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Fixed point theorems for psi–alpha–eta–expansive mappings in metric spaces

ผู้วิจัย

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หน้า

ABSTRACT

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In this project, we introduce the concept of generalized alpha-eta-psi-Geraghty contraction type mappings and prove the unique fixed point theorems for such mappings in alpha-eta-complete metric spaces without assuming the subadditivity of psi. We also give an example for supporting the result and present an application using our main result to obtain a solution of the integral equation.

We also we introduce the notion of modified (alpha-psi-varphi-theta)-rational contractive mappings where some conditions of Bianchini-Grandolfi gauge function varphi are omitted. We establish the existence of the unique fixed point theorems for such mappings which are triangular alpha-orbital admissible in alpha-complete b-metric spaces. Moreover, we also prove the unique common fixed point theorem for mappings T and g where T is a modified (alpha-psi-varphi-theta)-rational contractive mapping with respect to g. Our results extend the fixed point theorems in alpha-complete metric spaces to alpha-complete b-metric spaces.

Keywords: alpha-eta-complete metric spaces; alpha-eta-continuous mappings; triangular alpha-orbital admissible mappings; generalized alpha-eta-psi-Geraghty contraction type mappings.

บทคัดย่อ

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ชื่อโครงการ : ทฤษฎีบทจุดตรึงสำหรับการส่งไซ-แอลฟา-อีตา-แบบขยาย ในปริภูมิเมตริก

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ในโครงการนี้ผู้วิจัยได้แนะนำแนวคิดของการส่งชนิดคอนแทรกทีฟแอลฟา–อีตา–ไซ–เจราจตี วางนัยทั่วไป และพิสูจน์ทฤษฎีบทจุดตรึงสำหรับการส่งดังกล่าวในปริภูมิเมตริกแอลฟา–อีตา–บริบูรณ์โดยไม่ใช้การเป็นกึ่งบวกของฟังก์ชันไซ ผู้วิจัยได้ยกตัวอย่างเพื่อสนับสนุนผลลัพธ์และนำเสนอการประยุกต์โดยใช้ผลลัพธ์หลักเพื่อหาผลเฉลยของสมการอินทิกรัล

ผู้วิจัยได้แนะนำสัญลักษณ์ของการส่งโมดิฟาย (แอลฟา–ไซ–วาฬี–ซีตา)–คอนแทรกทีฟเศษส่วน โดยละบางเงื่อนไขของการส่งมาตรวัดเบียนชินี–แกรนโดฟี นอกจากนี้ผู้วิจัยยังได้สร้างทฤษฎีบทจุดตรึงสำหรับการส่งดังกล่าวซึ่งเป็นไตรแองกูลาแอลฟา–ออบิทอล แอดมิสสิเบิล ในปริภูมิแอลฟา–บริบูรณ์ บี–เมตริก และพิสูจน์ทฤษฎีบทจุดตรึงร่วมสำหรับการส่ง T และ g โดยที่ T เป็นการส่งโมดิฟาย (แอลฟา–ไซ–วาฬี–ซีตา)–คอนแทรกทีฟเศษส่วน ซึ่งสัมพันธ์กับ g ผลลัพธ์ที่ได้จะเป็นการขยายทฤษฎีบทจุดตรึงในปริภูมิเมตริกแอลฟา–บริบูรณ์ ไปยังปริภูมิแอลฟา–บริบูรณ์ บี–เมตริก

คำสำคัญ: ปริภูมิเมตริกแอลฟา–อีตา–บริบูรณ์; การส่งแอลฟา–อีตา–ต่อเนื่อง; การส่งไตรแองกูลาแอลฟา–ออบิทอล แอดมิสสิเบิล; การส่งชนิดคอนแทรกทีฟแอลฟา–อีตา–ไซ–เจราจตีวางนัยทั่วไป

CHAPTER I

EXECUTIVE SUMMARY

Let X be a set and $T : X \to X$ a mapping. The solutions we seek are represented by points invariant under T. These are the points satisfying

$$x = Tx. \tag{1}$$

Such points are said to be fixed under T or fixed points of T. The set of all solutions of (1) is called the fixed point set of T and denoted by Fix T. If the mapping T does not have a fixed point we often say that T is fixed point free.

Fundamental to the study of Fixed Point Theory is the attempt to identify conditions which may be imposed on the set X and/or the mapping T that will assure Fix $T \neq \varnothing$. Usually it is more efficient to study a family $\mathcal{T}$ of mapping satisfying some common conditions rather than an individual mapping. If all the mapping $T \in \mathcal{T}$ have fixed points, then we say that X has the fixed point property with respect to $\mathcal{T}$. The term " fixed point property" is often abbreviated as fpp, and if we are dealing with the fixed specific family $\mathcal{T}$ the words " with respect to $\mathcal{T}$ " are omitted.

Typically, a fixed point theorem has the following form.

Generic Theorem. *Let X be a set having structure* A *and let $\mathcal{T}$ be the family of mappings $T : X \to X$ satisfying condition* B. *Then each mapping $T \in \mathcal{T}$ has a fixed point.*

Fixed point theory has fascinated many researchers since 1922 with the celebrated Banach fixed point theorem. There exists a vast literature on the topic field and this is very active field of research at present. Fixed point theorems are very important tools for proving the existence and uniqueness of the solutions to various mathematical models (integral and partial equations, variational inequalities etc). It can be applied to, for examples, variational inequalities, optimization, and approximation theory. The fixed point theory has been continually studied by many researchers. It is well-known that the contractive-type conditions are very indispensable in the study of fixed point theory. The first important result on fixed points for contractive-type mappings was the well-known Banach-Caccioppoli theorem which was published in 1922. Later in 1968, Kannan studied a new type of contractive mappings. Since then, there have been many results related to mappings satisfying various types of contractive inequalities.

Recently, Samet et al. introduced a new category of contractive type mappings known as α-ψ contractive type mappings. The results obtained by Samet et al. extended and generalized the existing fixed point results in the literature, in particular the Banach contraction principle. Salimi et al. and Karapinar and Samet generalized the α-ψ contractive type mappings and obtained various fixed point theorems for this generalized class of contractive mappings. In most of papers have considered the α-ψ contractive type mapping for a nondecreasing mapping $\psi : [0, +\infty) \to [0, +\infty)$ with $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t \in (0, +\infty)$. The convergence of $\sum_{n=1}^{\infty} \psi^n(t)$ and nondecreasing condition for ψ are restrictive and it is a fact that such a mapping is differentiable almost everywhere and hence continuous why was one of our aims to write this article in order to consider a family of mappings $\psi : [0, +\infty) \to [0, +\infty)$ by relaxing nondecreasing condition and the convergence of the series $\sum_{n=1}^{\infty} \psi^n(t)$. This article inspired and motivated by above research works, we will introduce a new family of mappings on $[0, +\infty)$ and prove the fixed point theorems for mappings using properties of this new family in complete metric spaces. By applying our obtained results, we also assure the fixed point theorems in partially ordered complete metric spaces and give the applications to ordinary differential equations.

One of the most important results in fixed point theory is the Banach contraction principle introduced by Banach. There were many authors have studied and proved the results for fixed point theory by generalizing the Banach contraction principle in several directions. One of the remarkable result is Geraghty's theorem given by Geraghty. In 2013, Cho et al. introduced the notion of α-Geraghty contraction type mappings and assured the unique fixed point theorems for such mappings in complete metric spaces. Recently, Popescu defined the concept of triangular α-orbital admissible mappings and proved the unique fixed point theorems for the mentioned mappings which are generalized α-Geraghty contraction type mappings. On the other hand, Karapinar proved the existence of a unique fixed point for a triangular α-admissible mapping which is a generalized α-ψ-Geraghty contraction type mapping.

In this work, we introduce the notion of generalized α-η-ψ-Geraghty contraction type mappings in metric spaces. Moreover, we prove the unique fixed point theorems for generalized α-η-ψ-Geraghty contraction type mappings which are triangular α-orbital admissible mappings in the setting of α-η-complete metric spaces without assuming the subadditivity of ψ. Our results improve and generalize the results proved by Karapinar and Poposcu . Furthermore, we also give an example for supporting the result and present an application using our main result to obtain a solution of the integral equation.

Fixed point theory in metric spaces is one of the most important tools for proving the existence and uniqueness of the solutions to various mathematical models. Later in 1993 Czerwik, generalized the notion of metric spaces by introducing the notion of b-metric spaces. On the other hand, Samet et al. proved the fixed point theorems for α-admissible mappings which are α-φ-contractive mappings in complete metric spaces. Salimi et al. and Hussain et al. modified these notions and assured the fixed point theorems. Recently, Hussain et al. established fixed point theorems for modified α-φ-rational contractive mappings in α-complete metric spaces and proved the existence of solutions of integral equations.

In this project we extend the fixed point results in α-complete metric spaces proved by Hussian et al. to α-complete b-metric spaces by introducing the notion of modified $(\alpha$-ψ-φ-$\theta)$-rational contractive mappings where some conditions of Bianchini-Grandolfi gauge function φ are omitted. We establish the existence of the unique fixed point theorems for such mappings which are triangular α-orbital admissible. Moreover, we also prove the unique common fixed point theorem for mappings T and g where T is a modified (α-ψ-φ-θ)-rational contractive mapping with respect to g and is triangular g-α-admissible in the setting of α-complete b-metric spaces.

CHAPTER II

CONTENTS OF RESEARCH

In this project, we obtain two publications that published in the international journals as the followings:

1. Preeyaluk Chuadchawna, **Anchalee Kaewcharoen** and Somyot Plubtieng, Fixed point theorems for generalized alpha-eta-psi-Gerahty contraction type mappings in alpha-eta-complete partial metric spaces, Journal of Nonlinear Science and Applications, 9 (2016), 471-485. (Impact Factor := 1.176)

   (a) **Theorem** : Let (X, d) be a metric space. Assume that $\alpha, \eta : X \times X \to [0, \infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:

   (i) (X, d) is an α-η-complete metric space;

   (ii) T is a generalized α-η-ψ-Geraghty contraction type mapping;

   (iii) T is a triangular α-orbital admissible mapping with respect to η;

   (iv) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$;

   (v) T is an α-η-continuous mapping.

   Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

   (b) **Theorem** : Let (X, d) be a metric space. Assume that $\alpha, \eta : X \times X \to [0, \infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:

   (i) (X, d) is an α-η-complete metric space;

   (ii) T is a generalized α-η-ψ-Geraghty contraction type mapping;

   (iii) T is a triangular α-orbital admissible mapping with respect to η;

   (iv) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$;

   (v) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq \eta(x_{n(k)}, x^*)$ for all $k \in \mathbb{N}$.

   Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

   (c) **Corollary** : Let (X, d) be a complete metric space, $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:

   (i) T is a generalized α-Geraghty contraction type mapping;

   (ii) T is a triangular α-orbital admissible mapping;

(iii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;

(iv) T is a continuous mapping or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq 1$ for all $k \in \mathbb{N}$.

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

(d) **Theorem** : Let (X, d) be a complete metric space, $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:

(i) T is a triangular α-orbital admissible mapping;

(ii) if there exists $\beta \in \mathcal{F}$ such that

$$\alpha(x,y) \geq 1 \text{ implies } \psi(d(Tx,Ty)) \leq \beta(\psi(M(x,y)))\psi(M(x,y)) \text{ for all } x, y \in X,$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\} \text{ and } \psi \in \Psi';$$

(iii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;

(iv) T is a continuous mapping or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq 1$ for all $k \in \mathbb{N}$.

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

(e) **Theorem** : Let (X, d) be a complete metric space. Assume that $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:

(i) T is a triangular α-admissible mapping;

(ii) T is a generalized α-ψ-Geraghty contraction type mapping;

(iii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;

(iv) T is a continuous mapping or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq 1$ for all $k \in \mathbb{N}$.

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*. Then F has a coupled fixed point.

(f) **Theorem** : Let (X, d) be a metric space. Assume that $\alpha, \eta : X \times X \to [0, \infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:

(i) (X, d) is an α-η-complete metric space;

(ii) T is an α-η-ψ-Geraghty contraction type mapping;

(iii) T is a triangular α-orbital admissible mapping with respect to η;

(iv) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$;

(v) T is an α-η-continuous mapping.

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

(g) **Theorem** : Let (X, d) be a metric space. Assume that $\alpha, \eta : X \times X \to [0, \infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:

(i) (X, d) is an α-η-complete metric space;

(ii) T is an α-η-ψ-Geraghty contraction type mapping;

(iii) T is a triangular α-orbital admissible mapping with respect to η;

(iv) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$;

(v) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq \eta(x_{n(k)}, x^*)$ for all $k \in \mathbb{N}$.

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

(h) **Theorem** : Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that the metric space (X, d) is complete. Let $T : X \to X$ be a continuous and nondecreasing mapping with respect to $\preccurlyeq$. Assume that the following conditions hold:

(i) there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq k(d(x, y))$ for all $x, y \in X$ with $x \preccurlyeq y$;

(ii) there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;

(iii) T is continuous.

Then T has a fixed point.

(i) **Theorem** : Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that $T : X \to X$. Assume that the following conditions are satisfied:

(i) there exists $\beta \in \mathcal{F}$ such that

$$\psi(d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y))$$

for all $x, y \in X$ with $x \preceq y$ where $\psi \in \Psi'$;

(ii) there exists $x_1 \in X$ such that $x_1 \preceq Tx_1$;

(iii) T is nondecreasing;

(iv) either T is continuous or if $\{x_n\}$ is a nondecreasing sequence with $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all $k \in \mathbb{N}$.

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*. Further if for all $x \neq y \in X$, there exists $v \in X$ such that $x \preceq v$, $y \preceq v$ and $v \preceq Tv$, then T has a unique fixed point.

