,e; (b) the behavior of NCD for two algorithms in Figure 3d,e; and (c) the behavior of NCD for two algorithms in Figure 4d,e.



(a) Artsawang



(b) Mandril

Figure 6. (a,b) The behavior of NCD in motion blur and different different salt and pepper noise from 0% to 10%.

Table 2. The performance of the normalized color difference (NCD) in two images.

The Normalized Color Difference (NCD).				
	Kitkuan et al.'s Algorithm		Our Algorithm in Equation (19)	
n	Artsawang Image	Mandril Image	Artsawang Image	Mandril Image
1	0.99803	0.99842	0.99663	0.99731
50	0.99660	0.99730	0.99659	0.99727
100	0.99661	0.99729	0.99658	0.99726
200	0.99660	0.99728	0.99658	0.99726
300	0.99659	0.99727	0.99658	0.99726
400	0.99659	0.99727	0.99658	0.99726

5 Conclusion

In this project, we proposed a new Mann-type method combining both inertial terms and errors to solve the fixed point problem for a nonexpansive mapping. We also prove the strong convergence of the proposed algorithm under some sufficient conditions of involved parameters. We also apply results to approximate the solutions of the monotone inclusion problems. Furthermore, we also provided a numerical example to compare the proposed algorithm with other algorithms in the convex minimization problem. Finally, we use our method to solve image restoration problems.

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CHAPTER III

OUTPUT

ผลลัพธ์จากโครงการวิจัยที่ได้รับทุนจากงบประมาณรายได้มหาวิทยาลัยนเรศวร ประจำปี 2563

1. ผลงานวิจัยตีพิมพ์ในวารสารวิชาการนานาชาติ

- Natthaphon Artsawang and **Kasamsuk Ungchittrakool** (2020). Inertial Mann-type algorithm for a nonexpansive mapping to solve monotone inclusion and image restoration problems. Symmetry. 12, 750: 17 pages (ISI Impact Factor 2020 : 2.645)

2. การนำผลงานวิจัยไปใช้ประโยชน์

มีการนำไปใช้ประโยชน์ทั้งเชิงวิชาการและเชิงสาธารณะโดยทำให้มีการพัฒนาการเรียนการสอน โดยจะเห็นได้ว่าผลงานจากโครงการวิจัยนี้มีบทประยุกต์ไปใช้กับปัญหาการกู้คืนรูปภาพ (Image restoration) ให้มีประสิทธิภาพที่ดีกว่าเดิม รวมทั้งมีการสร้างเครือข่ายความร่วมมือในการทำวิจัย

ภาคผนวก

Inertial Mann-type algorithm for a nonexpansive mapping to solve monotone inclusion and image restoration problems

Natthaphon Artsawang and Kasamsuk Ungchittrakool

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Article

Inertial Mann-Type Algorithm for a Nonexpansive Mapping to Solve Monotone Inclusion and Image Restoration Problems

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Abstract: In this article, we establish a new Mann-type method combining both inertial terms and errors to find a fixed point of a nonexpansive mapping in a Hilbert space. We show strong convergence of the iterate under some appropriate assumptions in order to find a solution to an investigative fixed point problem. For the virtue of the main theorem, it can be applied to an approximately zero point of the sum of three monotone operators. We compare the convergent performance of our proposed method, the Mann-type algorithm without both inertial terms and errors, and the Halpern-type algorithm in convex minimization problem with the constraint of a non-zero asymmetric linear transformation. Finally, we illustrate the functionality of the algorithm through numerical experiments addressing image restoration problems.

Keywords: inertial method; Mann-type algorithm; monotone inclusion problem; nonexpansive mapping

MSC: 47H04; 47H10; 65K05; 90C25

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1. Introduction

Throughout this article, $\mathcal{H}$ is defined as a real Hilbert space with an inner product and corresponding norm which is denoted by the notations $\langle \cdot, \cdot \rangle$ and $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$, respectively. Let $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping, that is, $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in \mathcal{H}$. Given C a nonempty closed convex subset of $\mathcal{H}$. The set of all fixed points of the operator T is denoted by $\mathbf{Fix}(T) := \{x \in \mathcal{H} : Tx = x\}$. The metric projection of $\mathcal{H}$ onto C, $\mathbf{proj}_C : \mathcal{H} \to C$ is defined by $\mathbf{proj}_C(x) = \arg\min_{c \in C} \|x - c\|$ for all $x \in \mathcal{H}$ (see more detail in [1] and the references therein).

Problem: The fixed point problem for the mapping T is generally denoted as,

$$\text{find } x \in \mathcal{H} \text{ such that } x = Tx.$$

Many problems in the real world, such as optimal control problems, economic modelings, variational analysis, game theory, data analysis, etc. can be formed into the fixed point problem of nonexpansive mappings (see Bagiror et al.'s book [2] for more applications and recent developments). A solution of the fixed point problem for nonexpansive mappings was approximated by the iterative method which was introduced by Mann [3]. In addition, the "Mann Iteration" stated that

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad \forall n \geq 1, \tag{1}$$

where $x_1 \in \mathcal{H}$ and $(\alpha_n)_{n \geq 1}$ is a real sequence in $[0, 1]$. The weak convergent result of the iterative sequence $(x_n)_{n \geq 1}$ was obtained under control condition that $\sum_{n \geq 1} \alpha_n (1 - \alpha_n) = +\infty$ (see [4,5]).

To obtain the strong convergence for the fixed point solutions of nonexpansive mappings, one of the most important methods to solve the fixed point problem for a nonexpansive mapping was introduced by Halpern [6]:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad \forall n \geq 1, \tag{2}$$

where $x_1, u \in \mathcal{H}$ and $(\alpha_n)_{n\geq 1}$ is a real sequence in $[0,1]$. In direction to study and improve this algorithm in Equation (2), many results have been presented (see [7–14]). In 2000, Moudafi [15] proposed iterative method which involved the concept of viscosity to solve strong convergence of the iterate. Moreover, many authors are interested in studying and developing Moudafi's algorithm. The several methods that are in reference to this study are reviewed in the next extensively (see, for example, [7,16–20]). Recently, Bot et al. [21] proposed a new Mann-type algorithm **(MTA)** to solve the fixed point problem for a nonexpansive mapping and proved strong convergence of the iterate without using viscosity and projection method under some control conditions of parameters sequences. Their algorithm is defined by

$$\text{(MTA)}\ x_{n+1} = (1 - \alpha_n)\delta_n x_n + \alpha_n T\delta_n x_n, \quad \forall n \geq 1,$$

where $x_1 \in \mathcal{H}$ and $(\alpha_n)_{n\geq 0}$, $(\delta_n)_{n\geq 0}$ are sequences in $(0,1]$.

Polyak [22] firstly proposed an inertial extrapolation as an acceleration process to solve the smooth convex minimization problem. An inertial algorithm is a two-step iterative method and the next iterate is defined by making use of the previous two iterates. It is well known that combining an inertial term in an algorithm can accelerate the speed of convergence of the sequence generated by the algorithm. Subsequently, there are many authors who are interested in studying the inertial-type algorithm. We refer interested readers to [23–31] for more information. In 2015, Combettes and Yamada [32] presented a new Mann algorithm combining error term for solving a common fixed point of averaged nonexpansive mappings in a Hilbert space. By using the concept of the inertial method, the technique of Halpern method, and error terms, Shehu et al. [33] introduced an algorithm for solving a fixed point of a nonexpansive mapping, which is defined as follows:

$$\begin{cases} x_0, x_1 \in \mathcal{H}, \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = \alpha_n x_0 + \delta_n y_n + \gamma_n T y_n + e_n, \end{cases} \tag{3}$$

for all $n \geq 1$, where $(\theta_n)_{n\geq 0} \subseteq [0,\theta]$ with $\theta \in [0,1)$, $(\alpha_n)_{n\geq 0}$, $(\delta_n)_{n\geq 0}$ and $(\gamma_n)_{n\geq 0}$ are sequences in $(0,1]$ and $(e_n)_{n\geq 0}$ is a sequence in $\mathcal{H}$.

