รายงานวิจัยฉบับสมบูรณ์

เกณฑ์การทำให้เป็นเชิงเส้นสำหรับสมการเชิงอนุพันธ์สามัญอันดับห้า โดยใช้การแปลงแบบจุด

ดร.สุภาพร สุขเสริญ

กันยายน 2557

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ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์

กองทุนวิจัยมหาวิทยาลัยนเรศวร

บทคัดย่อ

ในงานวิจัยนี้ได้นำเสนอปัญหาการทำสมการเชิงอนุพันธ์สามัญอันดับห้าให้เป็นเชิงเส้นภายใต้การแปลงแบบจุด เราได้ค้นพบเงื่อนไขที่จำเป็นและเงื่อนไขที่เพียงพอสำหรับการลดรูปสมการเชิงอนุพันธ์สามัญไม่เชิงเส้นอันดับห้า ไปสู่สมการเชิงอนุพันธ์สามัญเชิงเส้นอันดับห้าอย่างง่าย ตลอดจนได้มาซึ่งกระบวนการการได้มาของการแปลงเชิงเส้นในรูปแบบชัดแจ้ง และตัวอย่างประกอบการใช้ทฤษฎีบทเพื่อความเข้าใจ

คำสำคัญ : ปัญหาการทำให้เป็นเชิงเส้น/ การแปลงแบบจุด/ สมการเชิงอนุพันธ์สามัญไม่เชิงเส้น

Abstract

We present the linearization problem of fifth-order ordinary differential equation under the point transformations. We found the necessary and sufficient conditions which allow the nonlinear fifth-order ordinary differential equation to be transformed to the simplest linear equation. Moreover, the procedure for obtaining the linearizing transformation are provided in explicit forms. The examples which is linearizable by our method is given.

Keyword : Linearization problem/ point transformation/ nonlinear ordinary differential equations

Executive Summary

1. ความสำคัญและที่มาของปัญหา

ในทางคณิตศาสตร์สมการไม่เชิงเส้น (nonlinear equation) เป็นสมการที่ไม่เป็นเชิงเส้น (linear) นั่นคือสมการที่ไม่สอดคล้องกับหลักการซ้อนทับ (superposition principle) หรือผลลัพธ์ที่ได้ไม่เป็นสัดส่วนโดยตรงกับปัจจัยที่ป้อนเข้าไป มีเทคนิคน้อยมากที่จะสามารถแก้ปัญหาของสมการไม่เชิงเส้นใดๆ ที่ซึ่งตัวแปรที่จะถูกแก้ไม่สามารถเขียนให้อยู่ในรูปผลรวมเชิงเส้นของส่วนประกอบอิสระ

ปัญหาไม่เชิงเส้นที่เกิดขึ้นเป็นที่สนใจของวิศวกร นักฟิสิกส์และนักคณิตศาสตร์ เนื่องจากระบบทางกายภาพส่วนใหญ่โดยเนื้อแท้แล้วมีลักษณะไม่เป็นเชิงเส้น สมการไม่เชิงเส้นเป็นสมการที่ค่อนข้างจะแก้ยากและก่อให้เกิดปรากฏการณ์ที่น่าสนใจหลายอย่างในธรรมชาติ ในการแก้ปัญหาที่เกี่ยวข้องกับสมการเชิงอนุพันธ์สามัญไม่เชิงเส้น เรามักจะลดความซับซ้อนของสมการ โดยการเปลี่ยนตัวแปรให้เหมาะสม หนึ่งในวิธีการพื้นฐานที่จะแก้ปัญหานี้อิงอยู่กับการแปลงสมการที่กำหนดให้ไปสู่สมการในรูปแบบมาตรฐานอื่น การแปลงอาจจะแปลงไปสู่สมการที่มีอันดับเท่ากัน หรืออันดับที่มากกว่า หรืออันดับที่น้อยกว่าก็ได้ โดยเฉพาะอย่างยิ่งความเป็นไปได้ที่ว่าสมการที่กำหนดให้สามารถแปลงไปสู่สมการเชิงเส้น ซึ่งเป็นสมการที่น่าสนใจที่สุด เนื่องด้วยคุณสมบัติพิเศษของสมการเชิงเส้นนั่นเอง การลดรูปของสมการเชิงอนุพันธ์สามัญไม่เชิงเส้นไปสู่สมการเชิงอนุพันธ์สามัญเชิงเส้น นอกจากทำให้กระบวนการแก้ปัญหาง่ายขึ้นแล้ว ยังยอมให้เราสร้างผลเฉลยที่แท้จริงของสมการเดิมด้วย

ปัญหาการจำแนก (classification problem) อีกประเภทหนึ่งคือ ปัญหาสมมูล (equivalence problem) สองสมการสมการเชิงอนุพันธ์ใดๆ จะกล่าวว่าสมมูลกัน (equivalent) ก็ต่อเมื่อ มีการแปลงซึ่งมีอินเวอร์ส (invertible transformation) ที่สามารถแปลงผลเฉลยใดๆ ของสมการหนึ่ง ไปสู่ผลเฉลยของสมการอื่นได้ ปัญหาการทำให้เป็นเชิงเส้น (linearization problem) เป็นกรณีเฉพาะของปัญหาสมมูลโดยที่หนึ่งในสมการที่พิจารณาเป็นสมการเชิงเส้น นี่เป็นส่วนหนึ่งที่สำคัญในการศึกษาสมการไม่เชิงเส้น

หนึ่งในแรงจูงใจหลักสำหรับการศึกษาปัญหาการทำให้เป็นเชิงเส้นคือ ความเป็นไปได้ในการหาผลเฉลยทั่วไป สังเกตเห็นได้ว่าหลังจากที่หาการแปลงเชิงเส้นได้แล้ว จะต้องแก้หาผลเฉลยจากระบบสมการเชิงอนุพันธ์สามัญเชิงเส้น

ปัญหาหลักในการแก้ปัญหาการทำให้เป็นเชิงเส้น มาจากการคำนวณที่ซับซ้อนของจำนวนที่มากมาย เนื่องจากความยุ่งยากของปัญหาดังกล่าวนี้ จึงยังไม่มีนักวิจัยท่านใดพยายามที่จะแก้ปัญหานี้สำหรับสมการไม่เชิงเส้นของสมการที่มีอันดับสูงกว่าสี่ แต่ถ้าเราสามารถแก้ปัญหาการทำให้เป็นเชิงเส้นของสมการเชิงอนุพันธ์สามัญอันดับห้า แล้วเราก็จะได้องค์ความรู้ใหม่ในการแก้ปัญหาในเชิงฟิสิกส์หรือวิศวกรรม

2. วัตถุประสงค์

2.1 หาเงื่อนไขที่จำเป็นและเงื่อนไขที่เพียงพอสำหรับการทำสมการเชิงอนุพันธ์สามัญอันดับห้า

$$y^{(5)} = F\left(x, y, y', y'', y''', y^{(4)}\right)$$

ให้สมมูลกับสมการเชิงเส้น

$$u^{(5)} = 0$$

โดยใช้การแปลงแบบจุด

$$t = \varphi(x, y), u = \psi(x, y)$$

2.2 หาการแปลงเชิงเส้น

2.3 หาตัวอย่างและการประยุกต์ใช้

2.4 สร้างโปรแกรมสำหรับตรวจสอบความเป็นเชิงเส้น

3. ระเบียบวิธีวิจัย

3.1 ศึกษาโครงสร้างของสมการเชิงอนุพันธ์สามัญอันดับห้า การแปลงในรูปแบบต่างๆ และผลงานวิจัยที่เกี่ยวข้องที่มีนักวิจัยทำมาก่อนหน้านี้

3.2 ค้นคว้าหาเอกสาร ตำรา วารสาร และ เอกสารสิ่งพิมพ์ที่เกี่ยวข้องกับงานวิจัยที่กำลังดำเนินการวิจัยอยู่จากแหล่งข้อมูลต่างๆ

3.3 หาเงื่อนไขที่จำเป็นสำหรับการทำให้เป็นเชิงเส้นโดยวิธีการ change of derivatives

3.4 หาเงื่อนไขที่เพียงพอสำหรับการทำให้เป็นเชิงเส้นโดยใช้ compatibility theory

3.5 หาการแปลงเชิงเส้น

3.6 สร้างโปรแกรมสำเร็จรูปในการทดสอบความเป็นเชิงเส้นโดยใช้โปรแกรม Reduce

3.7 หาตัวอย่าง และการนำไปประยุกต์ใช้

3.8 เขียนและพิมพ์ผลงานวิจัยเพื่อส่งตีพิมพ์

3.9 รายงานสรุปผลโครงการ

4. แผนการดำเนินงานวิจัย

กิจกรรม	เดือนที่											
	1	2	3	4	5	6	7	8	9	10	11	12
1. การศึกษาค้นคว้าเอกสารงานวิจัย	↔											
2. การหาเงื่อนไขที่จำเป็นสำหรับการทำให้เป็นเชิงเส้น	←	—	→									
3. การหาเงื่อนไขที่เพียงพอสำหรับการทำให้เป็นเชิงเส้น				←	—	—	→					
4. การหาการแปลงเชิงเส้น								↔				
5. การสร้างโปรแกรมสำเร็จรูป									↔			

6. การหาตัวอย่าง และการนำไปประยุกต์ใช้										↔		
7. การเขียนและพิมพ์ผลงานวิจัยเพื่อส่งตีพิมพ์											←—	—→
8. การเขียนรายงานสรุปผลโครงการ												↔

5. ตัวชี้วัดเพื่อการเมินผลสำเร็จของโครงการ

5.1 ตีพิมพ์ในวารสารระดับนานาชาติที่มีค่า Impact Factor **จำนวน 1 เรื่อง**

5.2 ได้สิ่งประดิษฐ์ อุปกรณ์ เครื่องมือ หรืออื่นๆ เช่น ฐานข้อมูล Software ที่สามารถนำไปใช้ประโยชน์ได้ต่อไป จำนวน 1 ผลงาน

เนื้อหางานวิจัย

(ดูรายระเอียดทั้งหมดดังเอกสารแนบหน้าถัดไป)

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Introduction

1.1 Introduction to the research problem and its significance

In mathematics, a nonlinear equation is an equation which is not linear; that is, an equation which does not satisfy the superposition principle, or whose output is not directly proportional to its input. Less technically, a nonlinear equation is any problem where the variables to be solved for cannot be written as a linear combination of independent components.

Nonlinear problems are of interest to engineers, physicists and mathematicians because most physical systems are inherently nonlinear in nature. Nonlinear equations are difficult to solve and give rise to interesting phenomena. While solving problems related to nonlinear ordinary differential equations, it is often expedient to simplify equations by a suitable change of variables. One of the fundamental methods to solve this relies upon the transformation of a given equation to another equation of standard form. The transformation may be to an equation of equal order or of greater or lesser order. In particular, the possibility that a given equation could be linearized, i.e., transformed to a linear equation, was a most attractive proposition due to the special properties of linear differential equations. The reduction of an ordinary differential equation to a linear ordinary differential equation besides simplification allows us to construct an exact solution of the original equation.

One type of the classification problem is the equivalence problem. Two equation of differential equations are said to be equivalent if there exists an invertible transformation which transforms any solution of one equation to a solution of the other equation and vice versa. The linearization problem is a particular case of the equivalence problem, where

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Introduction

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Nonlinear problems are of interest to engineers, physicists and mathematicians because most physical systems are inherently nonlinear in nature. Nonlinear equations are difficult to solve and give rise to interesting phenomena. While solving problems related to nonlinear ordinary differential equations, it is often expedient to simplify equations by a suitable change of variables. One of the fundamental methods to solve this relies upon the transformation of a given equation to another equation of standard form. The transformation may be to an equation of equal order or of greater or lesser order. In particular, the possibility that a given equation could be linearized, i.e., transformed to a linear equation, was a most attractive proposition due to the special properties of linear differential equations. The reduction of an ordinary differential equation to a linear ordinary differential equation besides simplification allows us to construct an exact solution of the original equation.

One type of the classification problem is the equivalence problem. Two equation of differential equations are said to be equivalent if there exists an invertible transformation which transforms any solution of one equation to a solution of the other equation and vice versa. The linearization problem is a particular case of the equivalence problem, where one of the equation is a linear equation. It is one of the essential parts in the study of nonlinear equations.

The main difficulty in solving the linearization problem comes from the large number of complicated calculations. Because of this difficulty, there are no one attempts to solve this problem for nonlinear equations higher than fourth. However if we can solve the linearization problem of fifth-order ordinary differential equations, then we should set a new process to solve the problems in Physics or Engineering.

1.2 Historical review

The problem of linearization of ordinary differential equations has a long history. It attracted attention of mathematicians such as S. Lie and E. Cartan. The first linearization problem for ordinary differential equations was solved by Lie [1]. He found the general form of all ordinary differential equations of second-order that can be reduced to a linear equation by changing the independent and dependent variables. He showed that any linearizable second-order equation should be at most cubic in the first-order derivative and provided a linearization test in terms of its coefficients. The linearization criterion is written through relative invariants of the equivalence group. Liouville [2] and Tresse [3] treated the equivalence problem for second-order ordinary differential equations in terms of relative invariants of the equivalence group of point transformations. There are other approaches for solving the linearization problem of a second-order ordinary differential equation. For example, one was developed by Cartan [4]. The idea of his approach was to associate with every differential equation a uniquely defined geometric structure of a certain form.

In 1993, Bocharov, Sokolov and Svinolupov [5] considered the linearization problem of third-order with respect to point transformations. Grebot [6] studied the linearization of third-order ordinary differential equations by means of a restricted class of point transformations, namely $t = \varphi(x)$, $u = \psi(x, y)$. However, the problem was not completely solved. Complete criteria for linearization by means of point transformations were obtained by Ibragimov and Meleshko [7].

In 2008, Ibragimov, Meleshko and Suksern [8] solved the linearization problem for fourth-order ordinary differential equations by using point transformation.

Nowadays, the linearization problem of fifth-order ordinary differential equations via point transformations still be an unsolved one. Therefore, this is the first project for solving this problem.

Chapter 2

Basic Concepts

2.1 Definitions and theorems used in the research

Definition 2.1.1. A transformation

$$\begin{aligned} t &= \varphi(x,y), \\ u &= \psi(x,y), \end{aligned} \tag{2.1}$$

where φ and ψ are sufficiently smooth functions is called a *point transformation*. If $\varphi_y = 0$, a transformation (2.1) is called a fiber preserving transformation.

Definition 2.1.2. Two equations are call *equivalent* if there is an invertible transformation which transforms one equation into another.

Definition 2.1.3. The problem of finding all equations, which are equivalent to a given equation is called an *equivalence problem*. If the given equation is a linear equation, then the equivalence problem is called a *linearization problem*.

Definition 2.1.4. If $F(u,v)$ and $G(u,v)$ are differentiable in a region, the *Jacobian determinant*, or briefly the *Jacobian*, of F and G with respect to u and v is the second-order functional determinant defined by

$$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}.$$

Similarly, the third-order determinant

$$\frac{\partial(F,G,H)}{\partial(u,v,w)} = \begin{vmatrix} F_u & F_v & F_w \\ G_u & G_v & G_w \\ H_u & H_v & H_w \end{vmatrix}$$

is called the *Jacobian* of F, G and H with respect to u, v and w.

can be transformed to the form

$$u'' = 0$$

by point transformation (2.1).

Theorem 2.1.8. *(Laguerre-Forsyth canonical form) Any kth-order linear ordinary differential equations*

$$y^{(k)} + \sum_{i=0}^{k-1} a_i(x) y^{(i)} = 0, \qquad k \geq 3$$

can be transformed to the form

$$u^{(k)} + \sum_{i=0}^{k-3} a_i(x) u^{(i)} = 0 \tag{2.2}$$

by point transformation (2.1).

Note that (2.2) is called the *Laguerre-Forsyth canonical form* of kth-order linear ordinary differential equations.

2.2 Transformation of derivatives

For simplicity of understanding, let us consider second-order ordinary differential equation

$$y'' = F(x, y, y'). \tag{2.3}$$

An invertible change of the independent and dependent variables (2.1) leads equation (2.3) into the equation

$$u'' = f(t, u, u'). \tag{2.4}$$

Notice that we require the Jacobian

$$\Delta = \frac{\partial(t,u)}{\partial(x,y)} = \frac{\partial(\varphi,\psi)}{\partial(x,y)} = \varphi_x\psi_y - \varphi_y\psi_x \neq 0.$$

in the neighborhood of a.

First of all, we have to change $y(x)$ to $u(t)$. Assume that we know a solution of equation (2.3), i.e.

$$y = y(x).$$

The transformed function $u(t)$ is found from equation

$$t = \varphi(x, y(x)).$$

Since $\varphi'(x,y(x)) = \varphi_x + y'\varphi_y \in C$ (i.e. $\varphi \in C^1$) and $\Delta\ (\varphi(x,y(x))) = \varphi_x + y'\varphi_y \neq 0$ then by the virtue of Inverse Function Theorem one finds

$$x = \alpha(t).$$

Thus, one obtains

$$u(t) = \psi(\alpha(t), y(\alpha(t))). \tag{2.5}$$

The transformation of the first derivatives can be found as follows. Let us differentiate (2.5) with respect to t

$$u'(t) = \frac{du}{dt} = \frac{\partial\psi}{\partial x}\frac{d\alpha}{dt} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}\frac{d\alpha}{dt} = (\psi_x + y'\psi_y)\frac{d\alpha}{dt}. \tag{2.6}$$

Since $t = \varphi(\alpha(t), y(\alpha(t)))$ then

$$\frac{dt}{dt} = \frac{\partial\varphi}{\partial x}\frac{d\alpha}{dt} + \frac{\partial\varphi}{\partial y}\frac{dy}{dx}\frac{d\alpha}{dt}$$

or

$$\frac{d\alpha}{dt} = \frac{1}{(\varphi_x + y'\varphi_y)}. \tag{2.7}$$

Substituting equation (2.7) into equation (2.6), one obtains

$$u'(t) = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y} = \frac{D_x\psi}{D_x\varphi} = g(x, y(x), y'(x)).$$

where $D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \ldots$ is the *total derivative*. So that the first prolongation of transformation (2.1) is $u' = g(x,y,y')$.

Next, to find the transformation of second-order derivative. Consider

$$\begin{aligned}
u''(t) &= \frac{\partial g}{\partial x}\frac{d\alpha}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dx}\frac{d\alpha}{dt} + \frac{\partial g}{\partial y'}\frac{dy'}{dx}\frac{d\alpha}{dt} \\
&= (g_x + y'g_y + y''g_{y'})\frac{d\alpha}{dt} \\
&= \frac{g_x + y'g_y + y''g_{y'}}{\varphi_x + y'\varphi_y} \\
&= \frac{D_x g}{D_x\varphi} \\
&= h(x, y(x), y'(x), y''(x)).
\end{aligned}$$

So that the second prolongation of transformation (2.1) is $u'' = h(x,y,y',y'')$.

Moreover, we can rewrite equation (2.4) as

$$h(x,y,y',y'') = f(\varphi(x,y), \psi(x,y), g(x,y,y')). \tag{2.8}$$

We see that if we find u from equation (2.4), then we can find y from equation (2.8). Hence $u'' = f(t,u,u')$ and $y'' = F(x,y,y')$ are equivalent.

2.3 The Lie linearization test

Theorem 2.3.1. ([1], S. Lie) *Any second-order ordinary differential equations (2.3) obtained from a linear equation*

$$u'' = 0 \tag{2.9}$$

by a point transformation (2.1) has to be either to the form

$$y'' + a(x,y)\,y'^3 + b(x,y)\,y'^2 + c(x,y)\,y' + d(x,y) = 0, \tag{2.10}$$

where

$$\begin{aligned} a &= \Delta^{-1}\left(\varphi_y\psi_{yy} - \varphi_{yy}\psi_y\right), \\ b &= \Delta^{-1}\left(\varphi_x\psi_{yy} - \varphi_{yy}\psi_x + 2\left(\varphi_y\psi_{xy} - \varphi_{xy}\psi_y\right)\right), \\ c &= \Delta^{-1}\left(\varphi_y\psi_{xx} - \varphi_{xx}\psi_y + 2\left(\varphi_x\psi_{xy} - \varphi_{xy}\psi_x\right)\right), \\ d &= \Delta^{-1}\left(\varphi_x\psi_{xx} - \varphi_{xx}\psi_x\right) \end{aligned} \tag{2.11}$$

and $\Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0$.

Proof.

Since $u = \psi(x,y)$, thus

$$\begin{aligned} u'(t) &= \frac{D_x\psi}{D_x\varphi} \\ &= \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y} \\ &= g(x,y,y'), \end{aligned}$$

$$\begin{aligned} u''(t) &= \tfrac{D_x g}{D_x\varphi} \\ &= \tfrac{g_x + y' g_y + y'' g_{y'}}{\varphi_x + y'\varphi_y} \\ &= P(x,y,y',y'') \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} g_x &= \frac{(\varphi_x + y'\varphi_y)\frac{\partial}{\partial x}(\psi_x + y'\psi_y) - (\psi_x + y'\psi_y)\frac{\partial}{\partial x}(\varphi_x + y'\varphi_y)}{(\varphi_x + y'\varphi_y)^2} \\ &= \frac{(\varphi_x + y'\varphi_y)\left(\psi_{xx} + \left(y'\frac{\partial}{\partial x}\psi_y + \psi_y\frac{\partial}{\partial x}y'\right)\right) - (\psi_x + y'\psi_y)\left(\varphi_{xx} + \left(y'\frac{\partial}{\partial x}\varphi_y + \varphi_y\frac{\partial}{\partial x}y'\right)\right)}{(\varphi_x + y'\varphi_y)^2} \\ &= \frac{(\varphi_x + y'\varphi_y)(\psi_{xx} + y'\psi_{xy} + \psi_y(0)) - (\psi_x + y'\psi_y)(\varphi_{xx} + y'\varphi_{xy} + \varphi_y(0))}{(\varphi_x + y'\varphi_y)^2} \\ &= \frac{(\varphi_x + y'\varphi_y)(\psi_{xx} + y'\psi_{xy}) - (\psi_x + y'\psi_y)(\varphi_{xx} + y'\varphi_{xy})}{(\varphi_x + y'\varphi_y)^2}, \end{aligned}$$

$$\begin{aligned}
g_y &= \frac{(\varphi_x + y'\varphi_y)\frac{\partial}{\partial y}(\psi_x + y'\psi_y) - (\psi_x + y'\psi_y)\frac{\partial}{\partial y}(\varphi_x + y'\varphi_y)}{(\varphi_x + y'\varphi_y)^2} \\
&= \frac{(\varphi_x + y'\varphi_y)\left(\psi_{xy} + \left(y'\frac{\partial}{\partial y}\psi_y + \psi_y\frac{\partial}{\partial y}y'\right)\right) - (\psi_x + y'\psi_y)\left(\varphi_{xy} + \left(y'\frac{\partial}{\partial y}\varphi_y + \varphi_y\frac{\partial}{\partial y}y'\right)\right)}{(\varphi_x + y'\varphi_y)^2} \\
&= \frac{(\varphi_x + y'\varphi_y)(\psi_{xy} + y'\psi_{yy} + \psi_y(0)) - (\psi_x + y'\psi_y)(\varphi_{xy} + y'\varphi_{yy} + \varphi_y(0))}{(\varphi_x + y'\varphi_y)^2} \\
&= \frac{(\varphi_x + y'\varphi_y)(\psi_{xy} + y'\psi_{yy}) - (\psi_x + y'\psi_y)(\varphi_{xy} + y'\varphi_{yy})}{(\varphi_x + y'\varphi_y)^2},
\end{aligned}$$

$$\begin{aligned}
g_{y'} &= \frac{(\varphi_x + y'\varphi_y)\frac{\partial}{\partial y'}(\psi_x + y'\psi_y) - (\psi_x + y'\psi_y)\frac{\partial}{\partial y'}(\varphi_x + y'\varphi_y)}{(\varphi_x + y'\varphi_y)^2} \\
&= \frac{(\varphi_x + y'\varphi_y)\left(0 + \left(y'\frac{\partial}{\partial y'}\psi_y + \psi_y\frac{\partial}{\partial y'}y'\right)\right) - (\psi_x + y'\psi_y)\left(0 + \left(y'\frac{\partial}{\partial y'}\varphi_y + \varphi_y\frac{\partial}{\partial y'}y'\right)\right)}{(\varphi_x + y'\varphi_y)^2} \\
&= \frac{(\varphi_x + y'\varphi_y)(0 + y'(0) + \psi_y(1)) - (\psi_x + y'\psi_y)(0 + y'(0) + \varphi_y(1))}{(\varphi_x + y'\varphi_y)^2} \\
&= \frac{(\varphi_x + y'\varphi_y)(\psi_y) - (\psi_x + y'\psi_y)(\varphi_y)}{(\varphi_x + y'\varphi_y)^2}
\end{aligned}$$

and $D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + ...$ is total derivatives.
Replacing $g_x, g_y, g_{y'}$ into equation (2.12) one gets

$$\begin{aligned}
u'' = \ & (y''(\varphi_x\psi_y - \varphi_y\psi_x) + y'^3(\varphi_y\psi_{yy} - \varphi_{yy}\psi_y) + y'^2(\varphi_x\psi_{yy} - \varphi_{yy}\psi_x \\
&+2(\varphi_y\psi_{xy} - \varphi_{xy}\psi_y)) + y'(\varphi_y\psi_{xx} - \varphi_{xx}\psi_y \\
&+2(\varphi_x\psi_{xy} - \varphi_{xy}\psi_x)) + \varphi_x\psi_{xx} - \varphi_{xx}\psi_x)/(\varphi_x + y'\varphi_y)^3.
\end{aligned} \tag{2.13}$$

