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Coupled and coincidence points for monotone operators in partial metric spaces

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บทคัดย่อ(ภาษาไทย)

ในงานวิจัยนี้คณะผู้วิจัยได้สร้างผลลัพธ์จุดตรึงคู่สำหรับการส่งแบบหดตัวอย่างอ่อนวางนัยทั่วไปที่มีคุณสมบัติทางเดียวผสมในปริภูมิเมตริกอันดับบางส่วน ผลลัพธ์ของงานวิจัยที่ได้คือเป็นทฤษฎีบทจุดตรึงวางนัยทั่วไปกว่าของผลลัพธ์งานวิจัยของ S. Alsulami, N. Hussain และ A. Alotaibi เรื่อง Coupled Fixed and Coincidence Points for Monotone Operators in Partial Metric Spaces ในวารสาร Fixed Point Theory and Applications ปี 2012.

บทคัดย่อ(ภาษาอังกฤษ)

In this paper, we establish coupled fixed point results for generalized weakly contractive mappings having the mixed monotone property in ordered partial metric spaces. The results on fixed point theorems are generalizations of the recent results of Alsulami, Hussain and Alotaibi [S. Alsulami, N. Hussain and A. Alotaibi, Coupled Fixed and Coincidence Points for Monotone Operators in Partial Metric Spaces, Fixed Point Theory and Applications 2012, 2012:173].

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CHAPTER 1

INTRODUCTION

The existence and uniqueness of fixed and common fixed point theorems of operators has been a subject of great interest since Banach [1] proved the Banach contraction principle in 1922. Many authors generalized the Banach contraction principle in various spaces such as quasi-metric spaces, generalized metric spaces, cone metric spaces and fuzzy metric spaces. Matthews [2] introduced the notion of partial metric spaces in such a way that each object does not necessarily have to have a zero distance from itself and proved a modified version of the Banach contraction principle. Afterwards, many authors proved many existing fixed point theorems in partial metric spaces (see [5]-[44] for examples).

In recent times, fixed point theory has developed rapidly in partially ordered metric spaces, that is, metric spaces endowed with a partial ordering. Some of these works are noted in [11, 17, 18, 36]. Bhaskar and Lakshmikantham [18] introduced the concept of a coupled fixed point and the mixed monotone property. Furthermore, they proved some coupled fixed point theorems for mappings which satisfy the mixed monotone property and gave some applications in the existence and uniqueness of a solution for a periodic boundary value problem. After the publication of this work, several coupled fixed point and coincidence point results have appeared in the recent literature. Works noted in [31, 38, 39] are some examples of these works.

We recall below the definition of partial metric space and some of its properties.

Definition 1.4. [2] A partial metric on a nonempty set X is a function $p : X \times X \longrightarrow \mathbb{R}_0^+$ such that for all $x, y, z \in X$:

(p1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$, (p2) $p(x,x) \leq p(x,y)$,

(p3) $p(x,y) = p(y,x)$,

(p4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x,y) = 0$, then from (p1) and (p2), $x = y$. But if $x = y$, $p(x,y)$ may not be 0. The function $p(x,y) = \max\{x,y\}$ for all $x,y \in \mathbb{R}^+$ defines a partial metric on $\mathbb{R}^+$. Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$. If p is a partial metric on X, then the function $d_p : X \times X \longrightarrow \mathbb{R}^+$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X.

Definition 1.5. Let (X,p) be a partial metric space. Then

(i) A sequence $\{x_n\}$ in a partial metric space (X,p) **converges** to a point $x \in X$ if and only if $p(x,x) = \lim_{n\longrightarrow\infty} p(x,x_n)$.

(ii) A sequence $\{x_n\}$ in a partial metric space (X,p) is called a **Cauchy sequence** iff $\lim_{n,m\longrightarrow\infty} p(x_n,x_m)$ exists (and is finite).

(iii) A partial metric space (X,p) is said to be **complete** if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\longrightarrow\infty} p(x_n,x_m)$.

(iv) A subset A of a partial metric space (X,p) is **closed** if whenever $\{x_n\}$ is a sequence in A such that $\{x_n\}$ converges to some $x \in X$, then $x \in A$.

Remark 1.6. The limit in a partial metric space is not unique.

Theorem 1.1. *Let (Y, d') be a subspace of metric space (X, d). If (X, d) is a complete metric space and Y is a closed set in X, then (Y, d') is a complete metric space.*

Lemma 1.7. [2, 33] Let (X, p) be a partial metric space.

(a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_p).

(b) A partial metric space (X, p) is complete if and only if the metric space (X, d_p) is complete. Furthermore, $\lim_{n \longrightarrow \infty} d_p(x_n, x) = 0$ if and only if

$$p(x, x) = \lim_{n \longrightarrow \infty} p(x_n, x) = \lim_{n,m \longrightarrow \infty} p(x_n, x_m).$$

Let (X, p) be a partial metric. We endow the product space $X \times X$ with the partial metric q defined as follows:

$$\text{for } (x, y), (u, v) \in X \times X, \quad q((x, y), (u, v)) = p(x, u) + p(y, v).$$

A mapping $F : X \times X \longrightarrow X$ is said to be continuous at $(x, y) \in X \times X$ is for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$F(B_q((x, y), \delta)) \subset B_p((x, y), \varepsilon).$$

The concept of a mixed monotone property has been introduced by Bhaskar and Lakshmikantham [18].

Definition 1.8 ([18]). Let $(X, \preceq)$ be a partial ordered set. A mapping $F : X \times X \to X$ is said to be have *mixed monotone property* if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$

$$x_1, x_2 \in X, x_1 \preceq x_2 \quad \Rightarrow \quad F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \preceq y_2 \quad \Rightarrow \quad F(x, y_1) \succeq F(x, y_2).$$

Definition 1.9 ([18]). Let $F : X \times X \to X$. An element $(x, y) \in X \times X$ is said to be a *coupled fixed point* of a mapping F if

$$x = F(x, y) \text{ and } y = F(y, x).$$

Let Φ denote the set of all functions $\phi : [0, \infty) \longrightarrow [0, \infty)$ which satisfy

(ϕ1) ϕ is continuous and non-decreasing,

(ϕ2) $\phi(t) = 0$ if and only if $t = 0$,

(ϕ3) $\phi(t + s) \leq \phi(t) + \phi(s)$ for all $t, s \in [0, \infty)$,

(ϕ4) $\phi(\alpha t) \leq \alpha\phi(t)$ for all $\alpha \in (0, \infty)$.

and let Ψ denote the set of all functions $\psi : [0, \infty) \longrightarrow [0, \infty)$ which satisfy

$$\lim_{t \longrightarrow r} \psi(t) > 0 \text{ for all } r > 0 \text{ and } \lim_{t \longrightarrow 0^+} \psi(t) = 0.$$

Alsulami, Hussain and Alotaibi [4] proved some coupled fixed point results for (ϕ, φ)- weakly contractive mappings in ordered partial metric spaces. More precisely, they obtained the following results.

Theorem 1.2. [4, Theorem 3.1] *Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Suppose that there exists $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\phi(p(F(x,y), F(u,v))) \leq \frac{1}{2}\phi(p(x,u) + p(y,v)) - \psi\left(\frac{p(x,u) + p(y,v)}{2}\right) \quad (1.1)$$

for all $x, y, u, v \in X$ for which $x \succeq u$ and $y \preceq v$. If there exists $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0),$$

and also suppose either

(a) *F is continuous or*

(b) *X has the following property:*

(i) *if a non-decreasing sequence* $\{x_n\}$ *is such that* $x_n \to x$,*then* $x_n \preceq x$ *for all* n,

(ii) *if a non-increasing sequence* $\{y_n\}$ *is such that* $y_n \to y$,*then* $y \preceq y_n$ *for all* n, *then* F *has a coupled fixed point in* X, *that is there exists* $(x,y) \in X \times X$ *such that* $x = F(x,y)$ *and* $y = F(y,x)$.

Starting from the results in Alsulami, Hussain and Alotaibi [4], our main aim in this paper is to obtain more general coupled fixed point theorems for mixed monotone operators $F : X \longrightarrow X$ satisfying a contractive condition which is significantly more general that the corresponding conditions (1.1) in [4], thus extending many other related results in literature. We also provide illustrative example in support of our results.

CHAPTER 2

MAIN RESULTS

We start with an example which shows the weakness of Theorem 1.2.

Example 2.1. Let $X = \mathbb{R}^+$ be a set endowed with order $x \preceq y \Leftrightarrow x \leq y$. Let $p(x,y) = \max\{x,y\}$, then (X,p) is a partial metric space. Define the mapping $F : X \times X \to X$ by

$$F(x,y) = \frac{5x-2y}{8} \text{ for all } x,y \in X.$$

Then the following properties hold:

(1) F is mixed monotone;

(2) the condition (1.1) does not hold.