2. Preeyaluk Chuadchawna and **Anchalee Kaewcharoen**, Fixed point theorems for modified (alpha-psi-varphi-theta)-rational contractive mappings in alpha-complete b-metric spaces, 14 (2016), 215-235. (SJR:=Q4)

   (a) **Theorem** : Let (X,d) be an α-complete b-metric space, $\alpha : X \times X \to [0,\infty)$ and $T : X \to X$ is a modified $(\alpha$-ψ-φ-$\theta)$-rational contractive mapping. Assume that the following conditions hold:

      (i) T is triangular α-orbital admissible;

      (ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

      (iii) T is α-continuous.

   Then T has a fixed point.

   (b) **Theorem** : Let (X,d) be an α-complete b-metric space and $\alpha : X \times X \to [0,\infty)$. Suppose that $T : X \to X$ is a modified $(\alpha$-ψ-φ-$\theta)$-rational contractive mapping. Assume that the following conditions hold:

      (i) T is triangular α-orbital admissible;

      (ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

      (iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all $k \in \mathbb{N}$.

   Then T has a fixed point.

   (c) **Corollary** : Let (X,d) be an α-complete b-metric space where $\alpha : X \times X \to [0,\infty)$ and $T : X \to X$. Assume that there exists $L \geq 0$ such that for all $x, y \in X$,

$$\alpha(x,y) \geq 1 \text{ implies } s^3 d(Tx, Ty) \leq \varphi(M_b(x,y)) + L\theta(N_b(x,y)), \quad (1)$$

   where

$$M_b(x,y) = \max\{d(x,y), \frac{d(x,Tx)}{1+d(x,Tx)}, \frac{d(y,Ty)}{1+d(y,Ty)}, \frac{d(x,Ty)+d(y,Tx)}{2s}\},$$

$$N_b(x,y) = \min\{d(x,Tx), d(x,Ty), d(y,Tx)\}$$

   and $\varphi, \theta : [0,\infty) \to [0,\infty)$ are continuous functions such that $\theta(0) = 0$, $\varphi(t) < t$, $\theta(t) > 0$ for each $t > 0$ and φ in nondecreasing. Assume that the following conditions hold:

      (i) T is triangular α-orbital admissible;

      (ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

      (iii) T is α-continuous or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all $k \in \mathbb{N}$.

   Then T has a fixed point. Moreover, either $\alpha(u,v) \geq 1$ or $\alpha(v,u) \geq 1$ whenever $Tu = u$ and $Tv = v$. Then T has a unique fixed point.

(d) **Corollary** : Let (X,d) be an α-complete b-metric space where $\alpha : X \times X \to [0,\infty)$. Suppose that $T : X \to X$ is a mapping such that for all $x, y \in X$,

$$\alpha(x,y) \geq 1 \text{ implies } s^3 d(Tx,Ty) \leq M_b(x,y) - \varphi'(M_b(x,y)), \quad (2)$$

where

$$M_b(x,y) = \max\{d(x,y), \frac{d(x,Tx)}{1+d(x,Tx)}, \frac{d(y,Ty)}{1+d(y,Ty)}, \frac{d(x,Ty)+d(y,Tx)}{2s}\},$$

and $\varphi' : [0,\infty) \to [0,\infty)$ is continuous such that $\varphi'(0) = 0$, $\varphi'(t) < t$ for each $t > 0$ and φ' in nonincreasing Assume that the following conditions hold:

(i) T is triangular α-orbital admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is α-continuous or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all $k \in \mathbb{N}$.

Then T has a fixed point. Moreover, either $\alpha(u,v) \geq 1$ or $\alpha(v,u) \geq 1$ whenever $Tu = u$ and $Tv = v$. Then T has a unique fixed point.

(e) **Theorem** :Let (X,d) be an α-complete b-metric space and $T, g : X \to X$ be such that $TX \subseteq gX$ and suppose that gX is closed. Let $\alpha : X \times X \to [0,\infty)$ and T is a modified $(\alpha$-ψ-φ-$\theta)$-rational contractive mapping with respect to g. Assume that the following conditions hold:

(i) T is triangular g-α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$;

(iii) T is α-continuous with respect to g.

Then T and g have a coincidence point.

(f) **Theorem** : Let (X,d) be an α-complete b-metric space and $T, g : X \to X$ be such that $TX \subseteq gX$ and suppose that gX is closed. Let $\alpha : X \times X \to [0,\infty)$ and T is a modified $(\alpha$-ψ-φ-$\theta)$-rational contractive mapping with respect to g. Assume that the following conditions hold:

(i) T is triangular g-α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$;

(iii) if $\{gx_n\}$ is a sequence in X such that $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $gx_n \to gx$ as $n \to \infty$, then there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $\alpha(gx_{n(k)}, gx) \geq 1$ for all $k \in \mathbb{N}$.

Then T and g have a coincidence point.

(g) **Theorem** :Let (X, d) be an α-complete b-metric space with respect to g and $T, g : X \to X$ be such that $TX \subseteq gX$. Assume that gX is closed and there exist $\alpha : X \times X \to \mathbb{R}$ and $L \geq 0$ such that for all $x, y \in X$,

$$\alpha(x,y) \geq 1 \text{ implies } s^3 d(Tx,Ty) \leq \varphi(M_b(x,y)) + L\theta(N_b(x,y)), \quad (3)$$

where

$$M_b(x,y) = \max\{d(gx,gy), \frac{d(gx,Tx)}{1+d(gx,Tx)}, \frac{d(gy,Ty)}{1+d(gy,Ty)}, \frac{d(gx,Ty)+d(gy,Tx)}{2s}\},$$

$$N_b(x,y) = \min\{\frac{d(gx,Tx)}{1+d(g,Tx)}, \frac{d(gx,Ty)}{1+d(gx,Ty)}, \frac{d(gy,Tx)}{1+d(gy,Tx)}\}$$

and $\varphi, \theta : [0,\infty) \to [0,\infty)$ are continuous functions such that $\theta(0) = 0$, $\varphi(t) < t$, $\theta(t) > 0$ for each $t > 0$. Assume that the following conditions hold:

(i) T is triangular g-α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$;

(iii) T is α-continuous with respect to g or if $\{gx_n\}$ is a sequence in X such that $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $gx_n \to gx$ as $n \to \infty$, then there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $\alpha(gx_{n(k)}, gx) \geq 1$ for all $k \in \mathbb{N}$.

Then T and g have a coincidence point. Moreover, assume that the following conditions hold:

(iv) the pair $\{T, g\}$ is weakly compatible;

(v) either $\alpha(u,v) \geq 1$ or $\alpha(v,u) \geq 1$ whenever $Tu = gu$ and $Tv = gv$.

Then T and g have a unique common fixed point.

CHAPTER III

OUTPUT

ผลลัพธ์จากโครงการวิจัยที่ได้รับทุนจากงบประมาณแผ่นดินมหาวิทยาลัยนเรศวร ประจำปี 2559

1. ผลงานวิจัยตีพิมพ์ในวารสารวิชาการนานาชาติ

1.1 Preeyaluk Chaudchawna, **Anchalee Kaewcharoen** and Somyot Plubtieng, Fixed point theorems for generalized alpha-eta-psi-Gerahty contraction type mappings in alpha-eta-complete partial metric spaces, Journal of Nonlinear Science and Applications, 9 (2016), 471-485. (Impact Factor := 1.176)

1.2 Preeyaluk Chaudchawna and **Anchalee Kaewcharoen**, Fixed point theorems for modified (alpha-psi-varphi-theta)-rational contractive mappings in alpha-complete b-metric spaces, Thai Journal of Mathematics, 14 (2016), 215-235. (SJR:=Q4)

2. การนำผลงานวิจัยไปใช้ประโยชน์

มีการนำไปใช้ประโยชน์ทั้งเชิงวิชาการและเชิงสาธารณะโดยทำให้มีการพัฒนาการเรียนการสอน รวมทั้งมีการสร้างเครือข่ายความร่วมมือในการทำวิจัย

ภาคผนวก 1

Fixed point theorems for generalized alpha-eta-psi-Gerahty contraction type mappings in alpha-eta-complete partial metric spaces

Anchalee Kaewcharoen

Journal of Nonlinear Science and Applications

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Fixed point theorems for $(\alpha, \eta, \psi, \xi)$-contractive multi-valued mappings on α-η-complete partial metric spaces

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Abstract

In this paper, the notion of strictly $(\alpha, \eta, \psi, \xi)$-contractive multi-valued mappings is introduced where the continuity of ξ is relaxed. The existence of fixed point theorems for such mappings in the setting of α-η-complete partial metric spaces are provided. The results of the paper can be viewed as the extension of the recent results obtained in the literature. Furthermore, we assure the fixed point theorems in partial complete metric spaces endowed with an arbitrary binary relation and with a graph using our obtained results.

Keywords: α-η-complete partial metric spaces, α-η-continuity, $(\alpha, \eta, \psi, \xi)$-contractive multi-valued mappings, α-admissible multi-valued mappings with respect to η.
2010 MSC: 47H10, 54H25.

1. Introduction and Preliminaries

The metric fixed point theory is one of the most important tools for proving the existence and uniqueness of the solution to various mathematical models. There are many authors who have generalized the metric spaces in many directions. In 1994, Matthews [12] introduced the partial metric spaces and proved the Banach contraction principle in such spaces. Later on, the researchers have studied the fixed point theorems

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the mentioned mappings which are generalized α-Geraghty contraction type mappings. On the other hand, Karapinar [8] proved the existence of a unique fixed point for a triangular α-admissible mapping which is a generalized α-ψ-Geraghty contraction type mapping.
For the sake of convenience, we recall the Geraghty's theorem. Let $\mathcal{F}$ be the family of all functions $\beta : [0, \infty) \to [0, 1)$ satisfying the condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0.$$

Geraghty [4] proved the following unique fixed point theorem in a complete metric space.

Theorem 1.1 ([4]). *Let (X, d) be a complete metric space and $T : X \to X$. Suppose that there exists $\beta \in \mathcal{F}$ such that*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y) \text{ for all } x, y \in X.$$

Then T has a unique fixed point $x^* \in X$.

In 2012, Samet *et al.* [13] introduced the notion of α-admissible mappings.

Definition 1.2 ([13]). Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then T is said to be α-admissible if

$$\alpha(x, y) \geq 1 \text{ implies } \alpha(Tx, Ty) \geq 1.$$

Karapinar *et al.* [9] defined the concept of triangular α-admissible mappings.

Definition 1.3 ([9]). A mapping $T : X \to X$ is said to be triangular α-admissible if

(a) T is α-admissible;
(b) $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ imply $\alpha(x, y) \geq 1$.

The definitions of α-orbital admissible mappings and triangular α-orbital admissible mappings are defined by Popescu [12] in 2014.

Definition 1.4 ([12]). Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then T is said to be α-orbital admissible if

$$\alpha(x, Tx) \geq 1 \text{ implies } \alpha(Tx, T^2x) \geq 1.$$

Definition 1.5 ([12]). Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then T is said to be triangular α-orbital admissible if

(a) T is α-orbital admissible;
(b) $\alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1$ imply $\alpha(x, Ty) \geq 1$.

Remark 1.6. Every triangular α-admissible mapping is a triangular α-orbital admissible mapping. There exists a triangular α-orbital admissible mapping which is not a triangular α-admissible mapping. For more details see [12].

Popescu [12] gave the definition of generalized α-Geraghty contraction type mappings and proved the fixed point theorems for such mappings in complete metric spaces.

Definition 1.7 ([12]). Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is said to be a generalized α-Geraghty contraction type mapping if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,

$$\alpha(x, y)d(Tx, Ty) \leq \beta(M_T(x, y))M_T(x, y),$$

where

$$M_T(x, y) = \max\left\{d(x, y),\, d(x, Tx),\, d(y, Ty),\, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

Theorem 1.8 ([12]). *Let (X, d) be a complete metric space, $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:*

(i) *T is a generalized α-Geraghty contraction type mapping;*
(ii) *T is a triangular α-orbital admissible mapping;*
(iii) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;*
(iv) *T is a continuous mapping.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Recently, Karapinar [8] introduced the concept of α-ψ-Geraghty contraction type mappings in complete metric spaces.

Let Ψ denote the class of the functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(a) ψ is nondecreasing;
(b) ψ is continuous;
(c) $\psi(t) = 0$ if and only if $t = 0$;
(d) ψ is subadditive, that is $\psi(s + t) \leq \psi(s) + \psi(t)$.

Definition 1.9. Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is said to be a generalized α-ψ-Geraghty contraction type mapping if there exists $\beta \in \mathcal{F}$ such that

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)) \text{ for all } x, y \in X,$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\} \text{ and } \psi \in \Psi.$$

Theorem 1.10 ([8]). *Let (X, d) be a complete metric space, $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$. Assume that the following conditions are satisfied:*

(i) *T is a generalized α-ψ-Geraghty contraction type mapping;*
(ii) *T is a triangular α-admissible mapping;*
(iii) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;*
(iv) *T is a continuous mapping.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

On the other hand, Hussain *et al.* [6] introduced the concepts of α-η-complete metric spaces and α-η-continuous functions.

Definition 1.11 ([6]). Let (X, d) be a metric space and $\alpha, \eta : X \times X \to [0, +\infty)$. Then X is said to be α-η-complete if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ converges in X.

Example 1.12. Let $X = (0, \infty)$ and define a metric on X by $d(x, y) = |x - y|$ for all $x, y \in X$. Therefore X is not complete. Let Y be a closed subset of X. Define $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} (x + y)^3, & \text{if } x, y \in Y \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \eta(x, y) = 3x^2 y.$$

We will prove that (X, d) is an α-η-complete metric space. Suppose that $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is in Y. By the completeness of Y, there exists $x^* \in Y$ such that $x_n \to x^*$ as $n \to \infty$.

Definition 1.13 ([6]). Let (X,d) be a metric space and $\alpha,\eta : X \times X \to [0,+\infty)$. A mapping $T : X \to X$ is said to be an α-η-continuous mapping if for each sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ imply $Tx_n \to Tx$ as $n \to \infty$.