Since the jacobian $\Delta \neq 0$, then after replacing u'' into equation (2.9) one gets equation (2.10). □

Theorem 2.3.2. ([1], S. Lie) *Equation (2.10) is linearizable by point transformation (2.1) if and only if its coefficients satisfied the follows.*

(a) *If $\varphi_y = 0$ then the conditions are*

$$a = 0, c_y = 2b_x, d_{yy} - b_{xx} - b_x c + b_y d + d_y b = 0. \tag{2.14}$$

(b) *If $\varphi_y \neq 0$ then the conditions are*

$$\begin{aligned}
&3a_{xx} - 2b_{xy} + c_{yy} - 3a_x c + 3a_y d + 2b_x b - 3c_x a - c_y b + 6d_y a = 0, \\
&b_{xx} - 2c_{xy} + 3d_{yy} - 6a_x d + b_x c + 3b_y d - 2c_y c - 3d_x a + 3d_y b = 0.
\end{aligned} \tag{2.15}$$

Proof. We will find conditions from system of equation (2.11).
Case 1 : $\varphi_y = 0$. That is φ does not depend on y. From (2.11) one gets that

$$\begin{aligned}
a &= 0,\\
b &= (\varphi_x\psi_y)^{-1}(\varphi_x\psi_{yy}) = \frac{\varphi_x\psi_{yy}}{\varphi_x\psi_y},\\
c &= (\varphi_x\psi_y)^{-1}(-\varphi_{xx}\psi_y + 2\varphi_x\psi_{xy}) = \left(-\varphi_x{}^{-1}\varphi_{xx} + 2\psi_y{}^{-1}\psi_{xy}\right),\\
d &= (\varphi_x\psi_y)^{-1}(\varphi_x\psi_{xx} - \varphi_{xx}\psi_x) = \frac{\psi_{xx}}{\psi_y} - \frac{\varphi_{xx}\psi_x}{\varphi_x\psi_y}
\end{aligned}$$

thus

$$a = 0, \psi_{yy} = \psi_y b, \psi_{xy} = \frac{1}{2}\left(\varphi_x{}^{-1}\psi_y\varphi_{xx} + \psi_y c\right), \psi_{xx} = \varphi_x{}^{-1}\psi_x\varphi_{xx} + \psi_y d. \tag{2.16}$$

Mixing the derivatives:

- $(\psi_{xy})_y = (\psi_{yy})_x$

$$\frac{1}{2}\left[\frac{\varphi_{xx}\psi_{yy}}{\varphi_x} + c\psi_{yy} + c_y\psi_y\right] = \psi_{xy}b + \psi_y b_x$$

$$\frac{\varphi_{xx}\psi_{yy}}{\varphi_x} + c\psi_{yy} + c_y\psi_y = 2(\psi_{xy}b + \psi_y b_x)$$

replacing (2.16), one gets

$$\begin{aligned}
\frac{\varphi_{xx}\psi_y b}{\varphi_x} + c\psi_y b + c_y\psi_y &= 2\left(\frac{1}{2}\left(\frac{\psi_y\varphi_{xx}}{\varphi_x} + \psi_y c\right)b + \psi_y b_x\right)\\
&= \frac{\psi_y\varphi_{xx}b}{\varphi_x} + c\psi_y b + 2\psi_y b_x\\
c_y\psi_y &= 2\psi_y b_x\\
c_y &= 2b_x
\end{aligned}$$

- $(\psi_{xy})_x = (\psi_{xx})_y$

$$\varphi_x{}^{-2}\left(2\varphi_x\varphi_{xxx} - 3\varphi_{xx}^2\right) = 4(d_y + bd) - \left(2c_x + c^2\right). \tag{2.17}$$

Since $\varphi_y = 0$, then differentiating (2.17) respect to y one arrives at

$$d_{yy} - b_{xx} - b_x c + b_y d + d_y b = 0.$$

Hence a second-order ordinary differential equations in the form (2.10) can be linearizable by function $\varphi = \varphi(x)$ if and only if it's coefficients satisfied (2.14).

Case 2 : $\varphi_y \neq 0$.

Consider (2.11).

From $a = \Delta^{-1}(\varphi_y\psi_{yy} - \varphi_{yy}\psi_y)$, one gets

$$\psi_{yy} = \frac{(\varphi_{yy}\psi_y + a\Delta)}{\varphi_y}.$$

From $b = \Delta^{-1}(\varphi_x\psi_{yy} - \varphi_{yy}\psi_x + 2(\varphi_y\psi_{xy} - \varphi_{xy}\psi_y))$, one gets

$$\psi_{xy} = \frac{2\varphi_{xy}\varphi_y\psi_y - \varphi_{yy}\Delta - (a\varphi_x - b\varphi_y)\Delta}{2\varphi_y^2}.$$

From $c = \Delta^{-1}(\varphi_y\psi_{xx} - \varphi_{xx}\psi_y + 2(\varphi_x\psi_{xy} - \varphi_{xy}\psi_x))$, one gets

$$\psi_{xx} = \frac{2\varphi_{xy}\varphi_y\psi_x - \varphi_x\varphi_{yy}\psi_x - \varphi_x^2\psi_x a + \varphi_x\varphi_y\psi_x b + \varphi_y^2(\psi_y d - \psi_x c)}{\varphi_y^2}.$$

From $d = \Delta^{-1}(\varphi_x\psi_{xx} - \varphi_{xx}\psi_x)$, one gets

$$\varphi_{xx} = \left(\frac{2\varphi_{xy}\varphi_x\varphi_y - \varphi_x^2\varphi_{yy} - \varphi_x^3 a + \varphi_x^2\varphi_y b - \varphi_x\varphi_y^2 c + \varphi_y^3 d}{\varphi_y^2}\right).$$

Therefore, by the relation (2.11) we have

$$\begin{aligned}
\psi_{yy} &= \tfrac{(\varphi_{yy}\psi_y + a\Delta)}{\varphi_y},\\
\psi_{xy} &= \tfrac{2\varphi_{xy}\varphi_y\psi_y - \varphi_{yy}\Delta - (a\varphi_x - b\varphi_y)\Delta}{2\varphi_y^2},\\
\psi_{xx} &= \tfrac{2\varphi_{xy}\varphi_y\psi_x - \varphi_x\varphi_{yy}\psi_x - \varphi_x^2\psi_x a + \varphi_x\varphi_y\psi_x b + \varphi_y^2(\psi_y d - \psi_x c)}{\varphi_y^2},\\
\varphi_{xx} &= \left(\tfrac{2\varphi_{xy}\varphi_x\varphi_y - \varphi_x^2\varphi_{yy} - \varphi_x^3 a + \varphi_x^2\varphi_y b - \varphi_x\varphi_y^2 c + \varphi_y^3 d}{\varphi_y^2}\right).
\end{aligned} \tag{2.18}$$

Mixing the derivatives :

- $(\psi_{xy})_y = (\psi_{yy})_x$

$$\begin{aligned}
2\varphi_y\varphi_{yyy} &= 3\left(\varphi_{yy}^2 - 2\varphi_{xy}\varphi_y a + 2\varphi_x\varphi_{yy}a + \varphi_x^2 a^2\right) - 2\varphi_x\varphi_y(a_y + ab)\\
&\quad + \varphi_y^2\left(2b_y - 4a_x + 4ac - b^2\right)\\
\varphi_{yyy} &= (3\left(\varphi_{yy}^2 - 2\varphi_{xy}\varphi_y a + 2\varphi_x\varphi_{yy}a + \varphi_x^2 a^2\right) - 2\varphi_x\varphi_y(a_y + ab)\\
&\quad + \varphi_y^2\left(2b_y - 4a_x + 4ac - b^2\right))/(2\varphi_y)
\end{aligned}$$

- $(\psi_{xy})_x = (\psi_{xx})_y$

$$\begin{aligned}
6\varphi_y^2\varphi_{xyy} &= 3\left(4\varphi_{xy}\varphi_{yy}\varphi_y - \varphi_x\varphi_{yy}^2 + 2\varphi_x\varphi_{yy}\varphi_y b - 2\varphi_{xy}\varphi_y^2 b\right)\\
&\quad + 3\varphi_x^3 a^2 + 3\varphi_x\varphi_y^2\left(-2a_x + 2ac - b^2\right) + 2\varphi_y^3(-b_x + 2c_y + 3ad)\\
\varphi_{xyy} &= (3\left(4\varphi_{xy}\varphi_{yy}\varphi_y - \varphi_x\varphi_{yy}^2 + 2\varphi_x\varphi_{yy}\varphi_y b - 2\varphi_{xy}\varphi_y^2 b\right) + 3\varphi_x^3 a^2\\
&\quad + 3\varphi_x\varphi_y^2\left(-2a_x + 2ac - b^2\right) + 2\varphi_y^3(-b_x + 2c_y + 3ad))/(6\varphi_y^2).
\end{aligned}$$

Mixing the derivative again :

- $(\varphi_{xyy})_y = (\varphi_{yyy})_x$ and $(\varphi_{xx})_{yy} = (\varphi_{xyy})_x$ one obtains (2.15). □

Chapter 3

Methodology and Result

We consider the fifth-order ordinary differential equation

$$y^{(5)} = f\left(x, y, y', y'', y''', y^{(4)}\right). \tag{3.1}$$

We apply a point transformation

$$\begin{aligned} t &= \varphi(x, y), \\ u &= \psi(x, y) \end{aligned} \tag{3.2}$$

to equation (3.1).

We begin with investigating the necessary conditions for linearization. The general form of equation (3.1) that can be obtained from a linear ordinary differential equation

$$u^{(5)}(t) = 0 \tag{3.3}$$

by point transformation (3.2) is found on this step. In consequence, we identify two candidates for linearization.

3.1 Necessary conditions for linearization

Consider t and u as the new independent and dependent variable, respectively, one obtains the following transformation of the first-order derivative

$$u'(t) = \frac{D_x\psi}{D_x\varphi} = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y}, \tag{3.4}$$

where $\varphi_x = \partial\varphi/\partial x, \varphi_y = \partial\varphi/\partial y$, etc., and

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''} + y^{(4)}\frac{\partial}{\partial y'''} + y^{(5)}\frac{\partial}{\partial y^{(4)}} + \ldots$$

is the total derivative. Likewise, one obtains the transformation of derivatives of the second and higher order. Namely, denoting by $P(x, y, y')$ the right-hand side of equation (3.4),

$$P(x,y,y') = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y}$$

one has

$$\begin{aligned} u''(t) &= \frac{D_x P}{D_x \varphi} = \frac{P_x + y'P_y + y''P_{y'}}{\varphi_x + y'\varphi_y} \\ &= \frac{\Delta}{(\varphi_x + y'\varphi_y)^3}[y'' + \frac{1}{\Delta}(\varphi_y\psi_{yy} - \varphi_{yy}\psi_y)y'^3 + ...]. \end{aligned} \quad (3.5)$$

Denoting by $Q(x, y, y', y'')$ the right-hand side of equation (3.5),

$$Q(x,y,y',y'') = \frac{\Delta}{(\varphi_x + y'\varphi_y)^3}[y'' + \frac{1}{\Delta}(\varphi_y\psi_{yy} - \varphi_{yy}\psi_y)y'^3 + ...]$$

one has

$$\begin{aligned} u'''(t) &= \frac{D_x Q}{D_x \varphi} = \frac{Q_x + y'Q_y + y''Q_{y'} + y'''Q_{y''}}{\varphi_x + y'\varphi_y} \\ &= \frac{\Delta}{(\varphi_x + y'\varphi_y)^5}[(\varphi_x + y'\varphi_y)y''' - 3\varphi_y y''^2 + ...]. \end{aligned} \quad (3.6)$$

Denoting by $R(x, y, y', y'', y''')$ the right-hand side of equation (3.6),

$$R(x,y,y',y'',y''') = \frac{\Delta}{(\varphi_x + y'\varphi_y)^5}[(\varphi_x + y'\varphi_y)y''' - 3\varphi_y y''^2 + ...]$$

one has

$$\begin{aligned} u^{(4)}(t) &= \frac{D_x R}{D_x \varphi} = \frac{R_x + y'R_y + y''R_{y'} + y'''R_{y''} + y^{(4)}R_{y'''}}{\varphi_x + y'\varphi_y} \\ &= \frac{\Delta}{(\varphi_x + y'\varphi_y)^7}[(\varphi_x + y'\varphi_y)^2 y^{(4)} - 10(\varphi_x + y'\varphi_y)\varphi_y y'' y''' + ...]. \end{aligned} \quad (3.7)$$

Denoting by $S(x, y, y', y'', y''', y^{(4)})$ the right-hand side of equation (3.7),

$$S(x,y,y',y'',y''',y^{(4)}) = \frac{\Delta}{(\varphi_x + y'\varphi_y)^7}[(\varphi_x + y'\varphi_y)^2 y^{(4)} - 10(\varphi_x + y'\varphi_y)\varphi_y y'' y''' + ...]$$

hence,

$$\begin{aligned} u^{(5)}(t) &= \frac{D_x S}{D_x \varphi} = \frac{S_x + y'S_y + y''S_{y'} + y'''S_{y''} + y^{(4)}S_{y'''} + y^{(5)}S_{y^{(4)}}}{\varphi_x + y'\varphi_y} \\ &= \frac{\Delta}{(\varphi_x + y'\varphi_y)^9}[(\varphi_x + y'\varphi_y)^3 y^5 - 15(\varphi_x + y'\varphi_y)\varphi_y y'' y^{(4)} + ...]. \end{aligned} \quad (3.8)$$

Thus, equation (3.3) becomes

$$\frac{\Delta}{(\varphi_x + y'\varphi_y)^9}[(\varphi_x + y'\varphi_y)^3 y^5 - (15(\varphi_x + y'\varphi_y)\varphi_y y'' + ...)y^{(4)} + ...] = 0. \quad (3.9)$$

Here

$$\Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0$$

is Jacobian of the change of variables (3.2). It is manifest from equation (3.9) that the transformations (3.2) with $\varphi_y = 0$ and $\varphi_y \neq 0$, respectively, provide two distinctly different candidates for linearization.

If $\varphi_y = 0$ we work out the missing terms in equation (3.9), substitute the resulting expression in equation (3.3) and obtain the following equation

$$\begin{aligned} y^{(5)} + (A_1y' + A_0)y^{(4)} + (B_3y'' + B_2y'^2 + B_1y' + B_0)y''' \\ +(C_1y' + C_0)y''^{(2)} + (D_3y'^3 + D_2y'^2 + D_1y' + D_0)y'' \\ +E_5y'^5 + E_4y'^4 + E_3y'^3 + E_2y'^2 + E_1y' + E_0 = 0 \end{aligned} \quad (3.10)$$

where

$$A_1 = 5\psi_{yy}/\psi_y, \quad (3.11)$$
$$A_0 = -5(2\varphi_{xx}\psi_y - \varphi_x\psi_{xy})/(\varphi_x\psi_y), \quad (3.12)$$
$$B_3 = 10\psi_{yy}/\psi_y, \quad (3.13)$$
$$B_2 = 10\psi_{yyy}/\psi_y, \quad (3.14)$$
$$B_1 = -20(2\varphi_{xx}\psi_{yy} - \varphi_x\psi_{xyy})/(\varphi_x\psi_y), \quad (3.15)$$
$$B_0 = -5(2\varphi_{xxx}\varphi_x\psi_y - 9\varphi_{xx}^2\psi_y + 8\varphi_{xx}\varphi_x\psi_{xy} - 2\varphi_x^2\psi_{xxy})/(\varphi_x^2\psi_y), \quad (3.16)$$
$$C_1 = 15\psi_{yyy}/\psi_y, \quad (3.17)$$
$$C_0 = -15(2\varphi_{xx}\psi_{yy} - \varphi_x\psi_{xyy})/(\varphi_x\psi_y), \quad (3.18)$$
$$D_3 = 10\psi_{yyyy}/\psi_y, \quad (3.19)$$
$$D_2 = -30(2\varphi_{xx}\psi_{yyy} - \varphi_x\psi_{xyyy})/(\varphi_x\psi_y), \quad (3.20)$$
$$D_1 = 15(9\varphi_{xx}^2\psi_{yy} - 8\varphi_{xx}\varphi_x\psi_{xyy} + 2\varphi_x^2\psi_{xxyy} - 2\varphi_{xxx}\varphi_x\psi_{yy})/(\varphi_x^2\psi_y), \quad (3.21)$$
$$\begin{aligned} D_0 = & -5(\varphi_{xxxx}\varphi_x^2\psi_y - 12\varphi_{xxx}\varphi_{xx}\varphi_x\psi_y + 6\varphi_{xxx}\varphi_x^2\psi_{xy} + 21\varphi_{xx}^3\psi_y \\ & - 27\varphi_{xx}^2\varphi_x\psi_{xy} + 12\varphi_{xx}\varphi_x^2\psi_{xxy} - 2\varphi_x^3\psi_{xxxy})/(\varphi_x^3\psi_y), \end{aligned} \quad (3.22)$$
$$E_5 = \psi_{yyyyy}/\psi_y, \quad (3.23)$$
$$E_4 = -5(2\varphi_{xx}\psi_{yyyy} - \varphi_x\psi_{xyyyy})/(\varphi_x\psi_y), \quad (3.24)$$
$$E_3 = -5((2\varphi_{xxx}\varphi_x - 9\varphi_{xx}^2)\psi_{yyy} + 2(4\varphi_{xx}\psi_{xyyy} - \varphi_x\psi_{xxyyy})\varphi_x)/(\varphi_x^2\psi_y), \quad (3.25)$$

$$
\begin{aligned}
E_2 =& 5((6(2\varphi_{xx}\psi_{yy} - \varphi_x\psi_{xyy})\varphi_{xxx} - \varphi_{xxxx}\varphi_x\psi_{yy})\varphi_x - (21\varphi_{xx}^3\psi_{yy} \\
& - 27\varphi_{xx}^2\varphi_x\psi_{xyy} + 12\varphi_{xx}\varphi_x^2\psi_{xxyy} - 2\varphi_x^3\psi_{xxxyy}))/(\varphi_x^3\psi_y), \qquad (3.26)\\
E_1 =& (5(21\varphi_{xx}^4\psi_y - 42\varphi_{xx}^3\varphi_x\psi_{xy} + 27\varphi_{xx}^2\varphi_x^2\psi_{xxy} - 8\varphi_{xx}\varphi_x^3\psi_{xxxy} + \varphi_x^4\psi_{xxxxy} \\
& + 2\varphi_{xxx}^2\varphi_x^2\psi_y) - \psi_{xxxxx}\varphi_x^3\psi_y + 5(3\varphi_{xx}\psi_y - 2\varphi_x\psi_{xy})\varphi_{xxxx}\varphi_x^2 \\
& - 15(7\varphi_{xx}^2\psi_y - 8\varphi_{xx}\varphi_x\psi_{xy} + 2\varphi_x^2\psi_{xxy})\varphi_{xxx}\varphi_x)/(\varphi_x^4\psi_y), \qquad (3.27)\\
E_0 =& -(\psi_{xxxxx}\varphi_x^3\psi_x - 15\varphi_{xxxx}\varphi_{xx}\varphi_x^2\psi_x + 5\varphi_{xxxx}\varphi_x^3\psi_{xx} - 10\varphi_{xxx}^2\varphi_x^2\psi_x \\
& + 105\varphi_{xxx}\varphi_{xx}^2\varphi_x\psi_x - 60\varphi_{xxx}\varphi_{xx}\varphi_x^2\psi_{xx} + 10\varphi_{xxx}\varphi_x^3\psi_{xxx} - 105\varphi_{xx}^4\psi_x \\
& + 105\varphi_{xx}^3\varphi_x\psi_{xx} - 45\varphi_{xx}^2\varphi_x^2\psi_{xxx} + 10\varphi_{xx}\varphi_x^3\psi_{xxxx} - \varphi_x^4\psi_{xxxxx})/(\varphi_x^4\psi_y). \qquad (3.28)
\end{aligned}
$$

If $\varphi_y \neq 0$, we proceed likewise and setting $r(x,y) = \frac{\varphi_x}{\varphi_y}$, arrive at the following equation

$$
\begin{aligned}
y^{(5)} \quad & + \tfrac{1}{y'+r}[(-15y'' + F_2y'^2 + F_1y' + F_0)y^{(4)} - 10y'''^2] \\
& + \tfrac{1}{(y'+r)^2}[105y''^2 + (G_2y'^2 + G_1y' + G_0)y'' + H_4y'^4 \\
& + H_3y'^3 + H_2y'^2 + H_1y' + H_0]y''' + \tfrac{1}{(y'+r)^3}[-105y''^4 \\
& + (J_2y'^2 + J_1y' + J_0)y''^3 + (K_4y'^4 + K_3y'^3 + K_2y'^2 \\
& + K_1y' + K_0)y''^2 + (L_6y'^6 + L_5y'^5 + L_4y'^4 + L_3y'^3 \\
& + L_2y'^2 + L_1y' + L_0)y'' + M_9y'^9 + M_8y'^8 + M_7y'^7 + M_6y'^6 \\
& + M_5y'^5 + M_4y'^4 + M_3y'^3 + M_2y'^2 + M_1y' + M_0] = 0
\end{aligned} \qquad (3.29)
$$

where

$$
\begin{aligned}
F_2 =& -5(3\varphi_{yy}\Delta - \varphi_y\Delta_y)/(\varphi_y\Delta), \qquad (3.30)\\
F_1 =& 5((\Delta_x + \Delta_y r - 6r_y\Delta)\varphi_y - 6\varphi_{yy}r\Delta)/(\varphi_y\Delta), \qquad (3.31)\\
F_0 =& -5(((3r_y\Delta - \Delta_x)r + 3r_x\Delta)\varphi_y + 3\varphi_{yy}r^2\Delta)/(\varphi_y\Delta), \qquad (3.32)\\
G_2 =& 50(3\varphi_{yy}\Delta - \varphi_y\Delta_y)/(\varphi_y\Delta), \qquad (3.33)\\
G_1 =& -20((3\Delta_x + 2\Delta_y r - 18r_y\Delta)\varphi_y - 15\varphi_{yy}r\Delta)/(\varphi_y\Delta), \qquad (3.34)\\
G_0 =& 10(15\varphi_{yy}r^2\Delta + 21\varphi_y r_x\Delta + 15\varphi_y r_y r\Delta - 6\varphi_y\Delta_x r + \varphi_y\Delta_y r^2)/(\varphi_y\Delta), \qquad (3.35)\\
H_4 =& -5(4\varphi_{yyy}\varphi_y\Delta - 21\varphi_{yy}^2\Delta + 12\varphi_{yy}\varphi_y\Delta_y - 2\varphi_y^2\Delta_{yy})/(\varphi_y^2\Delta), \qquad (3.36)\\
H_3 =& -20(4\varphi_{yyy}\varphi_y r\Delta - 21\varphi_{yy}^2 r\Delta - 15\varphi_{yy}\varphi_y r_y\Delta + 3\varphi_{yy}\varphi_y\Delta_x \\
& + 9\varphi_{yy}\varphi_y\Delta_y r + 3\varphi_y^2 r_{yy}\Delta + 6\varphi_y^2 r_y\Delta_y - \varphi_y^2\Delta_{xy} - \varphi_y^2\Delta_{yy}r)/(\varphi_y^2\Delta), \qquad (3.37)
\end{aligned}
$$

$$H_2 = -10((6(r_{xy} + 2r_{yy}r - 6r_y^2)\Delta + 12r_y\Delta_x + 18r_y\Delta_y r - 4\Delta_{xy}r - \Delta_{xx} - \Delta_{yy}r^2 + 6r_x\Delta_y)\varphi_y^2 - 3((5r_x\Delta - 6\Delta_y r^2 + (25r_y\Delta - 6\Delta_x)r)\varphi_{yy}\varphi_y - (4\varphi_{yyy}\varphi_y - 21\varphi_{yy}^2)r^2\Delta))/(\varphi_y^2\Delta), \quad (3.38)$$