Indeed, we show that F does not satisfy condition (1.1). Assume on the contrary, that there exist ϕ and ψ, such that (1.1) holds. Therefore it implies that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that, for all $x \geq u$ and $y \leq v$, we have

$$\begin{aligned} \phi\left(\frac{5x-2y}{8}\right) &= \phi\left(\max\left\{\frac{5x-2y}{8}, \frac{5u-2v}{8}\right\}\right) \\ &= \phi(p(F(x,y),F(u,v))) \\ &\leq \frac{1}{2}\phi(p(x,u)+p(y,v)) - \psi\left(\frac{p(x,u)+p(y,v)}{2}\right) \\ &= \frac{1}{2}\phi(x+v) - \psi\left(\frac{x+v}{2}\right). \end{aligned}$$

Setting $x = 5, y = 1/2$ and $v = 1$ to the last inequality, we get, since $\psi(3) > 0$, that

$$\phi(3) \leq \left(\frac{1}{2}\right)\phi(6) - \psi(3) < \left(\frac{1}{2}\right)\phi(6) \leq \left(\frac{1}{2}\right)2\phi(3) = \phi(3)$$

which gives a contradiction. Hence F does not satisfy condition (1.1). Notice, however, that $(0,0) \in X^2$ is the coupled fixed point of F. □

We now state and prove our first result which successively guarantees the existence of a coupled fixed point and generalizes Theorem 3.1 in [4].

Theorem 2.3. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Suppose that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$M_F^{\phi,\psi}(x,y,u,v) \leq \phi(p(x,u)+p(y,v)) - 2\psi\left(\frac{p(x,u)+p(y,v)}{2}\right) \tag{2.1}$$

for all $x,y,u,v \in X$ for which $x \succeq u$ and $y \preceq v$, where

$$M_F^{\phi,\psi}(x,y,u,v) = \phi(p(F(x,y),F(u,v))) + \phi(p(F(y,x),F(v,u))).$$

If there exist two elements $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0,y_0) \text{ and } y_0 \succeq F(y_0,x_0),$$

and also suppose either

(a) *F is continuous or*

(b) *X has the following property:*

(i) *if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \preceq x$ for all n,*

(ii) *if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$, then $y \preceq y_n$ for all n,*

then F has a coupled fixed point in X, that is there exists $(x,y) \in X \times X$ such that $x = F(x,y)$ and $y = F(y,x)$.

Proof. Let $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0,y_0)$ and $y_0 \succeq F(y_0,x_0)$. We construct sequence $\{x_n\}$ and $\{y_n\}$ in X as

$$x_{n+1} = F(x_n,y_n) \text{ and } y_{n+1} = F(x_n,y_n) \text{ for all } n \geq 0. \tag{2.2}$$

Next, we show that

$$x_n \preceq x_{n+1} \text{ for all } n \geq 0 \tag{2.3}$$

and

$$y_n \succeq y_{n+1} \text{ for all } n \geq 0. \tag{2.4}$$

For this we shall use mathematical induction.

Let $n = 0$. Since $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$ and as $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$, we have $x_0 \preceq x_1$ and $y_0 \succeq y_1$. Thus (2.3) and (2.4) hold for $n = 0$.

Suppose now that (2.3) and (2.4) hold for some fixed $n \geq 0$, then, since $x_n \preceq x_{n+1}$ and $y_n \succeq y_{n+1}$, we have

$$x_{n+2} = F(x_{n+1}, y_{n+1}) \succeq F(x_n, y_{n+1}) \succeq F(x_n, y_n) = x_{n+1} \tag{2.5}$$

and

$$y_{n+2} = F(y_{n+1}, x_{n+1}) \preceq F(y_{n+1}, x_n) \preceq F(y_n, x_n) = y_{n+1}. \tag{2.6}$$

Using (2.5) and (2.6), we get

$$x_{n+1} \preceq x_{n+2} \text{ and } y_{n+1} \succeq y_{n+2}.$$

Hence, by the induction method we conclude that (2.3) and (2.4) hold for all $n \geq 0$. Therefore,

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots \tag{2.7}$$

and

$$y_0 \succeq y_1 \succeq y_2 \succeq \cdots \succeq y_n \succeq y_{n+1} \succeq \cdots. \tag{2.8}$$

For each $n \geq 0$, let

$$\xi_{n+1} = p(x_{n+1}, x_n) + p(y_n, y_{n+1}).$$

Since $x_n \succeq x_{n-1}$ and $y_n \preceq y_{n-1}$, using (2.32) and (2.2), we have

$$\begin{aligned} \phi(p(x_{n+1}, x_n)) + \phi(p(y_n, y_{n+1})) &= \phi(p(F(x_n, y_n), F(x_{n-1}, y_{n-1}))) \\ &\quad + \phi(p(F(y_{n-1}, x_{n-1}), F(y_n, x_n))) \\ &= M_F^{\phi,\psi}(x_n, y_n, x_{n-1}, y_{n-1}) \\ &\leq \phi(p(x_n, x_{n-1}) + p(y_n, y_{n-1})) \\ &\quad -2\psi\Big(\frac{p(x_n, x_{n-1}) + p(y_n, y_{n-1})}{2}\Big). \end{aligned} \tag{2.9}$$

By property (ϕ3), we have

$$\begin{aligned} \phi(p(x_{n+1}, x_n) + p(y_n, y_{n+1})) &\leq \phi(p(x_n, x_{n-1}) + p(y_n, y_{n-1})) \\ &\quad -2\psi\Big(\frac{p(x_n, x_{n-1}) + p(y_n, y_{n-1})}{2}\Big), \end{aligned} \tag{2.10}$$

which implies, since ψ is a non-negative function,

$$\phi(p(x_{n+1}, x_n) + p(y_n, y_{n+1})) \leq \phi(p(x_n, x_{n-1}) + p(y_n, y_{n-1})); \tag{2.11}$$

that is

$$\phi(\xi_{n+1}) \leq \phi(\xi_n) \quad \text{for all } n \geq 0. \tag{2.12}$$

Using the fact that ϕ is a non-decreasing, we get $\xi_{n+1} \leq \xi_n$ for all $n \geq 0$. This shows that $\{\xi_n\}$ is decreasing. Therefore is some $\xi \geq 0$ such that

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} [p(x_{n+1}, x_n) + p(y_{n+1}, y_n)] = \xi. \tag{2.13}$$

We shall prove that $\xi = 0$. Suppose, to the contrary, that $\xi > 0$. Then taking the limit as $n \to \infty$ (equivalently, $\xi_n \to \xi$) of both sides of (2.10) and remembering $\lim_{t\to r} \psi(t) > 0$ for all $r > 0$ and ϕ is continuous, we have

$$\begin{aligned} \phi(\xi) &= \lim_{n\to\infty} \phi(\xi_n) \leq \lim_{n\to\infty} \Big[\phi(\xi_{n-1}) - 2\psi\Big(\frac{\xi_{n-1}}{2}\Big)\Big] \\ &= \phi(\xi) - 2 \lim_{\xi_{n-1}\to\xi} \psi\Big(\frac{\xi_{n-1}}{2}\Big) < \phi(\xi), \end{aligned}$$

which gives a contradiction. Thus $\xi = 0$, that is,

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} [p(x_{n+1}, x_n) + p(y_{n+1}, y_n)] = 0. \tag{2.14}$$

$$= \phi(0) - 2\psi(0) = -2\psi(0) \leq 0,$$

which implies $\phi(p(F(x,y),F(x,y))) + \phi(p(F(y,x),F(y,x))) = 0$. Therefore, from the property ($\phi 2$), we get

$$p(F(x,y),F(x,y)) = 0 = p(F(y,x),F(y,x)). \tag{2.26}$$

We now show that $x = F(x,y)$ and $y = F(y,x)$. Suppose that the assumption (a) holds. For any given $\varepsilon > 0$, the commutativity of F at a point (x,y) implies that there exists $\xi > 0$ such that if $(u,v) \in X \times X$ with $q((x,y),(u,v)) < q((x,y),(x,y)) + \xi = \xi$, meaning that

$$p(x,u) + p(y,v) < p(x,x) + p(y,y) + \xi = \xi,$$

because $p(x,x) = p(y,y) = 0$, then we have

$$p(F(x,y),F(u,v)) < p(F(x,y),F(x,y)) + \frac{\varepsilon}{2}. \tag{2.27}$$

Since $\lim_{n\to+\infty} p(x_n,x) = \lim_{n\to+\infty} p(y_n,y) = 0$, and $\xi = \min\{\frac{\xi}{2},\frac{\varepsilon}{2}\} > 0$, there exist $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$,

$$p(x_n,x) < \xi \text{ and } p(y_n,y) < \xi,$$

which gives that

$$p(x_n,x) + p(y_n,y) < 2\xi < \xi.$$

Using (2.27), we get that

$$p(F(x,y),F(x_n,y_n)) < p(F(x,y),F(x,y)) + \frac{\varepsilon}{2}. \tag{2.28}$$

Then, for any $n \geq n_0$, we have

$$\begin{aligned} p(F(x,y),x) &\leq p(F(x,y),x_{n+1}) + p(x_{n+1},x) \\ &= p(F(x,y),F(x_n,y_n)) + p(x_{n+1},x) \\ &\leq p(F(x,y),F(x,y)) + \frac{\varepsilon}{2} + \xi \\ &\leq p(F(x,y),F(x,y)) + \varepsilon. \end{aligned} \tag{2.29}$$

From (2.26), we have

$$p(F(x,y),x) < \varepsilon.$$

Since ε is arbitrary, we can conclude that

$$p(F(x,y),x) = 0. \tag{2.30}$$

Similarly, we show that $p(F(y,x),y) = 0$. These together with (2.26) and (p1) imply that

$$F(x,y) = x \text{ and } F(y,x) = y.$$

Hence, (x,y) is a coupled fixed point of F.

