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The solving of the new generalized equilibrium problems with generalized monotone multivalued mappings

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บทคัดย่อ(ภาษาไทย)

จุดประสงค์ของงานวิจัยนี้คือเพื่อศึกษาระบบใหม่ของอสมการการแปรผันที่มีนัยทั่วไปในปริภูมิบานาค และวิธีการทำซ้ำสำหรับการหารสมาชิกร่วมของเซตของผลเฉลยของระบบอสมการการแปรผันที่มีนัยทั่วไปสำหรับสองการส่งและ เซตจุดตรึงร่วมสำหรับสมาชิกนับได้ของการหดตัวเทียมโดยแท้ ซึ่งได้ทฤษฎีบทการลู่อย่างเข้มของวิธีการทำซ้ำที่ไม่ต้องควบคุมเงื่อนไขลำดับความต่อเนื่องอย่างอ่อนบนการส่งคู่ในปริภูมิบานาค

บทคัดย่อ(ภาษาอังกฤษ)

The purpose of this research is to introduce a new system of general variational inequalities in Banach spaces and an iterative method for finding a common element of the set of solutions of the system of general variational inequalities for two relaxed cocoercive mappings and the set of common fixed points for a countable family of strict pseudo-contractions. The strong convergence theorems of the proposed iterative method are obtained without the control condition weakly sequentially continuous on the duality mapping in Banach spaces.

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CHAPTER 1

INTRODUCTION

Variational inequalities are being used as a mathematical programming tool in modeling a wide class of obstacle, unilateral, free, moving, equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. Using the projection technique one can establish the equivalence between the variational inequalities and the fixed point problem. This equivalence has played an important role in developing several numerical techniques for solving variational inequalities and the related optimization problem. For the physical formulation, applications, numerical methods and other aspects of the variational inequalities, see [11, 21, 22, 23, 24, 25, 26] and the references therein. Related to the variational inequalities is the problem of finding the common fixed points of the strict pseudo-contractions, which is the subject of current interest in functional analysis. It is natural to unify these two problems and find the common elements of the set of the solution of variational inequality and the set of the common fixed points of the strict pseudo-contractions.

Let E be a real Banach space and $U_E = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be uniformly convex if, for any $\epsilon \in (0,2]$, there exists $\delta > 0$ such that, for any $x, y \in U_E$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \left\|\frac{x+y}{2}\right\| \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space E is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in U_E$. It is also said to be uniformly smooth if the limit is attained uniformly for all $x, y \in U_E$. The norm of E is said to be Fréchet differentiable if, for any $x \in U_E$, the above limit is attained uniformly for all $y \in U_E$. The modulus of smoothness of E is defined by

$$\rho(\tau) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau\},$$

where $\rho : [0,\infty) \to [0,\infty)$ is a function. It is known that E is uniformly smooth if and only if $\lim_{\tau\to 0} \frac{\rho(\tau)}{\tau} = 0$. Let q be a fixed real number with $1 < q \leq 2$. A Banach space E is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$.

From [5], we know the following property:

Let q be a real number with $1 < q \leq 2$ and let E be a Banach space. Then E is q-uniformly smooth if and only if there exists a constant $K \geq 1$ such that

$$\|x+y\|^q + \|x-y\|^q \leq 2(\|x\|^q + \|Ky\|^q), \quad \forall x, y \in E.$$

The best constant K in the above inequality is called the q-uniformly smoothness constant of E (see [5] for more details).

Let E be a real Banach space and E^* the dual space of E. Let $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^*. For $q > 1$, the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}, \quad \forall x \in E.$$

In particular, $J = J_2$ is called the normalized duality mapping. It is known that $J_q(x) = \|x\|^{q-2}J(x)$ for all $x \in E$. If E is a Hilbert space, then $J = I$ (: the identity mapping). Note that

(1) E is a uniformly smooth Banach space if and only if J is single-valued and uniformly continuous on any bounded subset of E.

(2) All Hilbert spaces, L^p (or l^p) spaces ($p \geq 2$) and the Sobolev spaces W_m^p ($p \geq 2$) are 2-uniformly smooth, while L^p (or l^p) and W_m^p spaces ($1 < p \leq 2$) are p-uniformly smooth.

(3) Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. More precisely, L^p is $\min\{p, 2\}$-uniformly smooth for any $p > 1$.

Further, we have the following properties of the generalized duality mapping J_q:

1. $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \in E$ with $x \neq 0$,
2. $J_q(tx) = t^{q-1} J_q(x)$ for all $x \in E$ and $t \in [0, \infty)$,
3. $J_q(-x) = -J_q(x)$ for all $x \in E$.

It is known that if E is smooth, then J is single-valued, which is denoted by j . Recall that the duality mapping j is said to be weakly sequentially continuous if for each sequence $\{x_n\} \subset E$ with $x_n \longrightarrow x$ weakly, we have $j(x_n) \longrightarrow j(x)$ weakly-$*$. We know that if E admits a weakly sequentially continuous duality mapping, then E is smooth. For the details, see [13].

Let C be a nonempty closed convex subset of a smooth Banach space E. Recall that the following definitions of a nonlinear mapping $A : C \longrightarrow E$.

Definition 1.1. Let $A : C \longrightarrow E$ be a mapping.

1. A is said to be *accretive* if

$$\langle Ax - Ay, j(x-y) \rangle \geq 0$$

for all $x, y \in C$.

2. A is said to be *α-strongly accretive* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq \alpha\|x-y\|^2$$

for all $x, y \in C$.

3. A is said to be *α-inverse-strongly accretive* or *α-cocoercive* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq \alpha\|Ax-Ay\|^2$$

for all $x, y \in C$.

4. A is said to be *α-relaxed cocoercive* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq -\alpha\|Ax-Ay\|^2$$

for all $x, y \in C$.

5. A is said to be *(α, β)-relaxed cocoercive* if there exists positive constants α and β such that

$$\langle Ax - Ay, j(x-y)\rangle \geq (-\alpha)\|Ax-Ay\|^2 + \beta\|x-y\|^2$$

for all $x, y \in C$.

Remark 1.4. (1). Every α-strongly accretive mapping is an accretive mapping.

(2). Every α-strongly accretive mapping is an (β, α)-relaxed cocoercive mapping for any positive constant β but the converse is not true in general. Then the class of relaxed cocoercive operators is more general than the class of strongly accretive operators.

(3). Evidently, the definition of the inverse-strongly accretive operator is based on that of the inverse-strongly monotone operator in real Hilbert spaces (see, for example, [6]).

(4). The notion of the cocoercivity is applied in several directions, especially to solving variational inequality problems using the auxiliary problem principle and projection methods [35]. Several classes of relaxed cocoercive variational inequalities have been studied in [33, 34].

Let C be a nonempty closed and convex subset of a smooth Banach space E. We introduce the following system of general variational inequalities involving two different nonlinear mappings $A, B : C \longrightarrow E$:

Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0, \ \ \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, j(x - y^*)\rangle \geq 0, \ \ \forall x \in C, \end{cases} \tag{1.1}$$

where λ and μ are two positive real numbers.

As special cases of the problem (1.1), we have the following:

1. If $A = B$, then the problem (1.1) is reduced to the following:

   Find $(x^*, y^*) \in C \times C$ such that

   $$\begin{cases} \langle \lambda Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0, \ \ \forall x \in C, \\ \langle \mu Ax^* + y^* - x^*, j(x - y^*)\rangle \geq 0, \ \ \forall x \in C, \end{cases} \tag{1.2}$$

   where λ and μ are two positive real numbers. This system of variational inequalities was considered and studied by Noor [22, 21] using the auxiliary principle technique.

2. If $\lambda = \mu = 1$, then the problem (1.1) is reduced to the following:

   Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0, \ \forall x \in C, \\ \langle Bx^* + y^* - x^*, j(x - y^*)\rangle \geq 0, \ \forall x \in C. \end{cases} \tag{1.3}$$

This system of variational inequalities was considered and studied by Yao, Noor, Noor, Liou, and Yaqoob [39].

3. In real Hilbert spaces, the problem (1.1) is reduced to the following: Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0, \ \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^*\rangle \geq 0, \ \forall x \in C, \end{cases} \tag{1.4}$$

where λ and μ are two positive real numbers. The system (1.4) is introduced and studied by Ceng, Wang and Yao [10]. To illustrate the applications of this system, Zhu and Marcotte [41] considered the problem of finding $x^* \in C$ such that

$$\langle A(x^*), x - x^*\rangle \geq 0, \forall x \in E = C \cap \{x \in H : B(x) \leq 0\}, \tag{1.5}$$

where A is strongly monotone on E and $B(x) = \{f_1(x), f_2(x), \ldots, f_m(x)\}$ is a constraint mapping explicitly defined by the convex, Lipschitz continuous and continuously differentiable functions $f_i, i = 1, 2, \ldots, m$. Assume that there exists $x_0 \in C$ such that $f_i(x_0) < 0, i = 1, 2, \ldots, m$ (Slaters constraint qualification). Then the variational inequality (1.5) is equivalent to the KuhnTucker-like system

$$\begin{cases} \langle A(x^*) + (\nabla B(x^*))' y^*, x - x^*\rangle \geq 0, \ \forall x \in C, \\ \langle \mu - B(x^*), y - y^*\rangle \geq 0, \ \forall y \geq 0, \end{cases} \tag{1.6}$$

which is exactly the system of variational inequalities (1.4).

4. If $A = B$, $\lambda = \mu = 1$, and $x^* = y^*$ then the problem (1.1) is reduced to the following:

   Find $x^* \in C$ such that

$$\langle Ax^*, j(x - x^*)\rangle \geq 0, \ \ \forall x \in C. \tag{1.7}$$

The problem (1.7) is very interesting as it is connected with the fixed point problem for nonlinear mappings, the problem of finding a zero point of an accretive operator and so on. For the problem of finding a zero point of an accretive operator by the proximal point algorithm, see Kamimura and Takahashi [16] and the references therein. In [2], Aoyama, Iiduka and Takahashi [2] first considered the such problem in Banach spaces. In order to find a solution of problem (1.7), they proved the following theorem which is generalized simultaneously theorems of [6] and [12].

Theorem AIT. *Let E be a uniformly convex and 2-uniformly smooth Banach space and C a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, $\alpha > 0$ and A an α-inverse strongly-accretive operator of C into E with $S(C, A) \neq \emptyset$, where*

$$S(C, A) = \{x^* \in C : \langle Ax^*, J(x - x^*)\rangle \geq 0, \ x \in C\}.$$

If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen such that $\lambda_n \in [a, \frac{\alpha}{K^2}]$ for some $a > 0$ and $\alpha_n \in [b, c]$ for some b, c with $0 < b < c < 1$, then the sequence $\{x_n\}$ defined by the following manners:

$$\begin{cases} x_1 = x \in C, \\ y_n = Q_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)y_n, \ n \geq 1, \end{cases} \tag{1.8}$$

converges weakly to some element z of $S(C, A)$, where K is the 2-uniformly smoothness constant of E.