$$H_1 = 20(((\Delta_{xy}r + \Delta_{xx} + 17r_y^2\Delta - 4r_{yy}r\Delta)r - r_{xx}\Delta - 4r_{xy}r\Delta - 3(3\Delta_x + \Delta_y r)r_y r - (3(\Delta_x + \Delta_y r) - 19r_y\Delta)r_x)\varphi_y^2 - (3((3\Delta_x + \Delta_y r - 10r_y\Delta)r - 5r_x\Delta)\varphi_{yy}\varphi_y + (4\varphi_{yyy}\varphi_y - 21\varphi_{yy}^2)r^2\Delta)r)/(\varphi_y^2\Delta), \quad (3.39)$$

$$H_0 = -5(4\varphi_{yyy}\varphi_y r^4\Delta - 21\varphi_{yy}^2 r^4\Delta - 30\varphi_{yy}\varphi_y r_x r^2\Delta - 30\varphi_{yy}\varphi_y r_y r^3\Delta + 12\varphi_{yy}\varphi_y\Delta_x r^3 + 4\varphi_y^2 r_{xy}r^2\Delta + 4\varphi_y^2 r_{xx}r\Delta - 21\varphi_y^2 r_x^2\Delta - 34\varphi_y^2 r_x r_y r\Delta + 12\varphi_y^2 r_x\Delta_x r + 4\varphi_y^2 r_{yy}r^3\Delta - 17\varphi_y^2 r_y^2 r^2\Delta + 12\varphi_y^2 r_y\Delta_x r^2 - 2\varphi_y^2\Delta_{xx}r^2)/(\varphi_y^2\Delta), \quad (3.40)$$

$$J_2 = -75(3\varphi_{yy}\Delta - \varphi_y\Delta_y)/(\varphi_y\Delta), \quad (3.41)$$

$$J_1 = 15((7\Delta_x + 3\Delta_y r - 42r_y\Delta)\varphi_y - 30\varphi_{yy}r\Delta)/(\varphi_y\Delta), \quad (3.42)$$

$$J_0 = -15(15\varphi_{yy}r^2\Delta + 28\varphi_y r_x\Delta + 14\varphi_y r_y r\Delta - 7\varphi_y\Delta_x r + 2\varphi_y\Delta_y r^2)/(\varphi_y\Delta), \quad (3.43)$$

$$K_4 = (15(4\varphi_{yyy}\varphi_y\Delta - 21\varphi_{yy}^2\Delta + 12\varphi_{yy}\varphi_y\Delta_y - 2\varphi_y^2\Delta_{yy}))/(\varphi_y^2\Delta), \quad (3.44)$$

$$K_3 = 15(16\varphi_{yyy}\varphi_y r\Delta - 84\varphi_{yy}^2 r\Delta - 75\varphi_{yy}\varphi_y r_y\Delta + 15\varphi_{yy}\varphi_y\Delta_x + 33\varphi_{yy}\varphi_y\Delta_y r + 15\varphi_y^2 r_{yy}\Delta + 30\varphi_y^2 r_y\Delta_y - 5\varphi_y^2\Delta_{xy} - 3\varphi_y^2\Delta_{yy}r)/(\varphi_y^2\Delta), \quad (3.45)$$

$$K_2 = (45(3((5\Delta_x + 3\Delta_y r - 20r_y\Delta)r - 5r_x\Delta)\varphi_{yy}\varphi_y + 2(4\varphi_{yyy}\varphi_y - 21\varphi_{yy}^2)r^2\Delta + (6r_{xy}\Delta + 6r_x\Delta_y + 9r_{yy}r\Delta - 36r_y^2\Delta + 12r_y\Delta_x + 12r_y\Delta_y r - 3\Delta_{xy}r - \Delta_{xx})\varphi_y^2))/(\varphi_y^2\Delta), \quad (3.46)$$

$$K_1 = (-15(((6\Delta_{xx} - \Delta_{yy}r^2 + 3\Delta_{xy}r + 83r_y^2\Delta - 16r_{yy}r\Delta)r - 7r_{xx}\Delta - 22r_{xy}r\Delta - 3(17\Delta_x + \Delta_y r)r_y r + (133r_y\Delta - 21\Delta_x - 15\Delta_y r)r_x)\varphi_y^2 - (3((15\Delta_x + \Delta_y r - 45r_y\Delta)r - 30r_x\Delta)\varphi_{yy}\varphi_y + 4(4\varphi_{yyy}\varphi_y - 21\varphi_{yy}^2)r^2\Delta)r))/(\varphi_y^2\Delta), \quad (3.47)$$

$$K_0 = (15(4\varphi_{yyy}\varphi_y r^4\Delta - 21\varphi_{yy}^2 r^4\Delta - 45\varphi_{yy}\varphi_y r_x r^2\Delta - 30\varphi_{yy}\varphi_y r_y r^3\Delta + 15\varphi_{yy}\varphi_y\Delta_x r^3 - 3\varphi_{yy}\varphi_y\Delta_y r^4 + 4\varphi_y^2 r_{xy}r^2\Delta + 7\varphi_y^2 r_{xx}r\Delta - 42\varphi_y^2 r_x^2\Delta - 49\varphi_y^2 r_x r_y r\Delta + 21\varphi_y^2 r_x\Delta_x r - 3\varphi_y^2 r_x\Delta_y r^2 + 4\varphi_y^2 r_{yy}r^3\Delta - 17\varphi_y^2 r_y^2 r^2\Delta + 15\varphi_y^2 r_y\Delta_x r^2 - 3\varphi_y^2 r_y\Delta_y r^3 + \varphi_y^2\Delta_{xy}r^3 - 3\varphi_y^2\Delta_{xx}r^2))/(\varphi_y^2\Delta), \quad (3.48)$$

$$L_6 = -5(3\varphi_{yyyy}\varphi_y^2\Delta - 42\varphi_{yyy}\varphi_{yy}\varphi_y\Delta + 12\varphi_{yyy}\varphi_y^2\Delta_y + 84\varphi_{yy}^3\Delta - 63\varphi_{yy}^2\varphi_y\Delta_y + 18\varphi_{yy}\varphi_y^2\Delta_{yy} - 2\varphi_y^3\Delta_{yyy})/(\varphi_y^3\Delta), \quad (3.49)$$

$$L_5 = -15(6\varphi_{yyyy}\varphi_y^2 r\Delta - 84\varphi_{yyy}\varphi_{yy}\varphi_y r\Delta - 16\varphi_{yyy}\varphi_y^2 r_y\Delta + 4\varphi_{yyy}\varphi_y^2\Delta_x + 20\varphi_{yyy}\varphi_y^2\Delta_y r + 168\varphi_{yy}^3 r\Delta + 84\varphi_{yy}^2\varphi_y r_y\Delta - 21\varphi_{yy}^2\varphi_y\Delta_x - 105\varphi_{yy}^2\varphi_y\Delta_y r - 30\varphi_{yy}\varphi_y^2 r_{yy}\Delta - 60\varphi_{yy}\varphi_y^2 r_y\Delta_y + 12\varphi_{yy}\varphi_y^2\Delta_{xy} + 24\varphi_{yy}\varphi_y^2\Delta_{yy}r + 4\varphi_y^3 r_{yyy}\Delta + 12\varphi_y^3 r_{yy}\Delta_y + 12\varphi_y^3 r_y\Delta_{yy} - 2\varphi_y^3\Delta_{xyy} - 2\varphi_y^3\Delta_{yyy}r)/(\varphi_y^3\Delta), \quad (3.50)$$

$$
\begin{aligned}
L_4 =& -15(15\varphi_{yyyy}\varphi_y^2 r^2\Delta - 210\varphi_{yyy}\varphi_{yy}\varphi_y r^2\Delta - 8\varphi_{yyy}\varphi_y^2 r_x\Delta - 72\varphi_{yyy}\varphi_y^2 r_y r\Delta \\
&+ 20\varphi_{yyy}\varphi_y^2\Delta_x r + 40\varphi_{yyy}\varphi_y^2\Delta_y r^2 + 420\varphi_{yy}^3 r^2\Delta + 42\varphi_{yy}^2\varphi_y r_x\Delta \\
&+ 378\varphi_{yy}^2\varphi_y r_y r\Delta - 105\varphi_{yy}^2\varphi_y\Delta_x r - 210\varphi_{yy}^2\varphi_y\Delta_y r^2 - 30\varphi_{yy}\varphi_y^2 r_{xy}\Delta \\
&- 30\varphi_{yy}\varphi_y^2 r_x\Delta_y - 120\varphi_{yy}\varphi_y^2 r_{yy} r\Delta + 150\varphi_{yy}\varphi_y^2 r_y^2\Delta - 60\varphi_{yy}\varphi_y^2 r_y\Delta_x \\
&- 210\varphi_{yy}\varphi_y^2 r_y\Delta_y r + 48\varphi_{yy}\varphi_y^2\Delta_{xy} r + 6\varphi_{yy}\varphi_y^2\Delta_{xx} + 36\varphi_{yy}\varphi_y^2\Delta_{yy} r^2 \\
&+ 6\varphi_y^3 r_{xyy}\Delta + 12\varphi_y^3 r_{xy}\Delta_y + 6\varphi_y^3 r_x\Delta_{yy} + 14\varphi_y^3 r_{yyy} r\Delta - 66\varphi_y^3 r_{yy} r_y\Delta \\
&+ 12\varphi_y^3 r_{yy}\Delta_x + 36\varphi_y^3 r_{yy}\Delta_y r - 72\varphi_y^3 r_y^2\Delta_y + 24\varphi_y^3 r_y\Delta_{xy} + 30\varphi_y^3 r_y\Delta_{yy} r \\
&- 6\varphi_y^3\Delta_{xyy} r - 2\varphi_y^3\Delta_{xxy} - 2\varphi_y^3\Delta_{yyy} r^2)/(\varphi_y^3\Delta), \qquad (3.51)
\end{aligned}
$$

$$
\begin{aligned}
L_3 =& -10(30\varphi_{yyyy}\varphi_y^2 r^3\Delta - 420\varphi_{yyy}\varphi_{yy}\varphi_y r^3\Delta - 48\varphi_{yyy}\varphi_y^2 r_x r\Delta - 192\varphi_{yyy}\varphi_y^2 r_y r^2\Delta \\
&+ 60\varphi_{yyy}\varphi_y^2\Delta_x r^2 + 60\varphi_{yyy}\varphi_y^2\Delta_y r^3 + 840\varphi_{yy}^3 r^3\Delta + 252\varphi_{yy}^2\varphi_y r_x r\Delta \\
&+ 1008\varphi_{yy}^2\varphi_y r_y r^2\Delta - 315\varphi_{yy}^2\varphi_y\Delta_x r^2 - 315\varphi_{yy}^2\varphi_y\Delta_y r^3 - 150\varphi_{yy}\varphi_y^2 r_{xy} r\Delta \\
&- 15\varphi_{yy}\varphi_y^2 r_{xx}\Delta + 240\varphi_{yy}\varphi_y^2 r_x r_y\Delta - 45\varphi_{yy}\varphi_y^2 r_x\Delta_x - 135\varphi_{yy}\varphi_y^2 r_x\Delta_y r \\
&- 285\varphi_{yy}\varphi_y^2 r_{yy} r^2\Delta + 660\varphi_{yy}\varphi_y^2 r_y^2 r\Delta 315\varphi_{yy}\varphi_y^2 r_y\Delta_x r - 405\varphi_{yy}\varphi_y^2 r_y\Delta_y r^2 \\
&+ 108\varphi_{yy}\varphi_y^2\Delta_{xy} r^2 + 36\varphi_{yy}\varphi_y^2\Delta_{xx} r + 36\varphi_{yy}\varphi_y^2\Delta_{yy} r^3 + 24\varphi_y^3 r_{xyy} r\Delta \\
&- 108\varphi_y^3 r_{xy} r_y\Delta + 18\varphi_y^3 r_{xy}\Delta_x + 42\varphi_y^3 r_{xy}\Delta_y r + 6\varphi_y^3 r_{xxy}\Delta + 6\varphi_y^3 r_{xx}\Delta_y \\
&- 51\varphi_y^3 r_x r_{yy}\Delta - 114\varphi_y^3 r_x r_y\Delta_y + 18\varphi_y^3 r_x\Delta_{xy} + 18\varphi_y^3 r_x\Delta_{yy} r + 30\varphi_y^3 r_{yyy} r^2\Delta \\
&- 237\varphi_y^3 r_{yy} r_y r\Delta + 54\varphi_y^3 r_{yy}\Delta_x r + 60\varphi_y^3 r_{yy}\Delta_y r^2 + 216\varphi_y^3 r_y^3\Delta - 108\varphi_y^3 r_y^2\Delta_x \\
&- 210\varphi_y^3 r_y^2\Delta_y r + 90\varphi_y^3 r_y\Delta_{xy} r + 18\varphi_y^3 r_y\Delta_{xx} + 36\varphi_y^3 r_y\Delta_{yy} r^2 - 9\varphi_y^3\Delta_{xyy} r^2 \\
&- \varphi_y^3\Delta_{xxx} - 9\varphi_y^3\Delta_{xxy} r - \varphi_y^3\Delta_{yyy} r^3)/(\varphi_y^3\Delta), \qquad (3.52)
\end{aligned}
$$

$$
\begin{aligned}
L_2 =& -15(((3(5(r_{xyy} + r_{yyy} r)\Delta - 2\Delta_{xxy})r - 5(28\Delta_x + 17\Delta_y r)r_y^2 + 6(5\Delta_{xx} + \Delta_{yy} r^2 \\
&+ 8\Delta_{xy} r)r_y)r + (4(\Delta_x + 2\Delta_y r) - 25r_y\Delta)r_{xx} + (6(\Delta_{xx} + \Delta_{yy} r^2) + 241 r_y^2\Delta \\
&- (63 r_{yy} r\Delta + 76 r_y\Delta_x + 110 r_y\Delta_y r - 24\Delta_{xy} r))r_x + (r_{xxx} + 9 r_{xxy} r)\Delta \\
&- 2(\Delta_{xyy} r^2 + \Delta_{xxx})r + 191 r_y^3 r\Delta - 21 r_x^2\Delta_y + 2(10(2\Delta_x + \Delta_y r) - 71 r_y\Delta)r_{yy} r^2 \\
&- ((39 r_x + 127 r_y r)\Delta - 4(7\Delta_x + 5\Delta_y r)r)r_{xy})\varphi_y^3 - (3((10 r_{xx}\Delta + 50 r_{yy} r^2\Delta \\
&- 155 r_y^2 r\Delta - 12\Delta_{xx} r + 10(9\Delta_x + 5\Delta_y r)r_y r + 10(3(\Delta_x + \Delta_y r) - 13 r_y\Delta)r_x)r \\
&- (15 r_x^2\Delta + 2\Delta_{yy} r^4 - 40 r_{xy} r^2\Delta + 16\Delta_{xy} r^3))\varphi_{yy}\varphi_y^2 - (3(7(4(3r_x + 7r_y r)\Delta \\
&- 5(2\Delta_x + \Delta_y r)r)\varphi_{yy}^2\varphi_y + 5(\varphi_{yyyy}\varphi_y^2 + 28\varphi_{yy}^3)r^2\Delta) - 2(2(4(3r_x + 7r_y r)\Delta \\
&- 5(2\Delta_x + \Delta_y r)r)\varphi_y + 105\varphi_{yy} r^2\Delta)\varphi_{yyy}\varphi_y)r^2))/(\varphi_y^3\Delta), \qquad (3.53)
\end{aligned}
$$

$$
\begin{aligned}
L_1 =&15(((2((\Delta_{xxx} + \Delta_{xxy}r - 45r_y^3\Delta - 3r_{yyy}r^2\Delta - 3r_{xxy}\Delta)r - r_{xxx}\Delta - 3r_{xyy}r^2\Delta) \\
&+ 17(5\Delta_x + \Delta_y r)r_y^2 r - 12(\Delta_{xy}r + 2\Delta_{xx})r_y r)r + 7(3(\Delta_x + \Delta_y r) - 20r_y\Delta)r_x^2 \\
&- 4(5\Delta_x + \Delta_y r - 15r_y\Delta)r_{yy}r^3 - 2(2(2\Delta_x + \Delta_y r - 9r_y\Delta)r - 7r_x\Delta)r_{xx} \\
&- 2((2(5\Delta_x + \Delta_y r) - 33r_y\Delta)r - 25r_x\Delta)r_{xy}r - 2(6(\Delta_{xy}r + \Delta_{xx}) + 101r_y^2\Delta \\
&- 19r_{yy}r\Delta - (55\Delta_x + 17\Delta_y r)r_y)r_x r)\varphi_y^3 - (6((2(\Delta_{xy}r + 2\Delta_{xx}) + 35r_y^2\Delta \\
&- 10r_{yy}r\Delta)r^2 + 15r_x^2\Delta - 5r_{xx}r\Delta - 10r_{xy}r^2\Delta - 5(5\Delta_x + \Delta_y r)r_y r^2 \\
&- 5(3\Delta_x + \Delta_y r - 10r_y\Delta)r_x r)\varphi_{yy}\varphi_y^2 - (3(7((5\Delta_x + \Delta_y r - 12r_y\Delta)r \\
&- 8r_x\Delta)\varphi_{yy}^2\varphi_y - 2(\varphi_{yyyy}\varphi_y^2 + 28\varphi_{yy}^3)r^2\Delta) - 4(((5\Delta_x + \Delta_y r - 12r_y\Delta)r \\
&- 8r_x\Delta)\varphi_y - 21\varphi_{yy}r^2\Delta)\varphi_{yyy}\varphi_y)r^2)r)/(\varphi_y^3\Delta), \qquad (3.54)
\end{aligned}
$$

$$
\begin{aligned}
L_0 =& - 5(3\varphi_{yyyy}\varphi_y^2 r^6\Delta - 42\varphi_{yyy}\varphi_{yy}\varphi_y r^6\Delta - 24\varphi_{yyy}\varphi_y^2 r_x r^4\Delta - 24\varphi_{yyy}\varphi_y^2 r_y r^5\Delta \\
&+ 12\varphi_{yyy}\varphi_y^2\Delta_x r^5 + 84\varphi_{yy}^3 r^6\Delta + 126\varphi_{yy}^2\varphi_y r_x r^4\Delta + 126\varphi_{yy}^2\varphi_y r_y r^5\Delta \\
&- 63\varphi_{yy}^2\varphi_y\Delta_x r^5 - 30\varphi_{yy}\varphi_y^2 r_{xy}r^4\Delta - 30\varphi_{yy}\varphi_y^2 r_{xx}r^3\Delta + 135\varphi_{yy}\varphi_y^2 r_x^2 r^2\Delta \\
&+ 210\varphi_{yy}\varphi_y^2 r_x r_y r^3\Delta - 90\varphi_{yy}\varphi_y^2 r_x\Delta_x r^3 - 30\varphi_{yy}\varphi_y^2 r_{yy}r^5\Delta + 105\varphi_{yy}\varphi_y^2 r_y^2 r^4\Delta \\
&- 90\varphi_{yy}\varphi_y^2 r_y\Delta_x r^4 + 18\varphi_{yy}\varphi_y^2\Delta_{xx}r^4 + 3\varphi_y^3 r_{xyy}r^4\Delta - 33\varphi_y^3 r_{xy}r_x r^2\Delta \\
&- 33\varphi_y^3 r_{xy}r_y r^3\Delta + 12\varphi_y^3 r_{xy}\Delta_x r^3 + 3\varphi_y^3 r_{xxx}r^2\Delta + 3\varphi_y^3 r_{xxy}r^3\Delta - 42\varphi_y^3 r_{xx}r_x r\Delta \\
&- 33\varphi_y^3 r_{xx}r_y r^2\Delta + 12\varphi_y^3 r_{xx}\Delta_x r^2 + 84\varphi_y^3 r_x^3\Delta + 168\varphi_y^3 r_x^2 r_y r\Delta - 63\varphi_y^3 r_x^2\Delta_x r \\
&- 27\varphi_y^3 r_x r_{yy}r^3\Delta + 135\varphi_y^3 r_x r_y^2 r^2\Delta - 102\varphi_y^3 r_x r_y\Delta_x r^2 + 18\varphi_y^3 r_x\Delta_{xx}r^2 \\
&+ 3\varphi_y^3 r_{yyy}r^5\Delta - 30\varphi_y^3 r_{yy}r_y r^4\Delta + 12\varphi_y^3 r_{yy}\Delta_x r^4 + 45\varphi_y^3 r_y^3 r^3\Delta - 51\varphi_y^3 r_y^2\Delta_x r^3 \\
&+ 18\varphi_y^3 r_y\Delta_{xx}r^3 - 2\varphi_y^3\Delta_{xxx}r^3)/(\varphi_y^3\Delta), \qquad (3.55)
\end{aligned}
$$

$$
\begin{aligned}
M_9 =& - (\varphi_{yyyyy}\varphi_y^3\psi_y - 15\varphi_{yyyy}\varphi_{yy}\varphi_y^2\psi_y + 5\varphi_{yyyy}\varphi_y^3\psi_{yy} - 10\varphi_{yyy}^2\varphi_y^2\psi_y \\
&+ 105\varphi_{yyy}\varphi_{yy}^2\varphi_y\psi_y - 60\varphi_{yyy}\varphi_{yy}\varphi_y^2\psi_{yy} + 10\varphi_{yyy}\varphi_y^3\psi_{yyy} - 105\varphi_{yy}^4\psi_y \\
&+ 105\varphi_{yy}^3\varphi_y\psi_{yy} - 45\varphi_{yy}^2\varphi_y^2\psi_{yyy} + 10\varphi_{yy}\varphi_y^3\psi_{yyyy} - \varphi_y^4\psi_{yyyyy})/(\varphi_y^3\Delta), \qquad (3.56)
\end{aligned}
$$

$$
\begin{aligned}
M_8 =& - (9\varphi_{yyyyy}\varphi_y^4\psi_y r - 6\varphi_{yyyyy}\varphi_y^3\Delta - 135\varphi_{yyyy}\varphi_{yy}\varphi_y^3\psi_y r + 105\varphi_{yyyy}\varphi_{yy}\varphi_y^2\Delta \\
&+ 45\varphi_{yyyy}\varphi_y^4\psi_{yy}r - 30\varphi_{yyyy}\varphi_y^3\Delta_y - 90\varphi_{yyy}^2\varphi_y^3\psi_y r + 70\varphi_{yyy}^2\varphi_y^2\Delta \\
&+ 945\varphi_{yyy}\varphi_{yy}^2\varphi_y^2\psi_y r - 840\varphi_{yyy}\varphi_{yy}^2\varphi_y\Delta - 540\varphi_{yyy}\varphi_{yy}\varphi_y^3\psi_{yy}r + 420\varphi_{yyy}\varphi_{yy}\varphi_y^2\Delta_y \\
&+ 90\varphi_{yyy}\varphi_y^4\psi_{yyy}r - 60\varphi_{yyy}\varphi_y^3\Delta_{yy} - 945\varphi_{yy}^4\varphi_y\psi_y r + 945\varphi_{yy}^4\Delta + 945\varphi_{yy}^3\varphi_y^2\psi_{yy}r \\
&- 840\varphi_{yy}^3\varphi_y\Delta_y - 405\varphi_{yy}^2\varphi_y^3\psi_{yyy}r + 315\varphi_{yy}^2\varphi_y^2\Delta_{yy} + 90\varphi_{yy}\varphi_y^4\psi_{yyyy}r \\
&- 60\varphi_{yy}\varphi_y^3\Delta_{yyy} - 9\varphi_y^5\psi_{yyyyy}r + 5\varphi_y^4\Delta_{yyyy})/(\varphi_y^4\Delta), \qquad (3.57)
\end{aligned}
$$