Example 1.14. Let $X = [0,\infty)$ and define a metric on X by $d(x,y) = |x-y|$ for all $x, y \in X$. Assume that $T : X \to X$ and $\alpha,\eta : X \times X \to [0,+\infty)$ are defined by

$$Tx = \begin{cases} x^4, & \text{if } x \in [0,1] \\ \cos \pi x + 3, & \text{if } x \in (1,\infty), \end{cases} \quad , \quad \alpha(x,y) = \begin{cases} x^3 + y^3 + 1, & \text{if } x, y \in [0,1] \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad \eta(x,y) = x^3.$$

Therefore T is not continuous. We will prove that T is an α-η-continuous mapping. Let $\{x_n\}$ be a sequence in X such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. This implies that $x_n \in [0,1]$ and so $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_n^4 = x^4 = Tx$.

In this work, we introduce the notion of generalized α-η-ψ-Geraghty contraction type mappings in metric spaces. Moreover, we prove the unique fixed point theorems for generalized α-η-ψ-Geraghty contraction type mappings which are triangular α-orbital admissible mappings in the setting of α-η-complete metric spaces without assuming the subadditivity of ψ. Our results improve and generalize the results proved by Karapinar [8] and Poposcu [12]. Furthermore, we also give an example for supporting the result and present an application using our main result to obtain a solution of the integral equation.

2. Main results

Let Ψ' denote the class of the functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(a) ψ is nondecreasing;
(b) ψ is continuous;
(c) $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.1. Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0,\infty)$. Then T is said to be α-orbital admissible with respect to η if

$$\alpha(x,Tx) \geq \eta(x,Tx) \text{ implies } \alpha(Tx,T^2x) \geq \eta(Tx,T^2x).$$

Definition 2.2. Let $T : X \to X$ and $\alpha,\eta : X \times X \to [0,\infty)$. Then T is said to be triangular α-orbital admissible with respect to η if

1. T is α-orbital admissible with respect to η;
2. $\alpha(x,y) \geq \eta(x,y)$ and $\alpha(y,Ty) \geq \eta(y,Ty)$ imply $\alpha(x,Ty) \geq \eta(x,Ty)$.

Remark 2.3. If we suppose that $\eta(x,y) = 1$ for all $x, y \in X$, then Definition 2.1 reduces to Definition 1.4 and Definition 2.2 reduces to Definition 1.5.

We now prove the important lemma that will be used for proving our main results.

Lemma 2.4. *Let $T : X \to X$ be a triangular α-orbital admissible with respect to η. Assume that there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq \eta(x_1,Tx_1)$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then $\alpha(x_n,x_m) \geq \eta(x_n,x_m)$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Proof. Since $\alpha(x_1,Tx_1) \geq \eta(x_1,Tx_1)$ and T is α-orbital admissible with respect to η, we obtain that

$$\alpha(x_2,x_3) = \alpha(Tx_1,T(Tx_1)) \geq \eta(Tx_1,T(Tx_1)) = \eta(x_2,x_3).$$

By continuing the process as above, we have $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}$. Suppose that

$$\alpha(x_n,x_m) \geq \eta(x_n,x_m) \tag{2.1}$$

and we will prove that $\alpha(x_n, x_{m+1}) \geq \eta(x_n, x_{m+1})$, where $m > n$. Since $\alpha(x_m, x_{m+1}) \geq \eta(x_m, x_{m+1})$, we obtain that

$$\alpha(x_m, Tx_m) = \alpha(x_m, x_{m+1}) \geq \eta(x_m, x_{m+1}) = \eta(x_m, Tx_m). \tag{2.2}$$

By (2.1), (2.2) and triangular α-orbital admissibility of T, we have

$$\alpha(x_n, Tx_m) \geq \eta(x_n, Tx_m).$$

This implies that

$$\alpha(x_n, x_{m+1}) \geq \eta(x_n, x_{m+1}).$$

Hence $\alpha(x_n, x_m) \geq \eta(x_n, x_m)$ for all $m, n \in \mathbb{N}$ with $n < m$. □

We now introduce the concept of generalized α-η-ψ-Geraghty contraction type mappings and prove the fixed point theorems for such mappings.

Definition 2.5. Let (X, d) be a metric space and $\alpha, \eta : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is said to be a generalized α-η-ψ-Geraghty contraction type mapping if there exists $\beta \in \mathcal{F}$ such that $\alpha(x, y) \geq \eta(x, y)$ implies

$$\psi(d(Tx, Ty)) \leq \beta(\psi(M_T(x, y)))\psi(M_T(x, y)),$$

where

$$M_T(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\} \quad \text{and} \quad \psi \in \Psi'.$$

Remark 2.6. In Definition 2.5, if we take $\eta(x, y) = 1$ and $\psi(t) = t$, then it reduces to Definition 1.7.

Theorem 2.7. *Let (X, d) be a metric space. Assume that $\alpha, \eta : X \times X \to [0, \infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:*

(i) *(X, d) is an α-η-complete metric space;*
(ii) *T is a generalized α-η-ψ-Geraghty contraction type mapping;*
(iii) *T is a triangular α-orbital admissible mapping with respect to η;*
(iv) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$;*
(v) *T is an α-η-continuous mapping.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Proof. Let $x_1 \in X$ be such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. Define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Suppose that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, we have $x_{n_0} = x_{n_0+1} = Tx_{n_0}$. Then T has a fixed point. Hence we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. By Lemma 2.4, we have $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Since T is a generalized α-η-ψ-Geraghty contraction type mapping, we have

$$\begin{aligned} \psi(d(x_{n+1}, x_{n+2})) &= \psi(d(Tx_n, Tx_{n+1})) \\ &\leq \beta(\psi(M_T(x_n, x_{n+1})))\psi(M_T(x_n, x_{n+1})) \end{aligned} \tag{2.3}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} M_T(x_n, x_{n+1}) &= \max\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \frac{1}{2}(d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n))\} \\ &= \max\left\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+2})}{2} + \frac{d(x_{n+1}, x_{n+1})}{2}\right\} \\ &\leq \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \left[\frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}\right]\right\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}. \end{aligned}$$

If $\max\{d(x_n, x_{n+1}),\, d(x_{n+1}, x_{n+2})\} = d(x_{n+1}, x_{n+2})$, then

$$\begin{aligned}\psi(d(x_{n+1}, x_{n+2})) &\le \beta(\psi(M_T(x_n, x_{n+1})))\psi(M_T(x_n, x_{n+1}))\\ &\le \beta(\psi(M_T(x_n, x_{n+1})))\psi(d(x_{n+1}, x_{n+2})) < \psi(d(x_{n+1}, x_{n+2})),\end{aligned}$$

which is a contradiction. Thus we conclude that

$$\max\{d(x_n, x_{n+1}),\, d(x_{n+1}, x_{n+2})\} = d(x_n, x_{n+1}).$$

By (2.3), we get that $\psi(d(x_{n+1}, x_{n+2})) < \psi(d(x_n, x_{n+1}))$ for all $n \in \mathbb{N}$. Since ψ is nondecreasing, we have $d(x_{n+1}, x_{n+2}) \le d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Hence we deduce that the sequence $\{d(x_n, x_{n+1})\}$ is nonincreasing. Therefore, there exists $r \ge 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. We claim that $r = 0$. Suppose that $r > 0$. Then due to (2.3), we have

$$\psi(d(x_{n+1}, x_{n+2})) \le \beta(\psi(M_T(x_n, x_{n+1})))\psi(d(x_n, x_{n+1})).$$

Therefore

$$\frac{\psi(d(x_{n+1}, x_{n+2}))}{\psi(d(x_n, x_{n+1}))} \le \beta(\psi(M_T(x_n, x_{n+1}))) < 1.$$

This implies that $\lim_{n\to\infty} \beta(\psi(M_T(x_n, x_{n+1}))) = 1$. Since $\beta \in \mathcal{F}$, we have $\lim_{n\to\infty} \psi(M_T(x_n, x_{n+1})) = 0$, which yields

$$r = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.4}$$

This is a contradiction. Next, we will show that $\{x_n\}$ is a Cauchy sequence. Suppose that there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, there exists $m(k) > n(k) > k$ with $d(x_{n(k)}, x_{m(k)}) \ge \varepsilon$. Let $m(k)$ be the smallest number satisfying the condition above. Then we have $d(x_{n(k)}, x_{m(k)-1}) < \varepsilon$. Therefore

$$\varepsilon \le d(x_{n(k)}, x_{m(k)}) \le d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) < \varepsilon + d(x_{m(k)-1}, x_{m(k)}).$$

Letting $k \to \infty$, we have $\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon$. Since

$$|d(x_{n(k)}, x_{m(k)-1}) - d(x_{n(k)}, x_{m(k)})| \le d(x_{m(k)}, x_{m(k)-1}),$$

we have $\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)-1}) = \varepsilon$. Similarly, we obtain that

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)-1}) = \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon.$$

By Lemma 2.4, we have $\alpha(x_{n(k)-1}, x_{m(k)-1}) \ge \eta(x_{n(k)-1}, x_{m(k)-1})$. Thus we have

$$\begin{aligned}\psi(d(x_{n(k)}, x_{m(k)})) &= \psi(d(Tx_{n(k)-1}, Tx_{m(k)-1}))\\ &\le \beta(\psi(M_T(x_{n(k)-1}, x_{m(k)-1})))\psi(M_T(x_{n(k)-1}, x_{m(k)-1})),\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}M_T(x_{n(k)-1}, x_{m(k)-1}) &= \max\{d(x_{n(k)-1}, x_{m(k)-1}),\, d(x_{n(k)-1}, Tx_{n(k)-1}),\, d(x_{m(k)-1}, Tx_{m(k)-1}),\\ &\qquad \frac{1}{2}(d(x_{n(k)-1}, Tx_{m(k)-1}) + d(x_{m(k)-1}, Tx_{n(k)-1}))\}\\ &= \max\Big\{d(x_{n(k)-1}, x_{m(k)-1}),\, d(x_{n(k)-1}, x_{n(k)}),\, d(x_{m(k)-1}, x_{m(k)}),\\ &\qquad \frac{d(x_{n(k)-1}, x_{m(k)})}{2} + \frac{d(x_{m(k)-1}, x_{n(k)})}{2}\Big\}.\end{aligned}$$

Therefore

$$\lim_{k\to\infty} M_T(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon. \tag{2.6}$$

By (2.5) and (2.6), we have

$$1 = \frac{\lim_{k\to\infty} \psi(d(x_{n(k)}, x_{m(k)}))}{\lim_{k\to\infty} \psi(M_T(x_{n(k)-1}, x_{m(k)-1}))} \leq \lim_{k\to\infty} \beta(\psi(M_T(x_{n(k)-1}, x_{m(k)-1}))),$$

which implies $\lim_{k\to\infty} \beta(\psi(M_T(x_{n(k)-1}, x_{m(k)-1}))) = 1$. Consequently, we get $\lim_{k\to\infty} M_T(x_{n(k)-1}, x_{m(k)-1}) = 0$. Hence $\varepsilon = 0$ which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Since X is an α-η-complete metric space and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, there is $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Since T is α-η-continuous, we get $\lim_{n\to\infty} Tx_n = Tx^*$ and so $x^* = Tx^*$. Hence T has a fixed point. □

In following theorem, we replace the continuity of T by some suitable conditions.

Theorem 2.8. *Let (X, d) be a metric space. Assume that $\alpha, \eta : X \times X \to [0, \infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:*

(i) *(X, d) is an α-η-complete metric space;*
(ii) *T is a generalized α-η-ψ-Geraghty contraction type mapping;*
(iii) *T is a triangular α-orbital admissible mapping with respect to η;*
(iv) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$;*
(v) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq \eta(x_{n(k)}, x^*)$ for all $k \in \mathbb{N}$.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Proof. By the analogous proof as in Theorem 2.7, we can construct the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ converging to $x^* \in X$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. By (v), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq \eta(x_{n(k)}, x^*)$ for all $k \in \mathbb{N}$. Therefore

$$\begin{aligned} \psi(d(x_{n(k)+1}, Tx^*)) &= \psi(d(Tx_{n(k)}, Tx^*)) \\ &\leq \beta(\psi(M_T(x_{n(k)}, x^*)))\psi(M_T(x_{n(k)}, x^*)), \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} M_T(x_{n(k)}, x^*) &= \max\{d(x_{n(k)}, x^*),\ d(x_{n(k)}, Tx_{n(k)}),\ d(x^*, Tx^*), \\ &\qquad \frac{1}{2}(d(x_{n(k)}, Tx^*) + d(x^*, Tx_{n(k)}))\} \\ &= \max\{d(x_{n(k)}, x^*),\ d(x_{n(k)}, x_{n(k)+1}),\ d(x^*, Tx^*), \\ &\qquad \frac{1}{2}(d(x_{n(k)}, Tx^*) + d(x^*, x_{n(k)+1}))\}. \end{aligned}$$

Suppose that $Tx^* \neq x^*$. Letting $k \to \infty$ in the above inequality, we have

$$\lim_{k\to\infty} M_T(x_{n(k)}, x^*) = d(x^*, Tx^*).$$

From (2.7), we have

$$\frac{\psi(d(x_{n(k)+1}, Tx^*))}{\psi(M_T(x_{n(k)}, x^*))} \leq \beta(\psi(M_T(x_{n(k)}, x^*))) < 1.$$

Letting $k \to \infty$ in the above inequality, we obtain that $\lim_{k\to\infty} \beta(\psi(M_T(x_{n(k)}, x^*))) = 1$ and so $\lim_{k\to\infty} M_T(x_{n(k)}, x^*) = 0$. Hence $d(x^*, Tx^*) = 0$. This is a contradiction. It follows that $Tx^* = x^*$. □

For the uniqueness of a fixed point of a generalized α-η-ψ-contractive type mapping, we assume the suitable condition introduced by Popescu [12].