On the other hand, in [14], Hao obtained a strong convergence theorem for approximating the solutions of the generalized variational inequality problem (1.7) by using the following iterative algorithm:

$$\begin{cases} x_1 = u \in C, \\ y_n = \beta_n x_n + (1-\beta_n)Q_C(I-\lambda_n A)x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n, \ n \geq 1. \end{cases} \tag{1.9}$$

where Q_C is a sunny nonexpansive retraction from E onto C, $\{\alpha_n\}, \{\lambda_n\}$ and $\{\beta_n\}$ are appropriate real sequences in $[0,1]$. Very recently, motivated by Aoyama, Iiduka and Takahashi [2] and Hao [14], for solving the problem (1.3), Yao, Noor, Noor, Liou, and Yaqoob [39] established the equivalence between the system of variational inequalities (1.3) and a fixed point problem involving the nonexpansive mapping. This alternative equivalent formulation is used to suggest and analyze a modified extragradient method. Using the demi-closedness principle for nonexpansive mappings, they obtained the following strong convergence theorem of the proposed iterative method under some suitable conditions.

Theorem YNNLY-A. *Let C be a nonempty closed convex subset of a real smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. Let $A, B : C \longrightarrow E$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $G : C \longrightarrow C$ be a mapping defined by*

$$G(x) = Q_C[Q_C(x - Bx) - AQ_C(x - Bx)], \forall x \in C. \tag{1.10}$$

Then

(i).[39, Lemma 3.2] If E is real 2-uniformly smooth Banach space, $\alpha \geq K^2$ and $\beta \geq K^2$, then G is nonexpansive.

(ii). [39, Lemma 3.3] For given $x^*, y^* \in C$, (x^*, y^*) is a solution of the problem (1.3) if and only if x^* is a fixed point the mapping $G : C \longrightarrow C$ defined by (1.10), where $y^* = Q_C(x^* - Bx^*)$.

Theorem YNNLY-B [39, Theorem 3.1]. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E which admits a weakly sequentially continuous duality mapping and the smooth constant K. Let Q_C be the sunny nonexpansive retraction from E onto C. Let $A, B : C \longrightarrow E$ be α-inverse-strongly accretive with $\alpha \geq K^2$ and β-inverse-strongly accretive with $\beta \geq K^2$, respectively. Suppose the set of fixed point Ω of the mapping $G : C \longrightarrow C$ defined by* (1.10) *is nonempty. For fixed $u \in C$, let the sequence $\{x_n\}$ be generated iteratively by*

$$\begin{cases} x_1 \in C, \text{ chosen arbitrary}, \\ y_n = Q_C(x_n - Bx_n), \\ z_n = Q_C(y_n - Ay_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n z_n, \ n \geq 1, \end{cases} \tag{1.11}$$

where the real sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\} \subset (0,1)$ satisfy the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1, \forall n \geq 1;$

(ii) $\lim_{n\longrightarrow\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = +\infty;$

(iii) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n \leq 1.$

Then $\{x_n\}$ defined by (1.11) *converges strongly to $Q_\Omega u$, where Q_Ω is the sunny nonexpansive retraction of C onto Ω.*

All of the above bring us the following conjectures?.

Question

(i) Could we weaken the condition uniformly convex on Banach spaces?.

(ii) Could we remove the control condition "weakly sequentially continuous" on the duality mapping in Theorem YNNLY-B?.

(iii) Could we construct an iterative algorithm to approximate a common element of the set of solutions of general variational inequalities (1.1) for two relaxed cocoercive mappings and the set of common fixed points of a countable family of strict pseudo-contractions in Banach spaces?.

In this paper, motivated by Aoyama, Iiduka and Takahashi [2], Hao [14], and Yao, Noor, Noor, Liou, and Yaqoob [39], we introduce a new system of general variational inequalities in Banach spaces. We establish the equivalence between the system of variational inequalities (1.1) for two relaxed cocoercive mappings and fixed point problems involving a nonexpansive mapping. This alternative equivalent formulation is used to suggest and analyze a new iterative approximation method for solving the system of general variational inequalities for two relaxed cocoercive mappings and fixed point problems for a countable family of strict pseudo-contractions. The strong convergence theorems of the proposed iterative method are obtained without the control condition "weakly sequentially continuous" of the duality mapping on Banach spaces. The results presented in the paper improve some recent results of Aoyama, Iiduka and Takahashi [2], Hao [14], and Yao, Noor, Noor, Liou, and Yaqoob [39].

Now we collect some useful lemmas for proving the convergence results.

Lemma 1.5. *[1, Lemma 2.3] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n c_n + b_n, \forall n \geq 0,$$

where $\{\alpha_n\}, \{b_n\}, \{c_n\}$ satisfy the restrictions:

$$\sum_{n=0}^{\infty} \alpha_n = \infty; \sum_{n=0}^{\infty} b_n < \infty; \text{ and } \limsup_{n\to\infty} c_n \leq 0.$$

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.6. ([30]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose* *that* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all* $n \geq 0$ *and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

A mapping $T : C \longrightarrow C$ is said to be *ε-strictly pseudo-contractive*, if there exists a constant $\varepsilon \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + \varepsilon\|(I-T)x - (I-T)y\|^2, \ \forall x,y \in C.$$

Note that the class of ε-strictly pseudo-contractive mappings strictly includes the class of nonexpansive mappings which are mappings T on C such that

$$\|Tx - Ty\| \leq \|x-y\|,$$

for all $x, y \in C$. That is, T is nonexpansive if and only if T is 0-strict pseudo-contractive. We denote by $F(T) := \{x \in C : Tx = x\}$ the set of fixed points of T.

Definition 1.2. A countable family of mapping $\{T_n : C \longrightarrow C\}_{i=1}^{\infty}$ is called a *family of uniformly ε-strict pseudo-contractions*, if there exists a constant $\varepsilon \in [0,1)$ such that

$$\|T_n x - T_n y\|^2 \leq \|x-y\|^2 + \varepsilon\|(I-T_n)x - (I-T_n)y\|^2, \ \forall x,y \in C, \ \ \forall n \geq 1.$$

Lemma 1.7. ([9]) *Let* E *be a strictly convex Banach space. Let* T_1 *and* T_2 *be two nonexpansive mappings from* E *into itself with a common fixed point. Define a mapping* S *by*

$$Sx = \lambda T_1 x + (1-\lambda)T_2 x, \quad \forall x \in E,$$

where λ *is a constant in* $(0,1)$. *Then* S *is nonexpansive and* $F(S) = F(T_1) \cap F(T_2)$.

Lemma 1.8. ([40]) *Let* E *be a real 2-uniformly smooth Banach space and* $T : E \to E$ *a ε-strict pseudo-contraction. Then* $S := (1-\varepsilon/K^2)I + \varepsilon/K^2 T$ *is nonexpansive and* $F(T) = F(S)$.

Lemma 1.9. ([37]) *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, jx\rangle + 2\|Ky\|^2, \quad \forall x, y \in E.$$

Let D be a nonempty subset of C. A mapping $Q : C \longrightarrow D$ is said to be sunny if

$$Q(Qx + t(x - Qx)) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t = 0$. A mapping $Q : C \longrightarrow D$ is called a retraction if $Qx = x$ for all $x \in D$. Furthermore, Q is a sunny nonexpansive retraction from C onto D if Q is a retraction from C onto D which is also sunny and nonexpansive. A subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D. The following lemma concerns the sunny nonex- pansive retraction.

Lemma 1.10. ([28, 8]) *Let C be a closed convex subset of a smooth Banach space E. Let D be a nonempty subset of C and $Q : C \longrightarrow D$ be a retraction. Then Q is sunny and nonexpansive if and only if*

$$\langle u - Qu, j(y - Qu)\rangle \leq 0$$

for all $u \in C$ and $y \in D$.

Definition 1.3. Let $\{S_n\}$ be a family of mappings from a subset C of Banach space E into E with $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$. We say that $\{S_n\}$ satisfies the PU-condition if for each bounded subset D of C, there exists a continuous increasing and convex function $h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$h(0) = 0 \text{ and } \lim_{k,l \longrightarrow \infty} \sup_{z \in D} h(\|S_k z - S_l z\|) = 0. \tag{1.12}$$

Remark 1.11. The example of a sequence of mappings satisfying PU-condition is supported by Example 2.4.

Lemma 1.12. [27, Lemma 3.1] *Suppose that there exists a continuous increasing function $h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying* (1.12). *Then*
(i). *For each $x \in C$, $\{S_n x\}$ is a convergent sequence in C.*
(ii). *Let the mapping $S : C \longrightarrow C$ be defined by*

$$Sx = \lim_{n \longrightarrow \infty} S_n x, \text{ for all } x \in C. \tag{1.13}$$

Then $\lim_{n \longrightarrow \infty} \sup_{z \in D} h(\|Sz - S_n z\|) = 0$ for each bounded subset D of C.

Remark 1.13. If $\{S_n\}$ satisfies the PU-condition, then the facts (i) and (ii) in Lemma 1.12 hold.

Lemma 1.14. *[15, Lemma 3.2] Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E with the smooth constant K. Let $A : C \longrightarrow E$ be a L_A-Lipchitzian and relaxed (c, d)-cocoercive mapping. Then*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq (1 + 2\lambda c L_A^2 - 2\lambda d + 2\lambda^2 K^2 L_A^2)\|x - y\|^2. \tag{1.14}$$

If $\lambda \leq \frac{d - cL_A^2}{K^2 L_A^2}$, then $I - \lambda A$ is nonexpansive.

CHAPTER 2

MAIN RESULTS

In this section, we state and prove our main results.

Lemma 2.1. *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E with the smooth constant K. Let Q_C be the sunny nonexpansive retraction from E onto C. Let $A : C \longrightarrow E$ be a L_A-Lipchitzian and relaxed (c, d)-cocoercive mapping and $B : C \longrightarrow H$ a L_B-Lipchitzian and relaxed (c', d')-cocoercive mapping, where $\lambda \leq \frac{d-cL_A^2}{K^2L_A^2}$ and $\mu \leq \frac{d'-c'L_B^2}{K^2L_B^2}$. Define the mapping G by*

$$G(x) = Q_C[Q_C(x - \mu Bx) - \lambda A Q_C(x - \mu Bx)], \forall x \in C. \tag{2.1}$$

Then G is nonexpansive.