$$
\begin{aligned}
M_7 = &-(36\varphi_{yyyyyy}\varphi_y^4\psi_y r^2 - 48\varphi_{yyyyyy}\varphi_y^3 r\Delta - 540\varphi_{yyyyy}\varphi_{yy}\varphi_y^3\psi_y r^2 + 840\varphi_{yyyyy}\varphi_{yy}\varphi_y^2 r\Delta \\
&+ 180\varphi_{yyyyy}\varphi_y^4\psi_{yy} r^2 + 45\varphi_{yyyyy}\varphi_y^3 r_y\Delta - 15\varphi_{yyyyy}\varphi_y^3\Delta_x - 225\varphi_{yyyyy}\varphi_y^3\Delta_y r \\
&- 360\varphi_{yyy}^2\varphi_y^3\psi_y r^2 + 560\varphi_{yyy}^2\varphi_y^2 r\Delta + 3780\varphi_{yyy}\varphi_{yy}^2\varphi_y^2\psi_y r^2 - 6720\varphi_{yyy}\varphi_{yy}^2\varphi_y r\Delta \\
&- 2160\varphi_{yyy}\varphi_{yy}\varphi_y^3\psi_{yy} r^2 - 630\varphi_{yyy}\varphi_{yy}\varphi_y^2 r_y\Delta + 210\varphi_{yyy}\varphi_{yy}\varphi_y^2\Delta_x + 3150\varphi_{yyy}\varphi_{yy}\varphi_y^2\Delta_y r \\
&+ 360\varphi_{yyy}\varphi_y^4\psi_{yyy} r^2 + 120\varphi_{yyy}\varphi_y^3 r_{yy}\Delta + 240\varphi_{yyy}\varphi_y^3 r_y\Delta_y - 60\varphi_{yyy}\varphi_y^3\Delta_{xy} \\
&- 420\varphi_{yyy}\varphi_y^3\Delta_{yy} r - 3780\varphi_{yy}^4\varphi_y\psi_y r^2 + 7560\varphi_{yy}^4 r\Delta + 3780\varphi_{yy}^3\varphi_y^2\psi_{yy} r^2 \\
&+ 1260\varphi_{yy}^3\varphi_y r_y\Delta - 420\varphi_{yy}^3\varphi_y\Delta_x - 6300\varphi_{yy}^3\varphi_y\Delta_y r - 1620\varphi_{yy}^2\varphi_y^3\psi_{yyy} r^2 \\
&- 630\varphi_{yy}^2\varphi_y^2 r_{yy}\Delta - 1260\varphi_{yy}^2\varphi_y^2 r_y\Delta_y + 315\varphi_{yy}^2\varphi_y^2\Delta_{xy} + 2205\varphi_{yy}^2\varphi_y^2\Delta_{yy} r \\
&+ 360\varphi_{yy}\varphi_y^4\psi_{yyyy} r^2 + 150\varphi_{yy}\varphi_y^3 r_{yyy}\Delta + 450\varphi_{yy}\varphi_y^3 r_{yy}\Delta_y + 450\varphi_{yy}\varphi_y^3 r_y\Delta_{yy} \\
&- 90\varphi_{yy}\varphi_y^3\Delta_{xyy} - 390\varphi_{yy}\varphi_y^3\Delta_{yyy} r - 36\varphi_y^5\psi_{yyyyy} r^2 - 15\varphi_y^4 r_{yyyy}\Delta - 60\varphi_y^4 r_{yyy}\Delta_y \\
&- 90\varphi_y^4 r_{yy}\Delta_{yy} - 60\varphi_y^4 r_y\Delta_{yyy} + 10\varphi_y^4\Delta_{xyyy} + 30\varphi_y^4\Delta_{yyyy} r)/(\varphi_y^4\Delta), \qquad (3.58)
\end{aligned}
$$

$$
\begin{aligned}
M_6 = &-(84\varphi_{yyyyyy}\varphi_y^4\psi_y r^3 - 168\varphi_{yyyyyy}\varphi_y^3 r^2\Delta - 1260\varphi_{yyyyy}\varphi_{yy}\varphi_y^3\psi_y r^3 + 2940\varphi_{yyyyy}\varphi_{yy}\varphi_y^2 r^2\Delta \\
&+ 420\varphi_{yyyyy}\varphi_y^4\psi_{yy} r^3 + 15\varphi_{yyyyy}\varphi_y^3 r_x\Delta + 300\varphi_{yyyyy}\varphi_y^3 r_y r\Delta - 105\varphi_{yyyyy}\varphi_y^3\Delta_x r \\
&- 735\varphi_{yyyyy}\varphi_y^3\Delta_y r^2 - 840\varphi_{yyy}^2\varphi_y^3\psi_y r^3 + 1960\varphi_{yyy}^2\varphi_y^2 r^2\Delta + 8820\varphi_{yyy}\varphi_{yy}^2\varphi_y^2\psi_y r^3 \\
&- 23520\varphi_{yyy}\varphi_{yy}^2\varphi_y r^2\Delta - 5040\varphi_{yyy}\varphi_{yy}\varphi_y^3\psi_{yy} r^3 - 210\varphi_{yyy}\varphi_{yy}\varphi_y^2 r_x\Delta \\
&- 4200\varphi_{yyy}\varphi_{yy}\varphi_y^2 r_y r\Delta + 1470\varphi_{yyy}\varphi_{yy}\varphi_y^2\Delta_x r + 10290\varphi_{yyy}\varphi_{yy}\varphi_y^2\Delta_y r^2 \\
&+ 840\varphi_{yyy}\varphi_y^4\psi_{yyy} r^3 + 80\varphi_{yyy}\varphi_y^3 r_{xy}\Delta + 80\varphi_{yyy}\varphi_y^3 r_x\Delta_y + 760\varphi_{yyy}\varphi_y^3 r_{yy} r\Delta \\
&- 320\varphi_{yyy}\varphi_y^3 r_y^2\Delta + 160\varphi_{yyy}\varphi_y^3 r_y\Delta_x + 1440\varphi_{yyy}\varphi_y^3 r_y\Delta_y r - 380\varphi_{yyy}\varphi_y^3\Delta_{xy} r \\
&- 20\varphi_{yyy}\varphi_y^3\Delta_{xx} - 1280\varphi_{yyy}\varphi_y^3\Delta_{yy} r^2 - 8820\varphi_{yy}^4\varphi_y\psi_y r^3 + 26460\varphi_{yy}^4 r^2\Delta \\
&+ 8820\varphi_{yy}^3\varphi_y^2\psi_{yy} r^3 + 420\varphi_{yy}^3\varphi_y r_x\Delta + 8400\varphi_{yy}^3\varphi_y r_y r\Delta - 2940\varphi_{yy}^3\varphi_y\Delta_x r \\
&- 20580\varphi_{yy}^3\varphi_y\Delta_y r^2 - 3780\varphi_{yy}^2\varphi_y^3\psi_{yyy} r^3 - 420\varphi_{yy}^2\varphi_y^2 r_{xy}\Delta - 420\varphi_{yy}^2\varphi_y^2 r_x\Delta_y \\
&- 3990\varphi_{yy}^2\varphi_y^2 r_{yy} r\Delta + 1680\varphi_{yy}^2\varphi_y^2 r_y^2\Delta - 840\varphi_{yy}^2\varphi_y^2 r_y\Delta_x - 7560\varphi_{yy}^2\varphi_y^2 r_y\Delta_y r \\
&+ 1995\varphi_{yy}^2\varphi_y^2\Delta_{xy} r + 105\varphi_{yy}^2\varphi_y^2\Delta_{xx} + 6720\varphi_{yy}^2\varphi_y^2\Delta_{yy} r^2 + 840\varphi_{yy}\varphi_y^4\psi_{yyyy} r^3 \\
&+ 150\varphi_{yy}\varphi_y^3 r_{xyy}\Delta + 300\varphi_{yy}\varphi_y^3 r_{xy}\Delta_y + 150\varphi_{yy}\varphi_y^3 r_x\Delta_{yy} + 900\varphi_{yy}\varphi_y^3 r_{yyy} r\Delta \\
&- 1350\varphi_{yy}\varphi_y^3 r_{yy} r_y\Delta + 300\varphi_{yy}\varphi_y^3 r_{yy}\Delta_x + 2550\varphi_{yy}\varphi_y^3 r_{yy}\Delta_y r - 1500\varphi_{yy}\varphi_y^3 r_y^2\Delta_y \\
&+ 600\varphi_{yy}\varphi_y^3 r_y\Delta_{xy} + 2400\varphi_{yy}\varphi_y^3 r_y\Delta_{yy} r - 510\varphi_{yy}\varphi_y^3\Delta_{xyy} r - 60\varphi_{yy}\varphi_y^3\Delta_{xxy} \\
&- 1110\varphi_{yy}\varphi_y^3\Delta_{yyy} r^2 - 84\varphi_y^5\psi_{yyyyy} r^3 - 20\varphi_y^4 r_{xyyy}\Delta - 60\varphi_y^4 r_{xyy}\Delta_y - 60\varphi_y^4 r_{xy}\Delta_{yy} \\
&- 20\varphi_y^4 r_x\Delta_{yyy} - 85\varphi_y^4 r_{yyyy} r\Delta + 200\varphi_y^4 r_{yyy} r_y\Delta - 40\varphi_y^4 r_{yyy}\Delta_x - 320\varphi_y^4 r_{yyy}\Delta_y r \\
&+ 150\varphi_y^4 r_{yy}^2\Delta + 660\varphi_y^4 r_{yy} r_y\Delta_y - 120\varphi_y^4 r_{yy}\Delta_{xy} - 450\varphi_y^4 r_{yy}\Delta_{yy} r + 360\varphi_y^4 r_y^2\Delta_{yy} \\
&- 120\varphi_y^4 r_y\Delta_{xyy} - 280\varphi_y^4 r_y\Delta_{yyy} r + 50\varphi_y^4\Delta_{xyyy} r + 10\varphi_y^4\Delta_{xxyy} + 80\varphi_y^4\Delta_{yyyy} r^2)/(\varphi_y^4\Delta),
\end{aligned}
$$

(3.59)

$$
\begin{aligned}
M_5 = &-(126\varphi_{yyyyyy}\varphi_y^4\psi_y r^4 - 336\varphi_{yyyyyy}\varphi_y^3 r^3\Delta - 1890\varphi_{yyyyy}\varphi_{yy}\varphi_y^3\psi_y r^4 + 5880\varphi_{yyyyy}\varphi_{yy}\varphi_y^2 r^3\Delta \\
&+ 630\varphi_{yyyyy}\varphi_y^4\psi_{yy}r^4 + 90\varphi_{yyyyy}\varphi_y^3 r_x r\Delta + 855\varphi_{yyyyy}\varphi_y^3 r_y r^2\Delta - 315\varphi_{yyyyy}\varphi_y^3\Delta_x r^2 \\
&- 1365\varphi_{yyyyy}\varphi_y^3\Delta_y r^3 - 1260\varphi_{yyyy}^2\varphi_y^3\psi_y r^4 + 3920\varphi_{yyyy}^2\varphi_y^2 r^3\Delta + 13230\varphi_{yyyy}\varphi_{yy}^2\varphi_y^2\psi_y r^4 \\
&- 47040\varphi_{yyyy}\varphi_{yy}^2\varphi_y r^3\Delta - 7560\varphi_{yyyy}\varphi_{yy}\varphi_y^3\psi_{yy}r^4 - 1260\varphi_{yyyy}\varphi_{yy}\varphi_y^2 r_x r\Delta \\
&- 11970\varphi_{yyyy}\varphi_{yy}\varphi_y^2 r_y r^2\Delta + 4410\varphi_{yyyy}\varphi_{yy}\varphi_y^2\Delta_x r^2 + 19110\varphi_{yyyy}\varphi_{yy}\varphi_y^2\Delta_y r^3 \\
&+ 1260\varphi_{yyyy}\varphi_y^4\psi_{yyy}r^4 + 440\varphi_{yyyy}\varphi_y^3 r_{xy}r\Delta + 20\varphi_{yyyy}\varphi_y^3 r_{xx}\Delta - 260\varphi_{yyyy}\varphi_y^3 r_x r_y\Delta \\
&+ 60\varphi_{yyyy}\varphi_y^3 r_x\Delta_x + 420\varphi_{yyyy}\varphi_y^3 r_x\Delta_y r + 2060\varphi_{yyyy}\varphi_y^3 r_{yy}r^2\Delta - 1660\varphi_{yyyy}\varphi_y^3 r_y^2 r\Delta \\
&+ 900\varphi_{yyyy}\varphi_y^3 r_y\Delta_x r + 3660\varphi_{yyyy}\varphi_y^3 r_y\Delta_y r^2 - 1020\varphi_{yyyy}\varphi_y^3\Delta_{xy}r^2 - 120\varphi_{yyyy}\varphi_y^3\Delta_{xx}r \\
&- 2220\varphi_{yyyy}\varphi_y^3\Delta_{yy}r^3 - 13230\varphi_{yy}^4\varphi_y\psi_y r^4 + 52920\varphi_{yy}^4 r^3\Delta + 13230\varphi_{yy}^3\varphi_y^2\psi_{yy}r^4 \\
&+ 2520\varphi_{yy}^3\varphi_y r_x r\Delta + 23940\varphi_{yy}^3\varphi_y r_y r^2\Delta - 8820\varphi_{yy}^3\varphi_y\Delta_x r^2 - 38220\varphi_{yy}^3\varphi_y\Delta_y r^3 \\
&- 5670\varphi_{yy}^2\varphi_y^3\psi_{yyy}r^4 - 2310\varphi_{yy}^2\varphi_y^2 r_{xy}r\Delta - 105\varphi_{yy}^2\varphi_y^2 r_{xx}\Delta + 1365\varphi_{yy}^2\varphi_y^2 r_x r_y\Delta \\
&- 315\varphi_{yy}^2\varphi_y^2 r_x\Delta_x - 2205\varphi_{yy}^2\varphi_y^2 r_x\Delta_y r - 10815\varphi_{yy}^2\varphi_y^2 r_{yy}r^2\Delta + 8715\varphi_{yy}^2\varphi_y^2 r_y^2 r\Delta \\
&- 4725\varphi_{yy}^2\varphi_y^2 r_y\Delta_x r - 19215\varphi_{yy}^2\varphi_y^2 r_y\Delta_y r^2 + 5355\varphi_{yy}^2\varphi_y^2\Delta_{xy}r^2 + 630\varphi_{yy}^2\varphi_y^2\Delta_{xx}r \\
&+ 11655\varphi_{yy}^2\varphi_y^2\Delta_{yy}r^3 + 1260\varphi_{yy}\varphi_y^4\psi_{yyyy}r^4 + 750\varphi_{yy}\varphi_y^3 r_{xyy}r\Delta - 1125\varphi_{yy}\varphi_y^3 r_{xy}r_y\Delta \\
&+ 225\varphi_{yy}\varphi_y^3 r_{xy}\Delta_x + 1425\varphi_{yy}\varphi_y^3 r_{xy}\Delta_y r + 75\varphi_{yy}\varphi_y^3 r_{xxy}\Delta + 75\varphi_{yy}\varphi_y^3 r_{xx}\Delta_y \\
&- 525\varphi_{yy}\varphi_y^3 r_x r_{yy}\Delta - 1200\varphi_{yy}\varphi_y^3 r_x r_y\Delta_y + 225\varphi_{yy}\varphi_y^3 r_x\Delta_{xy} + 675\varphi_{yy}\varphi_y^3 r_x\Delta_{yy}r \\
&+ 2325\varphi_{yy}\varphi_y^3 r_{yyy}r^2\Delta - 6450\varphi_{yy}\varphi_y^3 r_{yy}r_y r\Delta + 1575\varphi_{yy}\varphi_y^3 r_{yy}\Delta_x r + 6150\varphi_{yy}\varphi_y^3 r_{yy}\Delta_y r^2 \\
&+ 1875\varphi_{yy}\varphi_y^3 r_y^3\Delta - 1125\varphi_{yy}\varphi_y^3 r_y^2\Delta_x - 6675\varphi_{yy}\varphi_y^3 r_y^2\Delta_y r + 2925\varphi_{yy}\varphi_y^3 r_y\Delta_{xy}r \\
&+ 225\varphi_{yy}\varphi_y^3 r_y\Delta_{xx} + 5400\varphi_{yy}\varphi_y^3 r_y\Delta_{yy}r^2 - 1215\varphi_{yy}\varphi_y^3\Delta_{xyy}r^2 - 15\varphi_{yy}\varphi_y^3\Delta_{xxx} \\
&- 315\varphi_{yy}\varphi_y^3\Delta_{xxy}r - 1815\varphi_{yy}\varphi_y^3\Delta_{yyy}r^3 - 126\varphi_y^5\psi_{yyyyy}r^4 - 90\varphi_y^4 r_{xyyy}r\Delta + 255\varphi_y^4 r_{xyy}r_y\Delta \\
&- 45\varphi_y^4 r_{xyy}\Delta_x - 255\varphi_y^4 r_{xyy}\Delta_y r + 240\varphi_y^4 r_{xy}r_{yy}\Delta + 540\varphi_y^4 r_{xy}r_y\Delta_y - 90\varphi_y^4 r_{xy}\Delta_{xy} \\
&- 240\varphi_y^4 r_{xy}\Delta_{yy}r - 15\varphi_y^4 r_{xxyy}\Delta - 30\varphi_y^4 r_{xxy}\Delta_y - 15\varphi_y^4 r_{xx}\Delta_{yy} + 75\varphi_y^4 r_x r_{yyy}\Delta \\
&+ 255\varphi_y^4 r_x r_{yy}\Delta_y + 285\varphi_y^4 r_x r_y\Delta_{yy} - 45\varphi_y^4 r_x\Delta_{xyy} - 75\varphi_y^4 r_x\Delta_{yyy}r - 210\varphi_y^4 r_{yyyy}r^2\Delta \\
&+ 870\varphi_y^4 r_{yyy}r_y r\Delta - 195\varphi_y^4 r_{yyy}\Delta_x r - 735\varphi_y^4 r_{yyy}\Delta_y r^2 + 660\varphi_y^4 r_{yy}^2 r\Delta - 1365\varphi_y^4 r_{yy}r_y^2\Delta \\
&+ 495\varphi_y^4 r_{yy}r_y\Delta_x + 2670\varphi_y^4 r_{yy}r_y\Delta_y r - 540\varphi_y^4 r_{yy}\Delta_{xy}r - 45\varphi_y^4 r_{yy}\Delta_{xx} - 960\varphi_y^4 r_{yy}\Delta_{yy}r^2 \\
&- 1080\varphi_y^4 r_y^3\Delta_y + 540\varphi_y^4 r_y^2\Delta_{xy} + 1335\varphi_y^4 r_y^2\Delta_{yy}r - 495\varphi_y^4 r_y\Delta_{xyy}r - 90\varphi_y^4 r_y\Delta_{xxy} \\
&- 555\varphi_y^4 r_y\Delta_{yyy}r^2 + 105\varphi_y^4\Delta_{xyyy}r^2 + 5\varphi_y^4\Delta_{xxxy} + 45\varphi_y^4\Delta_{xxyy}r + 125\varphi_y^4\Delta_{yyyy}r^3)/(\varphi_y^4\Delta),
\end{aligned}
\tag{3.60}
$$

$$
\begin{aligned}
M_4 = & -(126\varphi_{yyyyy}\varphi_y^4\psi_y r^5 - 420\varphi_{yyyyy}\varphi_y^3 r^4\Delta - 1890\varphi_{yyyy}\varphi_{yy}\varphi_y^3\psi_y r^5 + 7350\varphi_{yyyy}\varphi_{yy}\varphi_y^2 r^4\Delta \\
& + 630\varphi_{yyyy}\varphi_y^4\psi_{yy} r^5 + 225\varphi_{yyyy}\varphi_y^3 r_x r^2\Delta + 1350\varphi_{yyyy}\varphi_y^3 r_y r^3\Delta - 525\varphi_{yyyy}\varphi_y^3\Delta_x r^3 \\
& - 1575\varphi_{yyyy}\varphi_y^3\Delta_y r^4 - 1260\varphi_{yyy}^2\varphi_y^3\psi_y r^5 + 4900\varphi_{yyy}^2\varphi_y^2 r^4\Delta + 13230\varphi_{yyy}\varphi_{yy}^2\varphi_y^2\psi_y r^5 \\
& - 58800\varphi_{yyy}\varphi_{yy}^2\varphi_y r^4\Delta - 7560\varphi_{yyy}\varphi_{yy}\varphi_y^3\psi_{yy} r^5 - 3150\varphi_{yyy}\varphi_{yy}\varphi_y^2 r_x r^2\Delta \\
& - 18900\varphi_{yyy}\varphi_{yy}\varphi_y^2 r_y r^3\Delta + 7350\varphi_{yyy}\varphi_{yy}\varphi_y^2\Delta_x r^3 + 22050\varphi_{yyy}\varphi_{yy}\varphi_y^2\Delta_y r^4 \\
& + 1260\varphi_{yyy}\varphi_y^4\psi_{yyy} r^5 + 1000\varphi_{yyy}\varphi_y^3 r_{xy} r^2\Delta + 100\varphi_{yyy}\varphi_y^3 r_{xx} r\Delta - 60\varphi_{yyy}\varphi_y^3 r_x^2\Delta \\
& - 1180\varphi_{yyy}\varphi_y^3 r_x r_y r\Delta + 300\varphi_{yyy}\varphi_y^3 r_x\Delta_x r + 900\varphi_{yyy}\varphi_y^3 r_x\Delta_y r^2 + 3100\varphi_{yyy}\varphi_y^3 r_{yy} r^3\Delta \\
& - 3560\varphi_{yyy}\varphi_y^3 r_y^2 r^2\Delta + 2100\varphi_{yyy}\varphi_y^3 r_y\Delta_x r^2 + 5100\varphi_{yyy}\varphi_y^3 r_y\Delta_y r^3 - 1500\varphi_{yyy}\varphi_y^3\Delta_{xy} r^3 \\
& - 300\varphi_{yyy}\varphi_y^3\Delta_{xx} r^2 - 2400\varphi_{yyy}\varphi_y^3\Delta_{yy} r^4 - 13230\varphi_{yy}^4\varphi_y\psi_y r^5 + 66150\varphi_{yy}^4 r^4\Delta \\
& + 13230\varphi_{yy}^3\varphi_y^2\psi_{yy} r^5 + 6300\varphi_{yy}^3\varphi_y r_x r^2\Delta + 37800\varphi_{yy}^3\varphi_y r_y r^3\Delta - 14700\varphi_{yy}^3\varphi_y\Delta_x r^3 \\
& - 44100\varphi_{yy}^3\varphi_y\Delta_y r^4 - 5670\varphi_{yy}^2\varphi_y^3\psi_{yyy} r^5 - 5250\varphi_{yy}^2\varphi_y^2 r_{xy} r^2\Delta - 525\varphi_{yy}^2\varphi_y^2 r_{xx} r\Delta \\
& + 315\varphi_{yy}^2\varphi_y^2 r_x^2\Delta + 6195\varphi_{yy}^2\varphi_y^2 r_x r_y r\Delta - 1575\varphi_{yy}^2\varphi_y^2 r_x\Delta_x r - 4725\varphi_{yy}^2\varphi_y^2 r_x\Delta_y r^2 \\
& - 16275\varphi_{yy}^2\varphi_y^2 r_{yy} r^3\Delta + 18690\varphi_{yy}^2\varphi_y^2 r_y^2 r^2\Delta - 11025\varphi_{yy}^2\varphi_y^2 r_y\Delta_x r^2 - 26775\varphi_{yy}^2\varphi_y^2 r_y\Delta_y r^3 \\
& + 7875\varphi_{yy}^2\varphi_y^2\Delta_{xy} r^3 + 1575\varphi_{yy}^2\varphi_y^2\Delta_{xx} r^2 + 12600\varphi_{yy}^2\varphi_y^2\Delta_{yy} r^4 + 1260\varphi_{yy}\varphi_y^4\psi_{yyyy} r^5 \\
& + 1545\varphi_{yy}\varphi_y^3 r_{xyy} r^2\Delta - 495\varphi_{yy}\varphi_y^3 r_{xy} r_x\Delta - 4500\varphi_{yy}\varphi_y^3 r_{xy} r_y r\Delta + 1005\varphi_{yy}\varphi_y^3 r_{xy}\Delta_x r \\
& + 2745\varphi_{yy}\varphi_y^3 r_{xy}\Delta_y r^2 + 15\varphi_{yy}\varphi_y^3 r_{xxx}\Delta + 330\varphi_{yy}\varphi_y^3 r_{xxy} r\Delta - 315\varphi_{yy}\varphi_y^3 r_{xx} r_y\Delta \\
& + 60\varphi_{yy}\varphi_y^3 r_{xx}\Delta_x + 315\varphi_{yy}\varphi_y^3 r_{xx}\Delta_y r - 270\varphi_{yy}\varphi_y^3 r_x^2\Delta_y - 2130\varphi_{yy}\varphi_y^3 r_x r_{yy} r\Delta \\
& + 2565\varphi_{yy}\varphi_y^3 r_x r_y^2\Delta - 960\varphi_{yy}\varphi_y^3 r_x r_y\Delta_x - 4500\varphi_{yy}\varphi_y^3 r_x r_y\Delta_y r + 945\varphi_{yy}\varphi_y^3 r_x\Delta_{xy} r \\
& + 90\varphi_{yy}\varphi_y^3 r_x\Delta_{xx} + 1215\varphi_{yy}\varphi_y^3 r_x\Delta_{yy} r^2 + 3360\varphi_{yy}\varphi_y^3 r_{yyy} r^3\Delta - 12810\varphi_{yy}\varphi_y^3 r_{yy} r_y r^2\Delta \\
& + 3435\varphi_{yy}\varphi_y^3 r_{yy}\Delta_x r^2 + 8190\varphi_{yy}\varphi_y^3 r_{yy}\Delta_y r^3 + 6810\varphi_{yy}\varphi_y^3 r_y^3 r\Delta - 4665\varphi_{yy}\varphi_y^3 r_y^2\Delta_x r \\
& - 12105\varphi_{yy}\varphi_y^3 r_y^2\Delta_y r^2 + 5805\varphi_{yy}\varphi_y^3 r_y\Delta_{xy} r^2 + 1035\varphi_{yy}\varphi_y^3 r_y\Delta_{xx} r + 6660\varphi_{yy}\varphi_y^3 r_y\Delta_{yy} r^3 \\
& - 1575\varphi_{yy}\varphi_y^3\Delta_{xyy} r^3 - 75\varphi_{yy}\varphi_y^3\Delta_{xxx} r - 675\varphi_{yy}\varphi_y^3\Delta_{xxy} r^2 - 1875\varphi_{yy}\varphi_y^3\Delta_{yyy} r^4 \\
& - 126\varphi_y^5\psi_{yyyyy} r^5 - 168\varphi_y^4 r_{xyyy} r^2\Delta + 108\varphi_y^4 r_{xyy} r_x\Delta + 879\varphi_y^4 r_{xyy} r_y r\Delta - 177\varphi_y^4 r_{xyy}\Delta_x r \\
& - 441\varphi_y^4 r_{xyy}\Delta_y r^2 + 108\varphi_y^4 r_{xy}^2\Delta + 234\varphi_y^4 r_{xy} r_x\Delta_y + 852\varphi_y^4 r_{xy} r_{yy} r\Delta - 1296\varphi_y^4 r_{xy} r_y^2\Delta \\
& + 432\varphi_y^4 r_{xy} r_y\Delta_x + 1734\varphi_y^4 r_{xy} r_y\Delta_y r - 330\varphi_y^4 r_{xy}\Delta_{xy} r - 36\varphi_y^4 r_{xy}\Delta_{xx} - 384\varphi_y^4 r_{xy}\Delta_{yy} r^2 \\
& - 6\varphi_y^4 r_{xxxy}\Delta - 6\varphi_y^4 r_{xxx}\Delta_y - 57\varphi_y^4 r_{xxyy} r\Delta + 144\varphi_y^4 r_{xxy} r_y\Delta - 24\varphi_y^4 r_{xxy}\Delta_x\Delta_y r \\
& - 108\varphi_y^4 r_{xxy} + 66\varphi_y^4 r_{xx} r_{yy}\Delta + 150\varphi_y^4 r_{xx} r_y\Delta_y - 24\varphi_y^4 r_{xx}\Delta_{xy} - 51\varphi_y^4 r_{xx}\Delta_{yy} r \\
& + 63\varphi_y^4 r_x^2\Delta_{yy} + 267\varphi_y^4 r_x r_{yyy} r\Delta - 1200\varphi_y^4 r_x r_{yy} r_y\Delta + 204\varphi_y^4 r_x r_{yy}\Delta_x + 837\varphi_y^4 r_x r_{yy}\Delta_y r \\
& - 1446\varphi_y^4 r_x r_y^2\Delta_y + 456\varphi_y^4 r_x r_y\Delta_{xy} + 843\varphi_y^4 r_x r_y\Delta_{yy} r - 153\varphi_y^4 r_x\Delta_{xyy} r - 36\varphi_y^4 r_x\Delta_{xxy}
\end{aligned}
$$