Being motivated by the above facts, we intend to accelerate the speed of convergence by avoiding the viscosity concept, hence, we propose a Mann-type method combining both inertial terms and errors for finding a fixed point of a nonexpansive mapping in a Hilbert space.

Let a nonexpansive mapping T from $\mathcal{H}$ into itself be such that $\mathbf{Fix}(T) \neq \emptyset$. We propose the following algorithm.

$$\textbf{(Algorithm 1)}\ \begin{cases} x_0, x_1 \in C, \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = \delta_n y_n + \alpha_n(T\delta_n y_n - \delta_n y_n) + \varepsilon_n, \end{cases}$$

for all $n \geq 1$, where $(\theta_n)_{n\geq 0} \subseteq [0,\theta]$ with $\theta \in [0,1)$, $(\alpha_n)_{n\geq 0}$ and $(\delta_n)_{n\geq 0}$ are sequences in $(0,1]$ and $(\varepsilon_n)_{n\geq 0}$ is a sequence in $\mathcal{H}$.

On the other hand, for the set of all zeros of the sum of three monotone operators A, B, C as the following

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in Ax + Bx + Cx, \tag{4}$$

where A, B, C are maximal monotone operators on a Hilbert space $\mathcal{H}$ and C is δ-cocoercive with parameter δ. The problem in Equation (4) was considered by Davis and Yin [34] and it can be reformulated to the fixed point problem for nonexpansive mappings. Therefore, it is interesting to study the fixed point problem in order to apply for solving the zeros problem of maximal monotone operators.

For the applications, we can formulate the main problem, that is, the fixed point problem in order to apply in the case of finding a zero point of the sum of three maximal monotone operators. Furthermore, the convergence behavior between the algorithms obtained from Algorithm 1 are illustrated by some numerical experiment.

2. Preliminaries

This section gathers the results in real Hilbert spaces that are useful for this study, e.g., convergence analysis.

Lemma 1. [20] *Let $\mathcal{H}$ be a real Hilbert space. The conditions are verifiable, as follows.*

1. $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$ *for all* $x, y \in \mathcal{H}$,
2. $\|x+y\|^2 \leq \|x\|^2 + 2\langle x+y, y\rangle$ *for all* $x, y \in \mathcal{H}$,
3. $\|rx + (1-r)y\|^2 = r\|x\|^2 + (1-r)\|y\|^2 - r(1-r)\|x-y\|^2$ *for all* $r \in [0,1]$ *and* $x, y \in \mathcal{H}$.

Lemma 2. [14,35] *Let $(a_n)_{n\geq 1}$ and $(\mu_n)_{n\geq 0}$ be sequences of nonnegative real numbers and satisfy the inequality*

$$a_{n+1} \leq (1-\delta_n)a_n + \mu_n + \varepsilon_n \quad \forall n \geq 0,$$

where $0 \leq \delta_n \leq 1$ for all $n \geq 0$. Assume that $\sum_{k\geq 1} \varepsilon_n < +\infty$. Then, the following statement hold:

1. *If $\mu_n \leq c\delta_n$ (where $c \geq 0$), then $(a_n)_{n\geq 1}$ is bounded.*
2. *If $\sum_{n\geq 0} \delta_n = \infty$ and $\limsup_{n\to+\infty} \frac{\mu_n}{\delta_n} \leq 0$, then the sequence $(a_n)_{n\geq 0}$ converges to 0.*

Lemma 3. [1] *Let T be a nonexpansive operator from $\mathcal{H}$ into itself. Let $(x_n)_{n\geq 0}$ be a sequence in $\mathcal{H}$ and $x \in \mathcal{H}$ such that $x_n \rightharpoonup x$ as $n \to +\infty$ (i.e., $(x_n)_{n\geq 0}$ converges weakly to x) and $x_n - Tx_n \to 0$ as $n \to +\infty$ (i.e., $(x_n - Tx_n)_{n\geq 0}$ converges strongly to 0). Then, $x \in \mathbf{Fix}(T)$.*

Assumption 1. *Let $(\alpha_n)_{n\geq 0}$ and $(\delta_n)_{n\geq 0}$ be sequences in $(0,1]$ and let $(\varepsilon_n)_{n\geq 0}$ be a sequence in $\mathcal{H}$. Assume the conditions are verifiable, as follows.*

1. $\liminf_{n\to+\infty} \alpha_n > 0$ *and* $\sum_{n\geq 1} |\alpha_n - \alpha_{n-1}| < +\infty$,
2. $\lim_{n\to+\infty} \delta_n = 1$, $\sum_{n\geq 0}(1-\delta_n) = +\infty$ *and* $\sum_{n\geq 1} |\delta_n - \delta_{n-1}| < +\infty$,
3. $\sum_{n\geq 0} \|\varepsilon_n\| < +\infty$.

We have verified Assumption 1, as shown in the following remark.

Remark 1. *Let $z \in \mathcal{H}$. We set $\delta_n = 1 - \frac{1}{n+2}$, $\alpha_n = \frac{1}{4} - \frac{1}{(n+3)^2}$ and $\varepsilon_n = \frac{z}{(n+1)^3}$ for all $n \geq 0$. It is easy to see that Assumption 1 is satisfied.*

3. Main Results

This section discusses the convergence analysis of the proposed algorithm, beginning with given boundedness of our algorithm, as in the following lemma.

Lemma 4. *Let $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping such that $\mathbf{Fix}(T) \neq \emptyset$ and let $(x_n)_{n\geq 0}$ be generated by Algorithm 1. Let $(\theta_n)_{n\geq 0}$ be a sequence in $[0,\theta]$ with $\theta \in [0,1)$ such that $\sum_{n\geq 1} \theta_n \|x_n - x_{n-1}\| < +\infty$. Suppose Assumption 1 holds. Then, $(x_n)_{n\geq 0}$ is bounded.*

Proof. Let $n \in \mathbb{N}$ and a sequence $(z_n)_{n\geq 1}$ be defined by

$$z_{n+1} = \delta_n z_n + \alpha_n(T\delta_n z_n - \delta_n z_n) + \varepsilon_n.$$

By nonexpansiveness of T, we have

$$\begin{aligned} \|x_{n+1} - z_{n+1}\| &= \|(1-\alpha_n)\delta_n(y_n - z_n) + \alpha_n(T\delta_n y_n - T\delta_n z_n)\| \\ &\leq (1-\alpha_n)\delta_n\|y_n - z_n\| + \alpha_n\delta_n\|y_n - z_n\| \\ &= \delta_n\|y_n - z_n\| \\ &= \delta_n\|x_n - z_n + \theta_n(x_n - x_{n-1})\| \\ &\leq \delta_n\|x_n - z_n\| + \delta_n\theta_n\|x_n - x_{n-1}\| \\ &\leq \delta_n\|x_n - z_n\| + \theta_n\|x_n - x_{n-1}\|. \end{aligned} \tag{5}$$

By applying Lemma 2, we have $\lim\limits_{n\to+\infty}\|x_n - z_n\| = 0$.