Proof. From Lemma 1.14, we deduce that $I - \lambda A$, $I - \mu B$ and Q_C are nonexpansive mappings. Then, for any $x, y \in C$, we obtain

$$\begin{aligned} \|G(x) - G(y)\|^2 &= \|Q_C[Q_C(x - \mu Bx) - \lambda A Q_C(x - \mu Bx)] \\ &\quad - Q_C[Q_C(y - \mu By) - \lambda A Q_C(y - \mu By)]\|^2 \\ &\leq \|(I - \lambda A)Q_C(I - \mu B)x - (I - \lambda A)Q_C(I - \mu B)y\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

Hence G is nonexpansive on C. □

Lemma 2.2. *Let C be a nonempty closed convex subset of a real smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of the problem (1.1) if and only if x^* is a fixed point the mapping $G : C \longrightarrow C$ defined by (2.30), where $y^* = Q_C(x^* - \mu Bx^*)$, λ, μ are positive constants and $A, B : C \longrightarrow H$ are possibly nonlinear mappings.*

Proof. We can rewrite the problem (1.1) as

$$\begin{cases} \langle x^* - (y^* - \lambda A y^*), j(x - x^*)\rangle \geq 0, \ \forall x \in C, \\ \langle y^* - (x^* - \mu B x^*), j(x - y^*)\rangle \geq 0, \ \forall x \in C. \end{cases} \tag{2.2}$$

Applying Lemma 1.10, we can deduce that (2.2) is equivalent to

$$x^* = Q_C(y^* - \lambda A y^*) \text{ and } y^* = Q_C(x^* - \mu B x^*),$$

which is equivalent to

$$x^* = Q_C(Q_C(x^* - \mu B x^*) - \lambda A Q_C(x^* - \mu B x^*)).$$

Hence x^* is a fixed point the mapping G defined by (2.30). This completes the proof. □

Throughout this paper, the set of fixed points of the mapping G is denoted by $GVI(A, B, C)$.

Theorem 2.4. *Let C be a nonempty closed convex subset of a strictly convex and 2-uniformly smooth Banach space E with the smooth constant K. Let Q_C be the sunny nonexpansive retraction from E onto C. Let $A : C \longrightarrow E$ be a L_A-Lipchitzian and relaxed (c, d)-cocoercive mapping and $B : C \longrightarrow H$ a L_B-Lipchitzian and relaxed (c', d')-cocoercive mapping. Let $\{T_n : C \longrightarrow C\}_{i=1}^{\infty}$ be a countable family of uniformly ε-strict pseudo-contractions such that $\Omega := \cap_{n=1}^{\infty} F(T_n) \cap GVI(A, B, C) \neq \emptyset$. Define a mapping $S_n : C \longrightarrow C$ by*

$$S_n x = (1 - \frac{\varepsilon}{K^2})x + \frac{\varepsilon}{K^2} T_n x \text{ for all } x \in C \text{ and } n \geq 1.$$

Let the sequence $\{x_n\}$ be generated iteratively by

$$\begin{cases} x_1 = u \in C, \text{ chosen arbitrary}, \\ y_n = Q_C(x_n - \mu B x_n), \\ z_n = Q_C(y_n - \lambda A y_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\nu S_n x_n + (1 - \nu) z_n], \ n \geq 1, \end{cases} \tag{2.3}$$

where $\nu \in (0,1)$, $\lambda \in (0, \frac{d-cL_A^2}{K^2L_A^2})$ *and* $\mu \in (0, \frac{d'-c'L_B^2}{K^2L_B^2})$ *and the real sequences* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\} \subset (0,1)$ *satisfy the following conditions:*

(D1) $\alpha_n + \beta_n + \gamma_n = 1$;

(D2) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$;

(D3) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$.

Suppose that $\{S_n\}$ *satisfies the PU-condition. Let the mapping* $S : C \longrightarrow C$ *be defined by* (1.13) *and suppose that* $F(S) = \cap_{n=1}^{\infty} F(S_n)$. *Then* $\{x_n\}$ *defined by* (2.31) *converges strongly to* $Q_\Omega u$, *where* Q_Ω *is the sunny nonexpansive retraction of* C *onto* Ω.

Proof. Take $x^* \in \Omega$. Then

$$x^* = Q_C[Q_C(x^* - \mu Bx^*) - \lambda AQ_C(x^* - \mu Bx^*)].$$

Putting $y^* = P_C(x^* - \mu Bx^*)$, we have $x^* = Q_C(y^* - \mu Ay^*)$. From nonexpansivity of Q_C, $I - \lambda A$ and $I - \mu B$, we have

$$\begin{aligned} \|z_n - x^*\| &= \|Q_C(y_n - \lambda Ay_n) - Q_C(y^* - \lambda Ay^*)\| \\ &\leq \|(I - \lambda A)y_n - (I - \lambda A)y^*\| \\ &\leq \|y_n - y^*\| \\ &= \|Q_C(x_n - \mu Bx_n) - Q_C(x^* - \mu Bx^*)\| \\ &\leq \|(I - \mu B)x_n - (I - \mu B)x^*\| \\ &\leq \|x_n - x^*\|. \qquad (2.4) \end{aligned}$$

For each $n \in \mathbb{N}$, set

$$t_n = \nu S_n x_n + (1 - \nu) z_n.$$

It follows from Lemma 1.8 that S_n is a nonexpansive mapping such that $F(S_n) =$

$F(T_n)$ for all $n \geq 1$ and hence $\cap_{n=1}^{\infty} F(S_n) = \cap_{n=1}^{\infty} F(T_n)$. Hence

$$\begin{aligned} \|t_n - x^*\| &= \|\nu S_n x_n + (1-\nu) z_n - x^*\| \\ &\leq \nu \|S_n x_n - x^*\| + (1-\nu)\|z_n - x^*\| \\ &\leq \|x_n - x^*\|. \end{aligned} \tag{2.5}$$

It follows that

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\alpha_n u + \beta_n x_n + \gamma_n t_n - x^*\| \\ &\leq \alpha_n \|u - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|t_n - x^*\| \\ &\leq \alpha_n \|u - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\ &\leq \max\{\|u - x^*\|, \|x_n - x^*\|\}. \end{aligned} \tag{2.6}$$

It follows from the simple induction that $\|x_n - x^*\| \leq \max\{\|u - x^*\|, \|x_1 - x^*\|\}$ for all $n \geq 1$. Therefore, $\{x_n\}$ is bounded. Hence $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$ are also bounded. We observe that

$$\begin{aligned} \|z_{n+1} - z_n\| &= \|Q_C(y_{n+1} - \lambda A y_{n+1}) - Q_C(y_n - \lambda A y_n)\| \\ &\leq \|(y_{n+1} - \lambda A y_{n+1}) - (y_n - \lambda A y_n)\| \\ &\leq \|y_{n+1} - y_n\| \\ &= \|Q_C(x_{n+1} - \mu B x_{n+1}) - Q_C(x_n - \mu B x_n)\| \\ &\leq \|(x_{n+1} - \mu B x_{n+1}) - (x_n - \mu B x_n)\| \\ &\leq \|x_{n+1} - x_n\|. \end{aligned} \tag{2.7}$$

It follows from (2.7) that

$$\begin{aligned} \|t_{n+1} - t_n\| &= \|\nu S_{n+1} x_{n+1} + (1-\nu) z_{n+1} - (\nu S_n x_n + (1-\nu) z_n)\| \\ &= \|\nu S_{n+1} x_{n+1} - \nu S_{n+1} x_n + (1-\nu) z_{n+1} + \nu S_{n+1} x_n - \nu S_n x_n - (1-\nu) z_n\| \\ &\leq \nu \|S_{n+1} x_{n+1} - S_{n+1} x_n\| + (1-\nu)\|z_{n+1} - z_n\| + \nu \|S_{n+1} x_n - S_n x_n\| \\ &\leq \nu \|x_{n+1} - x_n\| + (1-\nu)\|x_{n+1} - x_n\| + \nu \omega_n \\ &= \|x_{n+1} - x_n\| + \nu \omega_n, \end{aligned} \tag{2.8}$$

where $\omega_n = \|S_{n+1}x_n - S_nx_n\|$. Next, we will prove that $\lim_{n\longrightarrow\infty}\omega_n = 0$. Indeed, Since $\{x_n\}$ is bounded, there exists a bounded subset D of C such that $\{x_n\} \subset D$. We observe that

$$\frac{1}{2}\omega_n = \frac{1}{2}\|S_{n+1}x_n - S_nx_n\| \leq \frac{1}{2}\|S_{n+1}x_n - Sx_n\| + \frac{1}{2}\|Sx_n - S_nx_n\|.$$

Since $\{S_n\}$ satisfies PU-condition, then there exists an increasing, continuous and convex function $h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying (1.12). Then

$$\begin{aligned} h(\frac{1}{2}\omega_n) &\leq \frac{1}{2}h(\|S_{n+1}x_n - Sx_n\|) + \frac{1}{2}h(\|Sx_n - S_nx_n\|) \\ &= \frac{1}{2}\sup_{z\in D} h(\|S_{n+1}z - Sz\|) + \frac{1}{2}\sup_{z\in D} h(\|S_nz - Sz\|). \end{aligned} \tag{2.9}$$

Applying Lemma 1.12 to the above inequality, we obtain that

$$\lim_{n\longrightarrow\infty} h(\frac{1}{2}\omega_n) = 0.$$

The properties of the function h implies that

$$\lim_{n\longrightarrow\infty} \omega_n = 0. \tag{2.10}$$

Putting

$$x_{n+1} = (1-\beta_n)e_n + \beta_nx_n, \quad \forall n \geq 1, \tag{2.11}$$

one sees that

$$\begin{aligned} e_{n+1} - e_n &= \frac{\alpha_{n+1}u + \gamma_{n+1}t_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_nu + \gamma_nt_n}{1-\beta_n} \\ &= \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right)u + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(t_{n+1} - t_n) \\ &\quad + \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right)t_n. \end{aligned} \tag{2.12}$$

Combining (2.11) and (2.12), we have

$$\begin{aligned} \|e_{n+1} - e_n\| - \|x_{n+1} - x_n\| &\leq \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|\|u\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(\|x_{n+1} - x_n\| + \nu\omega_n) \\ &\quad + \left|\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right|\|t_n\| - \|x_{n+1} - x_n\| \\ &\leq \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|(\|u\| + \|t_n\|) + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\nu\omega_n. \end{aligned} \tag{2.13}$$

It follows from the conditions (D2), (D3) and (2.10) that

$$\limsup_{n\to\infty}(\|e_{n+1}-e_n\|-\|x_{n+1}-x_n\|)\leq 0.$$

Hence, from Lemma 1.6, it follows that

$$\lim_{n\to\infty}\|e_n-x_n\|=0. \tag{2.14}$$

From (2.11), it follows that

$$\|x_{n+1}-x_n\|=(1-\beta_n)\|e_n-x_n\|.$$

Using (2.14) and the condition (D3), one sees that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{2.15}$$

