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The general iterative algorithms based on the viscosity approximation method for equilibrium and constrained convex minimization problem

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ชื่อผู้วิจัย ผศ.ดร. รัตนาพร วังคีรี
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บทคัดย่อ(ภาษาไทย)

วัตถุประสงค์ของงานวิจัยนี้ คือการศึกษาความต่อเนื่องของการส่งค่าผลเฉลยของปัญหาเชิงดุลยภาพเวกเตอร์อย่างเข้มวางนัยทั่วไปเชิงพารามิเตอร์ โดยใช้ความสัมพันธ์โฮเดอร์ งานวิจัยนี้ได้ขยายผลลัพธ์ต่าง ๆ ที่มีในเอกสารอ้างอิง

บทคัดย่อ(ภาษาอังกฤษ)

The aim of this research is to establish the continuity of the efficient solution mappings to a parametric generalized strong vector equilibrium problem, by using the Holder relation. Our result extends and improves some recent results in the references therein.

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CHAPTER 1

INTRODUCTION

It is well known that the vector equilibrium problem provides a unified model of several classes of problems, including, vector variational inequality problems, vector complementarity problems, vector optimization problems and vector saddle point problems, so on. There are many papers have discussed the existence results for different types of vector equilibrium problems (see[2, 1, 3]) and references therein. Stability may be understood as lower or upper semicontinuity, continuity, Hölder continuity. Nowadays, the stability of the solution mappings for parametric vector equilibrium problems has been intensively studied in various directions, (see also [5, 4, 6, 8, 9, 2, 3, 10, 12, 11, 7]). As far as we know, almost all papers discusses the semicontinuity and continuity results of the efficient solutions to a parametric weak vector equilibrium. In this paper, we focus on the semicontinuity and/or continuity results of solution mappings to a parametric strong vector equilibrium problem. In 2008, Gong and Yao [9] first discussed the lower semicontinuity of the efficient solution mappings to a parametric strong vector equilibrium problem with C-strict monotonicity of a vector-valued function, by using a scalarization method and density result. In 2009, Xu and Li [17] presented a new proof of lower semicontinuity of the set of efficient solutions to a parametric strong vector equilibrium problem, which is different from the one used in [9]. Moreover, they relaxed the C-strict monotonicity assumption to C-monotonicity assumption. In 2010, Chen and Li [8] discussed and improved the lower semicontinuity and continuity results of efficient solution mappings to a parametric strong vector equilibrium problem in [9], without the uniform compactness assumption. Recently, by using an idea of [9], Li et al.[13] established the continuity of solution mappings to a parametric generalized strong vector equilibrium problem involving set-valued mappings under an assumption which is different from the C-strict monotonicity. More recently, Zhang et al.[18] obtained the lower semicontinuity of solution mappings for parametric strong vector equilibrium problems under the Hölder-related assumptions which were introduced by [12, 5].

Motivated and inspried by Li et al.[13] and Zhang et al.[18], the aim of this paper is to establish the continuity of the efficient solution mappings to a parametric generalized strong vector equilibrium problem involving a set-valued mapping under the Hölder relation assumption.

We presents the efficient solutions to parametric generalized strong vector equilibrium problems and materials used in the rest of this work as follows.

Let X and Z be metric spaces, and let Y be a metric vector space. We denote $d_X(\cdot,\cdot)$, $d_Y(\cdot,\cdot)$ and $d_Z(\cdot,\cdot)$ the distance in three metric spaces. For $\bar{x} \in X$ and $\delta > 0$, we define $B_X(\bar{x},\delta) := \{x \in X : d_X(x,\bar{x}) < \delta\}$, the open ball with center $\bar{x}$ and radius $\delta > 0$. For any nonempty subset A in X, we set $d_X(x,A) := \inf_{a\in A} d_X(x,a)$, the distance from x to a subset A. We also assume that C is a pointed closed convex cone in Y with nonempty interior $intC$. Suppose that A is a nonempty subset of X and $F : A \times A \to 2^Y\backslash\{\emptyset\}$ is a set-valued mapping. We consider the following generalized vector equilibrium problem: The generalized strong vector equilibrium problem **(GSVEP)** of finding $x \in A$ such that

$$F(x,y) \cap (-C\backslash\{0_Y\}) = \emptyset,\ \forall y \in A.$$

When the set A and the mapping F are perturbed by a parameter which varies over a set Λ of Z, we have the following problem: The parametric generalized vector equilibrium problem **(PGSVEP)** of finding $x \in A(\mu)$ such that

$$F(x,y,\mu) \cap (-C\backslash\{0_Y\}) = \emptyset,\ \forall y \in A(\mu), \tag{1.1}$$

where $A : \Lambda \to 2^X\backslash\{\emptyset\}$ is a set-valued mapping, $F : B \times B \times \Lambda \subset X \times X \times Z \to 2^Y\backslash\{\emptyset\}$ is a set-valued mapping with $A(\Lambda) = \bigcup_{\mu\in\Lambda} A(\mu) \subset B$.