Theorem 2.9. *Suppose all assumptions of* Theorem 2.7 (*respectively* Theorem 2.8) *hold. Assume that for all $x \neq y \in X$, there exists $v \in X$ such that $\alpha(x,v) \geq \eta(x,v)$, $\alpha(y,v) \geq \eta(y,v)$ and $\alpha(v,Tv) \geq \eta(v,Tv)$. Then T has a unique fixed point.*

Proof. Suppose that x^* and y^* are two fixed points of T such that $x^* \neq y^*$. Then by assumption, there exists $v \in X$ such that $\alpha(x^*,v) \geq \eta(x^*,v)$, $\alpha(y^*,v) \geq \eta(y^*,v)$ and $\alpha(v,Tv) \geq \eta(v,Tv)$. Since T is triangular α-orbital admissible with respect to η, we have

$$\alpha(x^*,T^nv) \geq \eta(x^*,T^nv) \text{ and } \alpha(y^*,T^nv) \geq \eta(y^*,T^nv),$$

for all $n \in \mathbb{N}$. This implies that

$$\begin{aligned}\psi(d(x^*,T^{n+1}v)) &= \psi(d(Tx^*,TT^nv)) \\ &\leq \beta(\psi(M_T(x^*,T^nv)))\psi(M_T(x^*,T^nv)),\end{aligned}$$

for all $n \in \mathbb{N}$ where

$$\begin{aligned}M_T(x^*,T^nv) &= \max\{d(x^*,T^nv), d(x^*,Tx^*), d(T^nv,T^{n+1}v), \\ &\qquad \frac{1}{2}(d(x^*,T^{n+1}v) + d(T^nv,Tx^*)\} \\ &= \max\{d(x^*,T^nv), d(T^nv,T^{n+1}v), \frac{1}{2}(d(x^*,T^{n+1}v) + d(T^nv,x^*)\}.\end{aligned}$$

By Theorem 2.7, we deduce that $\{T^nv\}$ converges to a fixed point z^* of T. Taking $n \to \infty$ in the above inequality, we have

$$\lim_{n\to\infty} M_T(x^*,T^nv) = d(x^*,z^*).$$

We will prove that $x^* = z^*$. Suppose that $x^* \neq z^*$. Since

$$\frac{\psi(d(x^*,T^{n+1}v))}{\psi(M_T(x^*,T^nv))} \leq \beta(\psi(M_T(x^*,T^nv))),$$

we obtain that $\lim_{n\to\infty} \beta(\psi(M_T(x^*,T^nv))) = 1$. This implies that $\lim_{n\to\infty} M_T(x^*,T^nv) = 0$, and then $d(x^*,z^*) = 0$ which is a contradiction. Hence $x^* = z^*$. Similarly, we can prove that $y^* = z^*$. Thus $x^* = y^*$. It follows that T has a unique fixed point. □

In Theorem 2.7 and Theorem 2.8, if we put $\eta(x,y) = 1$ and $\psi(t) = t$, then we obtain the following result proved by Popescu [12].

Corollary 2.10 ([12]). *Let (X,d) be a complete metric space, $\alpha : X \times X \to [0,\infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:*

(i) *T is a generalized α-Geraghty contraction type mapping;*
(ii) *T is a triangular α-orbital admissible mapping;*
(iii) *there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq 1$;*
(iv) *T is a continuous mapping or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x^*) \geq 1$ for all $k \in \mathbb{N}$.*

Then T has a fixed point $x^* \in X$ and $\{T^nx_1\}$ converges to x^*.

By taking $\eta(x,y) = 1$ and the same techniques using in Theorem 2.7 and Theorem 2.8, we obtain the following result.

Corollary 2.11. *Let (X,d) be a complete metric space, $\alpha : X \times X \to [0,\infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:*

(i) *T is a triangular α-orbital admissible mapping;*
(ii) *if there exists $\beta \in \mathcal{F}$ such that*

$$\alpha(x,y) \geq 1 \text{ implies } \psi(d(Tx,Ty)) \leq \beta(\psi(M(x,y)))\psi(M(x,y)) \text{ for all } x,y \in X,$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\} \text{ and } \psi \in \Psi';$$

(iii) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;*
(iv) *T is a continuous mapping or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq 1$ for all $k \in \mathbb{N}$.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Consequently, we obtain that the following result proved by Karapinar [8].

Corollary 2.12 ([8]). *Let (X,d) be a complete metric space. Assume that $\alpha : X \times X \to [0,\infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:*

(i) *T is a triangular α-admissible mapping;*
(ii) *T is a generalized α-ψ-Geraghty contraction type mapping;*
(iii) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;*
(iv) *T is a continuous mapping or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq 1$ for all $k \in \mathbb{N}$.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

3. Consequences

Definition 3.1. Let (X,d) be a metric space and $\alpha,\eta : X \times X \to [0,\infty)$. A mapping $T : X \to X$ is said to be an α-η-ψ-Geraghty contraction type mapping if there exists $\beta \in \mathcal{F}$ such that $\alpha(x,y) \geq \eta(x,y)$ implies

$$\psi(d(Tx,Ty)) \leq \beta(\psi(d(x,y)))\psi(d(x,y)),$$

where $\psi \in \Psi'$.

Theorem 3.2. *Let (X,d) be a metric space. Assume that $\alpha,\eta : X \times X \to [0,\infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:*

(i) *(X,d) is an α-η-complete metric space;*
(ii) *T is an α-η-ψ-Geraghty contraction type mapping;*
(iii) *T is a triangular α-orbital admissible mapping with respect to η;*
(iv) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$;*
(v) *T is an α-η-continuous mapping.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Proof. Let $x_1 \in X$ be such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. As in the proof of Theorem 2.7, we can construct the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ converging to some $x^* \in X$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Since T is α-η-continuous, we have

$$x_{n+1} = Tx_n \to Tx^* \text{ as } n \to \infty.$$

Hence T has a fixed point . □

Theorem 3.3. *Let (X,d) be a metric space. Assume that $\alpha, \eta : X \times X \to [0,\infty)$ and $T : X \to X$. Suppose that the following conditions are satisfied:*

(i) *(X,d) is an α-η-complete metric space;*
(ii) *T is an α-η-ψ-Geraghty contraction type mapping;*
(iii) *T is a triangular α-orbital admissible mapping with respect to η;*
(iv) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$;*
(v) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq \eta(x_{n(k)}, x^*)$ for all $k \in \mathbb{N}$.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Proof. Let $x_1 \in X$ be such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. As in the proof of Theorem 2.7, we can construct the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ converging to some $x^* \in X$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. By (v), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq \eta(x_{n(k)}, x^*)$ for all $k \in \mathbb{N}$. It follows that

$$\begin{aligned}\psi(d(x_{n(k)+1}, Tx^*)) &= \psi(d(Tx_{n(k)}, Tx^*)) \\ &\leq \beta(\psi(d(x_{n(k)}, x^*)))\psi(d(x_{n(k)}, x^*)) \\ &< \psi(d(x_{n(k)}, x^*)).\end{aligned}$$

Letting $k \to \infty$ in above inequality, we obtain that $\psi(d(x^*, Tx^*)) \leq 0$. Thus $\psi(d(x^*, Tx^*)) = 0$. This implies that $d(x^*, Tx^*) = 0$. Hence $x^* = Tx^*$. □

Theorem 3.4. *Suppose all assumptions of* Theorem 3.2 (*respectively* Theorem 3.3) *hold. Assume that for all $x \neq y \in X$, there exists $v \in X$ such that $\alpha(x, v) \geq \eta(x, v)$, $\alpha(y, v) \geq \eta(y, v)$ and $\alpha(v, Tv) \geq \eta(v, Tv)$. Then T has a unique fixed point.*

Proof. Suppose that x^* and y^* are two fixed points of T such that $x^* \neq y^*$. Then by assumption, there exists $v \in X$ such that $\alpha(x^*, v) \geq \eta(x^*, v)$, $\alpha(y^*, v) \geq \eta(y^*, v)$ and $\alpha(v, Tv) \geq \eta(v, Tv)$. Since T is triangular α-orbital admissible with respect to η, we have

$$\alpha(x^*, T^n v) \geq \eta(x^*, T^n v) \quad \text{and} \quad \alpha(y^*, T^n v) \geq \eta(y^*, T^n v)$$

for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned}\psi(d(x^*, T^{n+1}v)) &= \psi(d(Tx^*, TT^n v)) \\ &\leq \beta(\psi(d(x^*, T^n v)))\psi(d(x^*, T^n v)) \qquad (3.1)\\ &< \psi(d(x^*, T^n v))\end{aligned}$$

for all $n \in \mathbb{N}$. Consequently, the sequence $\{\psi(d(x^*, T^n v))\}$ is nonincreasing, then there exists $r \geq 0$ such that $\lim_{n\to\infty} \psi(d(x^*, T^n v)) = r$. By (3.1) we have

$$\frac{\psi(d(x^*, T^{n+1}v))}{\psi(d(x^*, T^n v))} \leq \beta(\psi(d(x^*, T^n v))).$$

Letting limit $n \to \infty$, we have $\lim_{n\to\infty} \beta(\psi(d(x^*, T^n v))) = 1$ and then $\lim_{n\to\infty} \psi(d(x^*, T^n v)) = 0$. It follows that $\lim_{n\to\infty} d(x^*, T^n v) = 0$. Hence $\lim_{n\to\infty} T^n v = x^*$. Similarly, we can prove that $\lim_{n\to\infty} T^n v = y^*$. Hence $x^* = y^*$. □

Corollary 3.5 ([8]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that $T : X \to X$. Assume that the following conditions are satisfied:*

(i) *there exists $\beta \in \mathcal{F}$ such that*

$$\psi(d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y))$$

for all $x, y \in X$ with $x \preceq y$ where $\psi \in \Psi'$;

(ii) *there exists $x_1 \in X$ such that $x_1 \preceq Tx_1$;*

(iii) *T is nondecreasing;*

(iv) *either T is continuous or if $\{x_n\}$ is a nondecreasing sequence with $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all $k \in \mathbb{N}$.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*. Further if for all $x \neq y \in X$, there exists $v \in X$ such that $x \preceq v$, $y \preceq v$ and $v \preceq Tv$, then T has a unique fixed point.

Proof. Define functions $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x \preceq y \\ \frac{1}{4}, & \text{otherwise} \end{cases} \quad \text{and} \quad \eta(x, y) = \begin{cases} \frac{1}{2}, & \text{if } x \preceq y \\ 2, & \text{otherwise.} \end{cases}$$

Let $x, y \in X$ with $\alpha(x, y) \geq \eta(x, y)$. By (i), we have

$$\psi(d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y)).$$

This implies that T is an α-η-ψ-Geraghty contraction type mapping. Since X is complete metric space, we have X is α-η-complete metric space. By (ii), there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. Let $\alpha(x, Tx) \geq \eta(x, Tx)$, we have $x \preceq Tx$. Since T is nondecreasing, we obtain that $Tx \preceq T(Tx)$. Then $\alpha(Tx, T^2x) \geq \eta(Tx, T^2x)$. Let $\alpha(x, y) \geq \eta(x, y)$ and $\alpha(y, Ty) \geq \eta(y, Ty)$, so we have $x \preceq y$ and $y \preceq Ty$. It follows that $x \preceq Ty$. Then $\alpha(x, Ty) \geq \eta(x, Ty)$. Thus all conditions of Theorem 3.2 and Theorem 3.3 are satisfied. Hence T has a fixed point. □

We now give an example for supporting Theorem 3.2.

Example 3.6. Let $X = [0, \infty)$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Let $\beta(t) = \frac{1}{1+2t}$ for all $t > 0$ and $\beta(0) = 0$. Then $\beta \in \mathcal{F}$. Let $\psi(t) = \frac{1}{4}t$ and a mapping $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{2}{3}x, & \text{if } 0 \leq x \leq 1 \\ 2x, & \text{if } x > 1. \end{cases}$$

Also, we define functions $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } 0 \leq x, y \leq 1 \\ 0, & \text{otherwise,} \end{cases} \quad , \quad \eta(x, y) = \begin{cases} \frac{1}{4}, & \text{if } 0 \leq x, y \leq 1 \\ 2, & \text{otherwise.} \end{cases}$$

First, we will prove that (X, d) is an α-η-complete metric space. If $\{x_n\}$ is a Cauchy sequence such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then $\{x_n\} \subseteq [0, 1]$. Since $([0, 1], d)$ is a complete metric space, then the sequence $\{x_n\}$ converges in $[0, 1] \subseteq X$. Let $\alpha(x, Tx) \geq \eta(x, Tx)$. Thus $x \in [0, 1]$ and $Tx \in [0, 1]$ and so $T^2x = T(Tx) \in [0, 1]$. Then $\alpha(Tx, T^2x) \geq \eta(Tx, T^2x)$. Thus T is α-orbital admissible with respect to η. Let $\alpha(x, y) \geq \eta(x, y)$ and $\alpha(y, Ty) \geq \eta(y, Ty)$. We have $x, y, Ty \in [0, 1]$. This implies that $\alpha(x, Ty) \geq \eta(x, Ty)$. Hence T is triangular α-orbital admissible with respect to η. Let $\{x_n\}$ be a sequence such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$, for all $n \in \mathbb{N}$. Then $\{x_n\} \subseteq [0, 1]$ for all $n \in \mathbb{N}$. This implies that

$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} \frac{2}{3}x_n = \frac{2}{3}x = Tx$. That is T is α-η -continuous. It is clear that condition (iv) of Theorem 3.2 is satisfied with $x_1 = 1$ since $\alpha(1, T(1)) = \alpha(1, \frac{2}{3}) = 1 > \frac{1}{4} = \eta(1, \frac{2}{3}) = \eta(1, T(1))$. Finally, we will prove that T is an α-η-ψ-Geraghty contraction type mapping. Let $\alpha(x, y) \geq \eta(x, y)$. Therefore $x, y \in [0, 1]$. It follows that

$$\begin{aligned}
&\beta(\psi(d(x,y)))\psi(d(x,y)) - \psi(d(Tx,Ty)) \\
&= \beta(\frac{1}{4}(d(x,y))) \cdot \frac{1}{4}(d(x,y)) - \frac{1}{4}(d(Tx,Ty)) \\
&= \beta(\frac{1}{4}|x-y|) \cdot \frac{1}{4}|x-y| - \frac{1}{4}|Tx - Ty| \\
&= \frac{1}{1+\frac{1}{2}|x-y|} \cdot \frac{1}{4}|x-y| - \frac{1}{4}|\frac{2}{3}x - \frac{2}{3}y| \\
&= \frac{\frac{1}{4}|x-y|}{1+\frac{1}{2}|x-y|} - \frac{1}{6}|x-y| \\
&= \frac{|x-y|(3-2+|x-y|)}{6(2+|x-y|)} \\
&\geq 0.
\end{aligned} \tag{3.2}$$

Then we have $\psi(d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y))$. Thus all assumptions of Theorem 3.2 are satisfied. Hence T has a fixed point $x^* = 0$.