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\begin{aligned}
&- 111\varphi_y^4 r_x\Delta_{yyy}r^2 - 294\varphi_y^4 r_{yyyy}r^3\Delta + 1602\varphi_y^4 r_{yyy}r_y r^2\Delta - 399\varphi_y^4 r_{yyy}\Delta_x r^2 \\
&- 945\varphi_y^4 r_{yyy}\Delta_y r^3 + 1224\varphi_y^4 r_{yy}^2 r^2\Delta - 4329\varphi_y^4 r_{yy}r_y^2 r\Delta + 1839\varphi_y^4 r_{yy}r_y\Delta_x r \\
&+ 4470\varphi_y^4 r_{yy}r_y\Delta_y r^2 - 996\varphi_y^4 r_{yy}\Delta_{xy}r^2 - 189\varphi_y^4 r_{yy}\Delta_{xx}r - 1140\varphi_y^4 r_{yy}\Delta_{yy}r^3 \\
&+ 1296\varphi_y^4 r_y^4\Delta - 864\varphi_y^4 r_y^3\Delta_x - 3090\varphi_y^4 r_y^3\Delta_y r + 1812\varphi_y^4 r_y^2\Delta_{xy}r + 216\varphi_y^4 r_y^2\Delta_{xx} \\
&+ 2010\varphi_y^4 r_y^2\Delta_{yy}r^2 - 819\varphi_y^4 r_y\Delta_{xyy}r^2 - 24\varphi_y^4 r_y\Delta_{xxx} - 342\varphi_y^4 r_y\Delta_{xxy}r \\
&- 615\varphi_y^4 r_y\Delta_{yyy}r^3 + 121\varphi_y^4\Delta_{xyyy}r^3 + \varphi_y^4\Delta_{xxxx} + 21\varphi_y^4\Delta_{xxxy}r + 81\varphi_y^4\Delta_{xxyy}r^2 \\
&+ 126\varphi_y^4\Delta_{yyyy}r^4)/(\varphi_y^4\Delta), \qquad (3.61)
\end{aligned}
$$

$$
\begin{aligned}
M_3 = &- (84\varphi_{yyyyy}\varphi_y^4\psi_y r^6 - 336\varphi_{yyyyy}\varphi_y^3 r^5\Delta - 1260\varphi_{yyyy}\varphi_{yy}\varphi_y^3\psi_y r^6 + 5880\varphi_{yyyy}\varphi_{yy}\varphi_y^2 r^5\Delta \\
&+ 420\varphi_{yyyy}\varphi_y^4\psi_{yy}r^6 + 300\varphi_{yyyy}\varphi_y^3 r_x r^3\Delta + 1275\varphi_{yyyy}\varphi_y^3 r_y r^4\Delta - 525\varphi_{yyyy}\varphi_y^3\Delta_x r^4 \\
&- 1155\varphi_{yyyy}\varphi_y^3\Delta_y r^5 - 840\varphi_{yyy}^2\varphi_y^3\psi_y r^6 + 3920\varphi_{yyy}^2\varphi_y^2 r^5\Delta + 8820\varphi_{yyy}\varphi_{yy}^2\varphi_y^2\psi_y r^6 \\
&- 47040\varphi_{yyy}\varphi_{yy}^2\varphi_y r^5\Delta - 5040\varphi_{yyy}\varphi_{yy}\varphi_y^3\psi_{yy}r^6 - 4200\varphi_{yyy}\varphi_{yy}\varphi_y^2 r_x r^3\Delta \\
&- 17850\varphi_{yyy}\varphi_{yy}\varphi_y^2 r_y r^4\Delta + 7350\varphi_{yyy}\varphi_{yy}\varphi_y^2\Delta_x r^4 + 16170\varphi_{yyy}\varphi_{yy}\varphi_y^2\Delta_y r^5 \\
&+ 840\varphi_{yyy}\varphi_y^4\psi_{yyy}r^6 + 1200\varphi_{yyy}\varphi_y^3 r_{xy}r^3\Delta + 200\varphi_{yyy}\varphi_y^3 r_{xx}r^2\Delta - 240\varphi_{yyy}\varphi_y^3 r_x^2 r\Delta \\
&- 2120\varphi_{yyy}\varphi_y^3 r_x r_y r^2\Delta + 600\varphi_{yyy}\varphi_y^3 r_x\Delta_x r^2 + 1000\varphi_{yyy}\varphi_y^3 r_x\Delta_y r^3 \\
&+ 2800\varphi_{yyy}\varphi_y^3 r_{yy}r^4\Delta - 4040\varphi_{yyy}\varphi_y^3 r_y^2 r^3\Delta + 2600\varphi_{yyy}\varphi_y^3 r_y\Delta_x r^3 \\
&+ 4200\varphi_{yyy}\varphi_y^3 r_y\Delta_y r^4 - 1300\varphi_{yyy}\varphi_y^3\Delta_{xy}r^4 - 400\varphi_{yyy}\varphi_y^3\Delta_{xx}r^3 - 1660\varphi_{yyy}\varphi_y^3\Delta_{yy}r^5 \\
&- 8820\varphi_{yy}^4\varphi_y\psi_y r^6 + 52920\varphi_{yy}^4 r^5\Delta + 8820\varphi_{yy}^3\varphi_y^2\psi_{yy}r^6 + 8400\varphi_{yy}^3\varphi_y r_x r^3\Delta \\
&+ 35700\varphi_{yy}^3\varphi_y r_y r^4\Delta - 14700\varphi_{yy}^3\varphi_y\Delta_x r^4 - 32340\varphi_{yy}^3\varphi_y\Delta_y r^5 - 3780\varphi_{yy}^2\varphi_y^3\psi_{yyy}r^6 \\
&- 6300\varphi_{yy}^2\varphi_y^2 r_{xy}r^3\Delta - 1050\varphi_{yy}^2\varphi_y^2 r_{xx}r^2\Delta + 1260\varphi_{yy}^2\varphi_y^2 r_x^2 r\Delta + 11130\varphi_{yy}^2\varphi_y^2 r_x r_y r^2\Delta \\
&- 3150\varphi_{yy}^2\varphi_y^2 r_x\Delta_x r^2 - 5250\varphi_{yy}^2\varphi_y^2 r_x\Delta_y r^3 - 14700\varphi_{yy}^2\varphi_y^2 r_{yy}r^4\Delta + 21210\varphi_{yy}^2\varphi_y^2 r_y^2 r^3\Delta \\
&- 13650\varphi_{yy}^2\varphi_y^2 r_y\Delta_x r^3 - 22050\varphi_{yy}^2\varphi_y^2 r_y\Delta_y r^4 + 6825\varphi_{yy}^2\varphi_y^2\Delta_{xy}r^4 + 2100\varphi_{yy}^2\varphi_y^2\Delta_{xx}r^3 \\
&+ 8715\varphi_{yy}^2\varphi_y^2\Delta_{yy}r^5 + 840\varphi_{yy}\varphi_y^4\psi_{yyyy}r^6 + 1680\varphi_{yy}\varphi_y^3 r_{xyy}r^3\Delta - 1680\varphi_{yy}\varphi_y^3 r_{xy}r_x r\Delta \\
&- 7050\varphi_{yy}\varphi_y^3 r_{xy}r_y r^2\Delta + 1770\varphi_{yy}\varphi_y^3 r_{xy}\Delta_x r^2 + 2730\varphi_{yy}\varphi_y^3 r_{xy}\Delta_y r^3 + 60\varphi_{yy}\varphi_y^3 r_{xxx}r\Delta \\
&+ 570\varphi_{yy}\varphi_y^3 r_{xxy}r^2\Delta - 150\varphi_{yy}\varphi_y^3 r_{xx}r_x\Delta - 1110\varphi_{yy}\varphi_y^3 r_{xx}r_y r\Delta + 240\varphi_{yy}\varphi_y^3 r_{xx}\Delta_x r \\
&+ 510\varphi_{yy}\varphi_y^3 r_{xx}\Delta_y r^2 + 1275\varphi_{yy}\varphi_y^3 r_x^2 r_y\Delta - 225\varphi_{yy}\varphi_y^3 r_x^2\Delta_x - 855\varphi_{yy}\varphi_y^3 r_x^2\Delta_y r \\
&- 3420\varphi_{yy}\varphi_y^3 r_x r_{yy}r^2\Delta + 7710\varphi_{yy}\varphi_y^3 r_x r_y^2 r\Delta - 3390\varphi_{yy}\varphi_y^3 r_x r_y\Delta_x r - 6450\varphi_{yy}\varphi_y^3 r_x r_y\Delta_y r^2 \\
&+ 1530\varphi_{yy}\varphi_y^3 r_x\Delta_{xy}r^2 + 360\varphi_{yy}\varphi_y^3 r_x\Delta_{xx}r + 1110\varphi_{yy}\varphi_y^3 r_x\Delta_{yy}r^3 + 2940\varphi_{yy}\varphi_y^3 r_{yyy}r^4\Delta \\
&- 13590\varphi_{yy}\varphi_y^3 r_{yy}r_y r^3\Delta + 3990\varphi_{yy}\varphi_y^3 r_{yy}\Delta_x r^3 + 6510\varphi_{yy}\varphi_y^3 r_{yy}\Delta_y r^4 + 9765\varphi_{yy}\varphi_y^3 r_y^3 r^2\Delta \\
&- 7635\varphi_{yy}\varphi_y^3 r_y^2\Delta_x r^2 - 11445\varphi_{yy}\varphi_y^3 r_y^2\Delta_y r^3 + 5970\varphi_{yy}\varphi_y^3 r_y\Delta_{xy}r^3 + 1890\varphi_{yy}\varphi_y^3 r_y\Delta_{xx}r^2
\end{aligned}
$$

$$+ 4890\varphi_{yy}\varphi_y^3 r_y\Delta_{yy}r^4 - 1200\varphi_{yy}\varphi_y^3\Delta_{xyy}r^4 - 150\varphi_{yy}\varphi_y^3\Delta_{xxx}r^2 - 750\varphi_{yy}\varphi_y^3\Delta_{xxy}r^3$$
$$- 1260\varphi_{yy}\varphi_y^3\Delta_{yyy}r^5 - 84\varphi_y^5\psi_{yyyyy}r^6 - 168\varphi_y^4 r_{xyyy}r^3\Delta + 300\varphi_y^4 r_{xyy}r_x r\Delta$$
$$+ 1191\varphi_y^4 r_{xyy}r_y r^2\Delta - 273\varphi_y^4 r_{xyy}\Delta_x r^2 - 399\varphi_y^4 r_{xyy}\Delta_y r^3 + 304\varphi_y^4 r_{xy}^2 r\Delta + 64\varphi_y^4 r_{xy}r_{xx}\Delta$$
$$- 1237\varphi_y^4 r_{xy}r_x r_y\Delta + 195\varphi_y^4 r_{xy}r_x\Delta_x + 601\varphi_y^4 r_{xy}r_x\Delta_y r + 1200\varphi_y^4 r_{xy}r_{yy}r^2\Delta$$
$$- 3165\varphi_y^4 r_{xy}r_y^2 r\Delta + 1283\varphi_y^4 r_{xy}r_y\Delta_x r + 2121\varphi_y^4 r_{xy}r_y\Delta_y r^2 - 460\varphi_y^4 r_{xy}\Delta_{xy}r^2$$
$$- 124\varphi_y^4 r_{xy}\Delta_{xx}r - 316\varphi_y^4 r_{xy}\Delta_{yy}r^3 - \varphi_y^4 r_{xxxx}\Delta - 20\varphi_y^4 r_{xxxy}r\Delta + 31\varphi_y^4 r_{xxx}r_y\Delta$$
$$- 5\varphi_y^4 r_{xxx}\Delta_x - 19\varphi_y^4 r_{xxx}\Delta_y r - 84\varphi_y^4 r_{xxyy}r^2\Delta + 66\varphi_y^4 r_{xxy}r_x\Delta + 417\varphi_y^4 r_{xxy}r_y r\Delta$$
$$- 81\varphi_y^4 r_{xxy}\Delta_x r - 147\varphi_y^4 r_{xxy}\Delta_y r^2 + 70\varphi_y^4 r_{xx}r_x\Delta_y + 200\varphi_y^4 r_{xx}r_{yy}r\Delta - 391\varphi_y^4 r_{xx}r_y^2\Delta$$
$$+ 125\varphi_y^4 r_{xx}r_y\Delta_x + 405\varphi_y^4 r_{xx}r_y\Delta_y r - 76\varphi_y^4 r_{xx}\Delta_{xy}r - 10\varphi_y^4 r_{xx}\Delta_{xx} - 64\varphi_y^4 r_{xx}\Delta_{yy}r^2$$
$$- 288\varphi_y^4 r_x^2 r_{yy}\Delta - 700\varphi_y^4 r_x^2 r_y\Delta_y + 105\varphi_y^4 r_x^2\Delta_{xy} + 147\varphi_y^4 r_x^2\Delta_{yy}r + 384\varphi_y^4 r_x r_{yyy}r^2\Delta$$
$$- 2987\varphi_y^4 r_x r_{yy}r_y r\Delta + 621\varphi_y^4 r_x r_{yy}\Delta_x r + 1063\varphi_y^4 r_x r_{yy}\Delta_y r^2 + 2551\varphi_y^4 r_x r_y^3\Delta$$
$$- 1205\varphi_y^4 r_x r_y^2\Delta_x - 3179\varphi_y^4 r_x r_y^2\Delta_y r + 1234\varphi_y^4 r_x r_y\Delta_{xy}r + 190\varphi_y^4 r_x r_y\Delta_{xx}$$
$$+ 922\varphi_y^4 r_x r_y\Delta_{yy}r^2 - 192\varphi_y^4 r_x\Delta_{xyy}r^2 - 10\varphi_y^4 r_x\Delta_{xxx} - 114\varphi_y^4 r_x\Delta_{xxy}r - 84\varphi_y^4 r_x\Delta_{yyy}r^3$$
$$- 252\varphi_y^4 r_{yyyy}r^4\Delta + 1611\varphi_y^4 r_{yyy}r_y r^3\Delta - 441\varphi_y^4 r_{yyy}\Delta_x r^3 - 735\varphi_y^4 r_{yyy}\Delta_y r^4$$
$$+ 1232\varphi_y^4 r_{yy}^2 r^3\Delta - 5582\varphi_y^4 r_{yy}r_y^2 r^2\Delta + 2726\varphi_y^4 r_{yy}r_y\Delta_x r^2 + 3990\varphi_y^4 r_{yy}r_y\Delta_y r^3$$
$$- 964\varphi_y^4 r_{yy}\Delta_{xy}r^3 - 316\varphi_y^4 r_{yy}\Delta_{xx}r^2 - 820\varphi_y^4 r_{yy}\Delta_{yy}r^4 + 2633\varphi_y^4 r_y^4 r\Delta - 2251\varphi_y^4 r_y^3\Delta_x r$$
$$- 3465\varphi_y^4 r_y^3\Delta_y r^2 + 2333\varphi_y^4 r_y^2\Delta_{xy}r^2 + 674\varphi_y^4 r_y^2\Delta_{xx}r + 1595\varphi_y^4 r_y^2\Delta_{yy}r^3 - 696\varphi_y^4 r_y\Delta_{xyy}r^3$$
$$- 86\varphi_y^4 r_y\Delta_{xxx}r - 498\varphi_y^4 r_y\Delta_{xxy}r^2 - 420\varphi_y^4 r_y\Delta_{yyy}r^4 + 84\varphi_y^4\Delta_{xyyy}r^4 + 4\varphi_y^4\Delta_{xxxx}r$$
$$+ 34\varphi_y^4\Delta_{xxxy}r^2 + 74\varphi_y^4\Delta_{xxyy}r^3 + 84\varphi_y^4\Delta_{yyyy}r^5)/(\varphi_y^4\Delta), \quad (3.62)$$

$$M_2 =(((3r_{xxxx}r - 10r_{xx}^2)\Delta + 6(4r_{xxxy}\Delta - \Delta_{xxxx})r^2 + 96r_{xyyy}r^4\Delta - 26\Delta_{xxxy}r^3$$
$$- 712r_{yy}^2 r^4\Delta + 3(93\Delta_x + 115\Delta_y r - 312r_y\Delta)r_{yyy}r^4 + 120r_x^3\Delta_y + 6(3(2(9\Delta_{xyy}$$
$$+ 5\Delta_{yyy})r + 19\Delta_{xxy})r + 19\Delta_{xxx})r_y r^2 - 3((5r_x + 26r_y r)\Delta - (5\Delta_x + 7\Delta_y r)r)r_{xxx}$$
$$+ 2(1856r_y^2\Delta - 1017r_y\Delta_x - 1005r_y\Delta_y r + 258\Delta_{xy}r + 132\Delta_{xx} + 180\Delta_{yy}r^2)r_{yy}r^3$$
$$+ ((778r_y^2\Delta - 315r_y\Delta_x - 375r_y\Delta_y r + 30\Delta_{xx} + 36\Delta_{yy}r^2)r + 5(79r_y\Delta - 12\Delta_x$$
$$- 30\Delta_y r)r_x - 28(8r_{yy}\Delta - 3\Delta_{xy})r^2)r_{xx} - ((152r_{xx}\Delta + 1377r_y\Delta_x r + 1239r_y\Delta_y r^2$$
$$- 300\Delta_{xy}r^2 - (2291r_y\Delta - 465\Delta_x - 519\Delta_y r)r_x)r - (315r_x^2\Delta - 848r_{yy}r^3\Delta + 156\Delta_{xx}r^2$$
$$+ 144\Delta_{yy}r^4 + 2824r_y^2 r^2\Delta))r_{xy} - (((4(9(\Delta_{xxyy} + \Delta_{yyyy}r^2 + \Delta_{xyyy}r)r^2 + 76r_{xy}^2\Delta$$
$$- 15r_{xxyy}r\Delta - 33r_{yyyy}r^3\Delta + 532r_y^4\Delta) - 9(241\Delta_x + 215\Delta_y r)r_y^3 + 3(257\Delta_{xx} + 240\Delta_{yy}r^2$$
$$+ 479\Delta_{xy}r)r_y^2)r + 3(3(17r_x + 48r_y r)\Delta - (33\Delta_x + 31\Delta_y r)r)r_{xxy} + 3((99r_x + 268r_y r)\Delta$$
$$- (69\Delta_x + 67\Delta_y r)r)r_{xyy}r)r - (9(61r_{yy}\Delta - 25\Delta_{xy})r - (2005r_y^2\Delta - 600r_y\Delta_x$$

$$
\begin{aligned}
&- 1140r_y\Delta_y r + 45\Delta_{xx} + 108\Delta_{yy}r^2))r_x^2 + (3(95r_{yyy}r^2\Delta - 10\Delta_{xxx} - 42\Delta_{xxy}r \\
&- (805\Delta_x + 807\Delta_y r)r_y^2) - (2786r_y\Delta - 699\Delta_x - 657\Delta_y r)r_{yy}r - (36(3\Delta_{xyy} \\
&+ \Delta_{yyy}r)r^2 - 3643r_y^3\Delta - 6(191\Delta_{xy}r + 80\Delta_{xx} + 78\Delta_{yy}r^2)r_y))r_x r))\varphi_y^4 \\
&- (15(3((2r_{xxx}r^2 + 5r_x^3)\Delta - 10\Delta_{xxy}r^4) + 68r_{xyy}r^4\Delta + 464r_y^3r^3\Delta + 3(70r_y\Delta \\
&- 15\Delta_x - 21\Delta_y r)r_x^2 r + 2(87\Delta_x + 103\Delta_y r - 272r_y\Delta)r_{yy}r^4 - 2(15r_x\Delta \\
&- 13\Delta_y r^2 + 12(4r_y\Delta - \Delta_x)r)r_{xx}r + 24\varphi_y\psi_{yyyy}r^7 - ((2(18(\Delta_{xyy} + \Delta_{yyy}r)r^2 \\
&+ 5\Delta_{xxx} - 16r_{xxy}\Delta - 52r_{yyy}r^2\Delta) + (411\Delta_x + 397\Delta_y r)r_y^2 - 6(19\Delta_{xx} + 24\Delta_{yy}r^2 \\
&+ 37\Delta_{xy}r)r_y)r + 2(3(23r_x + 60r_y r)\Delta - (51\Delta_x + 49\Delta_y r)r)r_{xy} - (2(18(\Delta_{xx} \\
&+ \Delta_{yy}r^2) - 91r_{yy}r\Delta) + 561r_y^2\Delta - 294r_y\Delta_x - 290r_y\Delta_y r + 78\Delta_{xy}r)r_x)r^2)\varphi_{yy}\varphi_y^3 \\
&+ (5(2(2((2(15(\Delta_x + \Delta_y r) - 47r_y\Delta)r_x + (76r_{yy}r\Delta - 128r_y^2\Delta + 90r_y\Delta_x + 102r_y\Delta_y r \\
&- 33\Delta_{xy}r - 15\Delta_{xx} - 36\Delta_{yy}r^2)r)r + 2(5r_{xx}r - 9r_x^2 + 20r_{xy}r^2)\Delta)\varphi_y^2 - 3((3(24\varphi_y\psi_{yy}r^2 \\
&+ 112r_y\Delta - 49\Delta_x)r + 35(3r_x\Delta - 7\Delta_y r^2))\varphi_{yy}\varphi_y - 2(63\varphi_{yy}^2\varphi_y\psi_y r - 392\varphi_{yy}^2\Delta \\
&+ 6\varphi_y^3\psi_{yyy}r)r^2)r^2)\varphi_{yyy} - 3(2(7(5r_{xx}r - 9r_x^2 + 20r_{xy}r^2)\Delta + 54\varphi_y\psi_{yyy}r^5) + 7(2(15(\Delta_x \\
&+ \Delta_y r) - 47r_y\Delta)r_x + (76r_{yy}r\Delta - 128r_y^2\Delta + 90r_y\Delta_x + 102r_y\Delta_y r - 33\Delta_{xy}r - 15\Delta_{xx} \\
&- 36\Delta_{yy}r^2)r)r)\varphi_{yy}^2\varphi_y)\varphi_y - (4((9(105(\varphi_y\psi_y r - 7\Delta)\varphi_{yy}^4 + \varphi_y^5\psi_{yyyyy}r) - (9\varphi_{yyyyy}\varphi_y^2\psi_y r \\
&- 42\varphi_{yyyyy}\varphi_y\Delta - 90\varphi_{yyy}^2\varphi_y\psi_y r + 490\varphi_{yyy}^2\Delta)\varphi_y^2)r^2 - 105(3(16r_y\Delta - 7\Delta_x \\
&+ 3\varphi_y\psi_{yy}r^2)r + 5(3r_x\Delta - 7\Delta_y r^2))\varphi_{yy}^3\varphi_y) + 15(4((9\varphi_y\psi_y r - 49\Delta)\varphi_{yy} - 3\varphi_y^2\psi_{yy}r)r^2 \\
&- (15r_x\Delta + 48r_y r\Delta - 21\Delta_x r - 35\Delta_y r^2)\varphi_y)\varphi_{yyyy}\varphi_y^2)r^2)r^2))/(\varphi_y^4\Delta), \qquad (3.63)
\end{aligned}
$$