For each $\mu \in \Lambda$, let $S(\mu)$ denote the efficient solution set of **(PGSVEP)**, *i.e.*,

$$S(\mu) = \{x \in A(\mu) : F(x, y, \mu) \cap (-C \setminus \{0_Y\}) = \emptyset, \ \forall y \in A(\mu)\}.$$

Throughout of this paper, we always assume $S(\mu) \neq \emptyset$ for all $\mu \in \Lambda$. Now, we recall the definition of semicontinuity of set-valued mappings. Let Λ and X be two metric spaces, $F : \Lambda \to 2^X$ be a set-valued mapping, and $\bar{\lambda} \in \Lambda$.

Definition 1.4. [16]

(i) F is said to be lower semicontinuous(l.s.c.) at $\bar{\lambda}$ if for any open set U satisfying $U \cap F(\bar{\lambda}) \neq \emptyset$, there exists $\delta > 0$ such that $F(\lambda) \cap U \neq \emptyset$, for all $\lambda \in B(\bar{\lambda}, \delta)$.

(ii) F is said to be upper semicontinuous(u.s.c) at $\bar{\lambda}$ if for any open set U satisfying $F(\bar{\lambda}) \subset U$, there exists $\delta > 0$ such that $F(\lambda) \subset U$, for all $\lambda \in B(\bar{\lambda}, \delta)$.

Proposition 1.5. [14, 15]

(i) *F is l.s.c. at $\bar{\lambda}$ if and only if for any sequence $\{\lambda_n\} \subset \Lambda$ with $\lambda_n \to \bar{\lambda}$ and any $\bar{x} \in F(\bar{\lambda})$, there exists $x_n \in F(\lambda_n)$ such that $x_n \to \bar{x}$.*

(ii) *If F has compact values (i.e., $F(\lambda)$ is a compact set for each $\lambda \in \Lambda$), then F is u.s.c. at $\bar{\lambda}$ if and only if for any sequence $\{\lambda_n\} \subset \Lambda$ with $\lambda_n \to \bar{\lambda}$ and for any $x_n \in F(\lambda_n)$, there exists $\bar{x} \in F(\bar{\lambda})$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \bar{x}$.*

CHAPTER 2

MAIN RESULTS

In this chapter, we establish a sufficient condition for the the continuity of the efficient solution mapping to **(PGSVEP)**. Moreover, we give examples to illustrate our results.

Theorem 2.1. *Suppose that the following conditions are satisfied.*

(i) *$A(\cdot)$ is continuous with compact values on Λ;*

(ii) *$F(\cdot,\cdot,\cdot)$ is l.s.c. on $B \times B \times \Lambda$;*

(iii) *for each $\mu \in \Lambda, x \in A(\mu)\backslash S(\mu)$, there exists $y \in S(\mu)$ such that*

$$d_X(x,y) \leq \inf_{z \in F(x,y,\mu)} d_Y(z, Y\backslash -intC).$$

Then, $S(\cdot)$ is continuous on Λ.

Proof. We first prove that $S(\cdot)$ is lower semicontinuous on Λ. Suppose to the contrary that there exists $\mu_0 \in \Lambda$ such that $S(\cdot)$ is not l.s.c. at μ_0. Then there exists a sequence $\{\mu_n\}$ with $\mu_n \to \mu_0$ and $x_0 \in S(\mu_0)$ such that for any $x_n \in S(\mu_n)$, $x_n \nrightarrow x_0$. Since $x_0 \in S(\mu_0)$, we have $x_0 \in A(\mu_0)$ and

$$F(x_0, y, \mu_0) \cap (-C\backslash\{0_Y\}) = \emptyset,\ \forall y \in A(\mu_0). \tag{2.1}$$

By the lower semicontinuity of $A(\cdot)$, there exists a sequence $\{\bar{x}_n\} \subset A(\mu_n)$ such that $\bar{x}_n \to x_0$. Obviously, $\bar{x}_n \in A(\mu_n)\backslash S(\mu_n)$. It follows from (iii) that, there exists $y_n \in S(\mu_n)$ such that

$$d_X(\bar{x}_n, y_n) \leq \inf_{z \in F(\bar{x}_n, y_n, \mu_n)} d_Y(z, Y\backslash -intC). \tag{2.2}$$

Since $y_n \in A(\mu_n)$, by the upper semicontinuity and compactness of $A(\cdot)$, we get that there exists $y_0 \in A(\mu_0)$ and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \to y_0$, denoted by $\{y_i\}$. Using (2.2), we have

$$d_X(\bar{x}_i, y_i) \leq \inf_{z \in F(\bar{x}_i, y_i, \mu_i)} d_Y(z, Y \setminus -intC). \tag{2.3}$$