4. Applications to ordinary differential equations

The following ordinary differential equation is taken from Karapinar [8]:

$$\begin{cases} -\frac{d^2x}{dt^2} = f(t, x(t)), & t \in [0,1] \\ x(0) = x(1) = 0, \end{cases} \tag{4.1}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The Green function associated to (4.1) is defined by

$$G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1 \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

Let $C(I)$ be the space of all continuous functions defined on I where $I = [0, 1]$. Suppose that $d(x, y) = \|x - y\|_\infty = \sup_{t\in I} |x(t) - y(t)|$ for all $x, y \in C(I)$. It is well known that $(C(I), d)$ is a complete metric space.

Assume that the following conditions hold:

(i) there exists a function $\xi : \mathbb{R}^2 \to \mathbb{R}$ such that for all $a, b \in \mathbb{R}$ with $\xi(a, b) \geq 0$, we have $|f(t, a) - f(t, b)| \leq 8\ln(|a - b| + 1)$ for all $t \in I$;

(ii) there exists $x_1 \in C(I)$ such that for all $t \in I$,

$$\xi\left(x_1(t), \int_0^1 G(t,s)f(s,x_1(s))ds\right) \geq 0;$$

(iii) for all $t \in I$ and for all $x, y, z \in C(I)$,

$$\xi(x(t), y(t)) \geq 0 \text{ and } \xi(y(t), z(t)) \geq 0 \text{ imply } \xi(x(t), z(t)) \geq 0;$$

(iv) for all $t \in I$ and for all $x, y \in C(I)$,

$$\xi(x(t), y(t)) \geq 0 \text{ implies } \xi\left(\int_0^1 G(t,s)f(s,x(s))ds, \int_0^1 G(t,s)f(s,y(s))ds\right) \geq 0;$$

(v) if $\{x_n\}$ is a sequence in $C([0,1])$ such that $x_n \to x \in C([0,1])$ and $\xi(x_n(t), x_{n+1}(t)) \geq 0$ for all $n \in \mathbb{N}$ and for all $t \in I$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\xi(x_{n(k)}(t), x(t)) \geq 0$ for all $k \in \mathbb{N}$ and for all $t \in I$.

We now assure the existence of a solution of the above second order differential equation. The method for proving the following result is taken from [8] but is slightly different.

Theorem 4.1. *Suppose that conditions* (i)-(v) *are satisfied. Then* (4.1) *has at least one solution* $x^* \in C^2(I)$.

Proof. It is well known that $x^* \in C^2(I)$ is a solution of (4.1) if and only if $x^* \in C(I)$ is a solution of the integral equation (see [8]). Define a mapping $T : C(I) \to C(I)$ by

$$Tx(t) = \int_0^1 G(t,s)f(s,x(s))ds \text{ for all } t \in I.$$

Therefore the problem (4.1) is equivalent to finding $x^* \in C(I)$ that is a fixed point of T. Let $x, y \in C(I)$ such that $\xi(x(t), y(t)) \geq 0$ for all $t \in I$. From (i), we obtain that

$$\begin{aligned}
|Tx(t) - Ty(t)| &= \left| \int_0^1 G(t,s)[f(s,x(s)) - f(s,y(s))]ds \right| \\
&\leq \int_0^1 G(t,s)|f(s,x(s)) - f(s,y(s))|ds \\
&\leq 8\int_0^1 G(t,s)\ln(|x(s) - y(s)| + 1)ds \\
&\leq 8\int_0^1 G(t,s)\ln(d(x,y) + 1)ds \\
&\leq 8\ln(d(x,y) + 1)\Big(\sup_{t\in I}\int_0^1 G(t,s)ds\Big).
\end{aligned}$$

Since $\int_0^1 G(t,s)ds = -(t^2/2) + t/2$ for all $t \in I$, we have $\sup_{t\in I}\int_0^1 G(t,s)ds = \frac{1}{8}$. This implies that

$$d(Tx, Ty) \leq \ln(d(x,y) + 1).$$

Therefore

$$\ln(d(Tx,Ty) + 1) \leq \ln(\ln(d(x,y) + 1) + 1) = \frac{\ln(\ln(d(x,y) + 1) + 1)}{\ln(d(x,y) + 1)}\ln(d(x,y) + 1).$$

Define mappings $\psi : [0,\infty) \to [0,\infty)$ and $\beta : [0,\infty) \to [0,1)$ by

$$\psi(x) = \ln(x+1) \text{ and } \beta(x) = \begin{cases} \frac{\psi(x)}{x}, & \text{if } x \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $\psi : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing and ψ is positive in $(0,\infty)$ with $\psi(0) = 0$ and also $\psi(x) < x$. Moreover, we obtain that $\beta \in \mathcal{F}$ and

$$\psi(d(Tx,Ty)) \leq \beta(\psi(d(x,y)))\psi(d(x,y)$$

for all $x, y \in C(I)$ such that $\xi(x(t), y(t)) \geq 0$ for all $t \in I$.
Define $\alpha, \eta : C(I) \times C(I) \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } \xi(x(t),y(t)) \geq 0, t \in I \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \eta(x,y) = \begin{cases} \frac{1}{2}, & \xi(x(t),y(t)) \geq 0, t \in [0,1] \\ 2, & \text{otherwise.} \end{cases}$$

Let $x, y \in C(I)$ such that $\alpha(x,y) \geq \eta(x,y)$. It follows that $\xi(x(t), y(t)) \geq 0$ for all $t \in I$. This yields

$$\psi(d(Tx,Ty)) \leq \beta(\psi(d(x,y)))\psi(d(x,y)).$$

Therefore T is an α-η-ψ-Geraghty contraction type mapping. Using (iv), for each $x \in C(I)$ such that $\alpha(x, Tx) \geq \eta(x, Tx)$, we obtain that $\xi(Tx(t), T^2x(t)) \geq 0$. This implies that $\alpha(Tx, T^2x) \geq \eta(Tx, T^2x)$. Let $x, y \in C(I)$ such that $\alpha(x,y) \geq \eta(x.y)$ and $\alpha(y, Ty) \geq \eta(y.Ty)$. Thus

$$\xi(x(t), y(t)) \geq 0 \text{ and } \xi(y(t), Ty(t)) \geq 0 \text{ for all } t \in I.$$

By applying (iii), we obtain that $\xi(x(t), Ty(t)) \geq 0$ and so $\alpha(x, Ty) \geq \eta(x, Ty)$. It follows that T is triangular α-orbital admissible with respect to η. Using (ii), there exists $x_1 \in C(I)$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. Let $\{x_n\}$ be a sequence in $C(I)$ such that $x_n \to x \in C(I)$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. By (v), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\xi(x_{n(k)}(t), x(t)) \geq 0$. This implies that $\alpha(x_{n(k)}, x) \geq \eta(x_{n(k)}, x)$. Therefore all assumptions in Theorem 3.2 are satisfied. Hence T has a fixed point in $C(I)$. It follows that there exists $x^* \in C(I)$ such that $Tx^* = x^*$ is a solution of (4.1). □

Corollary 4.2. *Assume that the following conditions hold:*

(i) $f : [0,1] \times \mathbb{R} \to [0, \infty)$ *is continuous and nondecreasing;*

(ii) *for all* $t \in [0,1]$, *for all* $a, b \in \mathbb{R}$ *with* $a \leq b$, *we have*

$$|f(t,a) - f(t,b)| \leq 8 \ln(|a-b| + 1);$$

(iii) *there exists* $x_1 \in C([0,1])$ *such that for all* $t \in [0,1]$, *we have*

$$x_1(t) \leq \int_0^1 G(t,s) f(s, x_1(s)) ds.$$

Then (4.1) *has a solution in* $C^2([0,1])$.

Proof. Define a mapping $\xi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\xi(a,b) = b - a \quad \text{for all } a, b \in \mathbb{R}.$$

By the analogous proof as in Theorem 4.1, we obtain that (4.1) has a solution. □

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ภาคผนวก 2

Fixed point theorems for modified (alpha-psi-varphi-theta)-rational contractive mappings in alpha-complete b-metric spaces

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Fixed Point Theorems for Modified (α-ψ-φ-θ)-Rational Contractive Mappings in α-Complete b-Metric Spaces[1]

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Abstract : In this paper, we introduce the notion of modified (α-ψ-φ-θ)-rational contractive mappings where some conditions of Bianchini-Grandolfi gauge function φ are omitted. We establish the existence of the unique fixed point theorems for such mappings which are triangular α-orbital admissible in α-complete b-metric spaces. Moreover, we also prove the unique common fixed point theorem for mappings T and g where T is a modified (α-ψ-φ-θ)-rational contractive mapping with respect to g. Our results extend the fixed point theorems in α-complete metric spaces proved by Hussian et al.[N. Hussain, M. A. Kutbi, P. Salimi, Fixed point theory in α-complete metric spaces with applications, Abstr. Appl. Anal. (2014) Article ID 280817] to α-complete b-metric spaces.

Keywords : triangular α-orbital admissible mappings; α-complete b-metric spaces; α-continuous mappings; modified (α-ψ-φ-θ)-rational contractive mappings; common fixed points.

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1 Introduction

Fixed point theory in metric spaces is one of the most important tools for proving the existence and uniqueness of the solutions to various mathematical models. Later in 1993 Czerwik [1], generalized the notion of metric spaces by introducing the notion of b-metric spaces. On the other hand, Samet et al. [2] proved the fixed point theorems for α-admissible mappings which are α-φ-contractive mappings in complete metric spaces. Salimi et al. [3] and Hussain et al. [4] modified these notions and assured the fixed point theorems. Recently, Hussain et al. [5] established fixed point theorems for modified α-φ-rational contractive mappings in α-complete metric spaces and proved the existence of solutions of integral equations.

In this paper, we extend the fixed point results in α-complete metric spaces proved by Hussian et al. [5] to α-complete b-metric spaces by introducing the notion of modified $(\alpha$-ψ-φ-$\theta)$-rational contractive mappings where some conditions of Bianchini-Grandolfi gauge function φ are omitted. We establish the existence of the unique fixed point theorems for such mappings which are triangular α-orbital admissible. Moreover, we also prove the unique common fixed point theorem for mappings T and g where T is a modified $(\alpha$-ψ-φ-$\theta)$-rational contractive mapping with respect to g and is triangular g-α-admissible in the setting of α-complete b-metric spaces.

2 Preliminaries

We now recall some definitions and lemmas that will be used in the sequel.

In 2012, Samet et al. [2] introduced the notion of α-admissible mappings.

Definition 2.1 ([2]). Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then T is said to be *α-admissible* if for all $x, y \in X$,

$$\alpha(x,y) \geq 1 \text{ implies } \alpha(Tx,Ty) \geq 1.$$

Recently Hussain et al. [5] introduced the concept of modified α-φ-rational contractive mappings and proved the fixed point theorems for such mappings in α-complete metric spaces.

Definition 2.2. A function $\varphi : [0,\infty) \to [0,\infty)$ is called a *Bianchini-Grandolfi gauge function* [6] if the following conditions hold:

(i) φ is nondecreasing;

(ii) $\sum_{k=1}^{\infty} \varphi^k(t)$ converges for all $t > 0$.

We denote by Φ the set of all Bianchini-Grandolfi gauge functions.

Lemma 2.3 ([7]). *If $\varphi \in \Phi$, then the following statements hold:*

(i) $\varphi(t) < t$ *for all* $t > 0$;

(ii) φ *is continuous at 0;*

(iii) $\varphi(0) = 0$.

Definition 2.4 ([5]). Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is a *modified α-ψ-rational contractive mapping* if for all $x, y \in X$,

$$\alpha(x, y) \geq 1 \text{ implies } d(Tx, Ty) \leq \varphi(M(x, y)), \tag{2.1}$$

where

$$M(x, y) = \max\{d(x, y), \frac{d(x, Tx)}{1 + d(x, Tx)}, \frac{d(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty) + d(y, Tx)}{2}\},$$

and $\varphi \in \Phi$.

Theorem 2.5 ([5]). *Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$. Assume that the following conditions are satisfied:*

(i) X is an α-complete metric space;

(ii) T is a modified α-φ-rational contractive mapping;

(iii) T is an α-admissible mapping;

(iv) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(v) T is an α-continuous mapping.

Then T has a fixed point.

Recently, Popescu [8] studied the definitions of α-orbital admissible mappings and triangular α-orbital admissible mappings.

Definition 2.6 ([8]). Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then T is said to be *α-orbital admissible* if

$$\alpha(x, Tx) \geq 1 \quad \text{implies} \quad \alpha(Tx, T^2x) \geq 1.$$

Definition 2.7 ([8]). Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then T is said to be *triangular α-orbital admissible* if

(a) T is α-orbital admissible;

(b) $\alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1$ imply $\alpha(x, Ty) \geq 1$.

Lemma 2.8 ([8]). *Let $T : X \to X$ be a triangular α-orbital admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Then $\alpha(x_m, x_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$.*

Definition 2.9 ([1]). Let X be a nonempty set and let $s \geq 1$ a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a *b-metric* if for all $x, y, z \in X$,

(i) $d(x,y) = 0$, if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

Then the pair (X,d) is called a *b-metric space*.

Note that a metric space is evidently a b-metric space but the converse is not generally true. For more details see [9].