On the other hand, one has

$$x_{n+1}-x_n=\alpha_n(u-t_n)+(1-\gamma_n)(t_n-x_n).$$

It follows that

$$(1-\gamma_n)\|t_n-x_n\|\leq\|x_{n+1}-x_n\|+\alpha_n\|u-t_n\|.$$

From the conditions $0<\liminf_{n\longrightarrow\infty}\gamma_n\leq\limsup_{n\longrightarrow\infty}\gamma_n<1$, $\lim_{n\longrightarrow\infty}\alpha_n=0$ and (2.15), one sees that

$$\lim_{n\to\infty}\|t_n-x_n\|=0. \tag{2.16}$$

Next, we prove that

$$\limsup_{n\longrightarrow\infty}\langle u-Q_\Omega u, j(x_n-Q_\Omega u)\rangle\leq 0,$$

where Q_Ω is the sunny nonexpansive retraction from E onto Ω. Define a mapping W_n by

$$W_ny=\nu S_ny+(1-\nu)Q_C[(I-\lambda A)Q_C(I-\mu B)y], \forall y\in C, \forall n\geq 1.$$

In view of Lemma 2.1 and Lemma 1.7, we see that W_n is a nonexpansive mapping satisfying

$$F(W_n) = F(S_n) \cap F(Q_C[(I-\lambda A)Q_C(I-\mu B)y]) = F(S_n) \cap F(G). \qquad (2.17)$$

This implies that

$$\cap_{n=1}^{\infty} F(W_n) = \cap_{n=1}^{\infty}[F(S_n) \cap F(G)] = (\cap_{n=1}^{\infty} F(S_n)) \cap F(G) = F(S) \cap F(G).$$

From (2.16), it follows that

$$\lim_{n\to\infty} \|W_n x_n - x_n\| = 0. \qquad (2.18)$$

On the other hand, since $\{S_n\}$ satisfies the PU-condition, we have

$$\lim_{k,l\longrightarrow\infty} \sup_{y\in D} h(\|W_k y - W_l y\|) = \lim_{k,l\longrightarrow\infty} \sup_{y\in D} h(\nu\|S_k y - S_l y\|) = 0.$$

In virtue of Lemma 1.12, we obtain that $\{W_n y\}$ is a convergent sequence for all $y \in C$. So, let W be a mapping from C into itself defined by

$$Wy = \lim_{n\longrightarrow\infty} W_n y, \text{ for all } y \in C. \qquad (2.19)$$

Using Lemma 1.12, we have

$$\lim_{n\longrightarrow\infty} \sup_{y\in D} h(\|Wy - W_n y\|) = 0. \qquad (2.20)$$

Since

$$Wy = \lim_{n\longrightarrow\infty} W_n y = \lim_{n\longrightarrow\infty}[\nu S_n y + (1-\nu)Q_C[(I-\lambda A)Q_C(I-\mu B)y]] = \nu Sy + (1-\nu)Gy.$$

The nonexpansivity of S and G and Lemma 1.7 imply that W is nonexpansive such that

$$F(W) = F(S) \cap F(G) = \cap_{n=1}^{\infty} F(W_n) = \Omega.$$

Next, we observe that

$$\begin{aligned} h(\frac{1}{2}\|Wx_n - x_n\|) &\leq \frac{1}{2}h(\|Wx_n - W_n x_n\|) + \frac{1}{2}h(\|W_n x_n - x_n\|) \\ &\leq \sup_{y\in D} h(\|Wy - W_n y\|) + h(\|W_n x_n - x_n\|). \end{aligned}$$

Applying (2.20) and (2.18) to the last inequality, we have

$$\lim_{n\longrightarrow\infty} h(\frac{1}{2}\|Wx_n - x_n\|) = 0. \tag{2.21}$$

It follows from the properties of h that

$$\lim_{n\longrightarrow\infty} \|Wx_n - x_n\| = 0. \tag{2.22}$$

Let Q_Ω be the sunny nonexpansive retraction of C onto Ω. Now we show that

$$\limsup_{n\longrightarrow\infty} \langle u - Q_\Omega u, j(x_n - Q_\Omega u)\rangle \leq 0. \tag{2.23}$$

Let u_t be the fixed point of the contraction $z \mapsto tu + (1-t)Wz$, where $t \in (0,1)$. That is,

$$u_t = tu + (1-t)Wu_t.$$

It follows that

$$\|u_t - x_n\| = \|(1-t)(Wu_t - x_n) + t(u - x_n)\|.$$

On the other hand, we have

$$\begin{aligned}\|u_t - x_n\|^2 &\leq (1-t)^2\|Wu_t - x_n\|^2 + 2t\langle u - x_n, j(u_t - x_n)\rangle \\ &\leq (1 - 2t + t^2)\|u_t - x_n\|^2 + f_n(t) \\ &\quad + 2t\langle u - u_t, j(u_t - x_n)\rangle + 2t\|u_t - x_n\|^2,\end{aligned}$$

where

$$f_n(t) = (2\|u_t - x_n\| + \|x_n - Wx_n\|)\|x_n - Wx_n\| \to 0 \quad \text{as } n \to 0. \tag{2.24}$$

It follows that

$$\langle u_t - u, j(u_t - x_n)\rangle \leq \frac{t}{2}\|u_t - x_n\|^2 + \frac{1}{2t}f_n(t).$$

In view of (2.24), we arrive at

$$\limsup_{n\to\infty} \langle u_t - u, j(u_t - x_n)\rangle \leq \frac{t}{2}M, \tag{2.25}$$

where $M > 0$ is an appropriate constant such that $M \geq \|u_t - x_n\|^2$ for all $t \in (0,1)$ and $n \geq 1$. Letting $t \to 0$ in (2.25), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle u_t - u, j(u_t - x_n)\rangle \leq 0.$$

So, for any $\xi > 0$, there exists a positive number δ_1 with $t \in (0, \delta_1)$ such that

$$\limsup_{n\to\infty}\langle u_t - u, j(u_t - x_n)\rangle \leq \frac{\xi}{2}. \tag{2.26}$$

On the other hand, we see that $Q_{F(W)}u = \lim_{t\to 0} u_t$ and $F(W) = \Omega$. It follows that $u_t \to Q_\Omega u$ as $t \to 0$. This implies that there exists $\delta_2 > 0$, for $t \in (0, \delta_2)$, such that

$$\begin{aligned}
&|\langle u - Q_\Omega u, j(x_n - Q_\Omega u)\rangle - \langle u_t - u, j(u_t - x_n)\rangle| \\
&\leq |\langle u - Q_\Omega u, j(x_n - Q_\Omega u)\rangle - \langle u - Q_\Omega u, j(x_n - u_t)\rangle| \\
&\quad + |\langle u - Q_\Omega u, j(x_n - u_t)\rangle - \langle u_t - u, j(u_t - x_n)\rangle| \\
&\leq |\langle u - Q_\Omega u, j(x_n - Q_\Omega u) - j(x_n - u_t)\rangle| + |\langle u_t - Q_\Omega u, j(x_n - u_t)\rangle| \\
&\leq \|u - Q_\Omega u\|\|j(x_n - Q_\Omega u) - j(x_n - u_t)\| + \|u_t - Q_\Omega u\|\|x_n - u_t\| \\
&< \frac{\xi}{2}.
\end{aligned}$$

Choosing $\delta = \min\{\delta_1, \delta_2\}$, it follows that, for each $t \in (0, \delta)$,

$$\langle u - Q_\Omega u, j(x_n - Q_\Omega u)\rangle \leq \langle u_t - u, j(u_t - x_n)\rangle + \frac{\xi}{2},$$

which implies that

$$\limsup_{n\to\infty}\langle u - Q_\Omega u, j(x_n - Q_\Omega u)\rangle \leq \limsup_{n\to\infty}\langle u_t - u, j(u_t - x_n)\rangle + \frac{\xi}{2}.$$

It follows from (2.26) that

$$\limsup_{n\to\infty}\langle u - Q_\Omega u, j(x_n - Q_\Omega u)\rangle \leq \xi.$$

Since ξ is chosen arbitrarily, we have

$$\limsup_{n\to\infty}\langle u - Q_\Omega u, j(x_n - Q_\Omega u)\rangle \leq 0. \tag{2.27}$$

Finally, we have

$$\begin{aligned}
\|x_{n+1} - Q_\Omega u\|^2 &= \langle \alpha_n u + \beta_n x_n + \gamma_n t_n - Q_\Omega u, j(x_n - Q_\Omega u)\rangle \\
&= \alpha_n\langle u - Q_\Omega u, j(x_{n+1} - Q_\Omega u)\rangle + \beta_n\langle x_n - Q_\Omega u, j(x_{n+1} - Q_\Omega u)\rangle \\
&\quad + \gamma_n\langle z_n - Q_\Omega u, j(x_{n+1} - Q_\Omega u)\rangle \\
&\leq \frac{1}{2}\beta_n(\|x_n - Q_\Omega u\|^2 + \|x_{n+1} - Q_\Omega u\|^2) \\
&\quad + \alpha_n\langle u - Q_\Omega u, j(x_{n+1} - Q_\Omega u)\rangle \\
&\quad + \frac{1}{2}\gamma_n(\|t_n - Q_\Omega u\|^2 + \|x_{n+1} - Q_\Omega u\|^2) \\
&\leq \frac{1}{2}(1-\alpha_n)(\|x_n - Q_\Omega u\|^2 + \|x_{n+1} - Q_\Omega u\|^2) \\
&\quad + \alpha_n\langle u - Q_\Omega u, j(x_{n+1} - Q_\Omega u)\rangle, \qquad (2.28)
\end{aligned}$$

which implies that

$$\|x_{n+1} - Q_\Omega u\|^2 \leq (1-\alpha_n)\|x_n - Q_\Omega u\|^2 + 2\alpha_n\langle u - Q_\Omega u, j(x_{n+1} - Q_\Omega u)\rangle. \quad (2.29)$$

Applying Lemma 1.5 and (2.27) to the inequality (2.29), we conclude that $\{x_n\}$ converges strongly to $Q_\Omega u$. This completes the proof.

□

Remark 2.3. Theorem 2.4 mainly improves of [39, Theorem 3.1] in the following respects:

(a) From a uniformly convex Banach space to a strictly convex Banach space.

(b) From the class of inverse-strongly accretive mappings to the class of Lipchitzian and relaxed cocoercive mappings.

(c) We can remove the property weakly sequentially continuous on the duality mapping.

The following is an example of a sequence $\{T_n\}$ of nonexpansive mappings satisfying the PU-condition.