Since F is l.s.c. on $B \times B \times \Lambda$, for $(\bar{x}_i, y_i, \mu_i) \to (x_0, y_0, \mu_0)$ and any $z_0 \in F(x_0, y_0, \mu_0)$ there exists $z_i \in F(\bar{x}_i, y_i, \mu_i)$ such that $z_i \to z_0$. It follows from (2.3) that $d_X(\bar{x}_i, y_i) \leq d_Y(z_i, Y \setminus -intC)$. By the continuity of metric distance $d_Y(\cdot, Y \setminus -intC)$ we get that

$$d_X(x_0, y_0) \leq d_Y(z_0, Y \setminus -intC).$$

We want to show that $x_0 = y_0$. Assume that $x_0 \neq y_0$, then the last inequality implies that

$$0 < d_X(x_0, y_0) \leq d_Y(z_0, Y \setminus -intC).$$

Hence, $z_0 \in -intC \subset -C \setminus \{0\}$. This is a contradiction with (2.1), because of $z_0 \in F(x_0, y_0, \mu_0)$. Hence, we have $x_0 = y_0$. This is impossible by the contradiction assumption.

Next, we show that $S(\cdot)$ is u.s.c. on Λ. Suppose to the contrary that there exists some $\mu_0 \in \Lambda$ such that $S(\cdot)$ is not u.s.c. at μ_0. Then there exists an open set V satisfying $S(\mu_0) \subset V$, and sequences $\mu_n \longrightarrow \mu_0$ and $x_n \in S(\mu_n)$ such that $x_n \notin V$ for all $n \in \mathbb{N}$. Since $x_n \in A(\mu_n)$ and $A(\cdot)$ is u.s.c. at μ_0 with compact values, there is an $x_0 \in A(\mu_0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$, denoted by $\{x_k\}$.

We want to show that $x_0 \in S(\mu_0)$. Assume that $x_0 \notin S(\mu_0)$. Then by (iii) there exists $y_0 \in S(\mu_0)$ such that

$$d_X(x_0, y_0) \leq \inf_{z \in F(x_0, y_0, \mu_0)} d_Y(z, Y \setminus -intC). \tag{2.4}$$

Since A is l.s.c. at μ_0 and $y_0 \in A(\mu_0)$, we have that there exists $y_n \in A(\mu_n)$ such that $y_n \to y_0$. It follows form $x_n \in S(\mu_n)$ and $y_n \in A(\mu_n)$ that

$$F(x_n, y_n, \mu_n) \cap (-C\backslash\{0_Y\}) = \emptyset. \tag{2.5}$$

Since $F(\cdot,\cdot,\cdot)$ is l.s.c. at (x_0, y_0, μ_0) and $(x_k, y_k, \mu_k) \to (x_0, y_0, \mu_0)$, for $z_0 \in F(x_0, y_0, \mu_0)$, there exists $z_k \in F(x_k, y_k, \mu_k)$ such that $z_k \to z_0$. Notice that $z_k \notin -C\backslash\{0\}$, also $d_Y(z_k, Y\backslash - intC) = 0$ for all k. It follows from (2.4), (2.5) and continuity of metric distance $d_Y(\cdot, Y\backslash - intC)$ that

$$0 \leq d_X(x_0, y_0) \leq d_Y(z_0, Y\backslash - intC) = \lim_{k\longrightarrow\infty} d_Y(z_k, Y\backslash - intC) = 0. \tag{2.6}$$

That is $x_0 = y_0$, which leads to a contradiction. Hence $x_0 \in S(\mu_0) \in V$. This is impossible by the contradiction assumption. Therefore, we complete the proof. □

The following example illustrates that the condition the condition (iii) in Theorem 2.1 is essential.

Example 2.2. Let $X = Y = Z = \mathbb{R}$, $C = \mathbb{R}_+$, $\Lambda = [0,1]$, $A(\mu) = B = [0,1]$ and $F(x, y, \mu) = [\mu(y - x), +\infty)$.
It is not hard to check that the assumption (i) and (ii) in Theorem 2.1 are satisfied. After calculating, we get that

$$S(\mu) = \begin{cases} \{0\}, & \text{if } \mu \neq 0, \\ [0,1], & \text{if } \mu = 0. \end{cases}$$

We see that $S(\cdot)$ is not continuous at $\mu_0 = 0$, since the assumption (iii) is violated. Indeed, for each $\mu \in (0,1)$, for any $x \in A(\mu)\backslash S(\mu) = (0,1]$, putting $y = 0 \in S(\mu)$, we have

$$d(x,y) = x \text{ and } \inf_{z\in F(x,y,\mu)} d(z, Y\backslash - intC) = \inf_{z\in[-\mu x, +\infty)} d(z, [0, +\infty)) = 0.$$

This implies that, $d(x,y) = x > 0 = \inf_{z\in F(x,y,\mu)} d(z, Y\backslash - intC)$. Hence, the assumption (iii) is essential.