By the triangle inequality,

$$\begin{aligned} r_k^p &= d_p(x_{n_k}, x_{m_k}) + d_p(y_{n_k}, y_{m_k}) \\ &\leq d_p(x_{n_k}, x_{n_k+1}) + d_p(x_{n_k+1}, x_{m_k+1}) + d_p(x_{m_k+1}, x_{m_k}) \\ &+ d_p(y_{n_k}, y_{n_k+1}) + d_p(y_{n_k+1}, y_{m_k+1}) + d_p(y_{m_k+1}, y_{m_k}) \\ &= \xi_{n_k}^p + \xi_{m_k}^p + d_p(x_{n_k+1}, x_{m_k+1}) + d_p(y_{n_k+1}, y_{m_k+1}). \end{aligned}$$

Using the properties of ϕ, we have

$$\begin{aligned} \phi(r_k^p) &\leq \phi(\xi_{n_k}^p + \xi_{m_k}^p + d_p(x_{n_k+1}, x_{m_k+1}) + d_p(y_{n_k+1}, y_{m_k+1})) \\ &\leq \phi(\xi_{n_k}^p + \xi_{m_k}^p) + \phi(d_p(x_{n_k+1}, x_{m_k+1})) + \phi(d_p(y_{n_k+1}, y_{m_k+1})). \end{aligned} \quad (2.18)$$

Now, let

$$r_k = p(x_{n_k}, x_{m_k}) + p(y_{n_k}, y_{m_k}).$$

By the definition of r_k^p, we have

$$\begin{aligned} r_k^p &= d_p(x_{n_k}, x_{m_k}) + d_p(y_{n_k}, y_{m_k}) \\ &= 2p(x_{n_k}, x_{m_k}) - p(x_{n_k}, x_{n_k}) - p(x_{m_k}, x_{m_k}) \\ &+ 2p(y_{n_k}, y_{m_k}) - p(y_{n_k}, y_{n_k}) - p(y_{m_k}, y_{m_k}) \\ &= 2r_k - p(x_{n_k}, x_{n_k}) - p(x_{m_k}, x_{m_k}) - p(y_{n_k}, y_{n_k}) - p(y_{m_k}, y_{m_k}). \end{aligned} \quad (2.19)$$

In view of property (p2) and (2.14), we have

$$\begin{aligned} \lim_{k \to +\infty} p(x_{n_k}, x_{n_k}) &= \lim_{k \to +\infty} p(x_{m_k}, x_{m_k}) \\ &= \lim_{k \to +\infty} p(y_{n_k}, y_{n_k}) \\ &= \lim_{k \to +\infty} p(y_{m_k}, y_{m_k}) = 0. \end{aligned}$$

Therefore, letting $k \to +\infty$ in (2.19) and using (2.17), we get

$$\lim_{k \to +\infty} r_k = \frac{\varepsilon}{2}.$$

Since $x_{n_k} \succeq x_{m_k}$ and $y_{n_k} \preceq y_{m_k}$, we have

$$\begin{aligned}
\phi(d_p(x_{n_k+1}, x_{m_k+1})) + \phi(d_p(y_{n_k+1}, y_{m_k+1})) \\
&\leq \phi(2p(x_{n_k+1}, x_{m_k+1})) + \phi(2p(y_{n_k+1}, y_{m_k+1})) \\
&\leq 2\phi(p(x_{n_k+1}, x_{m_k+1})) + 2\phi(p(y_{n_k+1}, y_{m_k+1})) \\
&= 2\phi(p(F(x_{n_k}, y_{n_k})), p(F(x_{m_k}, y_{m_k}))) + 2\phi(p(F(y_{n_k}, x_{n_k})), p(F(y_{m_k}, x_{m_k}))) \\
&= 2M_F^{\phi,\psi}(x_{n_k}, y_{n_k}, x_{m_k}, y_{m_k}) \\
&\leq 2\phi(p((x_{n_k}, x_{m_k})) + p((y_{n_k}, y_{m_k}))) \\
&\quad -4\psi\Big(\frac{p((x_{n_k}, x_{m_k})) + p((y_{n_k}, y_{m_k}))}{2}\Big) \\
&= 2\phi(r_k) - 4\psi\Big(\frac{r_k}{2}\Big). \qquad (2.20)
\end{aligned}$$

Thus, from (2.18), we have

$$\phi(r_k^p) \leq \phi(\xi_{n_k}^p + \xi_{m_k}^p) + 2\phi(r_k) - 4\psi\Big(\frac{r_k}{2}\Big).$$

Letting $k \to +\infty$, and using the properties of ϕ and ψ together with the inequalities established above, we have

$$\begin{aligned}
\phi(\varepsilon) \leq \phi(0) + 2\phi(\frac{\varepsilon}{2}) - 4\lim_{k\to+\infty}\psi(\frac{r_k}{2}) &\leq \phi(\varepsilon) - 4\lim_{\frac{r_k}{2}\to\frac{\varepsilon}{4}}\psi(\frac{r_k}{2}) \\
&\leq \phi(\varepsilon) - 4\lim_{t\to\frac{\varepsilon}{4}}\psi(t) \\
&< \phi(\varepsilon) \qquad (2.21)
\end{aligned}$$

which is a contradiction. Therefore, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in the complete metric space (X, d_p). Thus, there are $x, y \in X$ such that

$$\lim_{n\to+\infty} d_p(x_n, x) = \lim_{n\to+\infty} d_p(y_n, y) = 0, \qquad (2.22)$$

which implies that

$$\begin{aligned}
\lim_{n\to+\infty} F(x_n, y_n) &= \lim_{n\to+\infty} x_n = x \\
\lim_{n\to+\infty} F(y_n, x_n) &= \lim_{n\to+\infty} y_n = y. \qquad (2.23)
\end{aligned}$$

Therefore, from Lemma 1.7 (b), using (2.14) and the property (p2), we have

$$p(x,x) = \lim_{n\to+\infty} p(x_n,x) = \lim_{n\to+\infty} p(x_n,x_n) = 0, \tag{2.24}$$

$$p(y,y) = \lim_{n\to+\infty} p(y_n,y) = \lim_{n\to+\infty} p(y_n,y_n) = 0. \tag{2.25}$$

On utilizing $p(x,x) = p(y,y) = 0$ in (2.32), we get

$$\begin{aligned}
&\phi(p(F(x,y),F(x,y))) + \phi(p(F(y,x),F(y,x))) \\
&\leq \phi(p(x,x)+p(y,y)) - 2\psi\Big(\frac{p(x,x)+p(y,y)}{2}\Big) \\
&= \phi(0) - 2\psi(0) = -2\psi(0) \leq 0,
\end{aligned}$$

which implies $\phi(p(F(x,y),F(x,y))) + \phi(p(F(y,x),F(y,x))) = 0$. Therefore, from the property (ϕ2), we get

$$p(F(x,y),F(x,y)) = 0 = p(F(y,x),F(y,x)). \tag{2.26}$$

We now show that $x = F(x,y)$ and $y = F(y,x)$. Suppose that the assumption (a) holds. For any given $\varepsilon > 0$, the commutativity of F at a point (x,y) implies that there exists $\xi > 0$ such that if $(u,v) \in X \times X$ with $q((x,y),(u,v)) < q((x,y),(x,y)) + \xi = \xi$, meaning that

$$p(x,u) + p(y,v) < p(x,x) + p(y,y) + \xi = \xi,$$

because $p(x,x) = p(y,y) = 0$, then we have

$$p(F(x,y),F(u,v)) < p(F(x,y),F(x,y)) + \frac{\varepsilon}{2}. \tag{2.27}$$

Since $\lim_{n\to+\infty} p(x_n,x) = \lim_{n\to+\infty} p(y_n,y) = 0$, and $\zeta = \min\{\frac{\xi}{2},\frac{\varepsilon}{2}\} > 0$, there exist $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$,

$$p(x_n,x) < \zeta \text{ and } p(y_n,y) < \zeta,$$

which gives that

$$p(x_n,x) + p(y_n,y) < 2\zeta < \xi.$$

Using (2.27), we get that

$$p(F(x,y),F(x_n,y_n)) < p(F(x,y),F(x,y)) + \frac{\varepsilon}{2}. \tag{2.28}$$

Then, for any $n \geq n_0$, we have

$$\begin{aligned} p(F(x,y),x) &\leq p(F(x,y),x_{n+1}) + p(x_{n+1},x) \\ &= p(F(x,y),F(x_n,y_n) + p(x_{n+1},x) \\ &\leq p(F(x,y),F(x,y)) + \frac{\varepsilon}{2} + \zeta \\ &\leq p(F(x,y),F(x,y)) + \varepsilon. \end{aligned} \tag{2.29}$$