Next, we expect that $(z_n)_{n\geq 1}$ is bounded. Let $x^* \in \mathbf{Fix}(T)$. It follows that

$$\begin{aligned} \|z_{n+1} - x^*\| &\leq \|\delta_n z_n + \alpha_n(T\delta_n z_n - \delta_n z_n + \varepsilon_n - x^*)\| \\ &\leq (1-\alpha_n)\|\delta_n z_n - x^*\| + \alpha_n\|T\delta_n z_n - x^*\| + \|\varepsilon_n\| \\ &\leq \|\delta_n z_n - x^*\| + \|\varepsilon_n\| \\ &= \|\delta_n(z_n - x^*) + (\delta_n - 1)x^*\| + \|\varepsilon_n\| \\ &\leq \delta_n\|z_n - x^*\| + (1-\delta_n)\|x^*\| + \|\varepsilon_n\|. \end{aligned} \tag{6}$$

Notice that $\sum_{n\geq 0}\varepsilon_n < +\infty$. We can apply Lemma 2 to obtain that $(z_n)_{n\geq 1}$ is bounded. Seeing that $\lim\limits_{n\to+\infty}\|x_n - z_n\| = 0$ and $(z_n)_{n\geq 1}$ is bounded, we get that $(x_n)_{n\geq 0}$ is bounded. □

Theorem 1. *Let $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping such that $\mathbf{Fix}(T) \neq \emptyset$ and let $(x_n)_{n\geq 0}$ be generated by Algorithm 1. Let $(\theta_n)_{n\geq 0}$ be a sequence in $[0,\theta]$ with $\theta \in [0,1)$ such that $\sum_{n\geq 1}\theta_n\|x_n - x_{n-1}\| < +\infty$. Suppose Assumption 1 holds. Then, the sequence $(x_n)_{n\geq 0}$ strongly converges to $x^* := \mathbf{proj}_{\mathbf{Fix}(T)}(0)$.*

Proof. From Lemma 4, we have $(x_n)_{n\geq 0}$ is bounded. Moreover, $(y_n)_{n\geq 1}$ is also bounded. Let $x^* := \mathbf{proj}_{\mathbf{Fix}(T)}(0)$. Then, $x^* \in \mathbf{Fix}(T)$. By using Lemma 1 and Equation (2), we get that

$$\begin{aligned} \|\delta_n y_n - x^*\|^2 &= \|\delta_n(y_n - x^*) + (\delta_n - 1)x^*\|^2 \\ &= \delta_n^2\|y_n - x^*\|^2 + 2\delta_n(1-\delta_n)\langle -x^*, y_n - x^*\rangle + (1-\delta_n)^2\|x^*\|^2 \\ &\leq \delta_n\|x_n - x^* + \theta_n(x_n - x_{n-1})\|^2 + (1-\delta_n)\left(2\delta_n\langle -x^*, y_n - x^*\rangle + (1-\delta_n)\|x^*\|^2\right) \\ &\leq \delta_n\|x_n - x^*\|^2 + 2\delta_n\langle\theta_n(x_n - x_{n-1}), y_n - x^*\rangle \\ &\quad + (1-\delta_n)\left(2\delta_n\langle -x^*, y_n - x^*\rangle + (1-\delta_n)\|x^*\|^2\right). \end{aligned} \tag{7}$$

By using Lemma 1 and the nonexpansiveness of T, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|\delta_n y_n + \alpha_n(T\delta_n y_n - \delta_n y_n) + \varepsilon_n - x^*\|^2 \\ &= \|(1-\alpha_n)(\delta_n y_n - x^*) + \alpha_n(T\delta_n y_n - x^*) + \varepsilon_n\|^2 \\ &\leq \|(1-\alpha_n)(\delta_n y_n - x^*) + \alpha_n(T\delta_n y_n - x^*)\|^2 + 2\langle\varepsilon_n, x_{n+1} - x^*\rangle \\ &= (1-\alpha_n)\|\delta_n y_n - x^*\|^2 + \alpha_n\|T\delta_n y_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|T\delta_n y_n - \delta_n y_n\|^2 \\ &\quad + 2\langle\varepsilon_n, x_{n+1} - x^*\rangle \\ &\leq \|\delta_n y_n - x^*\|^2 + 2\langle\varepsilon_n, x_{n+1} - x^*\rangle. \end{aligned} \tag{8}$$

Combining Equations (7) and (8), we obtain that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \delta_n \|x_n - x^*\|^2 + (1-\delta_n)\left(2\delta_n\langle -x^*, y_n - x^*\rangle + (1-\delta_n)\|x^*\|^2\right) \\
&\quad + 2\delta_n\langle \theta_n(x_n - x_{n-1}), y_n - x^*\rangle + 2\langle \varepsilon_n, x_{n+1} - x^*\rangle \\
&\leq \delta_n \|x_n - x^*\|^2 + (1-\delta_n)\left(2\delta_n\langle -x^*, y_n - x^*\rangle + (1-\delta_n)\|x^*\|^2\right) \\
&\quad + 2\delta_n\|y_n - x^*\|\left(\theta_n\|(x_n - x_{n-1})\|\right) + 2\|x_{n+1} - x^*\|(\|\varepsilon_n\|). \qquad (9)
\end{aligned}
$$

Next, we claim that $\|x_{n+1} - x_n\| \to 0$ as $n \to +\infty$. By the boundedness of a sequence $(y_n)_{n\geq 1}$ and the nonexpansiveness of T, we have

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= \|\delta_n y_n + \alpha_n(T\delta_n y_n - \delta_n y_n) + \varepsilon_n - (\delta_{n-1}y_{n_1} + \alpha_{n_1}(T\delta_{n-1}y_{n-1} - \delta_{n-1}y_{n-1}) + \varepsilon_{n-1})\| \\
&\leq \|(1-\alpha_n)(\delta_n y_n - \delta_{n-1}y_{n-1}) + (\alpha_n - \alpha_{n-1})\delta_{n-1}y_{n-1}\| \\
&\quad + \|\alpha_n(T\delta_n y_n - T\delta_{n-1}y_{n-1}) + (\alpha_n - \alpha_{n-1})T\delta_{n-1}y_{n-1}\| + \|\varepsilon_n - \varepsilon_{n-1}\| \\
&\leq \|\delta_n y_n - \delta_{n-1}y_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\|\delta_{n-1}y_{n-1}\| + \|T\delta_{n-1}y_{n-1}\|) + \|\varepsilon_n - \varepsilon_{n-1}\| \\
&\leq \|\delta_n y_n - \delta_{n-1}y_{n-1}\| + |\alpha_n - \alpha_{n-1}|C_1 + \|\varepsilon_n - \varepsilon_{n-1}\|, \qquad (10)
\end{aligned}
$$

where $C_1 > 0$. After that, we consider the term $\|\delta_n y_n - \delta_{n-1}y_{n-1}\|$ in the inequality in Equation (10).

Let us consider,

$$
\begin{aligned}
\|\delta_n y_n - \delta_{n-1}y_{n-1}\| &= \|\delta_n(y_n - y_{n-1}) + (\delta_n - \delta_{n-1})y_{n-1}\| \\
&\leq \delta_n\|y_n - y_{n-1}\| + |\delta_n - \delta_{n-1}|(\|y_{n-1}\|) \\
&\leq \delta_n\|x_n - x_{n-1}\| + \delta_n\theta_n\|x_n - x_{n-1}\| + \delta_n\theta_{n-1}\|x_{n-1} - x_{n-2}\| \\
&\quad + |\delta_n - \delta_{n-1}|(\|y_{n-1}\|) \\
&\leq \delta_n\|x_n - x_{n-1}\| + \theta_n\|x_n - x_{n-1}\| + \theta_{n-1}\|x_{n-1} - x_{n-2}\| \\
&\quad + |\delta_n - \delta_{n-1}|C_2, \qquad (11)
\end{aligned}
$$

where $C_2 > 0$. Combining Equations (10) and (11), we get that

$$
\begin{aligned}
\|x_{n+1} - x_n\| &\leq \delta_n\|x_n - x_{n-1}\| + \theta_n\|x_n - x_{n-1}\| + \theta_{n-1}\|x_{n-1} - x_{n-2}\| + |\alpha_n - \alpha_{n-1}|C_1 \\
&\quad + |\delta_n - \delta_{n-1}|C_2 + \|\varepsilon_n - \varepsilon_{n-1}\|. \qquad (12)
\end{aligned}
$$