In this paper, we use the following concepts in b-metric spaces.

Definition 2.10. Let (X,d) be a b-metric space and $\alpha : X \times X \to [0,+\infty)$. Then X is said to be an *α-complete b-metric space* if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ converges in X.

Definition 2.11. Let (X,d) be a b-metric space, $\alpha : X \times X \to [0,+\infty)$ and $T : X \to X$. Then T is said to be an *α-continuous mapping on* (X,d) if for every sequence $\{x_n\}$ with $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ implies $Tx_n \to Tx$ as $n \to \infty$.

In 2014, Rosa and Vetro [10] introduced the notion of triangular g-α-admissible mappings.

Definition 2.12. Let $T, g : X \to X$ and $\alpha : X \times X \to [0,\infty)$. Then T is said to be *triangular g-α-admissible* if

1. $\alpha(gx, gy) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$;
2. $\alpha(gx, gy) \geq 1$ and $\alpha(gy, gz) \geq 1$ imply $\alpha(gx, gz) \geq 1$.

Lemma 2.13 ([5]). *Let $T : X \to X$ be a triangular g-α-admissible. Assume that that there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$. Define a sequence $\{gx_n\}$ by $gx_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Then $\alpha(gx_m, gx_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$.*

Definition 2.14. Let $T, g : X \to X$. If $w = Tx = gx$ for some $x \in X$, then x is called a *coincidence point of T and g*, and w is called a point of coincidence of T and g.

Definition 2.15. Let $T, g : X \to X$. The pair $\{T, g\}$ is said to be *weakly compatible* if $Tgx = gTx$, whenever $Tx = gx$ for some x in X.

Abbas and Rhoades [11] proved the existence of the unique common fixed points of a pair of weakly compatible mappings by using the following proposition as a main tool.

Proposition 2.16 ([11]). *Let $T, g : X \to X$ and $\{T, g\}$ is weakly compatible. If T and g have a unique point of coincidence $w = Tx = gx$, then w is the unique common fixed point of T and g.*

3 Main results

In this section, unique fixed point theorems and unique common fixed point theorems in α-complete b-metric spaces and applications to integral equations are presented.

3.1 The unique fixed point theorems

We first introduce the concept of modified (α-ψ-φ-θ)-rational contractive mappings and prove the existence of fixed point theorems for such mappings.

Definition 3.1. Let (X, d) be a b-metric space and $\alpha : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is a *modified (α-ψ-φ-θ)-rational contractive mapping* if there exists $L \geq 0$ such that for all $x, y \in X$,

$$\alpha(x, y) \geq 1 \text{ implies } \psi(s^3 d(Tx, Ty)) \leq \varphi(\psi(M_b(x, y))) + L\theta(N_b(x, y)). \tag{3.1}$$

where

$$M_b(x, y) = \max\{d(x, y), \frac{d(x, Tx)}{1 + d(x, Tx)}, \frac{d(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty) + d(y, Tx)}{2s}\},$$

$$N_b(x, y) = \min\{d(x, Tx), d(x, Ty), d(y, Tx)\}$$

and $\psi, \varphi, \theta : [0, \infty) \to [0, \infty)$ are continuous functions with $\varphi(t) < t$, $\theta(t) > 0$ for each $t > 0$, φ is nondecreasing, $\theta(0) = 0$, $\psi(t) = 0$ if and only if $t = 0$ and ψ is increasing.

Theorem 3.2. *Let (X, d) be an α-complete b-metric space, $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$ is a modified (α-ψ-φ-θ)-rational contractive mapping. Assume that the following conditions hold:*

(i) T is triangular α-orbital admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is α-continuous.

Then T has a fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ in X by

$$x_{n+1} = Tx_n \text{ for all } n \in \mathbb{N}.$$

By Lemma 2.8, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N}. \tag{3.2}$$

If $x_N = x_{N+1}$ for some $N \in \mathbb{N}$, then T has a fixed point. Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since T is a modified (α-ψ-φ-θ)-rational contraction and by (3.2), we obtain that

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &\leq \psi(s^3 d(x_n, x_{n+1})) \\ &= \psi(s^3 d(Tx_{n-1}, Tx_n)) \\ &\leq \varphi(\psi(M_b(x_{n-1}, x_n))) + L\theta(N_b(x_{n-1}, x_n)) \end{aligned} \tag{3.3}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} N_b(x_{n-1}, x_n) &= \min\{d(x_{n-1}, Tx_{n-1}), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\} \\ &= \min\{d(x_{n-1}, x_n), d(x_{n-1}, x_{n+1}), d(x_n, x_n)\} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} M_b(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, Tx_{n-1})}{1 + d(x_{n-1}, Tx_{n-1})}, \frac{d(x_n, Tx_n)}{1 + d(x_n, Tx_n)}, \\ &\quad \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2s}\} \\ &= \max\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n)}{1 + d(x_{n-1}, x_n)}, \frac{d(x_n, x_{n+1})}{1 + d(x_n, x_{n+1})}, \\ &\quad \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2s}\} \\ &\leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2s}\}. \end{aligned}$$

Since

$$\frac{d(x_{n-1}, x_{n+1})}{2s} \leq \frac{s[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]}{2s},$$

it follows that

$$M_b(x_{n-1}, x_n) \leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \tag{3.4}$$

By (3.3) and (3.4), we obtain that

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &\leq \varphi(\psi(M_b(x_{n-1}, x_n))) + L\theta(N_b(x_{n-1}, x_n)) \\ &\leq \varphi(\psi(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\})). \end{aligned}$$

If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, we have

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &\leq \varphi(\psi(d(x_n, x_{n+1}))) \\ &< \psi(d(x_n, x_{n+1})), \end{aligned}$$

which is a contradiction. This implies that

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &\leq \varphi(\psi(d(x_{n-1}, x_n))) \\ &< \psi(d(x_{n-1}, x_n)), \end{aligned} \tag{3.5}$$

for each $n \in \mathbb{N}$. Since ψ is increasing, we get $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$ for each $n \in \mathbb{N}$. Therefore $\{d(x_n, x_{n+1})\}$ is a nonincreasing sequence. Consequently, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. We claim that $r = 0$. Assume that $r > 0$. Since ψ and φ are continuous, from (3.5), we have

$$\psi(r) \leq \varphi(\psi(r)) \leq \psi(r).$$

This implies that $\psi(r) = \varphi(\psi(r))$. Since $\varphi(t) < t$, for each $t > 0$, we obtain that

$$\psi(r) = \varphi(\psi(r)) < \psi(r),$$

which is a contradiction and therefore $r = 0$. It follows that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.6}$$

Next we will prove that the sequence $\{x_n\}$ is a Cauchy sequence. Suppose on the contrary, that there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, there exist two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) \geq k$ such that

$$d(x_{m(k)}, x_{n(k)}) \geq \varepsilon. \tag{3.7}$$

Let $n(k)$ be the smallest number satisfying (3.7). Thus

$$d(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{3.8}$$

By triangle inequality, (3.7) and (3.8), we obtain that

$$\begin{aligned} \varepsilon \leq d(x_{n(k)}, x_{m(k)}) &\leq sd(x_{n(k)}, x_{n(k)-1}) + sd(x_{n(k)-1}, x_{m(k)}) \\ &< sd(x_{n(k)}, x_{n(k)-1}) + s\varepsilon. \end{aligned}$$

By taking the upper limit as $k \to \infty$ and (3.6), we have

$$\varepsilon \leq \limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)}) \leq s\varepsilon. \tag{3.9}$$

Using triangle inequality again, we obtain that

$$\begin{aligned} \varepsilon \leq d(x_{m(k)}, x_{n(k)}) &\leq sd(x_{m(k)}, x_{n(k)+1}) + sd(x_{n(k)+1}, x_{n(k)}) \\ &\leq s^2 d(x_{m(k)}, x_{n(k)}) + s^2 d(x_{n(k)}, x_{n(k)+1}) + sd(x_{n(k)+1}, x_{n(k)}) \\ &\leq s^2 d(x_{m(k)}, x_{n(k)}) + (s^2 + s) d(x_{n(k)}, x_{n(k)+1}). \end{aligned}$$

From above inequality, we obtain that

$$\varepsilon \leq sd(x_{m(k)}, x_{n(k)+1}) + sd(x_{n(k)+1}, x_{n(k)}) \leq s^2 d(x_{m(k)}, x_{n(k)}) + (s^2+s) d(x_{n(k)}, x_{n(k)+1}).$$

Taking the upper limit as $k \to \infty$, by (3.6) and (3.9), we have

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) \leq s^2\varepsilon. \tag{3.10}$$

Similarly, we obtain that

$$\begin{aligned}\varepsilon \leq d(x_{n(k)}, x_{m(k)}) &\leq sd(x_{n(k)}, x_{m(k)+1}) + sd(x_{m(k)+1}, x_{m(k)}) \\ &\leq s^2 d(x_{n(k)}, x_{m(k)}) + s^2 d(x_{m(k)}, x_{m(k)+1}) + sd(x_{m(k)+1}, x_{m(k)}) \\ &\leq s^2 d(x_{n(k)}, x_{m(k)}) + (s^2+s) d(x_{m(k)}, x_{m(k)+1}).\end{aligned}$$

So from (3.6) and (3.9), we have

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)+1}) \leq s^2\varepsilon. \tag{3.11}$$

Since

$$d(x_{m(k)+1}, x_{n(k)}) \leq sd(x_{m(k)+1}, x_{n(k)+1}) + sd(x_{n(k)+1}, x_{n(k)}),$$

and by using (3.6) and (3.11), we get that

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}). \tag{3.12}$$

Using (3.6), (3.9), (3.10) and (3.11), we have

$$\begin{aligned}\limsup_{k\to\infty} M_b(x_{n(k)}, x_{m(k)}) &= \max\{\limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)}), \limsup_{k\to\infty} \frac{d(x_{n(k)}, Tx_{n(k)})}{1 + d(x_{n(k)}, Tx_{n(k)})}, \\ &\quad \limsup_{k\to\infty} \frac{d(x_{m(k)}, Tx_{m(k)})}{1 + d(x_{m(k)}, Tx_{m(k)})}, \\ &\quad \frac{\limsup_{k\to\infty} d(x_{n(k)}, Tx_{m(k)}) + \limsup_{k\to\infty} d(x_{m(k)}, Tx_{n(k)})}{2s}\} \\ &= \max\{\limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)}), \limsup_{k\to\infty} \frac{d(x_{n(k)}, x_{n(k)+1})}{1 + d(x_{n(k)}, x_{n(k)+1})}, \\ &\quad \limsup_{k\to\infty} \frac{d(x_{m(k)}, x_{m(k)+1})}{1 + d(x_{m(k)}, x_{m(k)+1})}, \\ &\quad \frac{\limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)+1}) + \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1})}{2s}\} \\ &\leq \max\{s\varepsilon, 0, 0, \frac{s^2\varepsilon + s^2\varepsilon}{2s}\} = s\varepsilon.\end{aligned}$$

This implies that

$$\limsup_{k\to\infty} M_b(x_{n(k)}, x_{m(k)}) \leq s\varepsilon. \tag{3.13}$$

By using the same argument as above, we have

$$\limsup_{k\to\infty} N_b(x_{n(k)}, x_{m(k)}) = 0. \tag{3.14}$$

Since T is a modified (α-ψ-φ-θ)-rational contraction, by using Lemma 2.8 and (3.12), we have

$$\begin{aligned}
\psi(s\varepsilon) = \psi(s^3 \cdot \frac{\varepsilon}{s^2}) &\leq \psi(s^3 \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1})) \\
&= \limsup_{k\to\infty} \psi(s^3 d(x_{m(k)+1}, x_{n(k)+1})) \\
&= \limsup_{k\to\infty} \psi(s^3 d(Tx_{m(k)}, Tx_{n(k)})) \\
&\leq \limsup_{k\to\infty} [\varphi(\psi(M_b(x_{m(k)}, x_{n(k)}))) + L\theta(N_b(x_{m(k)}, x_{n(k)}))] \\
&= \varphi(\psi(\limsup_{k\to\infty} M_b(x_{m(k)}, x_{n(k)}))) + L\theta(\limsup_{k\to\infty} N_b(x_{m(k)}, x_{n(k)})) \\
&\leq \varphi(\psi(s\varepsilon)) \\
&< \psi(s\varepsilon),
\end{aligned}$$

which is a contradiction. Then we can conclude that $\{x_n\}$ is a Cauchy sequence. From (3.2) and since X is an α-complete b-metric space, we have $\lim_{n\to\infty} x_n = x$ for some $x \in X$. Since T is α-continuous, we obtain that $\lim_{n\to\infty} Tx_n = Tx$. This implies that $\lim_{n\to\infty} d(x_{n+1}, Tx) = \lim_{n\to\infty} d(Tx_n, Tx) = 0$. Then T has a fixed point. □

Example 3.3. Let $X = [0, 6)$ and $d : X \times X \to \mathbb{R}$ defined by $d(x,y) = |x-y|^2$. Then d is a b-metric on X with $s = 2$. Define $T : X \to X$ by

$$T(x) = \begin{cases} \frac{\sqrt{2}}{6}x, & \text{if } x \in [0,1]\ ; \\ \frac{1}{2}x, & \text{if } x \in (1,6). \end{cases}$$

and define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x, y \in [0,1]\ ; \\ 0, & \text{if otherwise.} \end{cases}$$

Define $\psi, \varphi : [0,\infty) \to [0,\infty)$ by $\psi(t) = \frac{t}{2}$ and $\varphi(t) = \frac{4}{9}t$. For all $x, y \in X$ and $\alpha(x,y) \geq 1$, we have $x, y \in [0,1]$ and then

$$\begin{aligned}
\psi(s^3 d(Tx, Ty)) &= \frac{s^3 d(Tx, Ty)}{2} \\
&= \frac{2^3 |\frac{\sqrt{2}}{6}x - \frac{\sqrt{2}}{6}y|^2}{2} \\
&= 4\left|\frac{\sqrt{2}}{6}x - \frac{\sqrt{2}}{6}y\right|^2 \\
&= 4 \cdot \frac{2}{36}|x-y|^2 \\
&= \frac{4}{9}\frac{|x-y|^2}{2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{4}{9}\frac{d(x,y)}{2} \qquad (3.15)\\
&= \frac{4}{9}\psi(d(x,y))\\
&= \varphi(\psi(d(x,y))) \leq \varphi(\psi(M_b(x,y))).
\end{aligned}
$$

Then T is a modified (α-ψ-φ-θ)-rational contractive mapping. We next show that (X,d) is an α-complete b-metric. If $\{x_n\}$ is a Cauchy sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\{x_n\} \subseteq [0,1]$. Now, since $([0,1], d)$ is a complete b-metric space, then the sequence $\{x_n\}$ converges in $[0,1]$. We will show that T is α-continuous. If $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $x_n \in [0,1]$ for all $n \in \mathbb{N}$ and so

$$d(Tx_n, Tx) = |\frac{\sqrt{2}}{6}x_n - \frac{\sqrt{2}}{6}x|^2 = \frac{1}{18}|x_n - x|^2 = \frac{1}{18}d(x_n, x) \to 0 \text{ as } n \to \infty.$$

Let $\alpha(x, Tx) \geq 1$. Thus $x \in [0,1]$ and $Tx \in [0,1]$ and so $T^2x = T(Tx) \in [0,1]$. Then $\alpha(Tx, T^2x) \geq 1$. Thus T is α-orbital admissible. Let $\alpha(x,y) \geq 1$ and $\alpha(y, Ty) \geq 1$. We have $x, y, Ty \in [0,1]$. This implies that $\alpha(x, Ty) \geq 1$. Hence T is triangular α-orbital admissible. It is clear that condition(ii) of Theorem 3.2 is satisfied with $x_0 = 0$ since $\alpha(x_0, Tx_0) = \alpha(0, T(0)) = \alpha(0,0) = 1$. Thus all assumptions of Theorem 3.2 are satisfied and so T has a fixed point which is $x = 0$.