From (2.26), we have

$$p(F(x,y),x) < \varepsilon.$$

Since ε is arbitrary, we can conclude that

$$p(F(x,y),x) = 0. \tag{2.30}$$

Similarly, we show that $p(F(y,x),y) = 0$. These together with (2.26) and (p1) imply that

$$F(x,y) = x \text{ and } F(y,x) = y.$$

Hence, (x,y) is a coupled fixed point of F.

Finally, suppose that (b) holds. By (2.3), (2.22) and (2.24), we have $\{x_n\}$ is a non-decreasing sequence, $x_n \to x$ and $\{y_n\}$ is a non-increasing sequence, $y_n \to y$ as $n \to \infty$. Hence, by the assumption (b), we have for all $n \geq 0$,

$$x_n \preceq x \text{ and } y \preceq y_n. \tag{2.31}$$

By property (p4), we have

$$p(F(x,y),x) \leq p(F(x,y),x_{n+1}) + p(x_{n+1},x) = p(F(x,y),F(x_n,y_n)) + p(x_{n+1},x)$$

and

$$p(F(y,x),y) \leq p(F(y,x),y_{n+1}) + p(y_{n+1},y) = p(F(y,x),F(y_n,x_n)) + p(y_{n+1},y)$$

Therefore,

$$\begin{aligned}\phi(p(F(x,y),x)) + \phi(p(F(y,x),y)) \\ &\leq \phi(p(x_{n+1},x)) + \phi(p(y_{n+1},y)) + \phi(p(F(x,y),F(x_n,y_n))) + \phi(p(F(y,x),F(y_n,x_n))) \\ &\leq \phi(p(x_{n+1},x)) + \phi(p(y_{n+1},y)) + \phi(p(x,x_n) + p(y,y_n)) - 2\psi\Big(\frac{(p(x,x_n) + p(y,y_n))}{2}\Big).\end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, using (2.24) and (2.25) and the properties of ϕ and ψ, we get $\phi(p(x,F(x,y))) = 0 = \phi(p(y,F(y,x)))$, which implies

$$p(x,F(x,y)) = 0 = p(y,F(y,x)).$$

These together with (2.26), we have

$$x = F(x,y) \text{ and } y = F(y,x)$$

Hence, (x,y) is a coupled coincidence point of F. This complete the proof. □

Remark 2.2. Theorem 2.3 is more general than [4, Theorem 3.1], since the contractive condition (2.32) is weaker than (1.1), a fact which is clearly illustrated by the following example.

Example 2.3. Let us recall Example 2.1. Define the mappings
$\phi, \psi : [0,\infty) \to [0,\infty)$ by

$$\phi(t) = 2t \quad \text{and} \quad \psi(t) = \frac{t}{2} \quad \text{for all} \quad t \in [0,\infty),$$

We show that F, ϕ and ψ satisfy condition (2.32). For all $x \geq u$ and $y \leq v$, we observe that

$$\phi(p(F(x,y),F(u,v))) + \phi(p(F(y,x),F(v,u))) = \frac{5x-2y}{4} + \frac{5v-2u}{4},$$

$$\phi(p(x,u) + p(y,v)) = \phi(x+v) = 2(x+v),$$

and

$$2\psi\left(\frac{p(x,u) + p(y,v)}{2}\right) = 2\psi\left(\frac{x+v}{2}\right) = \frac{x+v}{2}.$$

Furthermore, we can have the following fact

$$\begin{aligned} x+v \geq -2u-2y &\Leftrightarrow 6x+6v \geq 5x-2y+5v-2u \\ &\Leftrightarrow \frac{3}{2}(x+v) \geq \frac{5x-2y}{4}+\frac{5v-2u}{4}. \end{aligned}$$

Therefore, we arrive that

$$\begin{aligned} \phi(p(F(x,y),F(u,v)))+\phi(p(F(y,x),F(v,u))) &= \frac{5x-2y}{4}+\frac{5v-2u}{4} \\ &\leq \frac{3}{2}(x+v) \\ &= 2(x+v)-\frac{x+v}{2} \\ &= \phi(p(x,u)+p(y,v)) \\ &\quad -2\psi\Big(\frac{p(x,u)+p(y,v)}{2}\Big). \end{aligned}$$

Hence F,ϕ and ψ satisfy (2.32). By Theorem 2.3, we conclude that F has a coupled fixed point in X. Moreover, $(0,0)\in X^2$ is a coupled fixed point of F. $\square$

As an immediate consequence of the above theorem, by taking $\phi(t)=t$, we have:

Corollary 2.4. *Let $(X,\preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X,d) is a complete partial metric space. Let $F: X\times X\to X$ be a mapping having the mixed monotone property on X. Assume that there exist two elements $x_0,y_0\in X$ with*

$$x_0 \preceq F(x_0,y_0) \text{ and } y_0 \succeq F(y_0,x_0).$$

Suppose there exist $\phi\in\Phi$ and $\psi\in\Psi$ such that

$$p(F(x,y),F(u,v)) \leq \frac{1}{2}(p(x,u)+p(y,v))-\psi\Big(\frac{p(x,u)+p(y,v)}{2}\Big)$$

for all $x,y,u,v\in X$ with $x\succeq u$ and $y\preceq v$. Suppose either

(a) *F is continuous or*

(b) *X has the following property:*

(i) *if a non-decreasing sequence* $x_n \to x$,*then* $x_n \preceq x$ *for all* n,

(ii) *if a non-decreasing sequence* $y_n \to y$,*then* $y \preceq y_n$ *for all* n.

Then there exist $x, y, \in X$ *such that*

$$x = F(x,y) \text{ and } y = F(y,x),$$

that is F has a coupled fixed point in X.

Moreover, if we take $\psi(t) = \frac{1-k}{2}t$ where $k \in [0,1)$ in Corollary 2.4, we get:

Corollary 2.5. *Let* $(X, \preceq)$ *be a partially ordered set and suppose there is a partial metric* p *on* X *such that* (X, d) *is a complete partial metric space. Let* $F : X \times X \to X$ *be a mapping having the mixed monotone property on* X. *Assume that there exist two elements* $x_0, y_0 \in X$ *with*

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0).$$

Suppose there exist $\phi \in \Phi$ *and* $\psi \in \Psi$ *such that*

$$p(F(x,y), F(u,v)) \leq \frac{1}{2}(p(x,u) + p(y,v))$$

for all $x, y, u, v \in X$ *with* $x \succeq u$ *and* $y \preceq v$. *Suppose either*

(a) *F is continuous or*

(b) *X has the following property:*

(i) *if a non-decreasing sequence* $x_n \to x$,*then* $x_n \preceq x$ *for all* n,

(ii) *if a non-decreasing sequence* $y_n \to y$,*then* $y \preceq y_n$ *for all* n.

Then there exist $x, y, \in X$ *such that*

$$x = F(x,y) \text{ and } y = F(y,x),$$

that is F has a coupled fixed point in X.

CHAPTER 3

CONCLUSION

1. Let $X = \mathbb{R}^+$ be a set endowed with order $x \preceq y \Leftrightarrow x \leq y$. Let $p(x,y) = \max\{x,y\}$, then (X,p) is a partial metric space. Define the mapping $F : X \times X \to X$ by

$$F(x,y) = \frac{5x-2y}{8} \text{ for all } x,y \in X.$$

Then the following properties hold:

(1) F is mixed monotone;

(2) the condition (1.1) does not hold.