Example 2.4. *Let C be a closed convex subset of a smooth Banach space E. Suppose that $\{S_k\}$ is a sequence of nonexpansive mappings of E into inself with a common fixed point. For each $n \in \mathbb{N}$, define $T_n : C \longrightarrow C$ by*

$$T_n x = \sum_{k=1}^{n} \beta_n^k S_k x, \ \ \forall x \in E,$$

where $\{\beta_n^k\}$ is a family of nonnegative numbers with indices $n, k \in \mathbb{N}$ with $k \leq n$ such that

(i) $\sum_{k=1}^{n} \beta_n^k = 1$ *for all* $n \in \mathbb{N}$;

(ii) $\lim_{n \longrightarrow \infty} \beta_n^k > 0$ *for every* $k \in \mathbb{N}$;

(iii) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\beta_{n+1}^k - \beta_n^k| < \infty$.

It follows from [1, Theorem 4.1], we have

(1) Each T_n is a nonexpansive mapping.

(2) For any bounded subset D of E,

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty.$$

Using [27, Remark 3.2], we obtain that

$$\lim_{k,l \longrightarrow \infty} \sup_{z \in D} h(\|T_k z - T_l z\|) = 0$$

for any continuous increasing and convex function $h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $h(0) = 0$. Hence $\{T_n\}$ satisfies the PU-condition.

CHAPTER 3

CONCLUSION

1. Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E with the smooth constant K. Let Q_C be the sunny nonexpansive retraction from E onto C. Let $A : C \longrightarrow E$ be a L_A-Lipchitzian and relaxed (c,d)-cocoercive mapping and $B : C \longrightarrow H$ a L_B-Lipchitzian and relaxed (c',d')-cocoercive mapping, where $\lambda \leq \frac{d-cL_A^2}{K^2L_A^2}$ and $\mu \leq \frac{d'-c'L_B^2}{K^2L_B^2}$. Define the mapping G by

$$G(x) = Q_C[Q_C(x-\mu Bx) - \lambda A Q_C(x-\mu Bx)], \forall x \in C. \qquad (2.30)$$

Then G is nonexpansive.

2. Let C be a nonempty closed convex subset of a real smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of the problem (1.1) if and only if x^* is a fixed point the mapping $G : C \longrightarrow C$ defined by (2.30), where $y^* = Q_C(x^* - \mu Bx^*)$, λ, μ are positive constants and $A, B : C \longrightarrow H$ are possibly nonlinear mappings.

3. Let C be a nonempty closed convex subset of a strictly convex and 2-uniformly smooth Banach space E with the smooth constant K. Let Q_C be the sunny nonexpansive retraction from E onto C. Let $A : C \longrightarrow E$ be a L_A-Lipchitzian and relaxed (c,d)-cocoercive mapping and $B : C \longrightarrow H$ a L_B-Lipchitzian and relaxed (c',d')-cocoercive mapping. Let $\{T_n : C \longrightarrow C\}_{i=1}^{\infty}$ be a countable family of uniformly ε-strict pseudo-contractions such that

$$\Omega := \cap_{n=1}^{\infty} F(T_n) \cap GVI(A,B,C) \neq \emptyset.$$

Define a mapping $S_n : C \longrightarrow C$ by

$$S_n x = (1 - \frac{\varepsilon}{K^2})x + \frac{\varepsilon}{K^2} T_n x \text{ for all } x \in C \text{ and } n \geq 1.$$

Let the sequence $\{x_n\}$ be generated iteratively by

$$\begin{cases} x_1 = u \in C, \text{ chosen arbitrary}, \\ y_n = Q_C(x_n - \mu B x_n), \\ z_n = Q_C(y_n - \lambda A y_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\nu S_n x_n + (1-\nu) z_n], \ n \geq 1, \end{cases} \tag{2.31}$$

where $\nu \in (0,1)$, $\lambda \in (0, \frac{d - cL_A^2}{K^2 L_A^2})$ and $\mu \in (0, \frac{d' - c' L_B^2}{K^2 L_B^2})$ and the real sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\} \subset (0,1)$ satisfy the following conditions:

(D1) $\alpha_n + \beta_n + \gamma_n = 1$;

(D2) $\lim_{n \longrightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$;

(D3) $0 < \liminf_{n \longrightarrow \infty} \beta_n \leq \limsup_{n \longrightarrow \infty} \beta_n < 1$.

Suppose that $\{S_n\}$ satisfies the PU-condition. Let the mapping $S : C \longrightarrow C$ be defined by (1.13) and suppose that $F(S) = \cap_{n=1}^{\infty} F(S_n)$. Then $\{x_n\}$ defined by (2.31) converges strongly to $Q_\Omega u$, where Q_Ω is the sunny nonexpansive retraction of C onto Ω.

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APPENDIX

68--JOCAAA--ACCEPT

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A NEW SYSTEM OF GENERAL VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS FOR A COUNTABLE FAMILY OF STRICT PSEUDO-CONTRACTIONS IN BANACH SPACES

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A NEW SYSTEM OF GENERAL VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS FOR A COUNTABLE FAMILY OF STRICT PSEUDO-CONTRACTIONS IN BANACH SPACES**

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Abstract. The purpose of this paper is to introduce a new system of general variational inequalities in Banach spaces and an iterative method for finding a common element of the set of solutions of the system of general variational inequalities for two relaxed cocoercive mappings and the set of common fixed points for a countable family of strict pseudo-contractions. The strong convergence theorems of the proposed iterative method are obtained without the control condition weakly sequentially continuous on the duality mapping in Banach spaces.

Keywords: General variational inequality; Relaxed cocoercive mapping, Strict pseudo-contraction, Weakly sequentially continuous, Fixed point, and Banach space.

AMS Subject Classification: 47H05, 47H09, 47J25, 65J15.

1. Introduction

Variational inequalities are being used as a mathematical programming tool in modeling a wide class of obstacle, unilateral, free, moving, equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. Using the projection technique one can establish the equivalence between the variational inequalities and the fixed point problem. This equivalence has played an important role in developing several numerical techniques for solving variational inequalities and the related optimization problem. For the physical formulation, applications, numerical methods and other aspects of the variational inequalities, see [11, 21, 22, 23, 24, 25, 26] and the references therein. Related to the variational inequalities is the problem of finding the common fixed points of the strict pseudo-contractions, which is the subject of current interest in functional analysis. It is natural to unify these two problems and find the common elements of the set of the solution of variational inequality and the set of the common fixed points of the strict pseudo-contractions.

Let E be a real Banach space and $U_E = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be uniformly convex if, for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U_E$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \left\|\frac{x+y}{2}\right\| \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space E is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in U_E$. It is also said to be uniformly smooth if the limit is attained uniformly for all $x, y \in U_E$. The norm of E is said to be Fréchet differentiable if, for any $x \in U_E$, the above limit is attained uniformly for all $y \in U_E$. The modulus of smoothness of E is defined by

$$\rho(\tau) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau\},$$

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where $\rho : [0, \infty) \to [0, \infty)$ is a function. It is known that E is uniformly smooth if and only if $\lim_{\tau \to 0} \frac{\rho(\tau)}{\tau} = 0$. Let q be a fixed real number with $1 < q \leq 2$. A Banach space E is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$.

From [5], we know the following property:
Let q be a real number with $1 < q \leq 2$ and let E be a Banach space. Then E is q-uniformly smooth if and only if there exists a constant $K \geq 1$ such that

$$\|x + y\|^q + \|x - y\|^q \leq 2(\|x\|^q + \|Ky\|^q), \quad \forall x, y \in E.$$

The best constant K in the above inequality is called the q-uniformly smoothness constant of E (see [5] for more details).

Let E be a real Banach space and E^* the dual space of E. Let $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^*. For $q > 1$, the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}, \quad \forall x \in E.$$

In particular, $J = J_2$ is called the normalized duality mapping. It is known that $J_q(x) = \|x\|^{q-2}J(x)$ for all $x \in E$. If E is a Hilbert space, then $J = I$ (: the identity mapping). Note that

(1) E is a uniformly smooth Banach space if and only if J is single-valued and uniformly continuous on any bounded subset of E.
(2) All Hilbert spaces, L^p (or l^p) spaces ($p \geq 2$) and the Sobolev spaces W^p_m ($p \geq 2$) are 2-uniformly smooth, while L^p (or l^p) and W^p_m spaces ($1 < p \leq 2$) are p-uniformly smooth.
(3) Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. More precisely, L^p is $\min\{p, 2\}$-uniformly smooth for any $p > 1$.

Further, we have the following properties of the generalized duality mapping J_q:

(i) $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \in E$ with $x \neq 0$,
(ii) $J_q(tx) = t^{q-1}J_q(x)$ for all $x \in E$ and $t \in [0, \infty)$,
(iii) $J_q(-x) = -J_q(x)$ for all $x \in E$.

It is known that if E is smooth, then J is single-valued, which is denoted by j . Recall that the duality mapping j is said to be weakly sequentially continuous if for each sequence $\{x_n\} \subset E$ with $x_n \longrightarrow x$ weakly, we have $j(x_n) \longrightarrow j(x)$ weakly-$*$. We know that if E admits a weakly sequentially continuous duality mapping, then E is smooth. For the details, see [13].

Let C be a nonempty closed convex subset of a smooth Banach space E. Recall that the following definitions of a nonlinear mapping $A : C \longrightarrow E$.

Definition 1.1. Let $A : C \longrightarrow E$ be a mapping.

(i) A is said to be *accretive* if
$$\langle Ax - Ay, j(x - y) \rangle \geq 0$$
for all $x, y \in C$.
(ii) A is said to be α-*strongly accretive* if there exists a constant $\alpha > 0$ such that
$$\langle Ax - Ay, j(x - y) \rangle \geq \alpha\|x - y\|^2$$
for all $x, y \in C$.
(iii) A is said to be α-*inverse-strongly accretive* or α-*cocoercive* if there exists a constant $\alpha > 0$ such that
$$\langle Ax - Ay, j(x - y) \rangle \geq \alpha\|Ax - Ay\|^2$$
for all $x, y \in C$.
(iv) A is said to be α-*relaxed cocoercive* if there exists a constant $\alpha > 0$ such that
$$\langle Ax - Ay, j(x - y) \rangle \geq -\alpha\|Ax - Ay\|^2$$
for all $x, y \in C$.
(v) A is said to be (α, β)-*relaxed cocoercive* if there exists positive constants α and β such that
$$\langle Ax - Ay, j(x - y) \rangle \geq (-\alpha)\|Ax - Ay\|^2 + \beta\|x - y\|^2$$
for all $x, y \in C$.

Remark 1.2. (1). Every α-strongly accretive mapping is an accretive mapping.

(2). Every α-strongly accretive mapping is an (β, α)-relaxed cocoercive mapping for any positive constant β but the converse is not true in general. Then the class of relaxed cocoercive operators is more general than the class of strongly accretive operators.

(3). Evidently, the definition of the inverse-strongly accretive operator is based on that of the inverse-strongly monotone operator in real Hilbert spaces (see, for example, [6]).