By applying Lemma 2 and Assumption 1, we can conclude that $\|x_{n+1} - x_n\| \to 0$ as $n \to +\infty$. In the following, we prove that $\|T\delta_n y_n - \delta_n y_n\| \to 0$ as $n \to +\infty$. We observe that

$$
\begin{aligned}
\|T\delta_n y_n - \delta_n y_n\| &= \|T\delta_n y_n - x_{n+1} + x_{n+1} - \delta_n y_n\| \\
&\leq \|T\delta_n y_n - x_{n+1}\| + \|x_{n+1} - \delta_n y_n\| \\
&= \|(1-\alpha_n)(T\delta_n y_n - \delta_n y_n) - \varepsilon_n\| + \|(1-\delta_n)x_{n+1} + \delta_n x_{n+1} - \delta_n y_n\| \\
&\leq (1-\alpha_n)\|T\delta_n y_n - \delta_n y_n\| + \|\varepsilon_n\| + (1-\delta_n)\|x_{n+1}\| + \delta_n\|x_{n+1} - y_n\| \\
&= (1-\alpha_n)\|T\delta_n y_n - \delta_n y_n\| + \|\varepsilon_n\| + (1-\delta_n)\|x_{n+1}\| + \delta_n\|x_{n+1} - x_n\| \\
&\quad + \delta_n\theta_n\|x_n - x_{n-1}\|. \qquad (13)
\end{aligned}
$$

It follows that

$$
\|T\delta_n y_n - \delta_n y_n\| \leq \frac{1}{\alpha_n}\left(\|\varepsilon_n\| + (1-\delta_n)\|x_{n+1}\| + \delta_n\|x_{n+1} - x_n\| + \delta_n\theta_n\|x_n - x_{n-1}\|\right). \qquad (14)
$$

Since $\lim_{n\to+\infty} \|x_{n+1} - x_n\| = 0$ and the properties of the sequences involved, we can conclude that $\lim_{n\to+\infty} \|T\delta_n y_n - \delta_n y_n\| = 0$.

To show that the sequence $(x_n)_{n\geq 0}$ strongly converges to x^*, it is sufficient to prove that

$$\limsup_{n\to+\infty} \langle -x^*, y_n - x^* \rangle \leq 0. \tag{15}$$

On the other hand, assume that the inequality in Equation (15) does not hold. Then, there exist a real number $k > 0$ and a subsequence $(y_{n_i})_{i\geq 1}$ such that

$$\langle -x^*, y_{n_i} - x^* \rangle \geq k > 0 \ \forall i \geq 1.$$

For $(y_n)_{n\geq 1}$ bounded on a Hilbert space $\mathcal{H}$, we can find a subsequence of $(y_n)_{n\geq 1}$ that weakly converges to a point $y \in \mathcal{H}$. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup y$ as $i \to +\infty$. Therefore,

$$0 < k \leq \lim_{i\to+\infty} \langle -x^*, y_{n_i} - x^* \rangle = \langle -x^*, y - x^* \rangle. \tag{16}$$

Notice that $\lim_{n\to+\infty} \delta_n = 1$. We get $\delta_{n_i} y_{n_i} \rightharpoonup y$ as $i \to +\infty$. Applying Lemma 3, we obtain that $y \in \mathbf{Fix}(T)$. With this, we have $\langle -x^*, y - x^* \rangle \leq 0$, which is a contradiction. Hence, the inequality in Equation (15) is verified. It follows that

$$\limsup_{n\to+\infty} \left(2\delta_n \langle -x^*, y_n - x^* \rangle + (1 - \delta_n)\|x^*\|^2 \right) \leq 0.$$

Using Lemma 2 and Equation (9), we can conclude that $\lim_{n\to+\infty} x_n = x^*$. Based on what is described above, the proof is complete. □

Remark 2. *The assumption of the sequence $(\theta_n)_{n\geq 0}$ in Theorem 1 is verified, if we choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{c_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise}, \end{cases}$$

and $\sum_{n\geq 0} c_n < +\infty$.

4. Applications

This section is devoted to discussing the applications of the algorithm proposed in this paper in the monotone inclusion problems.

The operator $K : \mathcal{H} \to \mathcal{H}$ is called a monotone if it satisfies $\langle Kx - Ky, x - y \rangle \geq 0$ for all $x, y \in \mathcal{H}$ and is said to be *δ-cocoercive* with $\delta > 0$ if there exists a positive real number δ such that $\langle Kx - Ky, x - y \rangle \geq \delta\|Kx - Ky\|^2$ for all $x, y \in \mathcal{H}$. The set of all zeros of the operator K is denoted by $\mathbf{zer}(K) := \{z \in \mathcal{H} : 0 = K(z)\}$.

Let L be a set-valued operator on $\mathcal{H}$ and its graph be denoted by $\mathbf{gra}(L) := \{(x, u) \in \mathcal{H} \times \mathcal{H} : u \in Lx\}$. The operator L is called *maximal monotone* if there exists no proper monotone extension of the graph of L. The operator L is said to be *ρ-strongly monotone* with $\rho > 0$ if $\langle x - y, u - v \rangle \geq \rho\|x - y\|^2$ for all $(x, u), (y, v) \in \mathbf{gra}(L)$.

The *resolvent* of the operator L is denoted by $J_L : \mathcal{H} \to 2^{\mathcal{H}}$ which is defined by $J_L := (Id + L)^{-1}$ where Id is the *identity operator* on $\mathcal{H}$. Furthermore, J_L is a single-valued operator when L is a maximal monotone operator.

Let C be a nonempty closed convex subset of $\mathcal{H}$. The *indicator function* is defined by

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise,} \end{cases}$$

for all $x \in \mathcal{H}$. We consider the monotone inclusion problem as follows:

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in Ax + Bx + Cx, \tag{17}$$

where $A : \mathcal{H} \to 2^{\mathcal{H}}$ and $B : \mathcal{H} \to 2^{\mathcal{H}}$ are maximal monotone operators and $C : \mathcal{H} \to \mathcal{H}$ is a δ-cocoercive operator with $\delta > 0$. We assume that $\mathbf{zer}(A + B + C) \neq \emptyset$. We propose the following algorithm for solving the problem in Equation (17).

$$\text{(Algorithm 2)} \begin{cases} a_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = J_{\mu B}(\delta_n a_n), \\ z_n = J_{\mu A}(2y_n - \delta_n a_n - \mu C y_n), \\ x_{n+1} = \delta_n a_n + \alpha_n(z_n - y_n) + \varepsilon_n, \end{cases}$$

for all $n \geq 1$, where $x_0, x_1 \in \mathcal{H}$, $\mu \in (0, 2\delta)$, $(\theta_n)_{n\geq 1} \subseteq [0, \theta]$ with $\theta \in [0, 1)$, and $(\alpha_n)_{n\geq 0}$ and $(\delta_n)_{n\geq 0}$ are sequences in $(0, 1]$ and $(\varepsilon_n)_{n\geq 0}$ is a sequence in $\mathcal{H}$.