Indeed, we show that F does not satisfy condition (1.1). Assume on the contrary, that there exist ϕ and ψ, such that (1.1) holds. Therefore it implies that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that, for all $x \geq u$ and $y \leq v$, we have

$$\begin{aligned}
\phi\left(\frac{5x-2y}{8}\right) &= \phi\left(\max\left\{\frac{5x-2y}{8}, \frac{5u-2v}{8}\right\}\right) \\
&= \phi(p(F(x,y),F(u,v))) \\
&\leq \frac{1}{2}\phi(p(x,u)+p(y,v)) - \psi\left(\frac{p(x,u)+p(y,v)}{2}\right) \\
&= \frac{1}{2}\phi(x+v) - \psi\left(\frac{x+v}{2}\right).
\end{aligned}$$

Setting $x = 5, y = 1/2$ and $v = 1$ to the last inequality, we get, since $\psi(3) > 0$, that

$$\phi(3) \leq \left(\frac{1}{2}\right)\phi(6) - \psi(3) < \left(\frac{1}{2}\right)\phi(6) \leq \left(\frac{1}{2}\right)2\phi(3) = \phi(3)$$

which gives a contradiction. Hence F does not satisfy condition (1.1). Notice, however, that $(0,0) \in X^2$ is the coupled fixed point of F.

2. Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X,p) is a complete partial metric space. Let $F : X \times X \to X$

be a mapping having the mixed monotone property on X. Suppose that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that

$$M_F^{\phi,\psi}(x,y,u,v) \leq \phi(p(x,u)+p(y,v)) - 2\psi\left(\frac{p(x,u)+p(y,v)}{2}\right) \quad (2.32)$$

for all $x,y,u,v \in X$ for which $x \succeq u$ and $y \preceq v$, where

$$M_F^{\phi,\psi}(x,y,u,v) = \phi(p(F(x,y),F(u,v))) + \phi(p(F(y,x),F(v,u))).$$

If there exist two elements $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0,y_0) \text{ and } y_0 \succeq F(y_0,x_0),$$

and also suppose either

(a) F is continuous or

(b) X has the following property:

(i) if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$,then $x_n \preceq x$ for all n,

(ii) if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$,then $y \preceq y_n$ for all n,

then F has a coupled fixed point in X, that is there exists $(x,y) \in X \times X$ such that $x = F(x,y)$ and $y = F(y,x)$.

3. Let us recall Example 2.1. Define the mappings $\phi, \psi : [0,\infty) \to [0,\infty)$ by

$$\phi(t) = 2t \text{ and } \psi(t) = \frac{t}{2} \text{ for all } t \in [0,\infty),$$

We show that F, ϕ and ψ satisfy condition (2.32). For all $x \geq u$ and $y \leq v$, we observe that

$$\phi(p(F(x,y),F(u,v))) + \phi(p(F(y,x),F(v,u))) = \frac{5x-2y}{4} + \frac{5v-2u}{4},$$

$$\phi(p(x,u)+p(y,v)) = \phi(x+v) = 2(x+v),$$

and

$$2\psi\left(\frac{p(x,u)+p(y,v)}{2}\right) = 2\psi\left(\frac{x+v}{2}\right) = \frac{x+v}{2}.$$

Furthermore, we can have the following fact

$$\begin{aligned} x+v \geq -2u-2y &\Leftrightarrow 6x+6v \geq 5x-2y+5v-2u \\ &\Leftrightarrow \frac{3}{2}(x+v) \geq \frac{5x-2y}{4}+\frac{5v-2u}{4}. \end{aligned}$$

Therefore, we arrive that

$$\begin{aligned} \phi(p(F(x,y),F(u,v)))+\phi(p(F(y,x),F(v,u))) &= \frac{5x-2y}{4}+\frac{5v-2u}{4} \\ &\leq \frac{3}{2}(x+v) \\ &= 2(x+v)-\frac{x+v}{2} \\ &= \phi(p(x,u)+p(y,v)) \\ &\quad -2\psi\left(\frac{p(x,u)+p(y,v)}{2}\right). \end{aligned}$$

Hence F,ϕ and ψ satisfy (2.32). By Theorem 2.3, we conclude that F has a coupled fixed point in X. Moreover, $(0,0) \in X^2$ is a coupled fixed point of F.

4. Let $(X,\preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X,d) is a complete partial metric space. Let $F: X\times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exist two elements $x_0, y_0 \in X$ with

$$x_0 \preceq F(x_0,y_0) \text{ and } y_0 \succeq F(y_0,x_0).$$

Suppose there exist $\phi\in\Phi$ and $\psi\in\Psi$ such that

$$p(F(x,y),F(u,v)) \leq \frac{1}{2}(p(x,u)+p(y,v)) - \psi\left(\frac{p(x,u)+p(y,v)}{2}\right)$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. Suppose either

(a) F is continuous or

(b) X has the following property:

(i) if a non-decreasing sequence $x_n \to x$,then $x_n \preceq x$ for all n,

(ii) if a non-decreasing sequence $y_n \to y$,then $y \preceq y_n$ for all n.

Then there exist $x, y, \in X$ such that

$$x = F(x, y) \text{ and } y = F(y, x),$$

that is F has a coupled fixed point in X.

5. Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, d) is a complete partial metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exist two elements $x_0, y_0 \in X$ with

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0).$$

Suppose there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that

$$p(F(x, y), F(u, v)) \leq \frac{1}{2}(p(x, u) + p(y, v))$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. Suppose either

(a) F is continuous or

(b) X has the following property:

(i) if a non-decreasing sequence $x_n \to x$,then $x_n \preceq x$ for all n,

(ii) if a non-decreasing sequence $y_n \to y$,then $y \preceq y_n$ for all n.

Then there exist $x, y, \in X$ such that

$$x = F(x, y) \text{ and } y = F(y, x),$$

that is F has a coupled fixed point in X.

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APPENDIX

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Coupled fixed points for generalized weakly contractive mappings in partial metric spaces

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Abstract

In this paper, we establish coupled fixed point results for generalized weakly contractive mappings having the mixed monotone property in ordered partial metric spaces. The results on fixed point theorems are generalizations of the recent results of Alsulami, Hussain and Alotaibi [S. Alsulami, N. Hussain and A. Alotaibi, Coupled Fixed and Coincidence Points for Monotone Operators in Partial Metric Spaces, Fixed Point Theory and Applications 2012, 2012:173].

Keywords: Coupled fixed point; Partial metric space; Generalized weakly contractive mapping; Coupled coincidence point

1. Introduction and Preliminaries

The existence and uniqueness of fixed and common fixed point theorems of operators has been a subject of great interest since Banach [1] proved the Banach contraction principle in 1922. Many authors generalized the Banach contraction principle in various spaces such as quasi-metric spaces, generalized metric spaces, cone metric spaces and fuzzy metric spaces. Matthews [2] introduced the notion of partial metric spaces in such a way that each object does not necessarily have to have a zero distance from itself and proved a modified version of the Banach contraction principle. Afterwards, many authors proved many existing fixed point theorems in partial metric spaces (see [5]-[44] for examples).

In recent times, fixed point theory has developed rapidly in partially ordered metric spaces, that is, metric spaces endowed with a partial ordering. Some of these works are noted in [11, 17, 18, 36]. Bhaskar and Lakshmikantham [18] introduced the concept of a coupled fixed point and the mixed monotone property. Furthermore, they proved some coupled fixed point theorems for mappings which satisfy the mixed monotone property and gave some applications in the existence and uniqueness of a solution for a periodic boundary value problem. After the publication of this work, several coupled fixed point and coincidence

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point results have appeared in the recent literature. Works noted in [31, 38, 39] are some examples of these works.

We recall below the definition of partial metric space and some of its properties.

Definition 1.1. [2] A partial metric on a nonempty set X is a function $p : X \times X \longrightarrow \mathbb{R}_0^+$ such that for all $x, y, z \in X$:

(p1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(p2) $p(x,x) \leq p(x,y)$,
(p3) $p(x,y) = p(y,x)$,
(p4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x,y) = 0$, then from (p1) and (p2), $x = y$. But if $x = y$, $p(x,y)$ may not be 0. The function $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$ defines a partial metric on $\mathbb{R}^+$. Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$. If p is a partial metric on X, then the function $d_p : X \times X \longrightarrow \mathbb{R}^+$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X.

Definition 1.2. Let (X,p) be a partial metric space. Then

(i) A sequence $\{x_n\}$ in a partial metric space (X,p) **converges** to a point $x \in X$ if and only if $p(x,x) = \lim_{n\longrightarrow\infty} p(x,x_n)$.
(ii) A sequence $\{x_n\}$ in a partial metric space (X,p) is called a **Cauchy sequence** iff $\lim_{n,m\longrightarrow\infty} p(x_n,x_m)$ exists (and is finite).
(iii) A partial metric space (X,p) is said to be **complete** if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\longrightarrow\infty} p(x_n,x_m)$.
(iv) A subset A of a partial metric space (X,p) is **closed** if whenever $\{x_n\}$ is a sequence in A such that $\{x_n\}$ converges to some $x \in X$, then $x \in A$.