Corollary 2.1. *Suppose that the following conditions are satisfied.*

(i) $A(\cdot)$ *is continuous with compact values on* Λ;

(ii) $F(\cdot,\cdot,\cdot)$ *is l.s.c. on* $B \times B \times \Lambda$;

(iii) *for each* $\mu \in \Lambda, x \in A(\mu)\backslash S(\mu)$, *there exists* $y \in S(\mu)$ *such that*

$$d_X(x,y) \leq \inf_{z \in F(x,y,\mu)} d_Y(z, Y\backslash -intC).$$

Then, $S(\cdot)$ *is l.s.c. on* Λ.

Remark 2.2. Corollary 2.1 discusses the lower semicontinuity of the solution mappings involving set-valued mappings, which is more general than the result in Theorem 3.1 of [18] in the case that F is a set-valued mapping. Furthermore, The continuity of F is relaxed to the lower semicontinuity. Compared with Theorem 3.1 of [13], assumption (iii) of Corollary 2.1 is different from assumption (v) in Theorem 3.1 in [13]. An inconvenience of assumption (iii) of Corollary 2.1 is that it requires to know the solution map. However, in some situations Corollary 2.1 is applicable while Theorem 3.1 in [13] is not, as shown by the following example.

Example 2.3. Let $X = Y = Z = \mathbb{R}$, $C = \mathbb{R}_+$, $\Lambda = [0,1]$, $A(\mu) = [\mu, 1]$ and $F(x,y,\mu) = [y - x, (1+\mu)(y-x)]$.

It follows from direct computation that, $S(\mu) = \{\mu\}$. Obviously, $S(\mu) = S_f(\mu)$ for all $f \in C^*\backslash\{0\}$, where $S_f(\mu) = \{x \in A(\mu) : \inf_{z \in F(x,y,\mu)} f(z) \geq 0, \ \forall y \in A(\mu)\}$ and C^* is a dual cone of C. It is easy to verify that assumptions (i)-(iv) and (vi) in Theorem 3.1 of [13] are satisfied. But F is not satisfied (v). Indeed, $f = 1 \in C^*\backslash\{0\}$, $x = 1 \in A(\mu)\backslash S_f(\mu) = (\mu, 1]$, we have $y \in \{\mu\} = S(\mu)$

$$F(x,y,\mu) + F(x,y,\mu) + B(0, |x-y|) = B(0, |1-\mu|) \nsubseteq -C.$$

Therefore, we cannot obtain that $S(\cdot)$ is l.s.c. on Λ. However, the assumption (iii) in Corollary 2.1 is satisfied. Indeed, for all $x \in (\mu, 1]$, there is only one $y = \mu \in S(\mu)$, we get that

$$d(x,y) = |x - \mu|,$$

and

$$\inf_{z \in F(x,y,\mu)} d(z, Y \setminus -intC) = \inf_{z \in [(1+\mu)(\mu - x), \mu - x]} d(z, [0, +\infty)) = |x - \mu|.$$

Hence, all assumptions of Corollary 2.1 are satisfied, and so $S(\cdot)$ is l.s.c. on Λ.

CHAPTER 3

CONCLUSION

1. Suppose that the following conditions are satisfied.

   (i) $A(\cdot)$ is continuous with compact values on Λ:

   (ii) $F(\cdot,\cdot,\cdot)$ is l.s.c. on $B \times B \times \Lambda$;

   (iii) for each $\mu \in \Lambda, x \in A(\mu)\backslash S(\mu)$, there exists $y \in S(\mu)$ such that

$$d_X(x,y) \leq \inf_{z\in F(x,y,\mu)} d_Y(z, Y\backslash - intC).$$

   Then, $S(\cdot)$ is continuous on Λ.

2. Let $X = Y = Z = \mathbb{R}$, $C = \mathbb{R}_+$, $\Lambda = [0,1]$, $A(\mu) = B = [0,1]$ and $F(x,y,\mu) = [\mu(y-x), +\infty)$.
   It is not hard to check that the assumption (i) and (ii) in Theorem 2.1 are satisfied. After calculating, we get that

$$S(\mu) = \begin{cases} \{0\}, & \text{if } \mu \neq 0, \\ [0,1], & \text{if } \mu = 0. \end{cases}$$

   We see that $S(\cdot)$ is not continuous at $\mu_0 = 0$, since the assumption (iii) is violated. Indeed, for each $\mu \in (0,1)$, for any $x \in A(\mu)\backslash S(\mu) = (0,1]$, putting $y = 0 \in S(\mu)$, we have

$$d(x,y) = x \text{ and } \inf_{z\in F(x,y,\mu)} d(z, Y\backslash - intC) = \inf_{z\in[-\mu x,+\infty)} d(z, [0,+\infty)) = 0.$$

   This implies that, $d(x,y) = x > 0 = \inf_{z\in F(x,y,\mu)} d(z, Y\backslash - intC)$. Hence, the assumption (iii) is essential.