The above iterative scheme can be rewritten as

$$\begin{aligned} x_{n+1} &= \delta_n a_n + \alpha_n [J_{\mu A} \circ (2J_{\mu B} - \mathrm{Id} - \mu C \circ J_{\mu B}) + Id - J_{\mu_k B}](\delta_n a_n) + \varepsilon_n \\ &= \delta_n a_n + \alpha_n (T\delta_n a_n - \delta_n a_n) + \varepsilon_n \end{aligned}$$

where $x_0, x_1 \in \mathcal{H}$, $a_n := x_n + \theta_n(x_n - x_{n-1})$ and

$$T := J_{\mu A} \circ (2J_{\mu B} - Id - \mu C \circ J_{\mu B}) + Id - J_{\mu B}. \tag{18}$$

The following proposition is the important tool for verifying the convergence of Algorithm 2 (see Proposition 2.1 in [34])

Proposition 1. *Let $T_1, T_2 : \mathcal{H} \to \mathcal{H}$ be two firmly nonexpansive operators and C be a δ-cocoercive operator with $\delta > 0$. Let $\mu \in (0, 2\delta)$. Then, operator $T := Id - T_2 + T_1 \circ (2T_2 - Id - \mu C \circ T_2)$ is α-averaged with coefficient $\alpha := \frac{2\delta}{4\delta - \mu} < 1$.*

In particular, the following inequality holds for all $z, w \in \mathcal{H}$

$$\|Tz - Tw\|^2 \leq \|z - w\|^2 - \frac{(1-\alpha)}{\alpha}\|(Id - T)z - (Id - T)w\|^2.$$

The following lemma is a characterization of $\mathbf{zer}(A + B + C)$.

Lemma 5. Lemma 2.2 in [34] *Let $A : \mathcal{H} \to 2^{\mathcal{H}}$ and $B : \mathcal{H} \to 2^{\mathcal{H}}$ be maximal monotone operators and $C : \mathcal{H} \to \mathcal{H}$ be a δ-cocoercive operator with $\delta > 0$. Suppose that $\mathbf{zer}(A + B + C) \neq \emptyset$. Then,*

$$\mathbf{zer}(A + B + C) = J_{\mu B}(\mathbf{Fix}(T)),$$

where $T := J_{\mu A} \circ (2J_{\mu B} - Id - \mu C \circ J_{\mu B}) + Id - J_{\mu B}$ with $\mu > 0$.

Remark 3.

1. *If we set $Cx = 0$ for all $x \in \mathcal{H}$ in Lemma 5, $\mathbf{zer}(A + B) = J_{\mu B}(\mathbf{Fix}(T))$, where $T := J_{\mu A} \circ (2J_{\mu B} - Id) + Id - J_{\mu B}$ with $\mu > 0$.*

2. *If we set* $Bx = 0$ *for all* $x \in \mathcal{H}$ *in Lemma 5,* $\mathbf{zer}(A + C) = \mathbf{Fix}(T)$*, where* $T := J_{\mu A} \circ (Id - \mu C)$ *with* $\mu > 0$.

Theorem 2. *Let* $A, B : \mathcal{H} \to 2^{\mathcal{H}}$ *be two maximal monotone operators and* $C : \mathcal{H} \to \mathcal{H}$ *be* δ*-cocoercive with* $\delta > 0$*. Suppose that* $\mathbf{zer}(A + B + C) \neq \emptyset$*. Let* $(\theta_n)_{n\geq 1}$ *be a sequence in* $[0, \theta]$ *with* $\theta \in [0, 1)$ *and* $\mu \in (0, 2\delta)$*. Let* $(x_n)_{n\geq 0}$*,* $(y_n)_{n\geq 1}$ *and* $(z_n)_{n\geq 1}$ *be generated by Algorithm 2. Assume that Assumption 1 holds and* $\sum_{n\geq 1} \theta_n \|x_n - x_{n-1}\| < +\infty$*. Then, the following statements are true:*

1. $(x_n)_{n\geq 0}$ *strongly converges to* $x^* := \mathbf{proj}_{\mathbf{Fix}(T)}(0)$*, where*
   $T := J_{\mu A} \circ (2J_{\mu B} - Id - \mu C \circ J_{\mu B}) + Id - J_{\mu B}$ *for some* $\mu > 0$.
2. $(y_n)_{n\geq 1}$ *and* $(z_n)_{n\geq 1}$ *strongly converge to* $J_{\mu B}(x^*) \in \mathbf{zer}(A + B + C)$.

Proof. Equation (1): Let $(x_n)_{n\geq 0}$ be generated by Algorithm 2. Then, the iterative method can be rewritten as

$$x_{n+1} = \delta_n a_n + \alpha_n (T\delta_n a_n - \delta_n a_n)$$

where $x_0, x_1 \in \mathcal{H}$, $a_n := x_n + \theta_n(x_n - x_{n-1})$ and $T := J_{\mu A} \circ (2J_{\mu B} - Id - \mu C \circ J_{\mu B}) + Id - J_{\mu B}$.

By applying Proposition 1, we get T is nonexpansive.

On the other hand, by Lemma 5, we obtain that

$$J_{\mu B}(\mathbf{Fix}(T)) = \mathbf{zer}(A + B + C) \neq \emptyset.$$

It means that $\mathbf{Fix}(T) \neq \emptyset$. By applying Theorem 1, we have the sequence $(x_n)_{n\geq 0}$ strongly converges to $x^* := \mathbf{proj}_{\mathbf{Fix}(T)}(0)$ as $n \to +\infty$.

Equation (2): The sequences $(a_n)_{n\geq 0}$ as Algorithm 2, and we obtain that $a_n \to x^*$ as $n \to +\infty$. Since $J_{\mu B}$ is continuous, we have $y_n \to J_{\mu B}(x^*) \in \mathbf{zer}(A + B + C)$. From the last line of Algorithm 2, we get that $\lim_{n\to+\infty} \|z_n - y_n\| = 0$. This proof is complete. □

Using similar arguments as in Theorem 2 and set $Cx = 0$ for all $x \in \mathcal{H}$, we can prove the following results.

Corollary 1. *Let* $A, B : \mathcal{H} \to 2^{\mathcal{H}}$ *be two maximal monotone operators and* $\mathbf{zer}(A + B)$ *be a nonempty set. We consider the following algorithm:*

$$(\forall n \geq 1) \begin{cases} a_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = J_{\mu B}(\delta_n a_n), \\ z_n = J_{\mu A}(2y_n - \delta_n a_n), \\ x_{n+1} = \delta_n a_n + \alpha_n(z_n - y_n) + \varepsilon_n, \end{cases}$$

where $x_0, x_1 \in \mathcal{H}$*,* $\mu \in (0, 2\delta)$*,* $(\theta_n)_{n\geq 1} \subseteq [0, \theta]$ *with* $\theta \in [0, 1)$*, and* $(\alpha_n)_{n\geq 0}$ *and* $(\delta_n)_{n\geq 0}$ *are sequences in* $(0, 1]$ *and* $(\varepsilon_n)_{n\geq 0}$ *is a sequence in* $\mathcal{H}$*. Assume that Assumption 1 holds and* $\sum_{n\geq 1} \theta_n \|x_n - x_{n-1}\| < +\infty$*. Then, the following statements hold:*

1. $(x_n)_{n\geq 0}$ *strongly converges to* $x^* := \mathbf{proj}_{\mathbf{Fix}(J_{\mu A} \circ (2J_{\mu B} - Id) + Id - J_{\mu B})}(0)$ *for some* $\mu > 0$.
2. $(y_n)_{n\geq 1}$ *and* $(z_n)_{n\geq 1}$ *strongly converge to* $J_{\mu B}(x^*) \in \mathbf{zer}(A + B)$.

Proof. It follows from the proof of Theorem 2. □

Using similar arguments as in Theorem 2 and setting $Bx = 0$ for all $x \in \mathcal{H}$, we can prove the following results.