Remark 1.3. The limit in a partial metric space is not unique.

Theorem 1.4. *Let (Y,d') be a subspace of metric space (X,d). If (X,d) is a complete metric space and Y is a closed set in X, then (Y,d') is a complete metric space.*

Lemma 1.5. [2, 33] Let (X,p) be a partial metric space.

(a) $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,d_p).

(b) A partial metric space (X, p) is complete if and only if the metric space (X, d_p) is complete. Furthermore, $\lim_{n\longrightarrow\infty} d_p(x_n, x) = 0$ if and only if

$$p(x, x) = \lim_{n\longrightarrow\infty} p(x_n, x) = \lim_{n,m\longrightarrow\infty} p(x_n, x_m).$$

Let (X, p) be a partial metric. We endow the product space $X \times X$ with the partial metric q defined as follows:

$$\text{for } (x, y), (u, v) \in X \times X, \quad q((x, y), (u, v)) = p(x, u) + p(y, v).$$

A mapping $F : X \times X \to X$ is said to be continuous at $(x, y) \in X \times X$ is for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$F(B_q((x, y), \delta)) \subset B_p((x, y), \varepsilon).$$

The concept of a mixed monotone property has been introduced by Bhaskar and Lakshmikantham [18].

Definition 1.6 ([18]). Let $(X, \preceq)$ be a partial ordered set. A mapping $F : X \times X \to X$ is said to be have *mixed monotone property* if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$

$$x_1, x_2 \in X, x_1 \preceq x_2 \quad \Rightarrow \quad F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \preceq y_2 \quad \Rightarrow \quad F(x, y_1) \succeq F(x, y_2).$$

Definition 1.7 ([18]). Let $F : X \times X \to X$. An element $(x, y) \in X \times X$ is said to be a *coupled fixed point* of a mapping F if

$$x = F(x, y) \text{ and } y = F(y, x).$$

Let Φ denote the set of all functions $\phi : [0, \infty) \to [0, \infty)$ which satisfy

(ϕ1) ϕ is continuous and non-decreasing,
(ϕ2) $\phi(t) = 0$ if and only if $t = 0$,
(ϕ3) $\phi(t + s) \leq \phi(t) + \phi(s)$ for all $t, s \in [0, \infty)$,
(ϕ4) $\phi(\alpha t) \leq \alpha\phi(t)$ for all $\alpha \in (0, \infty)$.

and let Ψ denote the set of all functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy

$$\lim_{t\to r} \psi(t) > 0 \text{ for all } r > 0 \text{ and } \lim_{t\to 0^+} \psi(t) = 0.$$

Alsulami, Hussain and Alotaibi [4] proved some coupled fixed point results for (ϕ, φ)- weakly contractive mappings in ordered partial metric spaces. More precisely, they obtained the following results.

Theorem 1.8. [4, Theorem 3.1] *Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Suppose that there exists $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\phi(p(F(x,y), F(u,v))) \leq \frac{1}{2}\phi(p(x,u) + p(y,v)) - \psi\left(\frac{p(x,u) + p(y,v)}{2}\right) \tag{1.1}$$

for all $x, y, u, v \in X$ for which $x \succeq u$ and $y \preceq v$. If there exists $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0),$$

and also suppose either
(a) *F is continuous or*
(b) *X has the following property:*
  (i) *if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \preceq x$ for all n,*
  (ii) *if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$, then $y \preceq y_n$ for all n,*
then F has a coupled fixed point in X, that is there exists $(x, y) \in X \times X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Starting from the results in Alsulami, Hussain and Alotaibi [4], our main aim in this paper is to obtain more general coupled fixed point theorems for mixed monotone operators $F : X \to X$ satisfying a contractive condition which is significantly more general that the corresponding conditions (1.1) in [4], thus extending many other related results in literature. We also provide illustrative example in support of our results.

2. Coupled fixed points for generalized weakly contractive mappings

We start with an example which shows the weakness of Theorem 1.8.

Example 2.1. Let $X = \mathbb{R}^+$ be a set endowed with order $x \preceq y \Leftrightarrow x \leq y$. Let $p(x, y) = \max\{x, y\}$, then (X, p) is a partial metric space. Define the mapping $F : X \times X \to X$ by

$$F(x,y) = \frac{5x - 2y}{8} \quad \text{for all } x, y \in X.$$

Then the following properties hold:
(1) F is mixed monotone;
(2) the condition (1.1) does not hold.
Indeed, we show that F does not satisfy condition (1.1). Assume on the contrary, that there exist ϕ and ψ, such that (1.1) holds. Therefore it implies that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that, for all $x \geq u$ and $y \leq v$, we have

$$\begin{aligned}\phi\left(\frac{5x - 2y}{8}\right) &= \phi\left(\max\left\{\frac{5x-2y}{8}, \frac{5u-2v}{8}\right\}\right)\\ &= \phi(p(F(x,y), F(u,v)))\end{aligned}$$

$$\begin{aligned} &\le \frac{1}{2}\phi(p(x,u)+p(y,v)) - \psi\left(\frac{p(x,u)+p(y,v)}{2}\right) \\ &= \frac{1}{2}\phi(x+v) - \psi\left(\frac{x+v}{2}\right). \end{aligned}$$

Setting $x = 5, y = 1/2$ and $v = 1$ to the last inequality, we get, since $\psi(3) > 0$, that

$$\phi(3) \le \left(\frac{1}{2}\right)\phi(6) - \psi(3) < \left(\frac{1}{2}\right)\phi(6) \le \left(\frac{1}{2}\right)2\phi(3) = \phi(3)$$

which gives a contradiction. Hence F does not satisfy condition (1.1). Notice, however, that $(0,0) \in X^2$ is the coupled fixed point of F. □

We now state and prove our first result which successively guarantees the existence of a coupled fixed point and generalizes Theorem 3.1 in [4].

Theorem 2.2. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X,p) is a complete partial metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Suppose that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$M_F^{\phi,\psi}(x,y,u,v) \le \phi(p(x,u)+p(y,v)) - 2\psi\left(\frac{p(x,u)+p(y,v)}{2}\right) \tag{2.1}$$

for all $x, y, u, v \in X$ for which $x \succeq u$ and $y \preceq v$, where

$$M_F^{\phi,\psi}(x,y,u,v) = \phi(p(F(x,y),F(u,v))) + \phi(p(F(y,x),F(v,u))).$$

If there exist two elements $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0,y_0) \text{ and } y_0 \succeq F(y_0,x_0),$$

and also suppose either
(a) *F is continuous or*
(b) *X has the following property:*
  (i) *if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \preceq x$ for all n,*
  (ii) *if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$, then $y \preceq y_n$ for all n,*
then F has a coupled fixed point in X, that is there exists $(x, y) \in X \times X$ such that $x = F(x,y)$ and $y = F(y,x)$.

Proof. Let $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0,y_0)$ and $y_0 \succeq F(y_0,x_0)$. We construct sequence $\{x_n\}$ and $\{y_n\}$ in X as

$$x_{n+1} = F(x_n,y_n) \text{ and } y_{n+1} = F(x_n,y_n) \text{ for all } n \ge 0. \tag{2.2}$$

Next, we show that

$$x_n \preceq x_{n+1} \text{ for all } n \ge 0 \tag{2.3}$$

and

$$y_n \succeq y_{n+1} \text{ for all } n \geq 0. \tag{2.4}$$

For this we shall use mathematical induction.

Let $n = 0$. Since $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$ and as $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$, we have $x_0 \preceq x_1$ and $y_0 \succeq y_1$. Thus (2.3) and (2.4) hold for $n = 0$.