$$
\begin{aligned}
M_1 = &- ((((9((\Delta_{xxyy} + \Delta_{yyyy}r^2)r + \Delta_{xxxy})r + 4\Delta_{xxxx} + 9\Delta_{xyyy}r^3 + 806r_y^4\Delta + 224r_{yy}^2r^2\Delta \\
&- 39r_{yyyy}r^3\Delta)r^2 + 20r_{xx}^2\Delta - 21r_{xxyy}r^3\Delta - 12r_{xxxy}r^2\Delta - 3r_{xxxx}r\Delta + 128r_{xy}^2r^2\Delta \\
&- 30r_{xyyy}r^4\Delta - 36(26\Delta_x + 15\Delta_y r)r_y^3r^2 + 3(141\Delta_{xy}r + 128\Delta_{xx} + 60\Delta_{yy}r^2)r_y^2r^2 \\
&+ 3(99r_y\Delta - 32\Delta_x - 30\Delta_y r)r_{yyy}r^4 - (1274r_y^2\Delta - 771r_y\Delta_x - 540r_y\Delta_y r + 144\Delta_{xy}r \\
&+ 111\Delta_{xx} + 90\Delta_{yy}r^2)r_{yy}r^3 - (339r_{yy}r\Delta - 1805r_y^2\Delta + 885r_y\Delta_x + 495r_y\Delta_y r - 135\Delta_{xy}r \\
&- 90\Delta_{xx} - 27\Delta_{yy}r^2)r_x^2)r - 15(7\Delta_x + 9\Delta_y r - 49r_y\Delta)r_x^3 + (111r_{yyy}r^2\Delta + 1838r_y^3\Delta \\
&- 1530r_y^2\Delta_x - 774r_y^2\Delta_y r + 414r_y\Delta_{xy}r + 390r_y\Delta_{xx} + 117r_y\Delta_{yy}r^2 - 27\Delta_{xyy}r^2 - 30\Delta_{xxx} \\
&- 54\Delta_{xxy}r - 9\Delta_{yyy}r^3 - (1201r_y\Delta - 351\Delta_x - 198\Delta_y r)r_{yy}r)r_x r^2 - 3((5\Delta_x + 3\Delta_y r \\
&- 21r_y\Delta)r - 10r_x\Delta)r_{xxx}r - 6((13\Delta_x + 9\Delta_y r - 46r_y\Delta)r - 21r_x\Delta)r_{xyy}r^3 - 3((17\Delta_x \\
&+ 9\Delta_y r - 63r_y\Delta)r - 36r_x\Delta)r_{xxy}r^2 - 3(3(12\Delta_{xxy} + 5\Delta_{yyy}r^2)r + 22\Delta_{xxx} \\
&+ 27\Delta_{xyy}r^2)r_y r^3 - ((488r_y^2\Delta + 36\Delta_{xy}r + 30\Delta_{xx} + 9\Delta_{yy}r^2 - 112r_{yy}r\Delta)r^2 + 105r_x^2\Delta \\
&- 15(17\Delta_x + 9\Delta_y r)r_y r^2 + 10(58r_y\Delta - 12\Delta_x - 9\Delta_y r)r_x r)r_{xx} - ((1127r_y^2\Delta + 90\Delta_{xy}r \\
&+ 84\Delta_{xx}36\Delta_{yy}r^2 - 304r_{yy}r\Delta)r^2 + 420r_x^2\Delta - 112r_{xx}r\Delta - 3(211\Delta_x + 117\Delta_y r)r_y r^2
\end{aligned}
$$

$$
\begin{aligned}
&+(1291r_y\Delta-345\Delta_x-171\Delta_y r)r_x r)r_{xy}r)\varphi_y^4-(15((9(\Delta_{xxy}+\Delta_{yyy}r^2)r+5\Delta_{xxx}\\
&+9\Delta_{xyy}r^2-166r_y^3\Delta-31r_{yyy}r^2\Delta)r^3-30r_x^3\Delta-13r_{xxy}r^3\Delta-4r_{xxx}r^2\Delta-22r_{xyy}r^4\Delta\\
&+4(41\Delta_x+27\Delta_y r)r_y^2r^3-3(55r_y\Delta-15\Delta_x-9\Delta_y r)r_x^2 r+(73r_{yy}r\Delta-264r_y^2\Delta\\
&+166r_y\Delta_x+90r_y\Delta_y r-27\Delta_{xy}r-24\Delta_{xx}-9\Delta_{yy}r^2)r_x r^2-(61\Delta_x+54\Delta_y r\\
&-176r_y\Delta)r_{yy}r^4-3(17\Delta_{xx}+12\Delta_{yy}r^2+21\Delta_{xy}r)r_y r^3-((16\Delta_x+9\Delta_y r-54r_y\Delta)r\\
&-30r_x\Delta)r_{xx}r-((43\Delta_x+27\Delta_y r-135r_y\Delta)r-72r_x\Delta)r_{xy}r^2-6\varphi_y\psi_{yyyy}r^7)\varphi_{yy}\varphi_y^3\\
&-(15(7((43r_y^2\Delta+9\Delta_{xy}r+6\Delta_{xx}+9\Delta_{yy}r^2-23r_{yy}r\Delta)r^2+12r_x^2\Delta-5r_{xx}r\Delta\\
&-14r_{xy}r^2\Delta-3(11\Delta_x+9\Delta_y r)r_y r^2+(41r_y\Delta-15\Delta_x-9\Delta_y r)r_x r)-27\varphi_y\psi_{yyy}r^5)\varphi_{yy}^2\varphi_y^2\\
&-((3(3(105(\varphi_y\psi_y r-8\Delta)\varphi_{yy}^4+\varphi_y^5\psi_{yyyyy}r)-(3\varphi_y\psi_y r-16\Delta)\varphi_{yyyyy}\varphi_y^3)+10(9\varphi_y\psi_y r\\
&-56\Delta)\varphi_{yyy}^2\varphi_y^2)r^2+105(4((7\Delta_x+9\Delta_y r-15r_y\Delta)r-6r_x\Delta)-9\varphi_y\psi_{yy}r^3)\varphi_{yy}^3\varphi_y\\
&+15(((7\Delta_x+9\Delta_y r-15r_y\Delta)r-6r_x\Delta)\varphi_y+((9\varphi_y\psi_y r-56\Delta)\varphi_{yy}\\
&-3\varphi_y^2\psi_{yy}r)r^2)\varphi_{yyyy}\varphi_y^2)r^2-5(4((43r_y^2\Delta+9\Delta_{xy}r+6\Delta_{xx}+9\Delta_{yy}r^2-23r_{yy}r\Delta)r^2\\
&+12r_x^2\Delta-5r_{xx}r\Delta-14r_{xy}r^2\Delta-3(11\Delta_x+9\Delta_y r)r_y r^2+(41r_y\Delta-15\Delta_x\\
&-9\Delta_y r)r_x r)\varphi_y^2-3(2(7((7\Delta_x+9\Delta_y r-15r_y\Delta)r-6r_x\Delta)-18\varphi_y\psi_{yy}r^3)\varphi_{yy}\varphi_y\\
&+(63\varphi_{yy}^2\varphi_y\psi_y r-448\varphi_{yy}^2\Delta+6\varphi_y^3\psi_{yyy}r)r^2)r^2)\varphi_{yyy}\varphi_y)r^2)r)/(\varphi_y^4\Delta), \qquad (3.64)
\end{aligned}
$$

$$
\begin{aligned}
M_0=&-(\varphi_{yyyyy}\varphi_y^4\psi_y r^9-6\varphi_{yyyyy}\varphi_y^3r^8\Delta-15\varphi_{yyyy}\varphi_{yy}\varphi_y^3\psi_y r^9+105\varphi_{yyyy}\varphi_{yy}\varphi_y^2r^8\Delta\\
&+5\varphi_{yyyy}\varphi_y^4\psi_{yy}r^9+15\varphi_{yyyy}\varphi_y^3r_x r^6\Delta+30\varphi_{yyyy}\varphi_y^3r_y r^7\Delta-15\varphi_{yyyy}\varphi_y^3\Delta_x r^7\\
&-15\varphi_{yyyy}\varphi_y^3\Delta_y r^8-10\varphi_{yyy}^2\varphi_y^3\psi_y r^9+70\varphi_{yyy}^2\varphi_y^2r^8\Delta+105\varphi_{yyy}\varphi_{yy}^2\varphi_y^2\psi_y r^9\\
&-840\varphi_{yyy}\varphi_{yy}^2\varphi_y r^8\Delta-60\varphi_{yyy}\varphi_{yy}\varphi_y^3\psi_{yy}r^9-210\varphi_{yyy}\varphi_{yy}\varphi_y^2r_x r^6\Delta-420\varphi_{yyy}\varphi_{yy}\varphi_y^2r_y r^7\Delta\\
&+210\varphi_{yyy}\varphi_{yy}\varphi_y^2\Delta_x r^7+210\varphi_{yyy}\varphi_{yy}\varphi_y^2\Delta_y r^8+10\varphi_{yyy}\varphi_y^4\psi_{yyy}r^9+40\varphi_{yyy}\varphi_y^3r_{xy}r^6\Delta\\
&+20\varphi_{yyy}\varphi_y^3r_{xx}r^5\Delta-60\varphi_{yyy}\varphi_y^3r_x^2r^4\Delta-140\varphi_{yyy}\varphi_y^3r_x r_y r^5\Delta+60\varphi_{yyy}\varphi_y^3r_x\Delta_x r^5\\
&+20\varphi_{yyy}\varphi_y^3r_x\Delta_y r^6+60\varphi_{yyy}\varphi_y^3r_{yy}r^7\Delta-120\varphi_{yyy}\varphi_y^3r_y^2r^6\Delta+100\varphi_{yyy}\varphi_y^3r_y\Delta_x r^6\\
&+60\varphi_{yyy}\varphi_y^3r_y\Delta_y r^7-20\varphi_{yyy}\varphi_y^3\Delta_{xy}r^7-20\varphi_{yyy}\varphi_y^3\Delta_{xx}r^6-20\varphi_{yyy}\varphi_y^3\Delta_{yy}r^8\\
&-105\varphi_{yy}^4\varphi_y\psi_y r^9+945\varphi_{yy}^4r^8\Delta+105\varphi_{yy}^3\varphi_y^2\psi_{yy}r^9+420\varphi_{yy}^3\varphi_y r_x r^6\Delta+840\varphi_{yy}^3\varphi_y r_y r^7\Delta\\
&-420\varphi_{yy}^3\varphi_y\Delta_x r^7-420\varphi_{yy}^3\varphi_y\Delta_y r^8-45\varphi_{yy}^2\varphi_y^3\psi_{yyy}r^9-210\varphi_{yy}^2\varphi_y^2r_{xy}r^6\Delta\\
&-105\varphi_{yy}^2\varphi_y^2r_{xx}r^5\Delta+315\varphi_{yy}^2\varphi_y^2r_x^2r^4\Delta+735\varphi_{yy}^2\varphi_y^2r_x r_y r^5\Delta-315\varphi_{yy}^2\varphi_y^2r_x\Delta_x r^5\\
&-105\varphi_{yy}^2\varphi_y^2r_x\Delta_y r^6-315\varphi_{yy}^2\varphi_y^2r_{yy}r^7\Delta+630\varphi_{yy}^2\varphi_y^2r_y^2r^6\Delta-525\varphi_{yy}^2\varphi_y^2r_y\Delta_x r^6\\
&-315\varphi_{yy}^2\varphi_y^2r_y\Delta_y r^7+105\varphi_{yy}^2\varphi_y^2\Delta_{xy}r^7+105\varphi_{yy}^2\varphi_y^2\Delta_{xx}r^6+105\varphi_{yy}^2\varphi_y^2\Delta_{yy}r^8\\
&+10\varphi_{yy}\varphi_y^4\psi_{yyyy}r^9+45\varphi_{yy}\varphi_y^3r_{xyy}r^6\Delta-195\varphi_{yy}\varphi_y^3r_{xy}r_x r^4\Delta-300\varphi_{yy}\varphi_y^3r_{xy}r_y r^5\Delta
\end{aligned}
$$

$$
\begin{aligned}
&+105\varphi_{yy}\varphi_y^3 r_{xy}\Delta_x r^5 + 45\varphi_{yy}\varphi_y^3 r_{xy}\Delta_y r^6 + 15\varphi_{yy}\varphi_y^3 r_{xxx} r^4\Delta + 30\varphi_{yy}\varphi_y^3 r_{xxy} r^5\Delta \\
&-150\varphi_{yy}\varphi_y^3 r_{xx} r_x r^3\Delta - 165\varphi_{yy}\varphi_y^3 r_{xx} r_y r^4\Delta + 60\varphi_{yy}\varphi_y^3 r_{xx}\Delta_x r^4 + 15\varphi_{yy}\varphi_y^3 r_{xx}\Delta_y r^5 \\
&+225\varphi_{yy}\varphi_y^3 r_x^3 r^2\Delta + 600\varphi_{yy}\varphi_y^3 r_x^2 r_y r^3\Delta - 225\varphi_{yy}\varphi_y^3 r_x^2\Delta_x r^3 - 45\varphi_{yy}\varphi_y^3 r_x^2\Delta_y r^4 \\
&-180\varphi_{yy}\varphi_y^3 r_x r_{yy} r^5\Delta + 690\varphi_{yy}\varphi_y^3 r_x r_y^2 r^4\Delta - 510\varphi_{yy}\varphi_y^3 r_x r_y\Delta_x r^4 - 150\varphi_{yy}\varphi_y^3 r_x r_y\Delta_y r^5 \\
&+45\varphi_{yy}\varphi_y^3 r_x\Delta_{xy} r^5 + 90\varphi_{yy}\varphi_y^3 r_x\Delta_{xx} r^4 + 15\varphi_{yy}\varphi_y^3 r_x\Delta_{yy} r^6 + 60\varphi_{yy}\varphi_y^3 r_{yyy} r^7\Delta \\
&-360\varphi_{yy}\varphi_y^3 r_{yy} r_y r^6\Delta + 135\varphi_{yy}\varphi_y^3 r_{yy}\Delta_x r^6 + 90\varphi_{yy}\varphi_y^3 r_{yy}\Delta_y r^7 + 360\varphi_{yy}\varphi_y^3 r_y^3 r^5\Delta \\
&-390\varphi_{yy}\varphi_y^3 r_y^2\Delta_x r^5 - 180\varphi_{yy}\varphi_y^3 r_y^2\Delta_y r^6 + 105\varphi_{yy}\varphi_y^3 r_y\Delta_{xy} r^6 + 135\varphi_{yy}\varphi_y^3 r_y\Delta_{xx} r^5 \\
&+60\varphi_{yy}\varphi_y^3 r_y\Delta_{yy} r^7 - 15\varphi_{yy}\varphi_y^3\Delta_{xyy} r^7 - 15\varphi_{yy}\varphi_y^3\Delta_{xxx} r^5 - 15\varphi_{yy}\varphi_y^3\Delta_{xxy} r^6 \\
&-15\varphi_{yy}\varphi_y^3\Delta_{yyy} r^8 - \varphi_y^5\psi_{yyyyy} r^9 - 4\varphi_y^4 r_{xyyy} r^6\Delta + 21\varphi_y^4 r_{xyy} r_x r^4\Delta + 39\varphi_y^4 r_{xyy} r_y r^5\Delta \\
&-12\varphi_y^4 r_{xyy}\Delta_x r^5 - 6\varphi_y^4 r_{xyy}\Delta_y r^6 + 20\varphi_y^4 r_{xy}^2 r^4\Delta + 24\varphi_y^4 r_{xy} r_{xx} r^3\Delta - 105\varphi_y^4 r_{xy} r_x^2 r^2\Delta \\
&-237\varphi_y^4 r_{xy} r_x r_y r^3\Delta + 75\varphi_y^4 r_{xy} r_x\Delta_x r^3 + 19\varphi_y^4 r_{xy} r_x\Delta_y r^4 + 44\varphi_y^4 r_{xy} r_{yy} r^5\Delta \\
&-172\varphi_y^4 r_{xy} r_y^2 r^4\Delta + 107\varphi_y^4 r_{xy} r_y\Delta_x r^4 + 39\varphi_y^4 r_{xy} r_y\Delta_y r^5 - 10\varphi_y^4 r_{xy}\Delta_{xy} r^5 \\
&-16\varphi_y^4 r_{xy}\Delta_{xx} r^4 - 4\varphi_y^4 r_{xy}\Delta_{yy} r^6 - \varphi_y^4 r_{xxxx} r^3\Delta - 2\varphi_y^4 r_{xxxy} r^4\Delta + 15\varphi_y^4 r_{xxx} r_x r^2\Delta \\
&+16\varphi_y^4 r_{xxx} r_y r^3\Delta - 5\varphi_y^4 r_{xxx}\Delta_x r^3 - \varphi_y^4 r_{xxx}\Delta_y r^4 - 3\varphi_y^4 r_{xxyy} r^5\Delta + 21\varphi_y^4 r_{xxy} r_x r^3\Delta \\
&+30\varphi_y^4 r_{xxy} r_y r^4\Delta - 9\varphi_y 4 r_{xxy}\Delta_x r^4 - 3\varphi_y^4 r_{xxy}\Delta_y r^5 + 10\varphi_y^4 r_{xx}^2 r^2\Delta - 105\varphi_y^4 r_{xx} r_x^2 r\Delta \\
&-185\varphi_y^4 r_{xx} r_x r_y r^2\Delta + 60\varphi_y^4 r_{xx} r_x\Delta_x r^2 + 10\varphi_y^4 r_{xx} r_x\Delta_y r^3 + 22\varphi_y^4 r_{xx} r_{yy} r^4\Delta \\
&-101\varphi_y^4 r_{xx} r_y^2 r^3\Delta + 65\varphi_y^4 r_{xx} r_y\Delta_x r^3 + 15\varphi_y^4 r_{xx} r_y\Delta_y r^4 - 4\varphi_y^4 r_{xx}\Delta_{xy} r^4 - 10\varphi_y^4 r_{xx}\Delta_{xx} r^3 \\
&-\varphi_y^4 r_{xx}\Delta_{yy} r^5 + 105\varphi_y^4 r_x^4\Delta + 315\varphi_y^4 r_x^3 r_y r\Delta - 105\varphi_y^4 r_x^3\Delta_x r - 15\varphi_y^4 r_x^3\Delta_y r^2 \\
&-78\varphi_y^4 r_x^2 r_{yy} r^3\Delta + 430\varphi_y^4 r_x^2 r_y^2 r^2\Delta - 285\varphi_y^4 r_x^2 r_y\Delta_x r^2 - 55\varphi_y^4 r_x^2 r_y\Delta_y r^3 \\
&+15\varphi_y^4 r_x^2\Delta_{xy} r^3 + 45\varphi_y^4 r_x^2\Delta_{xx} r^2 + 3\varphi_y^4 r_x^2\Delta_{yy} r^4 + 18\varphi_y^4 r_x r_{yyy} r^5\Delta - 202\varphi_y^4 r_x r_{yy} r_y r^4\Delta \\
&+69\varphi_y^4 r_x r_{yy}\Delta_x r^4 + 22\varphi_y^4 r_x r_{yy}\Delta_y r^5 + 326\varphi_y^4 r_x r_y^3 r^3\Delta - 320\varphi_y^4 r_x r_y^2\Delta_x r^3 - 86\varphi_y^4 r_x r_y^2\Delta_y r^4 \\
&+46\varphi_y^4 r_x r_y\Delta_{xy} r^4 + 100\varphi_y^4 r_x r_y\Delta_{xx} r^3 + 13\varphi_y^4 r_x r_y\Delta_{yy} r^5 - 3\varphi_y^4 r_x\Delta_{xyy} r^5 - 10\varphi_y^4 r_x\Delta_{xxx} r^3 \\
&-6\varphi_y^4 r_x\Delta_{xxy} r^4 - \varphi_y^4 r_x\Delta_{yyy} r^6 - 5\varphi_y^4 r_{yyyy} r^7\Delta + 40\varphi_y^4 r_{yyy} r_y r^6\Delta - 14\varphi_y^4 r_{yyy}\Delta_x r^6 \\
&-10\varphi_y^4 r_{yyy}\Delta_y r^7 + 30\varphi_y^4 r_{yy}^2 r^6\Delta - 180\varphi_y^4 r_{yy} r_y^2 r^5\Delta + 119\varphi_y^4 r_{yy} r_y\Delta_x r^5 + 60\varphi_y^4 r_{yy} r_y\Delta_y r^6 \\
&-16\varphi_y^4 r_{yy}\Delta_{xy} r^6 - 19\varphi_y^4 r_{yy}\Delta_{xx} r^5 - 10\varphi_y^4 r_{yy}\Delta_{yy} r^7 + 120\varphi_y^4 r_y^4 r^4\Delta - 154\varphi_y^4 r_y^3\Delta_x r^4 \\
&-60\varphi_y^4 r_y^3\Delta_y r^5 + 47\varphi_y^4 r_y^2\Delta_{xy} r^5 + 71\varphi_y^4 r_y^2\Delta_{xx} r^4 + 20\varphi_y^4 r_y^2\Delta_{yy} r^6 - 9\varphi_y^4 r_y\Delta_{xyy} r^6 \\
&-14\varphi_y^4 r_y\Delta_{xxx} r^4 - 12\varphi_y^4 r_y\Delta_{xxy} r^5 - 5\varphi_y^4 r_y\Delta_{yyy} r^7 + \varphi_y^4\Delta_{xyyy} r^7 + \varphi_y^4\Delta_{xxxx} r^4 \\
&+\varphi_y^4\Delta_{xxxy} r^5 + \varphi_y^4\Delta_{xxyy} r^6 + \varphi_y^4\Delta_{yyyy} r^8)/(\varphi_y^4\Delta). \qquad (3.65)
\end{aligned}
$$

Theorem 3.1.1. *Any fifth-order ordinary differential equation linearizable by a point transformation has to be one of the forms either equation (3.10) and (3.29).*

3.2 Sufficient conditions for linearization

3.2.1 The first class of linearizable equations

In this case, the linearizing transformation (3.2) must be a fiber preserving transformation, i.e.,

$$t = \varphi(x), \quad u = \psi(x, y) \tag{3.66}$$

For obtaining sufficient conditions, one has to solve the compatibility problem. Considering the representations of the coefficients A_i, B_i, C_i, D_i and E_i through the unknown functions φ and ψ. We first rewrite the expressions (3.11) and (3.12) for A_1 and A_0 in the following form

$$\psi_{yy} = \psi_y A_1/5, \tag{3.67}$$

$$\varphi_{xx} = (5\psi_{xy} - \psi_y A_0)\varphi_x/(10\psi_y). \tag{3.68}$$

Differentiating equation (3.68) with respect to y, one obtains the condition

$$A_{0y} = A_{1x}. \tag{3.69}$$

From equations (3.13), (3.14), (3.15), (3.17), (3.18), (3.19), (3.20), (3.23) and (3.24) one gets conditions

$$B_3 = 2A_1, \tag{3.70}$$

$$A_{1y} = -(2A_1^2 - 5B_2)/10, \tag{3.71}$$

$$A_{1x} = -(4A_0A_1 - 5B_1)/20, \tag{3.72}$$

$$B_2 = 2C_1/3, \tag{3.73}$$

$$B_1 = 4C_0/3, \tag{3.74}$$

$$C_{1y} = -(2A_1C_1 - 15D_3)/10, \tag{3.75}$$

$$C_{0y} = -(2A_1C_0 - 5D_2)/10, \tag{3.76}$$

$$D_{3y} = -(A_1D_3 - 50E_5)/5, \tag{3.77}$$

$$D_{2y} = -(A_1D_2 - 30E_4)/5, \tag{3.78}$$

respectively. From equation (3.16) we have,

$$\psi_{xxy} = -(20A_{0x}\psi_y^2 - 125\psi_{xy}^2 + 10\psi_{xy}\psi_y A_0 + 7\psi_y^2 A_0^2 - 20\psi_y^2 B_0)/(100\psi_y).$$

Comparing the mixed derivative $(\psi_{xxy})_y = (\psi_{yy})_{xx}$ one obtains the condition

$$B_{0y} = -2(15A_{0x}A_1 - 25C_{0x} + 3A_0^2A_1 - 5A_0C_0)/75. \tag{3.79}$$

Equations (3.21), (3.22), (3.25), (3.26) and (3.27) provide the conditions

$$C_{0x} = (30A_{0x}A_1 + 6A_0^2A_1 - 10A_0C_0 - 15A_1B_0 + 25D_1)/50, \tag{3.80}$$

$$A_{0xx} = -(60A_{0x}A_0 - 75B_{0x} + 8A_0^3 - 30A_0B_0 + 50D_0)/50, \tag{3.81}$$

$$D_{1y} = -(A_1D_1 - 15E_3)/5, \tag{3.82}$$

$$\begin{aligned} D_{0y} = &(60A_{0x}A_0A_1 - 100A_{0x}C_0 - 75B_{0x}A_1 + 125D_{1x} + 12A_0^3A_1 - 20A_0^2C_0 \\ &- 45A_0A_1B_0 + 25A_0D_1 - 75A_1D_0 + 50B_0C_0 + 375E_2)/750, \end{aligned} \tag{3.83}$$

$$\begin{aligned} B_{0xx} = &2(175A_{0x}^2 + 70A_{0x}A_0^2 - 325A_{0x}B_0 - 75B_{0x}A_0 + 500D_{0x} + 7A_0^4 \\ &- 65A_0^2B_0 + 100A_0D_0 + 100B_0^2 - 625E_1)/375, \end{aligned} \tag{3.84}$$

respectively.