Corollary 2. *Let $A : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator and $C : \mathcal{H} \to \mathcal{H}$ a δ-cocoercive operator with $\delta > 0$ and $\mathbf{zer}(A + C) \neq \emptyset$. Let $\mu \in (0, 2\delta)$ and $(x_n)_{n\geq 0}$ be generated by the following iterative scheme*

$$\begin{cases} x_0, x_1 \in \mathcal{H}, \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (1 - \alpha_n)\delta_n y_n + \alpha_n J_{\mu A}(\delta_n y_n - \mu C \delta_n y_n) + \varepsilon_n, \end{cases} \tag{19}$$

for all $n \geq 1$, where $(\theta_n)_{n\geq 1} \subseteq [0, \theta]$ with $\theta \in [0, 1)$, and $(\alpha_n)_{n\geq 0}$ and $(\delta_n)_{n\geq 0}$ are sequences in $(0, 1]$ and $(\varepsilon_n)_{n\geq 0}$ is a sequence in $\mathcal{H}$. Assume that Assumption 1 holds and $\sum_{n\geq 1} \theta_n \|x_n - x_{n-1}\| < +\infty$.

Then, the sequence $(x_n)_{n\geq 0}$ strongly converges to a point $\mathbf{proj}_{\mathbf{zer}(A+C)}(0)$.

5. Numerical Experiments

To illustrate the behavior of the proposed iterative method, we provide a numerical example in a convex minimization problem and compare the convergence performance of the proposed algorithm with some algorithms in the literature. Moreover, we also employed our algorithm in the context of image restoration problems. All the experiments were implemented in MATLAB R2016b running on a MacBook Air 13-inch, Early 2017 with a 1.8 GHz Intel Core i5 processor and 8 GB 1600 MHz DDR3 memory.

5.1. Convex Minimization Problems

In this subsection, we present some comparisons among Algorithm 2, MTA, and Shehu et al.'s algorithm in Equation (3) (Algorithm 3.1 in [33]) in convex minimization problem.

Example 1. Let $f : \mathbb{R}^s \to \mathbb{R}$ be defied by $f(x) = \|x\|_1$ for all $x \in \mathbb{R}^s$, $g : \mathbb{R}^s \to \mathbb{R}$ be defined by indicator function $g(x) = \delta_W(x)$ with $W := \{x : \mathbf{A}x = \mathbf{b}\}$ for all $x \in \mathbb{R}^s$, where $\mathbf{A} : \mathbb{R}^s \to \mathbb{R}^l$ is a non-zero linear transformation, $\mathbf{b} \in \mathbb{R}^l$ and $s > l$; and $h : \mathbb{R}^s \to \mathbb{R}$ be defined by $h(x) = \frac{1}{2}\|x\|_2^2$ for all $x \in \mathbb{R}^s$. Since $s > l$, we get that A is an asymmetric transformation. Find the solution of the following problem:

$$\begin{aligned} &\text{minimize } \|x\|_1 + \delta_W(x) + \frac{1}{2}\|x\|_2^2 \\ &\text{subject to } x \in \mathbb{R}^s. \end{aligned} \tag{20}$$

The problem in Equation (20) can be written in the form of the problem in Equation (17) as:

$$\text{find } x \in \mathbb{R}^s \text{ such that } 0 \in \partial\|x\|_1 + \partial\delta_W(x) + \nabla h(x), \tag{21}$$

where $A = \partial\|\cdot\|_1$, $B = \partial\delta_W(\cdot)$ and $C = \nabla h(\cdot)$.

In this setting, we have $J_{\mu\partial\delta_W}(x) = x + \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}(\mathbf{b} - \mathbf{A}x)$,

$$J_{\mu\partial\|\cdot\|_1}(x) = (\max\{0, 1 - \frac{\mu}{|x^1|}\}x_1, \max\{0, 1 - \frac{\mu}{|x^2|}\}x_2, \ldots, \max\{0, 1 - \frac{\mu}{|x^s|}\}x_s),$$

and $\nabla h(x) = x$, where $x = (x^1, x^2, \ldots, x^s) \in \mathbb{R}^s$.

We begin with the problem by random vectors $z, x_0, x_1 \in \mathbb{R}^s$ and $\mathbf{b} \in \mathbb{R}^l$ and matrix $\mathbf{A} \in \mathbb{R}^{l\times s}$. Next, we compare the performance of Algorithm 2 with two remained performance. The parameters that are used in our algorithm are chosen as follows: $\alpha_n = 1 - \frac{1}{(n+2)^2}$, $\delta_n = 1 - \frac{1}{n+2}$, $\varepsilon_n = \frac{z}{(100n)^2}$, and

$$\theta_n = \begin{cases} \min\left\{\frac{1}{2}, \frac{1}{(n+1)^2\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \frac{1}{2}, & \text{otherwise.} \end{cases} \tag{22}$$

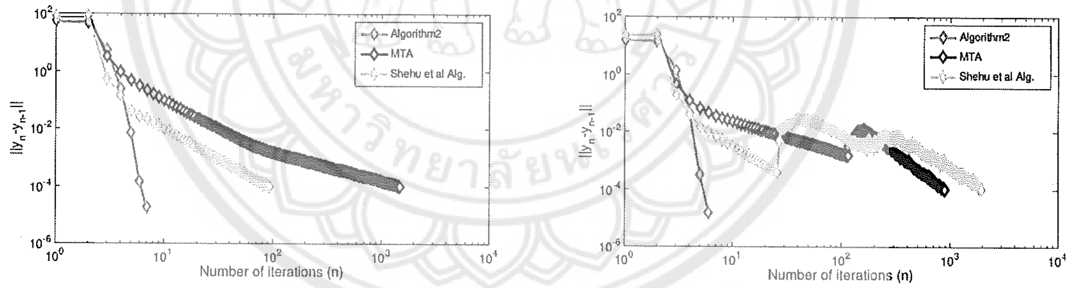
We choose $\alpha_n = \frac{1}{n+1}$, $\delta_n = \gamma_n = \frac{1}{2(n+1)}$ and $e_n = \varepsilon_n$ for Shehu et al.'s algorithm in Equation (3) in [33]. We obtain the CPU times (seconds) and the number of iterations by using the stopping criteria : $\|y_n - y_{n-1}\| \leq 10^{-4}$.

In Table 1, we present a comparison among the numerical results of Algorithm 2, MTA, and Shehu et al.'s algorithm in Equation (3) in different sizes of matrix **A**. The smallest number of iterations is generated by Algorithm 2 for all sizes of matrix **A**. Moreover, Algorithm 2 requires the least CPU computation time to reach the optimality tolerance for all cases.

Table 1. Comparison: Algorithm 2, MTA and Shehu et al.'s algorithm in Equation (3).

(l,s)	Algorithm 2		MTA		Shehu et al.'s Algorithm Equation (3)	
	CPU Time (s)	**Iterations**	**CPU Time (s)**	**Iterations**	**CPU Time (s)**	**Iterations**
(20,700)	0.0218	7	0.0428	278	0.0756	626
(20,800)	0.0189	7	0.0914	350	0.1745	796
(20,7000)	0.0302	7	1.7751	1273	0.0977	53
(20,8000)	0.0308	6	1.2419	1290	0.0671	54
(200,7000)	0.0365	8	1.9452	858	4.6538	2028
(200,8000)	0.0406	7	2.5115	977	0.1425	53
(500,7000)	0.0403	7	4.1647	892	8.3620	1956
(500,8000)	0.0548	8	4.3239	813	9.0929	1835
(1000,7000)	0.0703	7	6.7954	786	14.1693	1751
(1000,8000)	0.0728	7	7.8302	825	16.3752	1784
(3000,7000)	0.1597	7	18.0559	779	44.8129	1940
(3000,8000)	0.1763	7	22.3514	841	49.6872	1891
(100,80,000)	0.1376	8	26.6863	1489	1.5926	94
(1000,80,000)	0.6949	8	344.7048	3289	9.4181	93

Figure 1 shows the behavior of $\|y_k - y_{k-1}\|$ for Algorithm 2, MTA, and Shehu et al.'s algorithm in Equation (3) in two different choices of (l,s). We can observe that by using our algorithm the behavior of the red line, and Algorithm 2 has the best performance.