Suppose now that (2.3) and (2.4) hold for some fixed $n \geq 0$, then, since $x_n \preceq x_{n+1}$ and $y_n \succeq y_{n+1}$, we have

$$x_{n+2} = F(x_{n+1}, y_{n+1}) \succeq F(x_n, y_{n+1}) \succeq F(x_n, y_n) = x_{n+1} \tag{2.5}$$

and

$$y_{n+2} = F(y_{n+1}, x_{n+1}) \preceq F(y_{n+1}, x_n) \preceq F(y_n, x_n) = y_{n+1}. \tag{2.6}$$

Using (2.5) and (2.6), we get

$$x_{n+1} \preceq x_{n+2} \text{ and } y_{n+1} \succeq y_{n+2}.$$

Hence, by the induction method we conclude that (2.3) and (2.4) hold for all $n \geq 0$. Therefore,

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots \tag{2.7}$$

and

$$y_0 \succeq y_1 \succeq y_2 \succeq \cdots \succeq y_n \succeq y_{n+1} \succeq \cdots. \tag{2.8}$$

For each $n \geq 0$, let

$$\xi_{n+1} = p(x_{n+1}, x_n) + p(y_n, y_{n+1}).$$

Since $x_n \succeq x_{n-1}$ and $y_n \preceq y_{n-1}$, using (2.1) and (2.2), we have

$$\begin{aligned} \phi(p(x_{n+1}, x_n)) + \phi(p(y_n, y_{n+1})) &= \phi(p(F(x_n, y_n), F(x_{n-1}, y_{n-1}))) \\ &\quad + \phi(p(F(y_{n-1}, x_{n-1}), F(y_n, x_n))) \\ &= M_F^{\phi,\psi}(x_n, y_n, x_{n-1}, y_{n-1}) \\ &\leq \phi(p(x_n, x_{n-1}) + p(y_n, y_{n-1})) \\ &\quad -2\psi\Big(\frac{p(x_n, x_{n-1}) + p(y_n, y_{n-1})}{2}\Big). \end{aligned} \tag{2.9}$$

By property ($\phi3$), we have

$$\begin{aligned} \phi(p(x_{n+1}, x_n) + p(y_n, y_{n+1})) &\leq \phi(p(x_n, x_{n-1}) + p(y_n, y_{n-1})) \\ &\quad -2\psi\Big(\frac{p(x_n, x_{n-1}) + p(y_n, y_{n-1})}{2}\Big), \end{aligned} \tag{2.10}$$

which implies, since ψ is a non-negative function,

$$\phi(p(x_{n+1}, x_n) + p(y_n, y_{n+1})) \leq \phi(p(x_n, x_{n-1}) + p(y_n, y_{n-1})); \tag{2.11}$$

that is

$$\phi(\xi_{n+1}) \leq \phi(\xi_n) \quad \text{for all } n \geq 0. \tag{2.12}$$

Using the fact that ϕ is a non-decreasing, we get $\xi_{n+1} \leq \xi_n$ for all $n \geq 0$. This shows that $\{\xi_n\}$ is decreasing. Therefore is some $\xi \geq 0$ such that

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} [p(x_{n+1}, x_n) + p(y_{n+1}, y_n)] = \xi. \tag{2.13}$$

We shall prove that $\xi = 0$. Suppose, to the contrary, that $\xi > 0$. Then taking the limit as $n \to \infty$ (equivalently, $\xi_n \to \xi$) of both sides of (2.10) and remembering $\lim_{t\to r} \psi(t) > 0$ for all $r > 0$ and ϕ is continuous, we have

$$\begin{aligned}\phi(\xi) &= \lim_{n\to\infty} \phi(\xi_n) \leq \lim_{n\to\infty} \left[\phi(\xi_{n-1}) - 2\psi\Big(\frac{\xi_{n-1}}{2}\Big)\right] \\ &= \phi(\xi) - 2 \lim_{\xi_{n-1}\to\xi} \psi\Big(\frac{\xi_{n-1}}{2}\Big) < \phi(\xi),\end{aligned}$$

which gives a contradiction. Thus $\xi = 0$, that is,

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} [p(x_{n+1}, x_n) + p(y_{n+1}, y_n)] = 0. \tag{2.14}$$

Let

$$\xi_n^p = d_p(x_{n+1}, x_n) + d_p(y_n, y_{n+1})$$

for all $n \in \mathbb{N}$. From the definition of d_p, it is clear that $\xi_n^p \leq 2\xi_n$ for all $n \in \mathbb{N}$. Using (2.14), we get

$$\lim_{n\to+\infty} \xi_n^p = \lim_{n\to+\infty} [d_p(x_{n+1}, x_n) + d_p(y_{n+1}, y_n)] = 0.$$

Now, we prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in the partial metric space (X, p). From Lemma 1.5 (a), it is sufficient to prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in the metric space (X, d_p). Suppose, to the contrary, that at least one of $\{x_n\}$ or $\{y_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{x_{n_k}\}$, $\{x_{m_k}\}$ of $\{x_n\}$ and $\{y_{n_k}\}$, $\{y_{m_k}\}$ of $\{y_n\}$ with $n_k > m_k \geq k$ such that

$$d_p(x_{n_k}, x_{m_k}) + d_p(y_{n_k}, y_{m_k}) \geq \varepsilon. \tag{2.15}$$

Further, corresponding to m_k, we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (2.15). Then

$$d_p(x_{n_k-1}, x_{m_k}) + d_p(y_{n_k-1}, y_{m_k}) < \varepsilon. \tag{2.16}$$

Using (2.15), (2.16) and the triangle inequality, we have

$$\begin{aligned}\varepsilon &\leq r_k^p := d_p(x_{n_k}, x_{m_k}) + d_p(y_{n_k}, y_{m_k}) \\ &\leq d_p(x_{n_k}, x_{n_k-1}) + d_p(x_{n_k-1}, x_{m_k}) + d_p(y_{n_k}, y_{n_k-1}) + d_p(y_{n_k-1}, y_{m_k}) \\ &\leq d_p(x_{n_k}, x_{n_k-1}) + d_p(y_{n_k}, y_{n_k-1}) + \varepsilon.\end{aligned}$$

Letting $k \to \infty$ and using (2.14), we get

$$\lim_{k\to\infty} r_k^p = \lim_{k\to\infty} [d_p(x_{n_k}, x_{m_k}) + d_p(y_{n_k}, y_{m_k})] = \varepsilon. \tag{2.17}$$

By the triangle inequality,

$$\begin{aligned}r_k^p &= d_p(x_{n_k}, x_{m_k}) + d_p(y_{n_k}, y_{m_k}) \\ &\leq d_p(x_{n_k}, x_{n_k+1}) + d_p(x_{n_k+1}, x_{m_k+1}) + d_p(x_{m_k+1}, x_{m_k}) \\ &+ d_p(y_{n_k}, y_{n_k+1}) + d_p(y_{n_k+1}, y_{m_k+1}) + d_p(y_{m_k+1}, y_{m_k}) \\ &= \xi_{n_k}^p + \xi_{m_k}^p + d_p(x_{n_k+1}, x_{m_k+1}) + d_p(y_{n_k+1}, y_{m_k+1}).\end{aligned}$$

Using the properties of ϕ, we have

$$\begin{aligned}\phi(r_k^p) &\leq \phi(\xi_{n_k}^p + \xi_{m_k}^p + d_p(x_{n_k+1}, x_{m_k+1}) + d_p(y_{n_k+1}, y_{m_k+1})) \\ &\leq \phi(\xi_{n_k}^p + \xi_{m_k}^p) + \phi(d_p(x_{n_k+1}, x_{m_k+1})) + \phi(d_p(y_{n_k+1}, y_{m_k+1})).\end{aligned} \tag{2.18}$$

Now, let

$$r_k = p(x_{n_k}, x_{m_k}) + p(y_{n_k}, y_{m_k}).$$

By the definition of r_k^p, we have

$$\begin{aligned}r_k^p &= d_p(x_{n_k}, x_{m_k}) + d_p(y_{n_k}, y_{m_k}) \\ &= 2p(x_{n_k}, x_{m_k}) - p(x_{n_k}, x_{n_k}) - p(x_{m_k}, x_{m_k}) \\ &+ 2p(y_{n_k}, y_{m_k}) - p(y_{n_k}, y_{n_k}) - p(y_{m_k}, y_{m_k}) \\ &= 2r_k - p(x_{n_k}, x_{n_k}) - p(x_{m_k}, x_{m_k}) - p(y_{n_k}, y_{n_k}) - p(y_{m_k}, y_{m_k}).\end{aligned} \tag{2.19}$$

In view of property (p2) and (2.14), we have

$$\begin{aligned}\lim_{k\to+\infty} p(x_{n_k}, x_{n_k}) &= \lim_{k\to+\infty} p(x_{m_k}, x_{m_k}) \\ &= \lim_{k\to+\infty} p(y_{n_k}, y_{n_k}) \\ &= \lim_{k\to+\infty} p(y_{m_k}, y_{m_k}) = 0.\end{aligned}$$