Consider the form of ψ_{yy}:

$$\begin{aligned} \psi_{yy} &= \frac{\psi_y A_1}{5} \\ \frac{\psi_{yy}}{\psi_y} &= \frac{A_1}{5} \\ \int \frac{1}{\psi_y} d\psi_y &= \int \frac{A_1}{5} dy \\ ln\psi_y &= \int \frac{A_1}{5} dy + K_1(x) \\ \psi_y &= e^{\int \frac{A_1}{5} dy + K_1(x)} \\ \psi_y &= e^{\int \frac{A_1}{5} dy} e^{K_1(x)} \\ \psi_y &= \omega_1(x,y)\psi_1(x), \end{aligned} \tag{3.85}$$

where $\omega_1(x,y) = e^{\int \frac{A_1}{5} dy}$ and $\psi_1(x) = e^{K_1(x)}$. Since $\psi_y \neq 0$ then ψ_1 and ω_1 can not be zero. From $\omega_1 = e^{\int \frac{A_1}{5} dy}$, we found these relations

$$\begin{aligned} ln\,\omega_1 &= \int \frac{A_1}{5} dy \\ \int \frac{1}{\omega_1} d\omega_1 &= \int \frac{A_1}{5} dy \\ \frac{\omega_{1y}}{\omega_1} &= \frac{A_1}{5} \end{aligned}$$

thus

$$A_1 = 5\frac{\omega_{1y}}{\omega_1}.$$

Relations $(A_1)_x = A_{1x}$ and $(A_1)_y = A_{1y}$ provide the conditions

$$\begin{aligned} \omega_{1xy} &= (15\omega_{1x}\omega_{1y} - 3A_0\omega_{1y}\omega_1 + C_0\omega_1^2)/(15\omega_1), \\ \omega_{1yy} &= C_1\omega_1/15, \end{aligned}$$

respectively, and the relation (3.85) satisfied the condition $(\psi_y)_y = \psi_{yy}$. Comparing the relation $(\psi_y)_{xx} = \psi_{xxy}$, one has the equation

$$\begin{aligned} \psi_{1xx} =& (-20A_{0x}\omega_1^2\psi_1^2 - 100\omega_{1xx}\omega_1\psi_1^2 + 125\omega_{1x}^2\psi_1^2 + 50\omega_{1x}\psi_{1x}\omega_1\psi_1 - 10\omega_{1x}A_0\omega_1\psi_1^2 \\ &+ 125\psi_{1x}^2\omega_1^2 - 10\psi_{1x}A_0\omega_1^2\psi_1 - 7A_0^2\omega_1^2\psi_1^2 + 20B_0\omega_1^2\psi_1^2)/(100\omega_1^2\psi_1), \qquad (3.86) \end{aligned}$$

and satisfying the condition $(\psi_{1xx})_y = 0$. Consider

$$\begin{aligned} \psi_y &= \omega_1(x,y)\psi_1(x) \\ \psi &= \int \omega_1(x,y)\psi_1(x)dy + \psi_2(x) \\ &= \psi_1(x)\int \omega_1(x,y)dy + \psi_2(x) \end{aligned}$$

or

$$\psi = \psi_1(x)\omega_2(x,y) + \psi_2(x), \qquad (3.87)$$

where $\omega_2(x,y) = \int \omega_1(x,y)dy$. Because of

$$\int \omega_1(x,y)dy = \omega_2(x,y),$$

then

$$\omega_1(x,y) = \omega_{2y}.$$

Since $\omega_1 \neq 0$ then $\omega_{2y} \neq 0$. Substituting ω_1 into ω_{1xy} and ω_{1yy} we obtain the condition

$$\begin{aligned} \omega_{2xyy} &= ((-3\omega_{2yy}A_0 + \omega_{2y}C_0)\omega_{2y} + 15\omega_{2xy}\omega_{2yy})/(15\omega_{2y}), \\ \omega_{2yyy} &= \omega_{2y}C_1/15 \end{aligned}$$

respectively, and these satisfied the relations $(\psi)_y = \psi_y$, $(\psi)_{yy} = \psi_{yy}$ and $(\psi)_{xxy} = \psi_{xxy}$. From equation (3.28), setting $\mu_1(x,y)$, $\mu_2(x,y)$, $\mu_3(x,y)$ as equations (A.1), (A.2) and

(A.3),[1] then we obtain

$$\begin{aligned}\psi_{2xxxxx} =&(-140625\psi_{1x}^4\psi_{2x}\omega_{2y}^5 + 1125000\psi_{1x}^3\psi_{2xx}\omega_{2y}^5\psi_1 + 112500\psi_{1x}^3\psi_{2x}\omega_{2y}^4\psi_1(-5\omega_{2xy}\\&+\omega_{2y}A_0) - 2250000\psi_{1x}^2\psi_{2xxx}\omega_{2y}^5\psi_1^2 + 675000\psi_{1x}^2\psi_{2xx}\omega_{2y}^4\psi_1^2(5\omega_{2xy}-\omega_{2y}A_0)\\&+11250\psi_{1x}^2\psi_{2x}\omega_{2y}^3\psi_1^2(-40A_{0x}\omega_{2y}^2 - 75\omega_{2xy}^2 + 30\omega_{2xy}\omega_{2y}A_0 - 11\omega_{2y}^2A_0^2\\&+20\omega_{2y}^2B_0) + 1500000\psi_{1x}\psi_{2xxxx}\omega_{2y}^5\psi_1^3 + 900000\psi_{1x}\psi_{2xxx}\omega_{2y}^4\psi_1^3(-5\omega_{2xy}+\omega_{2y}A_0)\\&+75000\psi_{1x}\psi_{2xx}\omega_{2y}^3\psi_1^3(16A_{0x}\omega_{2y}^2 + 45\omega_{2xy}^2 - 18\omega_{2xy}\omega_{2y}A_0 + 5\omega_{2y}^2A_0^2\\&-8\omega_{2y}^2B_0) + 1500\psi_{1x}\psi_{2x}\omega_{2y}^2\psi_1^3(200A_{0x}\omega_{2xy}\omega_{2y}^2 - 40A_{0x}\omega_{2y}^3A_0 + 375\omega_{2xy}^3\\&-225\omega_{2xy}^2\omega_{2y}A_0 + 85\omega_{2xy}\omega_{2y}^2A_0^2 - 100\omega_{2xy}\omega_{2y}^2B_0 - 11\omega_{2y}^3A_0^3 + 20\omega_{2y}^3A_0B_0\\&-2\mu_3) + 300000\psi_{2xxxxx}\omega_{2y}^4\psi_1^4(5\omega_{2xy}-\omega_{2y}A_0) + 30000\psi_{2xxx}\omega_{2y}^3\psi_1^4(-20A_{0x}\omega_{2y}^2\\&-75\omega_{2xy}^2 + 30\omega_{2xy}\omega_{2y}A_0 - 7\omega_{2y}^2A_0^2 + 10\omega_{2y}^2B_0) + 3000\psi_{2xx}\omega_{2y}^2\mu_3\psi_1^4\\&+25\psi_{2x}\omega_{2y}\mu_2\psi_1^4 + 16\mu_1\psi_1^5)/(300000\omega_{2y}^5\psi_1^4).\end{aligned} \quad (3.88)$$

Differentiating ψ_{2xxxxx} with respect to y, we get

$$18000\psi_{1x}\psi_{2x}\omega_{2y}^2\mu_4 - 18000\psi_{2xx}\omega_{2y}^2\mu_4\psi_1 + 150\psi_{2x}\omega_{2y}\mu_5\psi_1 + \mu_6\psi_1^2 = 0, \quad (3.89)$$

where $\mu_4(x,y)$, $\mu_5(x,y)$ and $\mu_6(x,y)$ are defined as equations (A.4), (A.5) and (A.6).[2] Further analysis of the compatibility depends on value of μ_4: it is separated into two cases $\mu_4 = 0$ and $\mu_4 \neq 0$.

3.2.1 Case $\mu_4 = 0$

From equation (3.89), one obtains the equation

$$150\psi_{2x}\omega_{2y}\mu_5 + \mu_6\psi_1 = 0. \quad (3.90)$$

3.2.1.1 Case $\mu_5 = 0$

From equation (3.90), one obtains the condition

$$\mu_6 = 0. \quad (3.91)$$

3.2.1.2 Case $\mu_5 \neq 0$

From equation (3.90), one obtains the derivative

$$\psi_{2x} = -\mu_6\psi_1/(150\omega_{2y}\mu_5). \quad (3.92)$$

Differentiating equation (3.92) with respect to y, one gets the condition

$$\mu_{6y} = \mu_6(\mu_{5y}\omega_{2y} + \omega_{2yy}\mu_5)/(\omega_{2y}\mu_5). \quad (3.93)$$

[1]See Appendix
[2]See Appendix

From relation $(\psi_{2x})_{xxxx} = \psi_{2xxxxxx}$, one obtains equation

$$60\psi_{1x}\omega_{2y}\mu_5\mu_7 + \mu_8\psi_1 = 0 \tag{3.94}$$

where μ_7 and μ_8 are defined as equations (A.7) and (A.8).[3]

3.2.1.2.1 Case $\mu_7 = 0$

From equation (3.94), one obtains the condition

$$\mu_8 = 0. \tag{3.95}$$

3.2.1.2.2 Case $\mu_7 \neq 0$

From equation (3.94), one obtains the derivative

$$\psi_{1x} = -\mu_8\psi_1/(60\omega_{2y}\mu_5\mu_7). \tag{3.96}$$

Differentiating $\psi_{1x}$with respect to y, one obtains the condition

$$\begin{aligned}\mu_{8x} =&(19200A_{0x}^2\omega_{2y}^4\mu_5^4\mu_6\mu_8 - 132000A_{0x}\omega_{2xy}^2\omega_{2y}^2\mu_5^4\mu_6\mu_8\\
&- 24000A_{0x}\omega_{2xy}\omega_{2y}^3A_0\mu_5^4\mu_6\mu_8 + 192000A_{0x}\omega_{2xxy}\omega_{2y}^3\mu_5^4\mu_6\mu_8\\
&+ 17760A_{0x}\omega_{2y}^4A_0^2\mu_5^4\mu_6\mu_8 - 38400A_{0x}\omega_{2y}^4B_0\mu_5^4\mu_6\mu_8 + 2880A_{0x}\omega_{2y}^2\mu_5^2\mu_7^2\\
&- 480\mu_{3x}\omega_{2y}\mu_5^4\mu_6\mu_8 + 1200\mu_{5x}\omega_{2y}\mu_7\mu_8 - 607500\omega_{2xy}^4\mu_5^4\mu_6\mu_8\\
&+ 270000\omega_{2xy}^3\omega_{2y}A_0\mu_5^4\mu_6\mu_8 + 540000\omega_{2xy}^2\omega_{2xxy}\omega_{2y}\mu_5^4\mu_6\mu_8\\
&- 42600\omega_{2xy}^2\omega_{2y}^2A_0^2\mu_5^4\mu_6\mu_8 + 12000\omega_{2xy}^2\omega_{2y}^2B_0\mu_5^4\mu_6\mu_8 - 18000\omega_{2xy}^2\mu_5^2\mu_7^2\\
&- 216000\omega_{2xy}\omega_{2xxy}\omega_{2y}^2A_0\mu_5^4\mu_6\mu_8 - 11280\omega_{2xy}\omega_{2y}^3A_0^3\mu_5^4\mu_6\mu_8\\
&+ 33600\omega_{2xy}\omega_{2y}^3A_0B_0\mu_5^4\mu_6\mu_8 + 1440\omega_{2xy}\omega_{2y}A_0\mu_5^2\mu_7^2 + 1920\omega_{2xy}mu3\mu_5^4\mu_6\mu_8\\
&+ 1800\omega_{2xy}\mu_5\mu_7\mu_8 + 60000\omega_{2xxy}\omega_{2y}^3A_0^2\mu_5^4\mu_6\mu_8 - 96000\omega_{2xxy}\omega_{2y}^3B_0\mu_5^4\mu_6\mu_8\\
&+ 14400\omega_{2xxy}\omega_{2y}\mu_5^2\mu_7^2 + 3540\omega_{2y}^4A_0^4\mu_5^4\mu_6\mu_8 - 14880\omega_{2y}^4A_0^2B_0\mu_5^4\mu_6\mu_8\\
&+ 14400\omega_{2y}^4B_0^2\mu_5^4\mu_6\mu_8 + 1008\omega_{2y}^2A_0^2\mu_5^2\mu_7^2 - 2880\omega_{2y}^2B_0\mu_5^2\mu_7^2\\
&- 96\omega_{2y}A_0\mu_3\mu_5^4\mu_6\mu_8 - 120\omega_{2y}A_0\mu_5\mu_7\mu_8 - 1920\mu_1\mu_5^5\mu_8 + 20\mu_2\mu_5^4\mu_6\mu_8\\
&- 5\mu_8^2)/(240\omega_{2y}\mu_5\mu_7).\end{aligned} \tag{3.97}$$

The relation $(\psi_{1x})_x = \psi_{1xx}$ provides the condition

$$\mu_{8y} = \mu_8(\mu_{5y}\omega_{2y}\mu_7 + \mu_{7y}\omega_{2y}\mu_5 + \omega_{2yy}\mu_5\mu_7)/(\omega_{2y}\mu_5\mu_7). \tag{3.98}$$

[3]See Appendix

3.2.2 Case $\mu_4 \neq 0$

From equation (3.89), one obtains the derivative

$$\psi_{2xx} = (18000\psi_{1x}\psi_{2x}\omega_{2y}^2\mu_4 + 150\psi_{2x}\omega_{2y}\mu_5\psi_1 + \mu_6\psi_1^2)/(18000\omega_{2y}^2\mu_4\psi_1). \quad (3.99)$$

Differentiating ψ_{2xx} with respect to y, then we obtain the equation

$$150\psi_{2x}\omega_{2y}\mu_9 + \mu_{10}\psi_1 = 0, \quad (3.100)$$

where μ_9 and μ_{10} are defined as equations (A.9) and (A.10).[4]

3.2.2.1 Case $\mu_9 = 0$

From equation (3.100), one obtains the condition

$$\mu_{10} = 0. \quad (3.101)$$

3.2.2.2 Case $\mu_9 \neq 0$

From equation (3.100), one obtains the derivative

$$\psi_{2x} = -\mu_{10}\psi_1/(150\omega_{2y}\mu_9). \quad (3.102)$$

Differentiating ψ_{2x} with respect to y, one obtains the condition

$$\mu_{10y} = \mu_{10}(\mu_{9y}\omega_{2y} + \omega_{2yy}\mu_9)/(\omega_{2y}\mu_9). \quad (3.103)$$

Substituting ψ_{2x} into ψ_{2xx}, we obtain the condition

$$\mu_{10x} = (120\mu_{9x}\omega_{2y}\mu_{10}\mu_4 + 120\omega_{2xy}\mu_{10}\mu_4\mu_9 + \mu_{10}\mu_5\mu_9 - \mu_6\mu_9^2)/(120\omega_{2y}\mu_4\mu_9). \quad (3.104)$$

Substituting ψ_{2x} into ψ_{2xxxxx}, we get the relation

$$5\psi_{1x}\omega_{2y}\mu_{11}\mu_4\mu_9\mu_{12} + \mu_{12}\psi_1 = 0. \quad (3.105)$$

where μ_{11} and μ_{12} are defined as equations (A.11) and (A.12).[5]

3.2.2.2.1 Case $\mu_{11} = 0$

From equation (3.105), one obtains the condition

$$\mu_{12} = 0. \quad (3.106)$$

[4]See Appendix

[5]See Appendix

Case $\mu_{11} \neq 0$

From equation (3.105), one obtains the derivative

$$\psi_{1x} = -\mu_{12}\psi_1/(5\omega_{2y}\mu_{11}\mu_4\mu_9\omega_2). \tag{3.107}$$

The relations $(\psi_{1x})_y = 0$ and $(\psi_{1x})_x = \psi_{1xx}$ yield the conditions

$$\begin{aligned}\mu_{12y} =&\mu_{12}(\mu_{11y}\omega_{2y}\mu_4\mu_9\omega_2 + \mu_{4y}\omega_{2y}\mu_{11}\mu_9\omega_2 + \mu_{9y}\omega_{2y}\mu_{11}\mu_4\omega_2\\ &+ \omega_{2yy}\mu_{11}\mu_4\mu_9\omega_2 + \omega_{2y}^2\mu_{11}\mu_4\mu_9)/(\omega_{2y}\mu_{11}\mu_4\mu_9\omega_2),\end{aligned} \tag{3.108}$$

$$\begin{aligned}\mu_{12x} =&\mu_{12}(691200A_{0x}^2\omega_{2y}^4\mu_{10}\mu_4^4\mu_9\omega_2^2 - 4752000A_{0x}\omega_{2xy}^2\omega_{2y}^2\mu_{10}\mu_4^4\mu_9\omega_2^2\\ &- 864000A_{0x}\omega_{2xy}\omega_{2y}^3A_0\mu_{10}\mu_4^4\mu_9\omega_2^2 + 6912000A_{0x}\omega_{2xxy}\omega_{2y}^3\mu_{10}\mu_4^4\mu_9\omega_2^2\\ &+ 639360A_{0x}\omega_{2y}^4A_0^2\mu_{10}\mu_4^4\mu_9\omega_2^2 - 1382400A_{0x}\omega_{2y}^4B_0\mu_{10}\mu_4^4\mu_9\omega_2^2\\ &- 17280\mu_{3x}\omega_{2y}\mu_{10}\mu_4^4\mu_9\omega_2^2 + 20\mu_{4x}\omega_{2y}\mu_{11}\mu_9\omega_2 + 10\mu_{9x}\omega_{2y}\mu_{11}\mu_4\omega_2\\ &- 21870000\omega_{2xy}^4\mu_{10}\mu_4^4\mu_9\omega_2^2 + 9720000\omega_{2xy}^3\omega_{2y}A_0\mu_{10}\mu_4^4\mu_9\omega_2^2\\ &+ 19440000\omega_{2xy}^2\omega_{2xxy}\omega_{2y}\mu_{10}\mu_4^4\mu_9\omega_2^2 - 1533600\omega_{2xy}^2\omega_{2y}^2A_0^2\mu_{10}\mu_4^4\mu_9\omega_2^2\\ &+ 432000\omega_{2xy}^2\omega_{2y}^2B_0\mu_{10}\mu_4^4\mu_9\omega_2^2 - 7776000\omega_{2xy}\omega_{2xxy}\omega_{2y}^2A_0\mu_{10}\mu_4^4\mu_9\omega_2^2\\ &- 406080\omega_{2xy}\omega_{2y}^3A_0^3\mu_{10}\mu_4^4\mu_9\omega_2^2 + 1209600\omega_{2xy}\omega_{2y}^3A_0B_0\mu_{10}\mu_4^4\mu_9\omega_2^2\\ &+ 69120\omega_{2xy}\mu_{10}\mu_3\mu_4^4\mu_9\omega_2^2 + 35\omega_{2xy}\mu_{11}\mu_4\mu_9\omega_2 + 2160000\omega_{2xxy}\omega_{2y}^3A_0^2\mu_{10}\mu_4^4\mu_9\omega_2^2\\ &- 3456000\omega_{2xxy}\omega_{2y}^3B_0\mu_{10}\mu_4^4\mu_9\omega_2^2 + 10\omega_{2x}\omega_{2y}\mu_{11}\mu_4\mu_9 + 127440\omega_{2y}^4A_0^4\mu_{10}\mu_4^4\mu_9\omega_2^2\\ &- 535680\omega_{2y}^4A_0^2B_0\mu_{10}\mu_4^4\mu_9\omega_2^2 + 518400\omega_{2y}^4B_0^2\mu_{10}\mu_4^4\mu_9\omega_2^2\\ &- 3456\omega_{2y}A_0\mu_{10}\mu_3\mu_4^4\mu_9\omega_2^2 - 2\omega_{2y}A_0\mu_{11}\mu_4\mu_9\omega_2 - 69120\mu_1\mu_4^4\mu_9^2\omega_2^2\\ &+ 720\mu_{10}\mu_2\mu_4^4\mu_9\omega_2^2)/(5\omega_{2y}\mu_{11}\mu_4\mu_9\omega_2),\end{aligned} \tag{3.109}$$

respectively.

All obtained results can be summarized in the following theorem

Theorem 3.2.1. *Sufficient conditions for equation (3.10) to be linearizable via the fiber-preserving transformation (3.66) are as follows.*

(a) *If $\mu_4 = 0$ and $\mu_5 = 0$, then the conditions are (3.69), (3.72), (3.71), (3.70), (3.74), (3.73), (3.76), (3.75), (3.78), (3.77), (3.79), (3.80), (3.82), (3.81), (3.83), (3.84) and (3.91).*

(b) *If $\mu_4 = 0$, $\mu_5 \neq 0$ and $\mu_7 = 0$, then the conditions are (3.69), (3.72), (3.71), (3.70), (3.74), (3.73), (3.76), (3.75), (3.78), (3.77), (3.79), (3.80), (3.82), (3.81), (3.83), (3.84), (3.93) and (3.95).*

(c) *If $\mu_4 = 0$, $\mu_5 \neq 0$ and $\mu_7 \neq 0$, then the conditions are (3.69), (3.72), (3.71), (3.70),* *(3.74), (3.73), (3.76), (3.75), (3.78), (3.77), (3.79), (3.80), (3.82), (3.81), (3.83), (3.84), (3.93), (3.97) and (3.98).*

(d) *If $\mu_4 \neq 0$ and $\mu_9 = 0$ then the conditions are (3.69), (3.72), (3.71), (3.70), (3.74), (3.73), (3.76), (3.75), (3.78), (3.77), (3.79), (3.80), (3.82), (3.81), (3.83), (3.84) and (3.101).*

(e) *If $\mu_4 \neq 0$, $\mu_9 \neq 0$ and $\mu_{11} = 0$, then the conditions are (3.69), (3.72), (3.71), (3.70), (3.74), (3.73), (3.76), (3.75), (3.78), (3.77), (3.79), (3.80), (3.82), (3.81), (3.83), (3.84), (3.103), (3.104) and (3.106).*

(f) *If $\mu_4 \neq 0$, $\mu_9 \neq 0$ and $\mu_{11} \neq 0$, then the conditions are (3.69), (3.72), (3.71), (3.70), (3.74), (3.73), (3.76), (3.75), (3.78), (3.77), (3.79), (3.80), (3.82), (3.81), (3.83), (3.84), (3.103), (3.104), (3.108) and (3.109).*

3.2.2 The second class of linearizable equations

The problem is: for the given coefficients $F_i, G_i, H_i, J_i, K_i, L_i$ and M_i of equation (3.29) find the integrability conditions for the functions $\varphi(x, y)$ and $\psi(x, y)$.