We next replace the α-continuity of the mapping T by some appropriate conditions.

Theorem 3.4. *Let (X,d) be an α-complete b-metric space and $\alpha : X \times X \to [0,\infty)$. Suppose that $T : X \to X$ is a modified (α-ψ-φ-θ)-rational contractive mapping. Assume that the following conditions hold:*

(i) T is triangular α-orbital admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all $k \in \mathbb{N}$.

Then T has a fixed point.

Proof. As in Theorem 3.2, we can construct the sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$, $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$. From condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\alpha(x_{n(k)}, x) \geq 1 \text{ for all } k \in \mathbb{N}. \qquad (3.16)$$

We claim that x is a fixed point of T. Assume that $d(x, Tx) > 0$. By triangle inequality, we obtain that

$$\begin{aligned} d(x, Tx) &\leq sd(x, x_{n(k)+1}) + sd(x_{n(k)+1}, Tx) \\ &= sd(x, x_{n(k)+1}) + sd(Tx_{n(k)}, Tx). \end{aligned}$$

Taking limit $k \to \infty$ in above inequality, we have

$$d(x, Tx) \leq \lim_{k\to\infty} sd(Tx_{n(k)}, Tx). \tag{3.17}$$

Since T is a modified (α-ψ-φ-θ)-rational contractive mapping, using (3.16) and (3.17), we have

$$\begin{aligned} \psi(s^2 d(x, Tx)) &\leq \lim_{k\to\infty} \psi(s^3 d(Tx_{n(k)}, Tx)) \\ &\leq \lim_{k\to\infty} [\varphi(\psi(M_b(x_{n(k)}, x))) + L\theta(N_b(x_{n(k)}, x))] \\ &\leq \varphi(\psi(\lim_{k\to\infty} M_b(x_{n(k)}, x))) + L\theta(\lim_{k\to\infty} N_b(x_{n(k)}, x)), \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} M_b(x_{n(k)}, x) &= \max\{d(x_{n(k)}, x), \frac{d(x_{n(k)}, Tx_{n(k)})}{1 + d(x_{n(k)}, Tx_{n(k)})}, \frac{d(x, Tx)}{1 + d(x, Tx)}, \\ &\qquad \frac{d(x_{n(k)}, Tx) + d(x, Tx_{n(k)})}{2s}\} \\ &= \max\{d(x_{n(k)}, x), \frac{d(x_{n(k)}, x_{n(k)+1})}{1 + d(x_{n(k)}, x_{n(k)+1})}, \frac{d(x, Tx)}{1 + d(x, Tx)}, \\ &\qquad \frac{d(x_{n(k)}, Tx) + d(x, x_{n(k)+1})}{2s}\} \\ &\leq \max\{d(x_{n(k)}, x), d(x_{n(k)}, x_{n(k)+1}), d(x, Tx) \\ &\qquad \frac{d(x_{n(k)}, Tx) + d(x, x_{n(k)+1})}{2s}\} \end{aligned}$$

and

$$\begin{aligned} N_b(x_{n(k)}, x) &= \min\{d(x_{n(k)}, Tx_{n(k)}), d(x_{n(k)}, Tx), d(x, Tx_{n(k)})\} \\ &= \min d(x_{n(k)}, x_{n(k)+1}), d(x_{n(k)}, Tx), d(x, x_{n(k)+1})\}. \end{aligned}$$

Taking limit as $k \to \infty$, we obtain that

$$\lim_{k\to\infty} M_b(x_{n(k)}, x) \leq \max\{d(x, Tx), \frac{d(x, Tx)}{2}\} = d(x, Tx)$$

and

$$\lim_{k\to\infty} N_b(x_{n(k)}, x) = 0.$$

Corollary 3.6. *Let (X,d) be an α-complete b-metric space where $\alpha : X \times X \to [0,\infty)$ and $T : X \to X$. Assume that there exists $L \geq 0$ such that for all $x, y \in X$,*

$$\alpha(x,y) \geq 1 \text{ implies } s^3 d(Tx,Ty) \leq \varphi(M_b(x,y)) + L\theta(N_b(x,y)), \tag{3.19}$$

where

$$M_b(x,y) = \max\{d(x,y), \frac{d(x,Tx)}{1+d(x,Tx)}, \frac{d(y,Ty)}{1+d(y,Ty)}, \frac{d(x,Ty)+d(y,Tx)}{2s}\},$$

$$N_b(x,y) = \min\{d(x,Tx), d(x,Ty), d(y,Tx)\}$$

and $\varphi, \theta : [0,\infty) \to [0,\infty)$ are continuous functions such that $\theta(0) = 0$, $\varphi(t) < t$, $\theta(t) > 0$ for each $t > 0$ and φ is nondecreasing. Assume that the following conditions hold:

(i) T is triangular α-orbital admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is α-continuous or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all $k \in \mathbb{N}$.

Then T has a fixed point. Moreover, either $\alpha(u,v) \geq 1$ or $\alpha(v,u) \geq 1$ whenever $Tu = u$ and $Tv = v$. Then T has a unique fixed point.

In Corollary 3.6, if $\varphi(t) = t - \varphi'(t)$ for all $t \in [0,\infty)$ where $\varphi' : [0,\infty) \to [0,\infty)$ is continuous such tha $\varphi'(t) < t$ for each $t > 0$ and φ' is nonincreasing and $L = 0$, then we obtain the following corollary.

Corollary 3.7. *Let (X,d) be an α-complete b-metric space where $\alpha : X \times X \to [0,\infty)$. Suppose that $T : X \to X$ is a mapping such that for all $x, y \in X$,*

$$\alpha(x,y) \geq 1 \text{ implies } s^3 d(Tx,Ty) \leq M_b(x,y) - \varphi'(M_b(x,y)), \tag{3.20}$$

where

$$M_b(x,y) = \max\{d(x,y), \frac{d(x,Tx)}{1+d(x,Tx)}, \frac{d(y,Ty)}{1+d(y,Ty)}, \frac{d(x,Ty)+d(y,Tx)}{2s}\}.$$

and $\varphi' : [0,\infty) \to [0,\infty)$ is continuous such that $\varphi'(0) = 0$, $\varphi'(t) < t$ for each $t > 0$ and φ' is nonincreasing Assume that the following conditions hold:

(i) T is triangular α-orbital admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is α-continuous or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all $k \in \mathbb{N}$.

Then T has a fixed point. Moreover, either $\alpha(u,v) \geq 1$ or $\alpha(v,u) \geq 1$ whenever $Tu = u$ and $Tv = v$. Then T has a unique fixed point.

3.2 The unique of common fixed point theorems

In this section, we introduce the concept of modified (α-ψ-φ-θ)-rational contractive mappings with respect to g and prove the the existence of unique common fixed point theorems in α-complete b-metric spaces.

Definition 3.8. Let (X,d) be a b-metric space, $\alpha : X \times X \to [0,\infty)$, and $T,g : X \to X$. We say that $T : X \to X$ is a *modified (α-ψ-φ-θ)-rational contractive mapping with respect to g* if there exists $L \geq 0$ such that for all $x,y \in X$,

$$\alpha(x,y) \geq 1 \text{ implies } \psi(s^3 d(Tx,Ty)) \leq \varphi(\psi(M_b(x,y))) + L\theta(N_b(x,y)), \quad (3.21)$$

where

$$M_b(x,y) = \max\{d(gx,gy), \frac{d(gx,Tx)}{1+d(gx,Tx)}, \frac{d(gy,Ty)}{1+d(gy,Ty)}, \frac{d(gx,Ty)+d(gy,Tx)}{2s}\},$$

$$N_b(x,y) = \min\{d(gx,Tx), d(gx,Ty), d(gy,Tx)\}$$

and $\psi,\varphi,\theta : [0,\infty) \to [0,\infty)$ are continuous functions with $\varphi(t) < t$, $\theta(t) > 0$ for each $t > 0$, φ is nondecreasing, $\theta(0) = 0$, $\psi(t) = 0$ if and only if $t = 0$ and ψ is increasing.

Definition 3.9. Let (X,d) be a b-metric space and $\alpha : X \times X \to [0,+\infty)$ and $T,g : X \to X$. Then T is said to be *α-continuous with respect to g*, if for each sequence $\{gx_n\}$ with $gx_n \to gx$ as $n \to \infty$, $\alpha(gx_n, gx_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, we have $Tx_n \to Tx$ as $n \to \infty$.

Theorem 3.10. *Let (X,d) be an α-complete b-metric space and $T,g : X \to X$ be such that $TX \subseteq gX$ and suppose that gX is closed. Let $\alpha : X \times X \to [0,\infty)$ and T is a modified (α-ψ-φ-θ)-rational contractive mapping with respect to g. Assume that the following conditions hold:*

(i) T is triangular g-α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$;

(iii) T is α-continuous with respect to g.

Then T and g have a coincidence point.

Proof. Let $x_0 \in X$ be such that $\alpha(gx_0, Tx_0) \geq 1$. Since $TX \subseteq gX$, we can construct a sequence $\{gx_n\}$ such that

$$gx_{n+1} = Tx_n \text{ for all } n \in \mathbb{N}.$$

By using Lemma 2.13, we have

$$\alpha(gx_n, gx_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N}. \quad (3.22)$$

By the analogous proof as in Theorem 3.2, we can prove that $\{gx_n\}$ is a Cauchy sequence. Since $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and X is an α-complete b-metric space, we have $\{gx_n\}$ converges to $z \in gX$. Thus there exists $x \in X$ such that $\lim_{n\to\infty} gx_n = gx$. Since T is α-continuous with respect to g, so $Tx = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_{n+1} = gx$. Then x is a coincidence point of T and g. □

We replace the α-continuity of the mapping T with respect to g by some appropriate conditions.

Theorem 3.11. *Let (X,d) be an α-complete b-metric space and $T, g : X \to X$ be such that $TX \subseteq gX$ and suppose that gX is closed. Let $\alpha : X \times X \to [0,\infty)$ and T is a modified (α-ψ-φ-θ)-rational contractive mapping with respect to g. Assume that the following conditions hold:*

(i) T is triangular g-α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$;

(iii) if $\{gx_n\}$ is a sequence in X such that $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $gx_n \to gx$ as $n \to \infty$, then there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $\alpha(gx_{n(k)}, gx) \geq 1$ for all $k \in \mathbb{N}$.

Then T and g have a coincidence point.

Proof. As in the proof of Theorem 3.10, we can construct the sequence $\{gx_n\}$ with $Tx_n = gx_{n+1}$ for all $n \in \mathbb{N}$, $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} gx_n = gx$. By (iii), there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $\alpha(gx_{n(k)}, gx) \geq 1$, for all $k \in \mathbb{N}$. By the analogous proof as in Theorem 3.4, we obtain that T and g have a coincidence point. □

For the uniqueness of a common fixed point, we add some appropriate conditions to the hypotheses.

Theorem 3.12. *Suppose that all hypotheses of Theorem 3.10 (respectively Theorem 3.11) hold. Assume that the following conditions hold:*

(i) the pair $\{T, g\}$ is weakly compatible;

(ii) either $\alpha(u,v) \geq 1$ or $\alpha(v,u) \geq 1$ whenever $Tu = gu$ and $Tv = gv$.

Then T and g have a unique common fixed point.