Therefore, letting $k \to +\infty$ in (2.19) and using (2.17), we get

$$\lim_{k\to+\infty} r_k = \frac{\varepsilon}{2}.$$

Since $x_{n_k} \succeq x_{m_k}$ and $y_{n_k} \preceq y_{m_k}$, we have

$\phi(d_p(x_{n_k+1}, x_{m_k+1})) + \phi(d_p(y_{n_k+1}, y_{m_k+1}))$

$$\begin{aligned}
&\leq \phi(2p(x_{n_k+1}, x_{m_k+1})) + \phi(2p(y_{n_k+1}, y_{m_k+1})) \\
&\leq 2\phi(p(x_{n_k+1}, x_{m_k+1})) + 2\phi(p(y_{n_k+1}, y_{m_k+1})) \\
&= 2\phi(p(F(x_{n_k}, y_{n_k})), p(F(x_{m_k}, y_{m_k}))) + 2\phi(p(F(y_{n_k}, x_{n_k})), p(F(y_{m_k}, x_{m_k}))) \\
&= 2M_F^{\phi,\psi}(x_{n_k}, y_{n_k}, x_{m_k}, y_{m_k}) \\
&\leq 2\phi(p((x_{n_k}, x_{m_k})) + p((y_{n_k}, y_{m_k}))) \\
&\quad -4\psi\Big(\frac{p((x_{n_k}, x_{m_k})) + p((y_{n_k}, y_{m_k}))}{2}\Big) \\
&= 2\phi(r_k) - 4\psi\Big(\frac{r_k}{2}\Big). \qquad (2.20)
\end{aligned}$$

Thus, from (2.18), we have

$$\phi(r_k^p) \leq \phi(\xi_{n_k}^p + \xi_{m_k}^p) + 2\phi(r_k) - 4\psi\Big(\frac{r_k}{2}\Big).$$

Letting $k \to +\infty$, and using the properties of ϕ and ψ together with the inequalities established above, we have

$$\begin{aligned}
\phi(\varepsilon) \leq \phi(0) + 2\phi(\frac{\varepsilon}{2}) - 4\lim_{k\to+\infty}\psi(\frac{r_k}{2}) &\leq \phi(\varepsilon) - 4\lim_{\frac{r_k}{2}\to\frac{\varepsilon}{4}}\psi(\frac{r_k}{2}) \\
&\leq \phi(\varepsilon) - 4\lim_{t\to\frac{\varepsilon}{4}}\psi(t) \\
&< \phi(\varepsilon) \qquad (2.21)
\end{aligned}$$

which is a contradiction. Therefore, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in the complete metric space (X, d_p). Thus, there are $x, y \in X$ such that

$$\lim_{n\to+\infty} d_p(x_n, x) = \lim_{n\to+\infty} d_p(y_n, y) = 0, \qquad (2.22)$$

which implies that

$$\begin{aligned}
\lim_{n\to+\infty} F(x_n, y_n) &= \lim_{n\to+\infty} x_n = x \\
\lim_{n\to+\infty} F(y_n, x_n) &= \lim_{n\to+\infty} y_n = y. \qquad (2.23)
\end{aligned}$$

Therefore, from Lemma 1.5 (b), using (2.14) and the property (p2), we have

$$p(x, x) = \lim_{n\to+\infty} p(x_n, x) = \lim_{n\to+\infty} p(x_n, x_n) = 0, \qquad (2.24)$$

$$p(y, y) = \lim_{n\to+\infty} p(y_n, y) = \lim_{n\to+\infty} p(y_n, y_n) = 0. \qquad (2.25)$$

On utilizing $p(x, x) = p(y, y) = 0$ in (2.1), we get

$$\begin{aligned}
&\phi(p(F(x, y), F(x, y))) + \phi(p(F(y, x), F(y, x))) \\
&\quad\leq \phi(p(x, x) + p(y, y)) - 2\psi\Big(\frac{p(x, x) + p(y, y)}{2}\Big)
\end{aligned}$$

Finally, suppose that (b) holds. By (2.3), (2.22) and (2.24), we have $\{x_n\}$ is a non-decreasing sequence, $x_n \to x$ and $\{y_n\}$ is a non-increasing sequence, $y_n \to y$ as $n \to \infty$. Hence, by the assumption (b), we have for all $n \geq 0$,

$$x_n \preceq x \text{ and } y \preceq y_n. \tag{2.31}$$

By property (p4), we have

$$p(F(x,y),x) \leq p(F(x,y),x_{n+1}) + p(x_{n+1},x) = p(F(x,y),F(x_n,y_n)) + p(x_{n+1},x)$$

and

$$p(F(y,x),y) \leq p(F(y,x),y_{n+1}) + p(y_{n+1},y) = p(F(y,x),F(y_n,x_n)) + p(y_{n+1},y)$$

Therefore,

$$\begin{aligned}
&\phi(p(F(x,y),x)) + \phi(p(F(y,x),y)) \\
&\leq \ \phi(p(x_{n+1},x)) + \phi(p(y_{n+1},y)) + \phi(p(F(x,y),F(x_n,y_n))) + \phi(p(F(y,x),F(y_n,x_n))) \\
&\leq \ \phi(p(x_{n+1},x)) + \phi(p(y_{n+1},y)) + \phi(p(x,x_n) + p(y,y_n)) - 2\psi\Big(\frac{(p(x,x_n) + p(y,y_n))}{2}\Big).
\end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, using (2.24) and (2.25) and the properties of ϕ and ψ, we get $\phi(p(x, F(x,y))) = 0 = \phi(p(y, F(y,x)))$, which implies

$$p(x, F(x,y)) = 0 = p(y, F(y,x)).$$

These together with (2.26), we have

$$x = F(x,y) \text{ and } y = F(y,x)$$

Hence, (x,y) is a coupled coincidence point of F. This complete the proof. □

Remark 2.3. Theorem 2.2 is more general than [4, Theorem 3.1], since the contractive condition (2.1) is weaker than (1.1), a fact which is clearly illustrated by the following example.

Example 2.4. Let us recall Example 2.1. Define the mappings $\phi, \psi : [0,\infty) \to [0,\infty)$ by

$$\phi(t) = 2t \quad \text{and} \quad \psi(t) = \frac{t}{2} \quad \text{for all } t \in [0,\infty).$$

We show that F, ϕ and ψ satisfy condition (2.1). For all $x \geq u$ and $y \leq v$, we observe that

$$\phi(p(F(x,y),F(u,v))) + \phi(p(F(y,x),F(v,u))) = \frac{5x-2y}{4} + \frac{5v-2u}{4},$$

$$\phi(p(x,u) + p(y,v)) = \phi(x+v) = 2(x+v),$$

and

$$2\psi\left(\frac{p(x,u)+p(y,v)}{2}\right) = 2\psi\left(\frac{x+v}{2}\right) = \frac{x+v}{2}.$$

Furthermore, we can have the following fact

$$\begin{aligned} x+v \geq -2u-2y &\Leftrightarrow 6x+6v \geq 5x-2y+5v-2u \\ &\Leftrightarrow \frac{3}{2}(x+v) \geq \frac{5x-2y}{4} + \frac{5v-2u}{4}. \end{aligned}$$

Therefore, we arrive that

$$\begin{aligned} \phi(p(F(x,y),F(u,v))) + \phi(p(F(y,x),F(v,u))) &= \frac{5x-2y}{4} + \frac{5v-2u}{4} \\ &\leq \frac{3}{2}(x+v) \\ &= 2(x+v) - \frac{x+v}{2} \\ &= \phi(p(x,u)+p(y,v)) - 2\psi\left(\frac{p(x,u)+p(y,v)}{2}\right). \end{aligned}$$

Hence F, ϕ and ψ satisfy (2.1). By Theorem 2.2, we conclude that F has a coupled fixed point in X. Moreover, $(0,0) \in X^2$ is a coupled fixed point of F. □

As an immediate consequence of the above theorem, by taking $\phi(t) = t$, we have:

Corollary 2.5. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, d) is a complete partial metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0).$$

Suppose there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that

$$p(F(x,y),F(u,v)) \leq \frac{1}{2}(p(x,u)+p(y,v)) - \psi\left(\frac{p(x,u)+p(y,v)}{2}\right)$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. Suppose either
(a) *F is continuous or*
(b) *X has the following property:*
 (i) *if a non-decreasing sequence $x_n \to x$,then $x_n \preceq x$ for all n,*
 (ii) *if a non-decreasing sequence $y_n \to y$,then $y \preceq y_n$ for all n.*
Then there exist $x, y, \in X$ such that

$$x = F(x,y) \text{ and } y = F(y,x),$$

that is F has a coupled fixed point in X.

Moreover, if we take $\psi(t) = \frac{1-k}{2}t$ where $k \in [0,1)$ in Corollary 2.5, we get:

Corollary 2.6. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, d) is a complete partial metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0).$$

Suppose there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that

$$p(F(x,y), F(u,v)) \leq \frac{1}{2}(p(x,u) + p(y,v))$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. Suppose either
(a) *F is continuous or*
(b) *X has the following property:*
  (i) *if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,*
  (ii) *if a non-decreasing sequence $y_n \to y$, then $y \preceq y_n$ for all n.*
Then there exist $x, y, \in X$ such that

$$x = F(x,y) \text{ and } y = F(y,x),$$

that is F has a coupled fixed point in X.

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