3. Suppose that the following conditions are satisfied.

   (i) $A(\cdot)$ is continuous with compact values on Λ;

(ii) $F(\cdot,\cdot,\cdot)$ is l.s.c. on $B \times B \times \Lambda$;

(iii) for each $\mu \in \Lambda, x \in A(\mu)\backslash S(\mu)$, there exists $y \in S(\mu)$ such that

$$d_X(x,y) \leq \inf_{z\in F(x,y,\mu)} d_Y(z, Y\backslash -intC).$$

Then, $S(\cdot)$ is l.s.c. on Λ.

4. Let $X = Y = Z = \mathbb{R}$, $C = \mathbb{R}_+$, $\Lambda = [0,1]$, $A(\mu) = [\mu,1]$ and $F(x,y,\mu) = [y-x,(1+\mu)(y-x)]$.

It follows from direct computation that, $S(\mu) = \{\mu\}$. Obviously, $S(\mu) = S_f(\mu)$ for all $f \in C^*\backslash\{0\}$, where $S_f(\mu) = \{x \in A(\mu) : \inf_{z\in F(x,y,\mu)} f(z) \geq 0,\ \forall y \in A(\mu)\}$ and C^* is a dual cone of C. It is easy to verify that assumptions (i)-(iv) and (vi) in Theorem 3.1 of [13] are satisfied. But F is not satisfied (v). Indeed, $f = 1 \in C^*\backslash\{0\}$, $x = 1 \in A(\mu)\backslash S_f(\mu) = (\mu,1]$, we have $y \in \{\mu\} = S(\mu)$

$$F(x,y,\mu) + F(x,y,\mu) + B(0,|x-y|) = B(0,|1-\mu|) \nsubseteq -C.$$

Therefore, we cannot obtain that $S(\cdot)$ is l.s.c. on Λ. However, the assumption (iii) in Corollary 2.1 is satisfied. Indeed, for all $x \in (\mu,1]$, there is only one $y = \mu \in S(\mu)$, we get that

$$d(x,y) = |x-\mu|,$$

and

$$\inf_{z\in F(x,y,\mu)} d(z, Y\backslash -intC) = \inf_{z\in[(1+\mu)(\mu-x),\mu-x]} d(z,[0,+\infty)) = |x-\mu|.$$

Hence, all assumptions of Corollary 2.1 are satisfied, and so $S(\cdot)$ is l.s.c. on Λ.

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APPENDIX

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Continuity of the solution mappings to parametric generalized vector equilibrium problems☆

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ABSTRACT

The aim of this paper is to establish the continuity of the efficient solution mappings to a parametric generalized strong vector equilibrium problem, by using the Hölder relation. Our result extends and improves some recent results in the references therein.

1. Introduction

It is well known that the vector equilibrium problem provides a unified model of several classes of problems, including vector variational inequality problems, vector complementarity problems, vector optimization problems, vector saddle point problems, and so on. There are many papers that have discussed the existence results for different types of vector equilibrium problems (see [1–3]) and references therein. Stability may be understood as lower or upper semicontinuity, continuity, Hölder continuity. Nowadays, the stability of the solution mappings for parametric vector equilibrium problems has been intensively studied in various directions (see also [4–8,1,3,9–12]). As far as we know, almost all papers discuss the semicontinuity and continuity results of the efficient solutions to a parametric weak vector equilibrium. In this paper, we focus on the semicontinuity and/or continuity results of solution mappings to a parametric strong vector equilibrium problem. In 2008, Gong and Yao [8] first discussed the lower semicontinuity of the efficient solution mappings to a parametric strong vector equilibrium problem with C-strict monotonicity of a vector-valued function, by using a scalarization method and density result. In 2009, Xu and Li [13] presented a new proof of lower semicontinuity of the set of efficient solutions to a parametric strong vector equilibrium problem, which is different from the one used in [8]. Moreover, they relaxed the C-strict monotonicity assumption to C-monotonicity assumption. In 2010, Chen and Li [7] discussed and improved the lower semicontinuity and continuity results of efficient solution mappings to a parametric strong vector equilibrium problem in [8], without the uniform compactness assumption. Recently, by using an idea of [8], Li et al. [14] established the continuity of solution mappings to a parametric generalized strong vector equilibrium problem involving set-valued mappings under an assumption which is different from the C-strict monotonicity. More recently, Zhang et al. [15] obtained the lower semicontinuity of solution mappings for parametric strong vector equilibrium problems under the Hölder-related assumptions which were introduced by [10,4].

Motivated and inspired by Li et al. [14] and Zhang et al. [15], the aim of this paper is to establish the continuity of the efficient solution mappings to a parametric generalized strong vector equilibrium problem involving a set-valued mapping under the Hölder relation assumption.

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The structure of the paper is as follows. Section 2 presents the efficient solutions to parametric generalized strong vector equilibrium problems and materials used in the rest of this paper. We establish, in Section 3, a sufficient condition for the continuity of the efficient solution mappings. Moreover, we give examples to illustrate our results.