(4). The notion of the cocoercivity is applied in several directions, especially to solving variational inequality problems using the auxiliary problem principle and projection methods [35]. Several classes of relaxed cocoercive variational inequalities have been studied in [33, 34].

Let C be a nonempty closed and convex subset of a smooth Banach space E. We introduce the following system of general variational inequalities involving two different nonlinear mappings $A, B : C \longrightarrow E$:

Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, j(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \mu B x^* + y^* - x^*, j(x - y^*)\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.1}$$

where λ and μ are two positive real numbers.

As special cases of the problem (1.1), we have the following:

(i) If $A = B$, then the problem (1.1) is reduced to the following:
Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, j(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \mu A x^* + y^* - x^*, j(x - y^*)\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.2}$$

where λ and μ are two positive real numbers. This system of variational inequalities was considered and studied by Noor [22, 21] using the auxiliary principle technique.

(ii) If $\lambda = \mu = 1$, then the problem (1.1) is reduced to the following:
Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle A y^* + x^* - y^*, j(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle B x^* + y^* - x^*, j(x - y^*)\rangle \geq 0, & \forall x \in C. \end{cases} \tag{1.3}$$

This system of variational inequalities was considered and studied by Yao, Noor, Noor, Liou, and Yaqoob [39].

(iii) In real Hilbert spaces, the problem (1.1) is reduced to the following:
Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle \mu B x^* + y^* - x^*, x - y^*\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.4}$$

where λ and μ are two positive real numbers. The system (1.4) is introduced and studied by Ceng, Wang and Yao [10]. To illustrate the applications of this system, Zhu and Marcotte [41] considered the problem of finding $x^* \in C$ such that

$$\langle A(x^*), x - x^*\rangle \geq 0, \forall x \in E = C \cap \{x \in H : B(x) \leq 0\}, \tag{1.5}$$

where A is strongly monotone on E and $B(x) = \{f_1(x), f_2(x), \ldots, f_m(x)\}$ is a constraint mapping explicitly defined by the convex, Lipschitz continuous and continuously differentiable functions $f_i, i = 1, 2, \ldots, m$. Assume that there exists $x_0 \in C$ such that $f_i(x_0) < 0, i = 1, 2, \ldots, m$ (Slaters constraint qualification). Then the variational inequality (1.5) is equivalent to the KuhnTucker-like system

$$\begin{cases} \langle A(x^*) + (\nabla B(x^*))' y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle \mu - B(x^*), y - y^*\rangle \geq 0, & \forall y \geq 0, \end{cases} \tag{1.6}$$

which is exactly the system of variational inequalities (1.4).

(iv) If $A = B$, $\lambda = \mu = 1$, and $x^* = y^*$ then the problem (1.1) is reduced to the following:
Find $x^* \in C$ such that

$$\langle Ax^*, j(x - x^*)\rangle \geq 0, \quad \forall x \in C. \tag{1.7}$$

The problem (1.7) is very interesting as it is connected with the fixed point problem for nonlinear mappings, the problem of finding a zero point of an accretive operator and so on. For the problem of finding a zero point of an accretive operator by the proximal point algorithm, see Kamimura and Takahashi [16] and the references therein. In [2], Aoyama, Iiduka and Takahashi [2] first considered the such problem in Banach spaces. In order to find a solution of problem (1.7), they proved the following theorem which is generalized simultaneously theorems of [6] and [12].

Theorem AIT. *Let E be a uniformly convex and 2-uniformly smooth Banach space and C a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, $\alpha > 0$ and A an α-inverse strongly-accretive operator of C into E with $S(C, A) \neq \emptyset$, where*

$$S(C, A) = \{x^* \in C : \langle Ax^*, J(x - x^*)\rangle \geq 0, \ x \in C\}.$$

If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen such that $\lambda_n \in [a, \frac{\alpha}{K^2}]$ for some $a > 0$ and $\alpha_n \in [b, c]$ for some b, c with $0 < b < c < 1$, then the sequence $\{x_n\}$ defined by the following manners:

$$\begin{cases} x_1 = x \in C, \\ y_n = Q_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) y_n, \ n \geq 1, \end{cases} \tag{1.8}$$

converges weakly to some element z of $S(C, A)$, where K is the 2-uniformly smoothness constant of E.

On the other hand, in [14], Hao obtained a strong convergence theorem for approximating the solutions of the generalized variational inequality problem (1.7) by using the following iterative algorithm:

$$\begin{cases} x_1 = u \in C, \\ y_n = \beta_n x_n + (1 - \beta_n) Q_C(I - \lambda_n A) x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n, \ n \geq 1. \end{cases} \tag{1.9}$$

where Q_C is a sunny nonexpansive retraction from E onto C, $\{\alpha_n\}, \{\lambda_n\}$ and $\{\beta_n\}$ are appropriate real sequences in $[0, 1]$. Very recently, motivated by Aoyama, Iiduka and Takahashi [2] and Hao [14], for solving the problem (1.3), Yao, Noor, Noor, Liou, and Yaqoob [39] established the equivalence between the system of variational inequalities (1.3) and a fixed point problem involving the nonexpansive mapping. This alternative equivalent formulation is used to suggest and analyze a modified extragradient method. Using the demi-closedness principle for nonexpansive mappings, they obtained the following strong convergence theorem of the proposed iterative method under some suitable conditions.

Theorem YNNLY-A. *Let C be a nonempty closed convex subset of a real smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. Let $A, B : C \longrightarrow E$ be α-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $G : C \longrightarrow C$ be a mapping defined by*

$$G(x) = Q_C[Q_C(x - Bx) - AQ_C(x - Bx)], \forall x \in C. \tag{1.10}$$

Then
(i).[39, Lemma 3.2] *If E is real 2-uniformly smooth Banach space, $\alpha \geq K^2$ and $\beta \geq K^2$, then G is nonexpansive.*
(ii). [39, Lemma 3.3] *For given $x^*, y^* \in C$, (x^*, y^*) is a solution of the problem* (1.3) *if and only if x^* is a fixed point the mapping $G : C \longrightarrow C$ defined by* (1.10), *where $y^* = Q_C(x^* - Bx^*)$.*

Theorem YNNLY-B [39, Theorem 3.1]. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E which admits a weakly sequentially continuous duality mapping and the smooth constant K. Let Q_C be the sunny nonexpansive retraction from E onto C. Let $A, B : C \longrightarrow E$ be α-inverse-strongly accretive with $\alpha \geq K^2$ and β-inverse-strongly accretive with $\beta \geq K^2$, respectively. Suppose the set of fixed point Ω of the mapping $G : C \longrightarrow C$ defined by* (1.10) *is nonempty. For fixed* $u \in C$, *let the sequence* $\{x_n\}$ *be generated iteratively by*

$$\begin{cases} x_1 \in C, \text{ chosen arbitrary,} \\ y_n = Q_C(x_n - Bx_n), \\ z_n = Q_C(y_n - Ay_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n z_n, \ n \geq 1, \end{cases} \tag{1.11}$$

where the real sequences $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\} \subset (0,1)$ *satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1, \forall n \geq 1;$
(ii) $\lim_{n \longrightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty;$
(iii) $0 < \liminf_{n \longrightarrow \infty} \beta_n \leq \limsup_{n \longrightarrow \infty} \beta_n \leq 1.$

Then $\{x_n\}$ *defined by* (1.11) *converges strongly to* $Q_\Omega u$, *where* Q_Ω *is the sunny nonexpansive retraction of* C *onto* Ω.

All of the above bring us the following conjectures?.

Question

(i) Could we weaken the condition uniformly convex on Banach spaces?.

(ii) Could we remove the control condition "weakly sequentially continuous" on the duality mapping in Theorem YNNLY-B?.

(iii) Could we construct an iterative algorithm to approximate a common element of the set of solutions of general variational inequalities (1.1) for two relaxed cocoercive mappings and the set of common fixed points of a countable family of strict pseudo-contractions in Banach spaces?.

In this paper, motivated by Aoyama, Iiduka and Takahashi [2], Hao [14], and Yao, Noor, Noor, Liou, and Yaqoob [39], we introduce a new system of general variational inequalities in Banach spaces. We establish the equivalence between the system of variational inequalities (1.1) for two relaxed cocoercive mappings and fixed point problems involving a nonexpansive mapping. This alternative equivalent formulation is used to suggest and analyze a new iterative approximation method for solving the system of general variational inequalities for two relaxed cocoercive mappings and fixed point problems for a countable family of strict pseudo-contractions. The strong convergence theorems of the proposed iterative method are obtained without the control condition "weakly sequentially continuous" of the duality mapping on Banach spaces. The results presented in the paper improve some recent results of Aoyama, Iiduka and Takahashi [2], Hao [14], and Yao, Noor, Noor, Liou, and Yaqoob [39].

2. Preliminaries

Now we collect some useful lemmas for proving the convergence results.

Lemma 2.1. [1, Lemma 2.3] *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying the property:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n c_n + b_n, \forall n \geq 0,$$

where $\{\alpha_n\}, \{b_n\}, \{c_n\}$ *satisfy the restrictions:*

$$\sum_{n=0}^{\infty} \alpha_n = \infty; \sum_{n=0}^{\infty} b_n < \infty; \text{ and } \limsup_{n \to \infty} c_n \leq 0.$$

Then $\lim_{n \to \infty} a_n = 0$.

Lemma 2.2. ([30]) *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* E *and* $\{\beta_n\}$ *a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose that* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all* $n \geq 0$ *and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

A mapping $T : C \longrightarrow C$ is said to be ε-*strictly pseudo-contractive*, if there exists a constant $\varepsilon \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + \varepsilon\|(I-T)x - (I-T)y\|^2, \ \forall x, y \in C.$$

Note that the class of ε-strictly pseudo-contractive mappings strictly includes the class of nonexpansive mappings which are mappings T on C such that

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$. That is, T is nonexpansive if and only if T is 0-strict pseudo-contractive. We denote by $F(T) := \{x \in C : Tx = x\}$ the set of fixed points of T.

Definition 2.3. A countable family of mapping $\{T_n : C \longrightarrow C\}_{i=1}^{\infty}$ is called a *family of uniformly ε-strict pseudo-contractions*, if there exists a constant $\varepsilon \in [0, 1)$ such that

$$\|T_n x - T_n y\|^2 \leq \|x - y\|^2 + \varepsilon\|(I - T_n)x - (I - T_n)y\|^2, \ \forall x, y \in C, \ \ \forall n \geq 1.$$

Lemma 2.4. ([9]) *Let E be a strictly convex Banach space. Let T_1 and T_2 be two nonexpansive mappings from E into itself with a common fixed point. Define a mapping S by*

$$Sx = \lambda T_1 x + (1 - \lambda)T_2 x, \quad \forall x \in E,$$

where λ is a constant in $(0, 1)$. Then S is nonexpansive and $F(S) = F(T_1) \cap F(T_2)$.