Proof. Assume that $Tu = gu$ and $Tv = gv$. We will show that $gu = gv$. Suppose that $gu \neq gv$. Therefore $\alpha(u,v) \geq 1$ or $\alpha(v,u) \geq 1$. Suppose that $\alpha(u,v) \geq 1$. It follows that

$$\psi(s^3 d(gu, gv)) = \psi(s^3 d(Tu, Tv)) \leq \varphi(\psi(M_b(u,v))) + L\theta(N_b(u,v)),$$

where

$$\begin{aligned} M_b(u,v) &= \max\{d(gu,gv), \frac{d(gu,Tu)}{1+d(gu,Tu)}, \frac{d(v,Tv)}{1+d(v,Tv)}, \frac{d(gu,Tv)+d(gv,Tu)}{2s}\} \\ &= \max\{d(gu,gv), \frac{d(gu,gu)}{1+d(gu,gu)}, \frac{d(gv,gv)}{1+d(gv,gv)}, \frac{d(gu,gv)+d(gv,gu)}{2s}\} \\ &= d(gu,gv) \end{aligned}$$

and

$$N_b(u,v) = \min\{d(gu,Tu), d(gu,Tv), d(gv,Tu)\} = 0.$$

This implies that

$$\begin{aligned}\psi(s^3(d(gu,gv))) &\leq \varphi(\psi(d(gu,gv)))\\ &< \psi(d(gu,gv))\end{aligned}$$

which is a contradiction because $s \geq 1$. Thus $gu = gv$. Similarly, if $\alpha(v,u) \geq 1$, we can prove that $gu = gv$. This implies that T and g have a unique point of coincidence. Since the pair $\{T,g\}$ is weakly compatible and by Theorem 2.16, we can conclude that T and g have a unique common fixed point. □

Corollary 3.13. *Let (X,d) be an α-complete b-metric space with respect to g and $T,g : X \to X$ be such that $TX \subseteq gX$. Assume that gX is closed and there exist $\alpha : X \times X \to \mathbb{R}$ and $L \geq 0$ such that for all $x,y \in X$,*

$$\alpha(x,y) \geq 1 \text{ implies } s^3 d(Tx,Ty) \leq \varphi(M_b(x,y)) + L\theta(N_b(x,y)), \tag{3.23}$$

where

$$M_b(x,y) = \max\{d(gx,gy), \frac{d(gx,Tx)}{1+d(gx,Tx)}, \frac{d(gy,Ty)}{1+d(gy,Ty)}, \frac{d(gx,Ty)+d(gy,Tx)}{2s}\},$$

$$N_b(x,y) = \min\{\frac{d(gx,Tx)}{1+d(g,Tx)}, \frac{d(gx,Ty)}{1+d(gx,Ty)}, \frac{d(gy,Tx)}{1+d(gy,Tx)}\}$$

and $\varphi, \theta : [0,\infty) \to [0,\infty)$ are continuous functions such that $\theta(0) = 0$, $\varphi(t) < t$, $\theta(t) > 0$ for each $t > 0$. Assume that the following conditions hold:

(i) T is triangular g-α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$;

(iii) T is α-continuous with respect to g or if $\{gx_n\}$ is a sequence in X such that $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $gx_n \to gx$ as $n \to \infty$, then there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $\alpha(gx_{n(k)}, gx) \geq 1$ for all $k \in \mathbb{N}$.

Then T and g have a coincidence point. Moreover, assume that the following conditions hold:

(iv) the pair $\{T,g\}$ is weakly compatible;

(v) either $\alpha(u,v) \geq 1$ or $\alpha(v,u) \geq 1$ whenever $Tu = gu$ and $Tv = gv$.

Then T and g have a unique common fixed point.

3.3 Applications to integral equations

In this section, we prove the existence of a solution of a nonlinear quadratic integral equation taken from Allahari et al. [12].

Let $C(I)$ be the set of all continuous functions defined on $I = [0,1]$ and $\rho : C(I) \times C(I) \to \mathbb{R}$ defined by

$$\rho(x,y) = \sup_{t \in I} |x(t) - y(t)| \text{ for } x, y \in C(I).$$

~~Let $p \geq 1$. We define $d : C(I) \times C(I) \to \mathbb{R}$ defined by~~

$$d(x,y) = (\rho(x,y))^p = (\sup_{t \in I} |x(t) - y(t)|)^p = \sup_{t \in I} |x(t) - y(t)|^p \text{ for all } x, y \in C(I).$$

It is well known that (X, d) is a complete b-metric space with $s = 2^{p-1}$ (see [13]). Let Γ be the set of functions $\gamma : [0, +\infty) \to [0, +\infty)$ which satisfy the following conditions:

(i) γ is nondecreasing and $(\gamma(t))^p \leq \gamma(t^p)$ for all $p \geq 1$;

(ii) There exists $\varphi : [0, +\infty) \to [0, +\infty)$ which is nonincreasing and continuous, $\varphi(t) < t$ for all $t > 0$ such that $\gamma(t) = t - \varphi(t)$ for all $t \in [0, +\infty)$.

Consider the nonlinear quadratic equation as follows:

$$x(t) = h(t) + \lambda \int_0^1 k(t,s) f(s, x(s)) ds, \; t \in I, \; \lambda \geq 0. \tag{3.24}$$

Suppose that the following conditions hold:

(A1) $h : I \to \mathbb{R}$ is continuous;

(A2) $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(t, x) \geq 0$ and there exist $L \geq 0$, $\gamma \in \Gamma$ and a function $\xi : \mathbb{R}^2 \to \mathbb{R}$ such that for all $t \in I$, for all $a, b \in \mathbb{R}$ with $\xi(a,b) \geq 0$,

$$|f(t,a) - f(t,b)| \leq L\gamma(|a - b|);$$

(A3) $k : I \times I \to \mathbb{R}$ is continuous at $t \in I$ for every $s \in I$ and measurable at $s \in I$ for all $t \in I$ such that $k(t,s) \geq 0$ and $\int_0^1 k(t,s) ds \leq K$;

(A4) $\lambda^p K^p L^p \leq \frac{1}{2^{3p-3}}$;

(A5) there exists $x_0 \in C(I)$ such that for all $t \in I$,

$$\xi(x_0(t), h(t) + \lambda \int_0^1 k(t,s) f(s, x_0(s)) ds) \geq 0;$$

(A6) for all $t \in I$ and for all $x, y, z \in C(I)$,

$$\xi(x(t), y(t)) \geq 0 \text{ and } \xi(y(t), z(t)) \geq 0 \text{ imply } \xi(x(t), z(t)) \geq 0;$$

(A7) for all $t \in I$ and for all $x, y \in C(I)$,

$$\xi(x(t), y(t)) \geq 0 \text{ implies } \xi(h(t)+\lambda \int_0^1 k(t,s)f(s,x(s))ds, h(t)+\lambda \int_0^1 k(t,s)f(s,y(s))ds) \geq 0;$$

(A8) if $\{x_n\}$ is a sequence in $C(I)$ such that $x_n \to x \in C(I)$ and $\xi(x_n(t), x_{n+1}(t)) \geq 0$ for all $n \in \mathbb{N}$ and for all $t \in I$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\xi(x_{n(k)}(t), x(t)) \geq 0$ for all $k \in \mathbb{N}$ and for all $t \in I$.

Theorem 3.14. *Under assumptions (A1)-(A8), the integral equation (3.24) has a solution in $C(I)$.*

Proof. Let $T : C(I) \to C(I)$ be defined by

$$T(x)(t) = h(t) + \lambda \int_0^1 k(t,s)f(s,x(s))ds \text{ for } t \in I.$$

Let $x, y \in C(I)$ such that $\xi(x(t), y(t)) \geq 0$ for all $t \in I$. Therefore

$$\begin{aligned} |T(x)(t) - T(y)(t)| &= |h(t) + \lambda \int_0^1 k(t,s)f(s,x(s))ds - h(t) - \lambda \int_0^1 k(t,s)f(s,y(s))ds| \\ &\leq \lambda \int_0^1 k(t,s)|f(s,x(s)) - f(s,y(s))|ds \\ &\leq \lambda \int_0^1 k(t,s)L\gamma(|x(s) - y(s)|)ds. \end{aligned}$$

Since γ is nondecreasing, we obtain that

$$\gamma(|x(s) - y(s)|) \leq \gamma(\sup_{s \in I}|x(s) - y(s)|) = \gamma(\rho(x,y)).$$

This implies that

$$|T(x)(t) - T(y)(t)| \leq \lambda K L\gamma(\rho(x,y)).$$

Therefore

$$\begin{aligned} d(Tx, Ty) &= \sup_{t \in I}|T(x)(t) - T(y)(t)|^p \\ &\leq [\lambda K L\gamma(\rho(x,y))]^p \\ &\leq \lambda^p K^p L^p \gamma(d(x,y)) \\ &\leq \lambda^p K^p L^p \gamma(M(x,y)) \\ &\leq \lambda^p K^p L^p [M(x,y) - \varphi(M(x,y))] \\ &\leq \frac{1}{2^{3p-3}}[M(x,y) - \varphi(M(x,y))], \end{aligned}$$

for all $x, y \in C(I)$ such that $\xi(x(t), y(t)) \geq 0$ for all $t \in I$. We next define $\alpha : C(I) \times C(I) \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } \xi(x(t), y(t)) \geq 0,\ t \in I \\ 0, & \text{otherwise.} \end{cases}$$

Let $x, y \in C(I)$ be such that $\alpha(x,y) \geq 1$. It follows that $\xi(x(t), y(t)) \geq 0$ for all $t \in I$. This yields

$$s^3 d(Tx, Ty) \leq M(x,y) - \varphi(M(x,y).$$

This implies that T satisfies the contractive condition in Corollary 3.7. Using (A7), for each $x \in C(I)$ such that $\alpha(x, Tx) \geq 1$ we obtain that $\xi(Tx(t), T^2x(t)) \geq 0$. This implies that $\alpha(Tx, T^2x) \geq 1$. Let $x, y \in C(I)$ be such that $\alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1$. Thus

$$\xi(x(t), y(t)) \geq 0 \text{ and } \xi(y(t), Ty(t)) \geq 0 \text{ for all } t \in I.$$

By applying (A6), we obtain that $\xi(x(t), Ty(t)) \geq 0$ and so $\alpha(x, Ty) \geq 1$. It follows that T is triangular α-orbital admissible. Using (A5), there exists $x_0 \in C(I)$ such that $\alpha(x_0, Tx_0) \geq 1$. Let $\{x_n\}$ be a sequence in $C(I)$ such that $x_n \to x \in C(I)$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. By (A8), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\xi(x_{n(k)}(t), x(t)) \geq 0$. This implies that $\alpha(x_{n(k)}, x) \geq 1$. Therefore all assumptions in Corollary 3.7 are satisfied. Hence T has a fixed point in $C(I)$ that is a solution of the integral equation (3.24). □

Corollary 3.15. *Assume that the following conditions hold:*

(i) $h : I \to \mathbb{R}$ *is a continuous;*

(ii) $f : I \times \mathbb{R} \to [0, \infty)$ *is continuous and nondecreasing and* $f(t, s) \geq 0$.

(iii) there exist $L \geq 0$ *and* $\gamma \in \Gamma$ *such that for all* $t \in I$, *for all* $a, b \in \mathbb{R}$ *with* $a \leq b$, *we have*

$$|f(t, a) - f(t, b)| \leq L\gamma(|a - b|);$$

(iv) $k : I \times I \to \mathbb{R}$ *is continuous at* $t \in I$ *for every* $s \in I$ *and measurable at* $s \in I$ *for all* $t \in I$ *such that* $k(t, s) \geq 0$ *and* $\int_0^1 k(t, s)ds \leq K$;

(v) $\gamma^p K^p L^p \leq \frac{1}{2^{3p-3}}$;

(vi) there exists $x_0 \in C([0, 1])$ *such that for all* $t \in I$, *we have*

$$x_0(t) \leq h(t) + \lambda \int_0^1 k(t, s) f(s, x_1(s))ds.$$

Then (3.24) has a solution in $C(I)$.

Proof. Define a mapping $\xi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\xi(a, b) = b - a \quad \text{for all } a, b \in \mathbb{R}.$$

By the analogous proof as in Theorem 3.14, we obtain that (3.24) has a solution in $C(I)$. □

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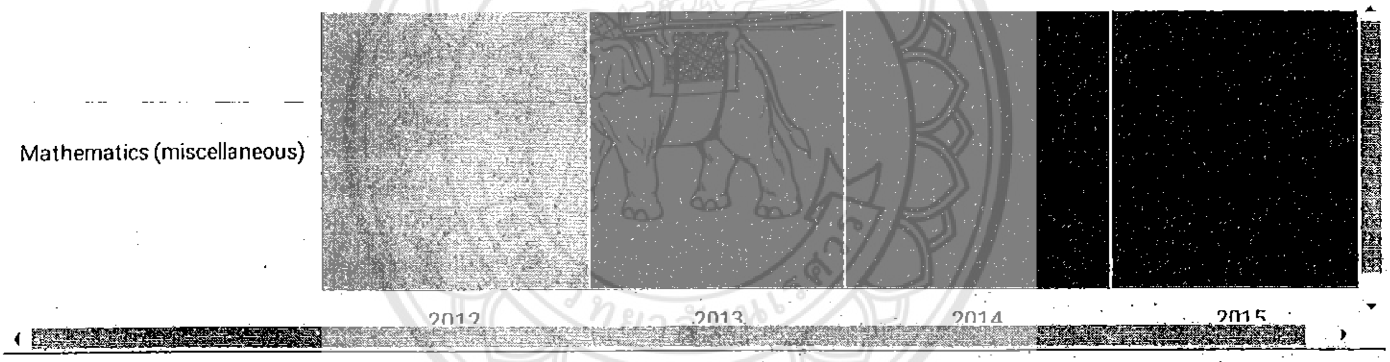

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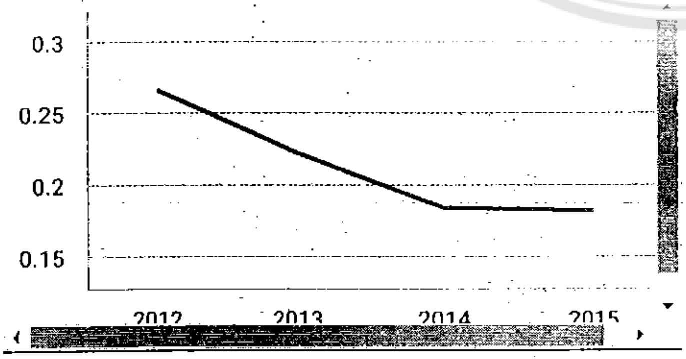

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