2. Preliminaries

Let X and Z be metric spaces, and let Y be a metric vector space. We denote by $d_X(\cdot,\cdot)$, $d_Y(\cdot,\cdot)$ and $d_Z(\cdot,\cdot)$ the distance in three metric spaces. For $\bar{x} \in X$ and $\delta > 0$, we define $B_X(\bar{x}, \delta) := \{x \in X : d_X(x, \bar{x}) < \delta\}$, the open ball with center $\bar{x}$ and radius $\delta > 0$. For any nonempty subset A in X, we set $d_X(x, A) := \inf_{a\in A} d_X(x, a)$, the distance from x to a subset A. We also assume that C is a pointed closed convex cone in Y with nonempty interior int C. Suppose that A is a nonempty subset of X and $F : A \times A \to 2^Y \setminus \{\emptyset\}$ is a set-valued mapping. We consider the following generalized vector equilibrium problem: the generalized strong vector equilibrium problem (*GSVEP*) of finding $x \in A$ such that

$$F(x, y) \cap (-C \setminus \{0_Y\}) = \emptyset, \quad \forall y \in A.$$

When the set A and the mapping F are perturbed by a parameter μ which varies over a set Λ of Z, we have the following problem: the parametric generalized vector equilibrium problem (*PGSVEP*) of finding $x \in A(\mu)$ such that

$$F(x, y, \mu) \cap (-C \setminus \{0_Y\}) = \emptyset, \quad \forall y \in A(\mu), \tag{2.1}$$

where $A : \Lambda \to 2^X \setminus \{\emptyset\}$ is a set-valued mapping, and $F : B \times B \times \Lambda \subset X \times X \times Z \to 2^Y \setminus \{\emptyset\}$ is a set-valued mapping with $A(\Lambda) = \bigcup_{\mu\in\Lambda} A(\mu) \subset B$.

For each $\mu \in \Lambda$, let $S(\mu)$ denote the efficient solution set of (*PGSVEP*), i.e.,

$$S(\mu) = \{x \in A(\mu) : F(x, y, \mu) \cap (-C \setminus \{0_Y\}) = \emptyset, \ \forall y \in A(\mu)\}.$$

Throughout of this paper, we always assume $S(\mu) \neq \emptyset$ for all $\mu \in \Lambda$. Now, we recall the definition of semicontinuity of set-valued mappings. Let Λ and X be two metric spaces, $F : \Lambda \to 2^X$ be a set-valued mapping, and $\bar{\lambda} \in \Lambda$.

Definition 2.1 (*[16]*).

(i) F is said to be lower semicontinuous (l.s.c.) at $\bar{\lambda}$ if for any open set U satisfying $U \cap F(\bar{\lambda}) \neq \emptyset$, there exists $\delta > 0$ such that $F(\lambda) \cap U \neq \emptyset$, for all $\lambda \in B(\bar{\lambda}, \delta)$.
(ii) F is said to be upper semicontinuous (u.s.c) at $\bar{\lambda}$ if for any open set U satisfying $F(\bar{\lambda}) \subset U$, there exists $\delta > 0$ such that $F(\lambda) \subset U$, for all $\lambda \in B(\bar{\lambda}, \delta)$.

Proposition 2.2 (*[17,18]*).

(i) *F is l.s.c. at $\bar{\lambda}$ if and only if for any sequence $\{\lambda_n\} \subset \Lambda$ with $\lambda_n \to \bar{\lambda}$ and any $\bar{x} \in F(\bar{\lambda})$, there exists $x_n \in F(\lambda_n)$ such that $x_n \to \bar{x}$.*
(ii) *If F has compact values (i.e., $F(\lambda)$ is a compact set for each $\lambda \in \Lambda$), then F is u.s.c. at $\bar{\lambda}$ if and only if for any sequence $\{\lambda_n\} \subset \Lambda$ with $\lambda_n \to \bar{\lambda}$ and for any $x_n \in F(\lambda_n)$, there exists $\bar{x} \in F(\bar{\lambda})$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \bar{x}$.*

3. Main results

In this section, we present the continuity of the efficient solution mapping to (*PGSVEP*).

Theorem 3.1. *Suppose that the following conditions are satisfied.*

(i) *$A(\cdot)$ is continuous with compact values on Λ;*
(ii) *$F(\cdot,\cdot,\cdot)$ is l.s.c. on $B \times B \times \Lambda$;*
(iii) *for each $\mu \in \Lambda$, $x \in A(\mu) \setminus S(\mu)$, there exists $y \in S(\mu)$ such that*

$$d_X(x, y) \leq \inf_{z\in F(x,y,\mu)} d_Y(z, Y \setminus -\text{int}\, C).$$

Then, $S(\cdot)$ is continuous on Λ.