(a) Case: $(l,s) = (100, 80,000)$ (b) Case: $(l,s) = (500, 7000)$

Figure 1. Illustration the behavior of $\|y_n - y_{n-1}\|$ for Algorithm 2, MTA, and Shehu et al.'s algorithm in Equation (3).

5.2. *Image Restoration Problems*

In this subsection, we apply the proposed algorithm, image restoration problems, which involves deblurring and denoising images. We consider the degradation model that represents an actual image restoration problems or through the least useful mathematical abstractions thereof.

$$y = Hx + w, \tag{23}$$

where y, H, x and w represent the degraded image, degradation operator, or blurring operator; original image; and noise operator, respectively.

The reconstructed image is obtained by solving the following regularized least-squares problem

$$\min_x \left\{ \frac{1}{2}\|Hx - y\|_2^2 + \mu\phi(x) \right\}, \tag{24}$$

where $\mu > 0$ is the regularization parameter and $\phi(\cdot)$ is the regularization functional. A well-known regularization function used to remove noise in the restoration problem is the l_1 norm, which is called Tikhonov regularization [36]. The problem in Equation (24) can be written in the form of the following problem as:

$$\text{find } x \in \arg\min_{x\in\mathbb{R}^k} \left\{ \frac{1}{2}\|Hx - y\|_2^2 + \mu\|x\|_1 \right\}, \tag{25}$$

where y is the degraded image and H is a bounded linear operator. Note that problem in Equation (25) is a spacial case of the problem in Equation (4) by setting $A = \partial f(\cdot)$, $B = 0$, and $C = \nabla L(\cdot)$ where $f(x) = \|x\|_1$ and $L(x) = \frac{1}{2}\|Hx - y\|_2^2$. This setting we have that $C(x) = \nabla L(x) = H^*(Hx - y)$, where H^* is a transpose of H. We begin the problem by choosing images and degrade them by random noise and different types of blurring. The random noise in this study is provided by Gaussian white noise of zero mean and 0.001 variance. We solve the problem in Equation (25) by using our algorithm in Corollary 2. We set $\alpha_n = 1 - \frac{1}{(n+1)^2}$, $\delta_n = 1 - \frac{1}{100n+1}$, $\mu = 0.001$, $\varepsilon_n = 0$ and θ_n is defined as Equation (22).

We compare our proposed algorithm with the inertial Mann-type algorithm that was introduced by Kitkuan et al. [30]. In Kitkuan et al.'s algorithm (Algorithm in Theorem 3.1 in [30]), we choose $\varsigma_n = \theta_n$, $\alpha_n = \frac{1}{n+1}$, $\lambda_n = 0.001$, and $h(x) = \frac{1}{12}\|x\|_2^2$. We assess the quality of the reconstructed image by using the signal to noise ratio (**SNR**) for monochrome images, which is defined by

$$\text{SNR}(n) = 20\log_{10}\frac{\|x\|_2^2}{\|x - x_n\|_2^2},$$

where x and x_n denote the original and the restored image at iteration n, respectively.

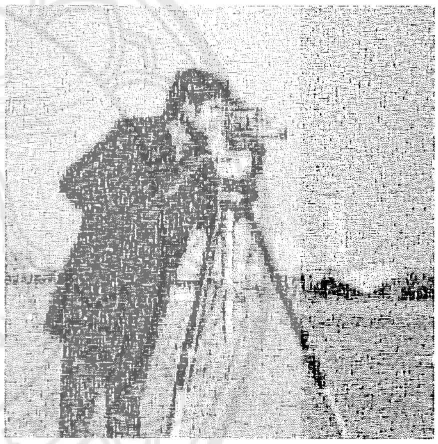
For color images, we estimate the quality of the reconstructed image by using the normalized color difference (**NCD**) [37] which is defined by

$$\text{NCD}(n) = \frac{\sum_{i=1}^{N}\sum_{j=1}^{M}\sqrt{(L_{i,j}^o - L_{i,j}(n))^2 + (u_{i,j}^o - u_{i,j}(n))^2 + (v_{i,j}^o - v_{i,j}(n))^2}}{\sum_{i=1}^{N}\sum_{j=1}^{M}\sqrt{(L_{i,j}^o)^2 + (u_{i,j}^o)^2 + (v_{i,j}^o)^2}},$$

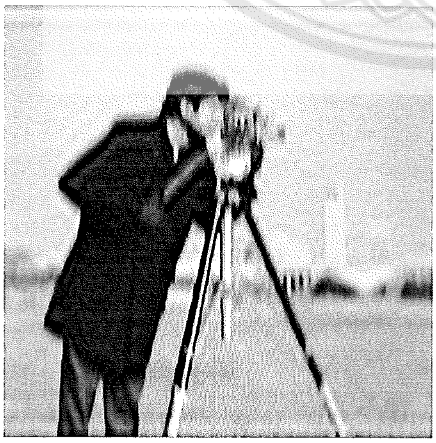
where i,j are indices of the sample position, N, M characterize an image size and $L_{i,j}^o$, $u_{i,j}^o$, $v_{i,j}^o$ and $L_{i,j}(n)$, $u_{i,j}(n)$, $v_{i,j}(n)$ are values of the perceived lightness and two representatives of chrominance related to the original and the restored image at iteration n, respectively. We generated the noised model in order to obviously see the differences between degraded and original figure as follows. Figure 2 firstly shows the original image. Secondly, the degraded image was corrupted by average blur (size 20 by 20) and Gaussian noise (zero mean and 0.001 variance). We randomly selected parameters which visibly showed the differences sharpness level and. Lastly, reconstructed images are shown. Figure 3 firstly shows the original image. Secondly, the degraded image was corrupted by Gaussian blur (size 20 by 20 with the standard deviation 20) and Gaussian noise (zero mean and 0.001 variance). With this point, we found that any adjustment of the standard deviation as much as small might not shown the difference between degraded and original figure. Lastly, reconstructed images are shown. Figure 4 firstly shows the original image. Secondly, the degraded image was corrupted by motion blur (the linear motion of a camera by 30 pixels with an angle of 60 degrees) and Gaussian noise (zero mean and 0.001 variance). We randomly selected parameters which visibly showed the differences sharpness level. Lastly, reconstructed images are shown. The comparisons between our proposed algorithm in Equation (19) and Kitkuan et al.'s algorithm (Algorithm in Theorem 3.1 in [30]) in image restoration problems are presented in Figure 5 and Table 2. Furthermore, we also present the comparison of Kitkuan et al.'s algorithm (Algorithm in Theorem 3.1 in [30]), our algorithm, and the well-known technique for image restoration which is Weiner filtering (**WF**) [38,39]. Figure 6 presents the comparative results of two degradation images 'Artsawang' and 'Mandril' corrupted by motion blur and different salt and pepper noise from 0% to 10%.

(a) camera man

(b) average blur and random noise



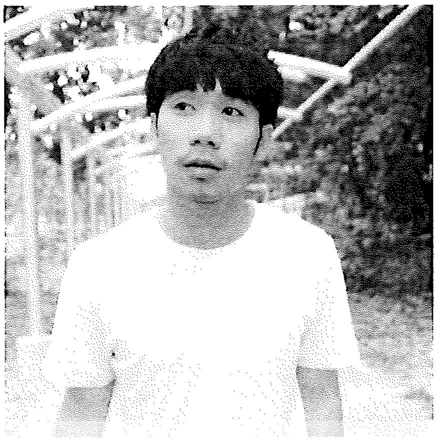
(c) Weiner Filtering



(d) Kitkuan et al.'s algorithm

(e) our algorithm

Figure 2. **(a)** The original image 'camera man'; **(b)** the images degraded by average blur and random noise (Gaussian noise); and **(c–e)** the reconstructed image by using Weiner filter, Kitkuan et al.'s algorithm, and our algorithm in Equation (19), respectively.