Lemma 2.5. ([40]) *Let E be a real 2-uniformly smooth Banach space and $T : E \to E$ a ε-strict pseudo-contraction. Then $S := (1 - \varepsilon/K^2)I + \varepsilon/K^2 T$ is nonexpansive and $F(T) = F(S)$.*

Lemma 2.6. ([37]) *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, jx\rangle + 2\|Ky\|^2, \quad \forall x, y \in E.$$

Let D be a nonempty subset of C. A mapping $Q : C \longrightarrow D$ is said to be sunny if

$$Q(Qx + t(x - Qx)) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t = 0$. A mapping $Q : C \longrightarrow D$ is called a retraction if $Qx = x$ for all $x \in D$. Furthermore, Q is a sunny nonexpansive retraction from C onto D if Q is a retraction from C onto D which is also sunny and nonexpansive. A subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D. The following lemma concerns the sunny nonex- pansive retraction.

Lemma 2.7. ([28, 8]) *Let C be a closed convex subset of a smooth Banach space E. Let D be a nonempty subset of C and $Q : C \longrightarrow D$ be a retraction. Then Q is sunny and nonexpansive if and only if*

$$\langle u - Qu, j(y - Qu)\rangle \leq 0$$

for all $u \in C$ and $y \in D$.

Definition 2.8. Let $\{S_n\}$ be a family of mappings from a subset C of Banach space E into E with $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$. We say that $\{S_n\}$ satisfies the PU-condition if for each bounded subset D of C, there exists a continuous increasing and convex function $h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$h(0) = 0 \text{ and } \lim_{k,l \longrightarrow \infty} \sup_{z \in D} h(\|S_k z - S_l z\|) = 0. \tag{2.1}$$

Remark 2.9. The example of a sequence of mappings satisfying PU-condition is supported by Example 3.5.

Lemma 2.10. [27, Lemma 3.1] *Suppose that there exists a continuous increasing function $h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying* (2.1). *Then*
(i). *For each $x \in C$, $\{S_n x\}$ is a convergent sequence in C.*
(ii). *Let the mapping $S : C \longrightarrow C$ be defined by*

$$Sx = \lim_{n \longrightarrow \infty} S_n x, \text{ for all } x \in C. \tag{2.2}$$

Then $\lim_{n \longrightarrow \infty} \sup_{z \in D} h(\|Sz - S_n z\|) = 0$ for each bounded subset D of C.

Remark 2.11. If $\{S_n\}$ satisfies the PU-condition, then the facts (i) and (ii) in Lemma 2.10 hold.

Lemma 2.12. [15, Lemma 3.2] *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E with the smooth constant K. Let $A : C \longrightarrow E$ be a L_A-Lipchitzian and relaxed (c,d)-cocoercive mapping. Then*

$$\|(I-\lambda A)x-(I-\lambda A)y\|^2 \leq (1+2\lambda cL_A^2-2\lambda d+2\lambda^2K^2L_A^2)\|x-y\|^2. \tag{2.3}$$

If $\lambda \leq \frac{d-cL_A^2}{K^2L_A^2}$, then $I-\lambda A$ is nonexpansive.

3. Main Results

In this section, we state and prove our main results.

Lemma 3.1. *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E with the smooth constant K. Let Q_C be the sunny nonexpansive retraction from E onto C. Let $A : C \longrightarrow E$ be a L_A-Lipchitzian and relaxed (c,d)-cocoercive mapping and $B : C \longrightarrow H$ a L_B-Lipchitzian and relaxed (c',d')-cocoercive mapping, where $\lambda \leq \frac{d-cL_A^2}{K^2L_A^2}$ and $\mu \leq \frac{d'-c'L_B^2}{K^2L_B^2}$. Define the mapping G by*

$$G(x) = Q_C[Q_C(x-\mu Bx)-\lambda AQ_C(x-\mu Bx)], \forall x \in C. \tag{3.1}$$

Then G is nonexpansive.

Proof. From Lemma 2.12, we deduce that $I-\lambda A$, $I-\mu B$ and Q_C are nonexpansive mappings. Then, for any $x,y \in C$, we obtain

$$\begin{aligned}\|G(x)-G(y)\|^2 &= \|Q_C[Q_C(x-\mu Bx)-\lambda AQ_C(x-\mu Bx)]-Q_C[Q_C(y-\mu By)-\lambda AQ_C(y-\mu By)]\|^2\\ &\leq \|(I-\lambda A)Q_C(I-\mu B)x-(I-\lambda A)Q_C(I-\mu B)y\|^2\\ &\leq \|x-y\|^2.\end{aligned}$$

Hence G is nonexpansive on C. □

Lemma 3.2. *Let C be a nonempty closed convex subset of a real smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of the problem (1.1) if and only if x^* is a fixed point the mapping $G : C \longrightarrow C$ defined by (3.1), where $y^* = Q_C(x^*-\mu Bx^*)$, λ, μ are positive constants and $A, B : C \longrightarrow H$ are possibly nonlinear mappings.*

Proof. We can rewrite the problem (1.1) as

$$\begin{cases}\langle x^*-(y^*-\lambda Ay^*), j(x-x^*)\rangle \geq 0, & \forall x \in C,\\ \langle y^*-(x^*-\mu Bx^*), j(x-y^*)\rangle \geq 0, & \forall x \in C.\end{cases} \tag{3.2}$$

Applying Lemma 2.7, we can deduce that (3.2) is equivalent to

$$x^* = Q_C(y^*-\lambda Ay^*) \text{ and } y^* = Q_C(x^*-\mu Bx^*),$$

which is equivalent to

$$x^* = Q_C(Q_C(x^*-\mu Bx^*)-\lambda AQ_C(x^*-\mu Bx^*)).$$

Hence x^* is a fixed point the mapping G defined by (3.1). This completes the proof. □

Throughout this paper, the set of fixed points of the mapping G is denoted by $GVI(A,B,C)$.

Theorem 3.3. *Let C be a nonempty closed convex subset of a strictly convex and 2-uniformly smooth Banach space E with the smooth constant K. Let Q_C be the sunny nonexpansive retraction from E onto C. Let $A : C \longrightarrow E$ be a L_A-Lipchitzian and relaxed (c,d)-cocoercive mapping and $B : C \longrightarrow H$ a L_B-Lipchitzian and relaxed (c',d')-cocoercive mapping. Let $\{T_n : C \longrightarrow C\}_{i=1}^{\infty}$ be a countable family of uniformly ε-strict pseudo-contractions such that $\Omega := \cap_{n=1}^{\infty} F(T_n) \cap GVI(A,B,C) \neq \emptyset$. Define a mapping $S_n : C \longrightarrow C$ by*

$$S_n x = (1-\frac{\varepsilon}{K^2})x + \frac{\varepsilon}{K^2}T_n x \text{ for all } x \in C \text{ and } n \geq 1.$$

Let the sequence $\{x_n\}$ be generated iteratively by

$$\begin{cases} x_1 = u \in C, \text{ chosen arbitrary},\\ y_n = Q_C(x_n-\mu Bx_n),\\ z_n = Q_C(y_n-\lambda Ay_n),\\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\nu S_n x_n + (1-\nu)z_n], \; n \geq 1,\end{cases} \tag{3.3}$$

where $\nu \in (0,1)$, $\lambda \in (0, \frac{d-cL_A^2}{K^2L_A^2})$ *and* $\mu \in (0, \frac{d'-c'L_B^2}{K^2L_B^2})$ *and the real sequences* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\} \subset (0,1)$ *satisfy the following conditions:*

(D1) $\alpha_n + \beta_n + \gamma_n = 1$;
(D2) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$;
(D3) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$.

Suppose that $\{S_n\}$ *satisfies the PU-condition. Let the mapping* $S : C \longrightarrow C$ *be defined by* (2.2) *and suppose that* $F(S) = \cap_{n=1}^{\infty} F(S_n)$. *Then* $\{x_n\}$ *defined by* (3.3) *converges strongly to* $Q_\Omega u$, *where* Q_Ω *is the sunny nonexpansive retraction of* C *onto* Ω.

Proof. Take $x^* \in \Omega$. Then

$$x^* = Q_C[Q_C(x^* - \mu Bx^*) - \lambda AQ_C(x^* - \mu Bx^*)].$$

Putting $y^* = P_C(x^* - \mu Bx^*)$, we have $x^* = Q_C(y^* - \mu Ay^*)$. From nonexpansivity of Q_C, $I - \lambda A$ and $I - \mu B$, we have

$$\begin{aligned} \|z_n - x^*\| &= \|Q_C(y_n - \lambda Ay_n) - Q_C(y^* - \lambda Ay^*)\| \\ &\leq \|(I - \lambda A)y_n - (I - \lambda A)y^*\| \\ &\leq \|y_n - y^*\| \\ &= \|Q_C(x_n - \mu Bx_n) - Q_C(x^* - \mu Bx^*)\| \\ &\leq \|(I - \mu B)x_n - (I - \mu B)x^*\| \\ &\leq \|x_n - x^*\|. \end{aligned} \tag{3.4}$$

For each $n \in \mathbb{N}$, set

$$t_n = \nu S_n x_n + (1-\nu)z_n.$$

It follows from Lemma 2.5 that S_n is a nonexpansive mapping such that $F(S_n) = F(T_n)$ for all $n \geq 1$ and hence $\cap_{n=1}^{\infty} F(S_n) = \cap_{n=1}^{\infty} F(T_n)$. Hence

$$\begin{aligned} \|t_n - x^*\| &= \|\nu S_n x_n + (1-\nu)z_n - x^*\| \\ &\leq \nu\|S_n x_n - x^*\| + (1-\nu)\|z_n - x^*\| \\ &\leq \|x_n - x^*\|. \end{aligned} \tag{3.5}$$

It follows that

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\alpha_n u + \beta_n x_n + \gamma_n t_n - x^*\| \\ &\leq \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|t_n - x^*\| \\ &\leq \alpha_n\|u - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\ &\leq \max\{\|u - x^*\|, \|x_n - x^*\|\}. \end{aligned} \tag{3.6}$$

It follows from the simple induction that $\|x_n - x^*\| \leq \max\{\|u - x^*\|, \|x_1 - x^*\|\}$ for all $n \geq 1$. Therefore, $\{x_n\}$ is bounded. Hence $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$ are also bounded. We observe that

$$\begin{aligned} \|z_{n+1} - z_n\| &= \|Q_C(y_{n+1} - \lambda Ay_{n+1}) - Q_C(y_n - \lambda Ay_n)\| \\ &\leq \|(y_{n+1} - \lambda Ay_{n+1}) - (y_n - \lambda Ay_n)\| \\ &\leq \|y_{n+1} - y_n\| \\ &= \|Q_C(x_{n+1} - \mu Bx_{n+1}) - Q_C(x_n - \mu Bx_n)\| \\ &\leq \|(x_{n+1} - \mu Bx_{n+1}) - (x_n - \mu Bx_n)\| \\ &\leq \|x_{n+1} - x_n\|. \end{aligned} \tag{3.7}$$