Proof. We first prove that $S(\cdot)$ is lower semicontinuous on Λ. Suppose on the contrary that there exists $\mu_0 \in \Lambda$ such that $S(\cdot)$ is not l.s.c. at μ_0. Then there exists a sequence $\{\mu_n\}$ with $\mu_n \to \mu_0$ and $x_0 \in S(\mu_0)$ such that for any $x_n \in S(\mu_n)$, $x_n \nrightarrow x_0$. Since $x_0 \in S(\mu_0)$, we have $x_0 \in A(\mu_0)$ and

$$F(x_0, y, \mu_0) \cap (-C \setminus \{0_Y\}) = \emptyset, \quad \forall y \in A(\mu_0). \tag{3.1}$$

By the lower semicontinuity of $A(\cdot)$, there exists a sequence $\{\bar{x}_n\} \subset A(\mu_n)$ such that $\bar{x}_n \to x_0$. Obviously, $\bar{x}_n \in A(\mu_n) \setminus S(\mu_n)$. It follows from (iii) that there exists $y_n \in S(\mu_n)$ such that

$$d_X(\bar{x}_n, y_n) \leq \inf_{z\in F(\bar{x}_n,y_n,\mu_n)} d_Y(z, Y \setminus -\text{int}\, C). \tag{3.2}$$

Since $y_n \in A(\mu_n)$, by the upper semicontinuity and compactness of $A(\cdot)$, we get that there exists $y_0 \in A(\mu_0)$ and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \to y_0$, denoted by $\{y_i\}$. Using (3.2), we have

$$d_X(\bar{x}_i, y_i) \leq \inf_{z \in F(\bar{x}_i, y_i, \mu_i)} d_Y(z, Y \setminus -\operatorname{int} C). \tag{3.3}$$

Since F is l.s.c. on $B \times B \times \Lambda$, for $(\bar{x}_i, y_i, \mu_i) \to (x_0, y_0, \mu_0)$ and any $z_0 \in F(x_0, y_0, \mu_0)$ there exists $z_i \in F(\bar{x}_i, y_i, \mu_i)$ such that $z_i \to z_0$. It follows from (3.3) that $d_X(\bar{x}_i, y_i) \leq d_Y(z_i, Y \setminus -\operatorname{int} C)$. By the continuity of metric distance $d_Y(\cdot, Y \setminus -\operatorname{int} C)$ we get that

$$d_X(x_0, y_0) \leq d_Y(z_0, Y \setminus -\operatorname{int} C).$$

We want to show that $x_0 = y_0$. Assume that $x_0 \neq y_0$, then the last inequality implies that

$$0 < d_X(x_0, y_0) \leq d_Y(z_0, Y \setminus -\operatorname{int} C).$$

Hence, $z_0 \in -\operatorname{int} C \subset -C \setminus \{0\}$. This is a contradiction with (3.1), because of $z_0 \in F(x_0, y_0, \mu_0)$. Hence, we have $x_0 = y_0$. This is impossible by the contradiction assumption.

Next, we show that $S(\cdot)$ is u.s.c. on Λ. Suppose on the contrary that there exists some $\mu_0 \in \Lambda$ such that $S(\cdot)$ is not u.s.c. at μ_0. Then there exists an open set V satisfying $S(\mu_0) \subset V$, and sequences $\mu_n \to \mu_0$ and $x_n \in S(\mu_n)$ such that $x_n \notin V$ for all $n \in \mathbb{N}$. Since $x_n \in A(\mu_n)$ and $A(\cdot)$ is u.s.c. at μ_0 with compact values, there is $x_0 \in A(\mu_0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$, denoted by $\{x_k\}$.

We want to show that $x_0 \in S(\mu_0)$. Assume that $x_0 \notin S(\mu_0)$. Then by (iii) there exists $y_0 \in S(\mu_0)$ such that

$$d_X(x_0, y_0) \leq \inf_{z \in F(x_0, y_0, \mu_0)} d_Y(z, Y \setminus -\operatorname{int} C). \tag{3.4}$$

Since A is l.s.c. at μ_0 and $y_0 \in A(\mu_0)$, we have that there exists $y_n \in A(\mu_n)$ such that $y_n \to y_0$. It follows from $x_n \in S(\mu_n)$ and $y_n \in A(\mu_n)$ that

$$F(x_n, y_n, \mu_n) \cap (-C \setminus \{0_Y\}) = \emptyset. \tag{3.5}$$

Since $F(\cdot, \cdot, \cdot)$ is l.s.c. at (x_0, y_0, μ_0) and $(x_k, y_k, \mu_k) \to (x_0, y_0, \mu_0)$, for $z_0 \in F(x_0, y_0, \mu_0)$, there exists $z_k \in F(x_k, y_k, \mu_k)$ such that $z_k \to z_0$. Notice that $z_k \notin -C \setminus \{0\}$, also $d_Y(z_k, Y \setminus -\operatorname{int} C) = 0$ for all k. It follows from (3.4), (3.5) and continuity of metric distance $d_Y(\cdot, Y \setminus -\operatorname{int} C)$ that

$$0 \leq d_X(x_0, y_0) \leq d_Y(z_0, Y \setminus -\operatorname{int} C) = \lim_{k \to \infty} d_Y(z_k, Y \setminus -\operatorname{int} C) = 0. \tag{3.6}$$

That is $x_0 = y_0$, which leads to a contradiction. Hence $x_0 \in S(\mu_0) \in V$. This is impossible by the contradiction assumption. Therefore, we complete the proof. □

The following example illustrates that the condition (iii) in Theorem 3.1 is essential.