(a) Artsawang

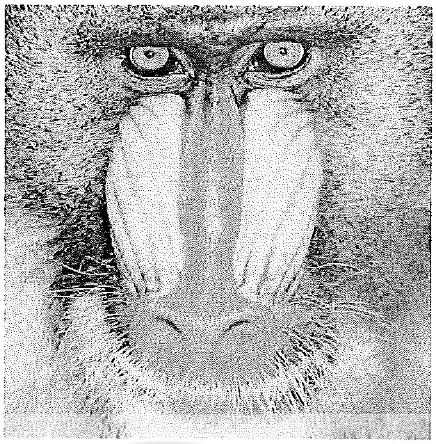
(**b**) Gaussian blur and random noise

(c) Weiner Filtering

(**d**) Kitkuan et al.'s algorithm

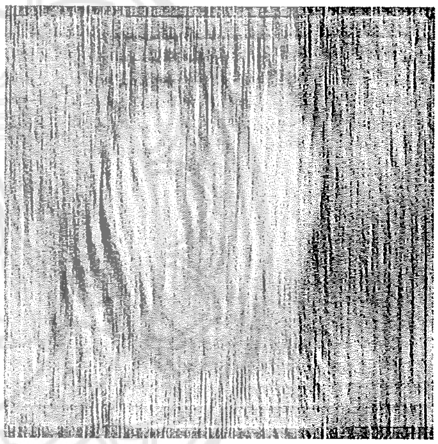
(**e**) our algorithm

Figure 3. (**a**) The original image 'Artsawang'; (**b**) the images degraded by Gaussian blur and random noise (Gaussian noise); and (**c–e**) the reconstructed image by using Weiner filter, Kitkuan et al.'s algorithm, and our algorithm in Equation (19), respectively.

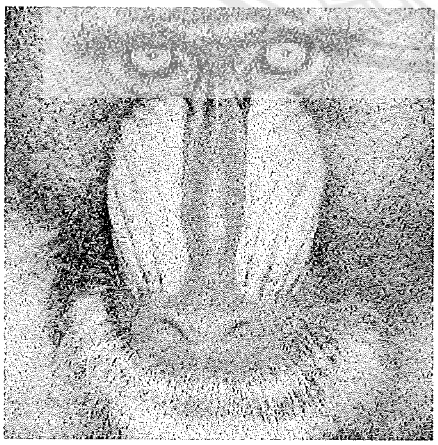


(a) Mandril

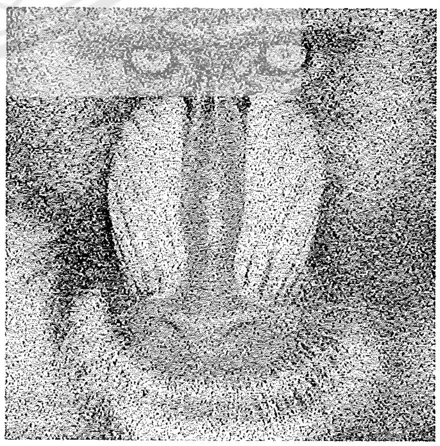
(b) motion blur and random noise



(c) Weiner Filtering

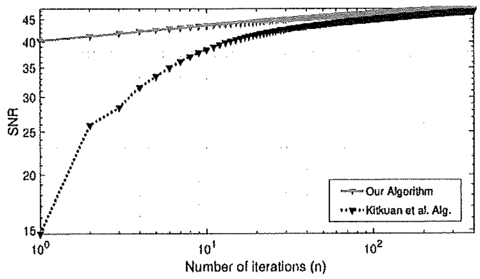


(d) Kitkuan et al.'s algorithm

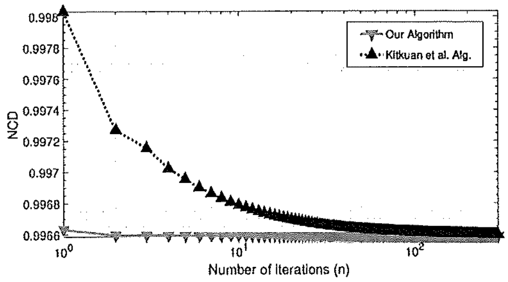


(e) our algorithm

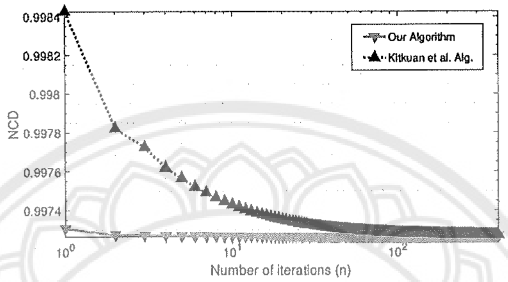
Figure 4. (**a**) The original image 'Mandril'; (**b**) the images degraded by motion blur and random noise (Gaussian noise); and (**c–e**) the reconstructed image by using Weiner filter, Kitkuan et al.'s algorithm, and our algorithm in Equation (19), respectively.



(**a**) camera man

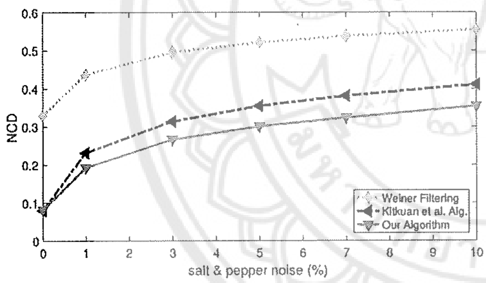


(**b**) Artsawang

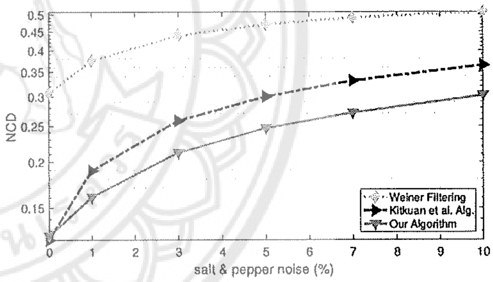


(c) Mandril

Figure 5. (**a**) The behavior of SNR for two algorithms in Figure 2d,e; (**b**) the behavior of NCD for two algorithms in Figure 3d,e; and (**c**) the behavior of NCD for two algorithms in Figure 4d,e.



(a) Artsawang



(b) Mandril

Figure 6. (a,b) The behavior of NCD in motion blur and different different salt and pepper noise from 0% to 10%.

Table 2. The performance of the normalized color difference (NCD) in two images.

	The Normalized Color Difference (NCD).			
	Kitkuan et al.'s Algorithm		**Our Algorithm in Equation (19)**	
n	**Artsawang Image**	**Mandril Image**	**Artsawang Image**	**Mandril Image**
1	0.99803	0.99842	0.99663	0.99731
50	0.99660	0.99730	0.99659	0.99727
100	0.99661	0.99729	0.99658	0.99726
200	0.99660	0.99728	0.99658	0.99726
300	0.99659	0.99727	0.99658	0.99726
400	0.99659	0.99727	0.99658	0.99726

6. Conclusions

In this paper, we propose a new Mann-type method combining both inertial terms and errors to solve the fixed point problem for a nonexpansive mapping. We also prove the strong convergence of the proposed algorithm under some sufficient conditions of involved parameters. We also apply results to approximate the solutions of the monotone inclusion problems. Furthermore, we also provide a numerical example to compare the proposed algorithm with other algorithms in the convex minimization problem. Finally, we use our method to solve image restoration problems.

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