It follows from (3.7) that

$$\begin{aligned} \|t_{n+1} - t_n\| &= \|\nu S_{n+1}x_{n+1} + (1-\nu)z_{n+1} - (\nu S_n x_n + (1-\nu)z_n)\| \\ &= \|\nu S_{n+1}x_{n+1} - \nu S_{n+1}x_n + (1-\nu)z_{n+1} + \nu S_{n+1}x_n - \nu S_n x_n - (1-\nu)z_n\| \\ &\leq \nu\|S_{n+1}x_{n+1} - S_{n+1}x_n\| + (1-\nu)\|z_{n+1} - z_n\| + \nu\|S_{n+1}x_n - S_n x_n\| \\ &\leq \nu\|x_{n+1} - x_n\| + (1-\nu)\|x_{n+1} - x_n\| + \nu\omega_n \\ &= \|x_{n+1} - x_n\| + \nu\omega_n, \end{aligned} \tag{3.8}$$

where $\omega_n = \|S_{n+1}x_n - S_nx_n\|$. Next, we will prove that $\lim_{n\longrightarrow\infty}\omega_n = 0$. Indeed, Since $\{x_n\}$ is bounded, there exists a bounded subset D of C such that $\{x_n\} \subset D$. We observe that

$$\frac{1}{2}\omega_n = \frac{1}{2}\|S_{n+1}x_n - S_nx_n\| \leq \frac{1}{2}\|S_{n+1}x_n - Sx_n\| + \frac{1}{2}\|Sx_n - S_nx_n\|.$$

Since $\{S_n\}$ satisfies PU-condition, then there exists an increasing, continuous and convex function $h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying (2.1). Then

$$\begin{aligned} h(\frac{1}{2}\omega_n) &\leq \frac{1}{2}h(\|S_{n+1}x_n - Sx_n\|) + \frac{1}{2}h(\|Sx_n - S_nx_n\|) \\ &= \frac{1}{2}\sup_{z\in D} h(\|S_{n+1}z - Sz\|) + \frac{1}{2}\sup_{z\in D} h(\|S_nz - Sz\|). \end{aligned} \tag{3.9}$$

Applying Lemma 2.10 to the above inequality, we obtain that

$$\lim_{n\longrightarrow\infty} h(\frac{1}{2}\omega_n) = 0.$$

The properties of the function h implies that

$$\lim_{n\longrightarrow\infty} \omega_n = 0. \tag{3.10}$$

Putting

$$x_{n+1} = (1-\beta_n)e_n + \beta_n x_n, \quad \forall n \geq 1, \tag{3.11}$$

one sees that

$$\begin{aligned} e_{n+1} - e_n &= \frac{\alpha_{n+1}u + \gamma_{n+1}t_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + \gamma_n t_n}{1-\beta_n} \\ &= \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right)u + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(t_{n+1} - t_n) \\ &\quad + \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right)t_n. \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12), we have

$$\begin{aligned} \|e_{n+1} - e_n\| - \|x_{n+1} - x_n\| &\leq \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|\|u\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(\|x_{n+1} - x_n\| + \nu\omega_n) \\ &\quad + \left|\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right|\|t_n\| - \|x_{n+1} - x_n\| \\ &\leq \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|(\|u\| + \|t_n\|) + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\nu\omega_n. \end{aligned} \tag{3.13}$$

It follows from the conditions (D2), (D3) and (3.10) that

$$\limsup_{n\to\infty}(\|e_{n+1} - e_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence, from Lemma 2.2, it follows that

$$\lim_{n\to\infty} \|e_n - x_n\| = 0. \tag{3.14}$$

From (3.11), it follows that

$$\|x_{n+1} - x_n\| = (1-\beta_n)\|e_n - x_n\|.$$

Using (3.14) and the condition (D3), one sees that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.15}$$

On the other hand, one has

$$x_{n+1} - x_n = \alpha_n(u - t_n) + (1-\gamma_n)(t_n - x_n).$$

It follows that

$$(1-\gamma_n)\|t_n - x_n\| \leq \|x_{n+1} - x_n\| + \alpha_n\|u - t_n\|.$$

From the conditions $0 < \liminf_{n\longrightarrow\infty}\gamma_n \leq \limsup_{n\longrightarrow\infty}\gamma_n < 1$, $\lim_{n\longrightarrow\infty}\alpha_n = 0$ and (3.15), one sees that

$$\lim_{n\to\infty} \|t_n - x_n\| = 0. \tag{3.16}$$

Next, we prove that

$$\limsup_{n\longrightarrow\infty}\langle u - Q_\Omega u, j(x_n - Q_\Omega u)\rangle \le 0,$$

where Q_Ω is the sunny nonexpansive retraction from E onto Ω. Define a mapping W_n by

$$W_n y = \nu S_n y + (1-\nu)Q_C[(I-\lambda A)Q_C(I-\mu B)y], \forall y \in C, \forall n \ge 1.$$

In view of Lemma 3.1 and Lemma 2.4, we see that W_n is a nonexpansive mapping satisfying

$$F(W_n) = F(S_n) \cap F(Q_C[(I-\lambda A)Q_C(I-\mu B)y]) = F(S_n) \cap F(G). \tag{3.17}$$

This implies that

$$\cap_{n=1}^{\infty} F(W_n) = \cap_{n=1}^{\infty}[F(S_n)\cap F(G)] = (\cap_{n=1}^{\infty}F(S_n))\cap F(G) = F(S)\cap F(G).$$

From (3.16), it follows that

$$\lim_{n\to\infty}\|W_n x_n - x_n\| = 0. \tag{3.18}$$

On the other hand, since $\{S_n\}$ satisfies the PU-condition, we have

$$\lim_{k,l\longrightarrow\infty}\sup_{y\in D} h(\|W_k y - W_l y\|) = \lim_{k,l\longrightarrow\infty}\sup_{y\in D} h(\nu\|S_k y - S_l y\|) = 0.$$

In virtue of Lemma 2.10, we obtain that $\{W_n y\}$ is a convergent sequence for all $y \in C$. So, let W be a mapping from C into itself defined by

$$Wy = \lim_{n\longrightarrow\infty} W_n y, \text{ for all } y \in C. \tag{3.19}$$

Using Lemma 2.10, we have

$$\lim_{n\longrightarrow\infty}\sup_{y\in D} h(\|Wy - W_n y\|) = 0. \tag{3.20}$$

Since

$$Wy = \lim_{n\longrightarrow\infty} W_n y = \lim_{n\longrightarrow\infty}[\nu S_n y + (1-\nu)Q_C[(I-\lambda A)Q_C(I-\mu B)y]] = \nu Sy + (1-\nu)Gy.$$

The nonexpansivity of S and G and Lemma 2.4 imply that W is nonexpansive such that

$$F(W) = F(S)\cap F(G) = \cap_{n=1}^{\infty}F(W_n) = \Omega.$$

Next, we observe that

$$h(\frac{1}{2}\|Wx_n - x_n\|) \le \frac{1}{2}h(\|Wx_n - W_n x_n\|) + \frac{1}{2}h(\|W_n x_n - x_n\|) \le \sup_{y\in D} h(\|Wy - W_n y\|) + h(\|W_n x_n - x_n\|).$$

Applying (3.20) and (3.18) to the last inequality, we have

$$\lim_{n\longrightarrow\infty} h(\frac{1}{2}\|Wx_n - x_n\|) = 0. \tag{3.21}$$

It follows from the properties of h that

$$\lim_{n\longrightarrow\infty}\|Wx_n - x_n\| = 0. \tag{3.22}$$

Let Q_Ω be the sunny nonexpansive retraction of C onto Ω. Now we show that

$$\limsup_{n\longrightarrow\infty}\langle u - Q_\Omega u, j(x_n - Q_\Omega u)\rangle \le 0. \tag{3.23}$$

Let u_t be the fixed point of the contraction $z \mapsto tu + (1-t)Wz$, where $t \in (0,1)$. That is,

$$u_t = tu + (1-t)Wu_t.$$

It follows that

$$\|u_t - x_n\| = \|(1-t)(Wu_t - x_n) + t(u - x_n)\|.$$

On the other hand, we have

$$\begin{aligned}\|u_t - x_n\|^2 &\le (1-t)^2\|Wu_t - x_n\|^2 + 2t\langle u - x_n, j(u_t - x_n)\rangle\\ &\le (1-2t+t^2)\|u_t - x_n\|^2 + f_n(t)\\ &\quad + 2t\langle u - u_t, j(u_t - x_n)\rangle + 2t\|u_t - x_n\|^2,\end{aligned}$$

where

$$f_n(t) = (2\|u_t - x_n\| + \|x_n - Wx_n\|)\|x_n - Wx_n\| \to 0 \quad \text{as } n \to 0. \tag{3.24}$$

□

Remark 3.4. Theorem 3.3 mainly improves of [39, Theorem 3.1] in the following respects:

(a) From a uniformly convex Banach space to a strictly convex Banach space.
(b) From the class of inverse-strongly accretive mappings to the class of Lipchitzian and relaxed cocoercive mappings.
(c) We can remove the property weakly sequentially continuous on the duality mapping.

The following is an example of a sequence $\{T_n\}$ of nonexpansive mappings satisfying the PU-condition.

Example 3.5. Let C be a closed convex subset of a smooth Banach space E. Suppose that $\{S_k\}$ is a sequence of nonexpansive mappings of E into inself with a common fixed point. For each $n \in \mathbb{N}$, define $T_n : C \longrightarrow C$ by

$$T_n x = \sum_{k=1}^{n} \beta_n^k S_k x, \ \ \forall x \in E,$$

where $\{\beta_n^k\}$ is a family of nonnegative numbers with indices $n, k \in \mathbb{N}$ with $k \leq n$ such that
(i) $\sum_{k=1}^{n} \beta_n^k = 1$ for all $n \in \mathbb{N}$;
(ii) $\lim_{n\longrightarrow\infty} \beta_n^k > 0$ for every $k \in \mathbb{N}$;
(iii) $\sum_{n=1}^{\infty}\sum_{k=1}^{n} |\beta_{n+1}^k - \beta_n^k| < \infty$.
It follows from [1, Theorem 4.1], we have
(1) Each T_n is a nonexpansive mapping.
(2) For any bounded subset D of E,

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty.$$

Using [27, Remark 3.2], we obtain that

$$\lim_{k,l\longrightarrow\infty} \sup_{z\in D} h(\|T_k z - T_l z\|) = 0$$

for any continuous increasing and convex function $h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $h(0) = 0$. Hence $\{T_n\}$ satisfies the PU-condition.

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