Example 3.2. Let $X = Y = Z = \mathbb{R}$, $C = \mathbb{R}_+$, $\Lambda = [0, 1]$, $A(\mu) = B = [0, 1]$ and $F(x, y, \mu) = [\mu(y - x), +\infty)$.

It is not hard to check that assumptions (i) and (ii) in Theorem 3.1 are satisfied. After calculating, we get that

$$S(\mu) = \begin{cases} \{0\}, & \text{if } \mu \neq 0, \\ [0, 1], & \text{if } \mu = 0. \end{cases}$$

We see that $S(\cdot)$ is not continuous at $\mu_0 = 0$, since assumption (iii) is violated. Indeed, for each $\mu \in (0, 1)$, for any $x \in A(\mu) \setminus S(\mu) = (0, 1]$, putting $y = 0 \in S(\mu)$, we have

$$d(x, y) = x \quad \text{and} \quad \inf_{z \in F(x, y, \mu)} d(z, Y \setminus -\operatorname{int} C) = \inf_{z \in [-\mu x, +\infty)} d(z, [0, +\infty)) = 0.$$

This implies that $d(x, y) = x > 0 = \inf_{z \in F(x,y,\mu)} d(z, Y \setminus -\operatorname{int} C)$. Hence, assumption (iii) is essential.

Corollary 3.3. *Suppose that the following conditions are satisfied.*

(i) *$A(\cdot)$ is continuous with compact values on Λ;*
(ii) *$F(\cdot, \cdot, \cdot)$ is l.s.c. on $B \times B \times \Lambda$;*
(iii) *for each $\mu \in \Lambda$, $x \in A(\mu) \setminus S(\mu)$, there exists $y \in S(\mu)$ such that*

$$d_X(x, y) \leq \inf_{z \in F(x, y, \mu)} d_Y(z, Y \setminus -\operatorname{int} C).$$

Then, $S(\cdot)$ is l.s.c. on Λ.

Remark 3.4. Corollary 3.3 discusses the lower semicontinuity of the solution mappings involving set-valued mappings, which is more general than the result in Theorem 3.1 of [15] in the case that F is a set-valued mapping. Furthermore, the continuity of F is relaxed to the lower semicontinuity. Compared with Theorem 3.1 of [14], assumption (iii) of Corollary 3.3 is different from assumption (v) in Theorem 3.1 in [14]. An inconvenience of assumption (iii) of Corollary 3.3 is that it requires to know the solution map. However, in some situations Corollary 3.3 is applicable while Theorem 3.1 in [14] is not, as shown by the following example.

Example 3.5. Let $X = Y = Z = \mathbb{R}$, $C = \mathbb{R}_+$, $\Lambda = [0, 1]$, $A(\mu) = [\mu, 1]$ and $F(x, y, \mu) = [y - x, (1 + \mu)(y - x)]$.

It follows from direct computation that, $S(\mu) = \{\mu\}$. Obviously, $S(\mu) = S_f(\mu)$ for all $f \in C^* \setminus \{0\}$, where $S_f(\mu) = \{x \in A(\mu) : \inf_{z \in F(x,y,\mu)} f(z) \geq 0,\ \forall y \in A(\mu)\}$ and C^* is a dual cone of C. It is easy to verify that assumptions (i)–(iv) and (vi) in Theorem 3.1 of [14] are satisfied. But F is not satisfied (v). Indeed, $f = 1 \in C^* \setminus \{0\}$, $x = 1 \in A(\mu) \setminus S_f(\mu) = (\mu, 1]$, we have $y \in \{\mu\} = S(\mu)$

$$F(x, y, \mu) + F(x, y, \mu) + B(0, |x - y|) = B(0, |1 - \mu|) \not\subseteq -C.$$

Therefore, we cannot obtain that $S(\cdot)$ is l.s.c. on Λ. However, assumption (iii) in Corollary 3.3 is satisfied. Indeed, for all $x \in (\mu, 1]$, there is only one $y = \mu \in S(\mu)$, we get that

$$d(x, y) = |x - \mu|,$$

and

$$\inf_{z \in F(x,y,\mu)} d(z, Y \setminus -\operatorname{int} C) = \inf_{z \in [(1+\mu)(\mu - x), \mu - x]} d(z, [0, +\infty)) = |x - \mu|.$$

Hence, all assumptions of Corollary 3.3 are satisfied, and so $S(\cdot)$ is l.s.c. on Λ.

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