Recall that according to our notations, the following equations hold

$$\varphi_x = r\varphi_y, \tag{3.110}$$

$$\psi_x = (\varphi_y\psi_y r - \Delta)/\varphi_y. \tag{3.111}$$

From equations (3.30) and (3.31), one obtains the derivatives

$$\varphi_{yy} = (\varphi_y(5\Delta_y - F_2\Delta))/(15\Delta), \tag{3.112}$$

$$\Delta_x = (30r_y\Delta + 5\Delta_y r + F_1\Delta - 2F_2 r\Delta)/5. \tag{3.113}$$

Comparing the mixed derivative $(\varphi_x)_{yy} = (\varphi_{yy})_x$ one gets the condition

$$r_{yy} = (-F_{1y} + F_{2x} + F_{2y}r + r_y F_2)/15. \tag{3.114}$$

From equations (3.32), (3.33), (3.34) and (3.35), one obtains the following conditions

$$r_x = (15r_y r - F_0 + F_1 r - F_2 r^2)/15, \tag{3.115}$$

$$G_2 = -10F_2, \tag{3.116}$$

$$G_1 = 4(-3F_1 + F_2 r), \tag{3.117}$$

$$G_0 = 2(-7F_0 + F_1 r), \tag{3.118}$$

respectively. Equation (3.36) provides derivative

$$\Delta_{yy} = (-60F_{2y}\Delta^2 + 175\Delta_y^2 - 10\Delta_y F_2\Delta - 17F_2^2\Delta^2 + 45H_4\Delta^2)/(150\Delta). \tag{3.119}$$

From equations (3.37), (3.38), (3.41), (3.42), (3.44), (3.43), (3.45), (3.46), (3.39), (3.40) and (3.47), one obtains the following conditions

$$F_{2x} =(20F_{2y}r - 4F_1F_2 + 8F_2^2r + 5H_3 - 20H_4r)/20, \tag{3.120}$$

$$F_{1x} =(30F_{1y}r - 14F_0F_2 - 6F_1^2 + 26F_1F_2r - 14F_2^2r^2 + 15H_2 - 30H_3r + 30H_4r^2)/30, \tag{3.121}$$

$$J_2 =15F_2, \tag{3.122}$$

$$J_1 =3(7F_1 - 4F_2r), \tag{3.123}$$

$$K_4 = - 3H_4, \tag{3.124}$$

$$J_0 =28F_0 - 7F_1r + F_2r^2, \tag{3.125}$$

$$K_3 =3(-5H_3 + 4H_4r)/4, \tag{3.126}$$

$$K_2 =9(-2H_2 + H_3r)/4, \tag{3.127}$$

$$F_{0x} =(60F_{0y}r - 40F_0F_1 + 52F_0F_2r + 28F_1^2r - 56F_1F_2r^2 + 28F_2^2r^3 + 45H_1 - 60H_2r + 60H_3r^2 - 60H_4r^3)/60, \tag{3.128}$$

$$H_0 =(7F_0^2 - 14F_0F_1r + 14F_0F_2r^2 + 7F_1^2r^2 - 14F_1F_2r^3 + 7F_2^2r^4 + 15H_1r - 15H_2r^2 + 15H_3r^3 - 15H_4r^4)/15, \tag{3.129}$$

$$K_1 =3(-7H_1 + 2H_2r)/4, \tag{3.130}$$

respectively. Comparing the mixed derivative $(\Delta_x)_{yy} = (\Delta_{yy})_x$ one obtains the condition

$$F_{1yy} =(-260F_{1y}F_2 + 600F_{2yy}r - 240F_{2y}F_1 + 1000F_{2y}F_2r + 300H_{3y} - 450H_{4x} - 750H_{4y}r + 140r_yF_2^2 - 300r_yH_4 - 76F_1F_2^2 + 152F_2^3r + 95F_2H_3 - 380F_2H_4r)/300. \tag{3.131}$$

From equations (3.49), (3.50), (3.48), (3.51), (3.52), (3.53), (3.54) and (3.55), one obtains the conditions

$$F_{2yy} =(-60F_{2y}F_2 + 75H_{4y} - 8F_2^3 + 30F_2H_4 - 50L_6)/50, \tag{3.132}$$

$$H_{4x} =(15H_{4y}r - 3F_1H_4 + 6F_2H_4r + 5L_5 - 30L_6r)/15, \tag{3.133}$$

$$K_0 =(-56F_0^2 + 112F_0F_1r - 112F_0F_2r^2 - 56F_1^2r^2 + 112F_1F_2r^3 - 56F_2^2r^4 - 105H_1r + 120H_2r^2 - 120H_3r^3 + 120H_4r^4)/20, \tag{3.134}$$

$$F_{0yy} =(-120F_{0y}F_2 - 140F_{1y}F_1 + 140F_{1y}F_2r - 100F_{2y}F_0 + 140F_{2y}F_1r - 140F_{2y}F_2r^2 + 150H_{2y} - 150H_{3y}r + 150H_{4y}r^2 + 140r_yF_1F_2 - 140r_yF_2^2r - 150r_yH_3 + 300r_yH_4r - 20F_0F_2^2 + 20F_0H_4 - 28F_1^2F_2 + 56F_1F_2^2r + 35F_1H_3 - 70F_1H_4r - 28F_2^3r^2 + 30F_2H_2 - 65F_2H_3r + 100F_2H_4r^2 - 50L_4 + 100L_5r - 150L_6r^2)/300, \tag{3.135}$$

$$\begin{aligned}H_{2x} =&(15H_{2y}r + 15H_{3x}r - 15H_{3y}r^2 - 8F_0H_3 + 16F_0H_4r - 3F_1H_2 + 11F_1H_3r \\ &- 16F_1H_4r^2 + 6F_2H_2r - 14F_2H_3r^2 + 16F_2H_4r^3 + 15L_3 - 40L_4r + 60L_5r^2 \\ &- 60L_6r^3)/15, \end{aligned} \tag{3.136}$$

$$\begin{aligned}H_{1x} =&(45H_{1y}r - 30H_{3x}r^2 + 30H_{3y}r^3 - 24F_0H_2 + 24F_0H_3r - 40F_0H_4r^2 - 9F_1H_1 \\ &+ 24F_1H_2r - 30F_1H_3r^2 + 40F_1H_4r^3 + 18F_2H_1r - 24F_2H_2r^2 + 36F_2H_3r^3 \\ &- 40F_2H_4r^4 + 60L_2 - 90L_3r + 110L_4r^2 - 150L_5r^3 + 150L_6r^4)/45, \end{aligned} \tag{3.137}$$

$$\begin{aligned}L_1 =&(-14F_0^2F_1 + 28F_0^2F_2r + 28F_0F_1^2r - 84F_0F_1F_2r^2 + 56F_0F_2^2r^3 + 35F_0H_1 \\ &- 70F_0H_2r + 105F_0H_3r^2 - 140F_0H_4r^3 - 14F_1^3r^2 + 56F_1^2F_2r^3 - 70F_1F_2^2r^4 \\ &- 35F_1H_1r + 70F_1H_2r^2 - 105F_1H_3r^3 + 140F_1H_4r^4 + 28F_2^3r^5 + 35F_2H_1r^2 \\ &- 70F_2H_2r^3 + 105F_2H_3r^4 - 140F_2H_4r^5 + 100L_2r - 150L_3r^2 + 200L_4r^3 \\ &- 250L_5r^4 + 300L_6r^5)/50, \end{aligned} \tag{3.138}$$

$$\begin{aligned}L_0 =&(56F_0^3 - 294F_0^2F_1r + 420F_0^2F_2r^2 + 420F_0F_1^2r^2 - 1092F_0F_1F_2r^3 + 672F_0F_2^2r^4 \\ &+ 315F_0H_1r - 630F_0H_2r^2 + 945F_0H_3r^3 - 1260F_0H_4r^4 - 182F_1^3r^3 + 672F_1^2F_2r^4 \\ &- 798F_1F_2^2r^5 - 315F_1H_1r^2 + 630F_1H_2r^3 - 945F_1H_3r^4 + 1260F_1H_4r^5 + 308F_2^3r^6 \\ &+ 315F_2H_1r^3 - 630F_2H_2r^4 + 945F_2H_3r^5 - 1260F_2H_4r^6 + 450L_2r^2 - 900L_3r^3 \\ &+ 1350L_4r^4 - 1800L_5r^5 + 2250L_6r^6)/450, \end{aligned} \tag{3.139}$$

respectively. Equation (3.56) provides the derivative

$$\begin{aligned}\psi_{yyyyy} =&(112500\varphi_y\psi_{yyyy}\Delta_y\Delta^3 - 22500\varphi_y\psi_{yyyy}F_2\Delta^4 - 112500\varphi_y\psi_{yyy}\Delta_y^2\Delta^2 \\ &+ 45000\varphi_y\psi_{yyy}\Delta_yF_2\Delta^3 + 2250\varphi_y\psi_{yyy}\Delta^4(-30F_{2y} - 8F_2^2 + 15H_4) \\ &+ 37500\varphi_y\psi_{yy}\Delta_y^3\Delta - 22500\varphi_y\psi_{yy}\Delta_y^2F_2\Delta^2 + 22500\varphi_y\psi_{yy}\Delta_y\Delta^3(4F_{2y} + F_2^2 - 2H_4) \\ &+ 750\varphi_y\psi_{yy}\Delta^4(12F_{2y}F_2 - 45H_{4y} + 2F_2^3 - 15F_2H_4 + 45L_6) - 3125\varphi_y\psi_y\Delta_y^4 \\ &+ 2500\varphi_y\psi_y\Delta_y^3F_2\Delta + 750\varphi_y\psi_y\Delta_y^2\Delta^2(-30F_{2y} - 7F_2^2 + 15H_4) \\ &+ 100\varphi_y\psi_y\Delta_y\Delta^3(-90F_{2y}F_2 + 225H_{4y} - 17F_2^3 + 90F_2H_4 - 225L_6) \\ &+ \varphi_y\psi_y\Delta^4(-2700F_{2y}^2 - 3780F_{2y}F_2^2 + 4050F_{2y}H_4 - 6750H_{4yy} - 450H_{4y}F_2 \\ &+ 6750L_{6y} - 653F_2^4 + 2610F_2^2H_4 - 900F_2L_6 - 2025H_4^2) + 33750M_9\Delta^5)/(33750\varphi_y\Delta^4). \end{aligned} \tag{3.140}$$

From equations (3.57), (3.58), (3.59) and (3.60), one obtains the conditions

$$\begin{aligned}H_{4yy} =&2(175F_{2y}^2 + 70F_{2y}F_2^2 - 325F_{2y}H_4 - 75H_{4y}F_2 + 500L_{6y} + 7F_2^4 - 65F_2^2H_4 \\ &+ 100F_2L_6 + 100H_4^2 + 625M_8 - 5625M_9r)/375, \end{aligned} \tag{3.141}$$

$$
\begin{aligned}
H_{3yy} =&(700F_{1y}F_{2y} + 140F_{1y}F_2^2 - 550F_{1y}H_4 - 700F_{2y}r_yF_2 + 140F_{2y}F_1F_2 - 475F_{2y}H_3 \\
&+ 400F_{2y}H_4r - 150H_{3y}F_2 + 750H_{4y}r_y + 750L_{5y} - 500L_{6y}r - 140r_yF_2^3 + 700r_yF_2H_4 \\
&- 1500r_yL_6 + 28F_1F_2^3 - 110F_1F_2H_4 - 95F_2^2H_3 + 80F_2^2H_4r + 150F_2L_5 - 100F_2L_6r \\
&+ 250H_3H_4 - 200H_4^2r + 1500M_7 - 7000M_8r + 9000M_9r^2)/375, \qquad (3.142)
\end{aligned}
$$

$$
\begin{aligned}
H_{2yy} =&(5600F_{0y}F_{2y} + 1120F_{0y}F_2^2 - 2800F_{0y}H_4 + 2800F_{1y}^2 - 5600F_{1y}r_yF_2 + 1120F_{1y}F_1F_2 \\
&- 3000F_{1y}H_3 + 1600F_{1y}H_4r - 5600F_{2y}r_yF_1 + 5600F_{2y}r_yF_2r + 1120F_{2y}F_1F_2r \\
&- 1120F_{2y}F_2^2r^2 - 2400F_{2y}H_2 + 1800F_{2y}H_3r - 400F_{2y}H_4r^2 - 1200H_{2y}F_2 + 6000H_{3y}r_y \\
&- 6000H_{4y}r_yr + 1400H_{4y}F_0 - 1400H_{4y}F_1r + 1400H_{4y}F_2r^2 + 4000L_{4y} - 2000L_{5y}r \\
&+ 2800r_y^2F_2^2 - 6000r_y^2H_4 - 2240r_yF_1F_2^2 + 2800r_yF_1H_4 + 1120r_yF_2^3r + 4200r_yF_2H_3 \\
&- 5600r_yF_2H_4r - 8000r_yL_5 + 12000r_yL_6r + 280F_0F_2H_4 - 1400F_0L_6 + 112F_1^2F_2^2 \\
&+ 224F_1F_2^3r - 600F_1F_2H_3 - 520F_1F_2H_4r + 1400F_1L_6r - 224F_2^4r^2 - 480F_2^2H_2 \\
&+ 360F_2^2H_3r + 760F_2^2H_4r^2 + 800F_2L_4 - 400F_2L_5r - 1400F_2L_6r^2 + 900H_2H_4 \\
&+ 575H_3^2 - 1300H_3H_4r + 200H_4^2r^2 + 9000M_6 - 27000M_7r + 24000M_8r^2)/3000, \\
&\qquad (3.143)
\end{aligned}
$$

$$
\begin{aligned}
H_{1yy} =&(8400F_{0y}F_{1y} - 8400F_{0y}r_yF_2 + 1680F_{0y}F_1F_2 - 2100F_{0y}H_3 - 8400F_{1y}r_yF_1 \\
&+ 8400F_{1y}r_yF_2r + 1680F_{1y}F_1F_2r - 1680F_{1y}F_2^2r^2 - 2400F_{1y}H_2 + 300F_{1y}H_3r+ \\
&1800F_{1y}H_4r^2 - 8400F_{2y}r_yF_0 + 8400F_{2y}r_yF_1r - 8400F_{2y}r_yF_2r^2 - 1680F_{2y}F_0F_1 \\
&+ 3360F_{2y}F_0F_2r + 1680F_{2y}F_1^2r - 5040F_{2y}F_1F_2r^2 + 3360F_{2y}F_2^2r^3 - 1500F_{2y}H_1 \\
&+ 600F_{2y}H_2r + 1500F_{2y}H_3r^2 - 3600F_{2y}H_4r^3 - 1800H_{1y}F_2 + 9000H_{2y}r_y - 9000H_{3y}r_yr \\
&+ 2100H_{3y}F_0 - 2100H_{3y}F_1r + 2100H_{3y}F_2r^2 + 9000H_{4y}r_yr^2 - 4200H_{4y}F_0r + 4200H_{4y}F_1r^2 \\
&- 4200H_{4y}F_2r^3 + 3000L_{3y} - 3000L_{5y}r^2 + 6000L_{6y}r^3 + 8400r_y^2F_1F_2 - 8400r_y^2F_2^2r \\
&- 9000r_y^2H_3 + 18000r_y^2H_4r - 1680r_yF_0F_2^2 - 1680r_yF_1^2F_2 + 1680r_yF_1F_2^2r \\
&+ 2100r_yF_1H_3 + 4200r_yF_2H_2 - 4200r_yF_2H_3r - 6000r_yL_4 + 6000r_yL_5r - 336F_0F_1F_2^2 \\
&+ 840F_0F_1H_4 + 672F_0F_2^3r + 420F_0F_2H_3 - 2520F_0F_2H_4r - 1400F_0L_5 + 4200F_0L_6r \\
&+ 672F_1^2F_2^2r - 840F_1^2H_4r - 1344F_1F_2^3r^2 - 480F_1F_2H_2 - 780F_1F_2H_3r + 3720F_1F_2H_4r^2 \\
&+ 1400F_1L_5r - 4200F_1L_6r^2 + 672F_2^4r^3 - 300F_2^2H_1 + 120F_2^2H_2r + 1140F_2^2H_3r^2 \\
&- 3240F_2^2H_4r^3 + 600F_2L_3 - 2000F_2L_5r^2 + 5400F_2L_6r^3 + 300H_1H_4 + 600H_2H_3 \\
&- 300H_2H_4r - 75H_3^2r - 1200H_3H_4r^2 + 1800H_4^2r^3 + 6000M_5 - 9000M_6r - 9000M_7r^2 \\
&+ 48000M_8r^3 - 108000M_9r^4)/4500, \qquad (3.144)
\end{aligned}
$$

respectively. Comparing the mixed derivative $(\psi_{yyyyy})_x = (\psi_x)_{yyyyy}$, we get

$$3750\varphi_y\psi_{yy}\lambda_1\Delta - 2500\varphi_y\psi_y\Delta_y\lambda_1 + 50\varphi_y\psi_y\Delta(15\lambda_{1y} - 2F_2\lambda_1) + \lambda_2\Delta^2 = 0, \qquad (3.145)$$

where λ_1 and λ_2 are defined as equations (A.13) and (A.14).[6] Further analysis of the compatibility depends on value of λ_1: it is separated into two cases $\lambda_1 = 0$ and $\lambda_1 \neq 0$.

3.2.1 Case $\lambda_1 = 0$

From equation (3.145), one obtains the condition

$$\lambda_2 = 0. \qquad (3.146)$$

From equation (3.61), one get equation

$$10\Delta_y\lambda_3 r^2 + \lambda_4\Delta = 0 \qquad (3.147)$$

where λ_3 and λ_4 are defined as equations (A.15) and (A.16).[7]

3.2.1.1 Case $\lambda_3 r = 0$

3.2.1.1.1 Case $\lambda_3 = 0, r \neq 0$

From equations (3.147), (3.62), (3.63), (3.64) and (3.65), one obtains the conditions

$$\lambda_4 = 0, \qquad (3.148)$$

$$\begin{aligned} L_{2x} = &(30L_{2y}r + 65L_{4x}r^2 - 65L_{4y}r^3 - 115L_{5x}r^3 + 115L_{5y}r^4 - 18F_0L_3 \\ &+ 18F_0L_4r + 21F_0L_5r^2 - 90F_0L_6r^3 - 6F_1L_2 + 18F_1L_3r - 5F_1L_4r^2 \\ &- 44F_1L_5r^3 + 90F_1L_6r^4 + 12F_2L_2r - 18F_2L_3r^2 - 8F_2L_4r^3 + 67F_2L_5r^4 \\ &- 90F_2L_6r^5 - 450M_3 + 900M_4r - 690M_5r^2 - 630M_6r^3 + 2985M_7r^4 \\ &- 5400M_8r^5 + 5400M_9r^6)/30, \end{aligned} \qquad (3.149)$$

$$\begin{aligned} M_2 = &(-40F_0^3F_2 - 144F_0^2F_1^2 + 696F_0^2F_1F_2r - 696F_0^2F_2^2r^2 + 180F_0^2H_2 \\ &- 540F_0^2H_3r + 1080F_0^2H_4r^2 + 288F_0F_1^3r - 1560F_0F_1^2F_2r^2 + 2544F_0F_1F_2^2r^3 \\ &+ 360F_0F_1H_1 - 1080F_0F_1H_2r + 2160F_0F_1H_3r^2 - 3600F_0F_1H_4r^3 \\ &- 1272F_0F_2^3r^4 - 720F_0F_2H_1r + 1800F_0F_2H_2r^2 - 3240F_0F_2H_3r^3 \\ &+ 5040F_0F_2H_4r^4 - 600F_0L_2 + 1800F_0L_3r - 3600F_0L_4r^2 \\ &+ 6000F_0L_5r^3 - 9000F_0L_6r^4 - 144F_1^4r^2 + 904F_1^3F_2r^3 - 1992F_1^2F_2^2r^4 \\ &- 360F_1^2H_1r + 900F_1^2H_2r^2 - 1620F_1^2H_3r^3 + 2520F_1^2H_4r^4 \end{aligned}$$

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[6] See Appendix

[7] See Appendix

$$+ 1848F_1F_2^3r^5 + 1080F_1F_2H_1r^2 - 2520F_1F_2H_2r^3 + 4320F_1F_2H_3r^4$$
$$- 6480F_1F_2H_4r^5 + 600F_1L_2r - 1800F_1L_3r^2 + 3600F_1L_4r^3$$
$$- 6000F_1L_5r^4 + 9000F_1L_6r^5 - 616F_2^4r^6 - 720F_2^2H_1r^3 + 1620F_2^2H_2r^4$$
$$- 2700F_2^2H_3r^5 + 3960F_2^2H_4r^6 - 600F_2L_2r^2 + 1800F_2L_3r^3 - 3600F_2L_4r^4$$
$$+ 6000F_2L_5r^5 - 9000F_2L_6r^6 - 225H_1^2 + 900H_1H_2r - 1350H_1H_3r^2$$
$$+ 1800H_1H_4r^3 - 900H_2^2r^2 + 2700H_2H_3r^3 - 3600H_2H_4r^4 - 2025H_3^2r^4$$
$$+ 5400H_3H_4r^5 - 3600H_4^2r^6 + 27000M_3r - 54000M_4r^2 + 90000M_5r^3$$
$$- 135000M_6r^4 + 189000M_7r^5 - 252000M_8r^6 + 324000M_9r^7)/9000, \quad (3.150)$$

$$M_1 =(56F_0^3F_1 - 152F_0^3F_2r - 312F_0^2F_1^2r + 1200F_0^2F_1F_2r^2 - 1032F_0^2F_2^2r^3$$
$$- 105F_0^2H_1 + 390F_0^2H_2r - 855F_0^2H_3r^2 + 1500F_0^2H_4r^3 + 456F_0F_1^3r^2$$
$$- 2232F_0F_1^2F_2r^3 + 3384F_0F_1F_2^2r^4 + 570F_0F_1H_1r - 1500F_0F_1H_2r^2$$
$$+ 2790F_0F_1H_3r^3 - 4440F_0F_1H_4r^4 - 1608F_0F_2^3r^5 - 930F_0F_2H_1r^2$$
$$+ 2220F_0F_2H_2r^3 - 3870F_0F_2H_3r^4 + 5880F_0F_2H_4r^5 - 600F_0L_2r$$
$$+ 1800F_0L_3r^2 - 3600F_0L_4r^3 + 6000F_0L_5r^4 - 9000F_0L_6r^5 - 200F_1^4r^3$$
$$+ 1184F_1^3F_2r^4 - 2496F_1^2F_2^2r^5 - 465F_1^2H_1r^2 + 1110F_1^2H_2r^3 - 1935F_1^2H_3r^4$$
$$+ 2940F_1^2H_4r^5 + 2240F_1F_2^3r^6 + 1290F_1F_2H_1r^3 - 2940F_1F_2H_2r^4$$
$$+ 4950F_1F_2H_3r^5 - 7320F_1F_2H_4r^6 + 600F_1L_2r^2 - 1800F_1L_3r^3$$
$$+ 3600F_1L_4r^4 - 6000F_1L_5r^5 + 9000F_1L_6r^6 - 728F_2^4r^7 - 825F_2^2H_1r^4$$
$$+ 1830F_2^2H_2r^5 - 3015F_2^2H_3r^6 + 4380F_2^2H_4r^7 - 600F_2L_2r^3 + 1800F_2L_3r^4$$
$$- 3600F_2L_4r^5 + 6000F_2L_5r^6 - 9000F_2L_6r^7 - 225H_1^2r + 900H_1H_2r^2$$
$$- 1350H_1H_3r^3 + 1800H_1H_4r^4 - 900H_2^2r^3 + 2700H_2H_3r^4 - 3600H_2H_4r^5$$
$$- 2025H_3^2r^5 + 5400H_3H_4r^6 - 3600H_4^2r^7 + 13500M_3r^2 - 36000M_4r^3$$
$$+ 67500M_5r^4 - 108000M_6r^5 + 157500M_7r^6 - 216000M_8r^7$$
$$+ 283500M_9r^8)/4500, \quad (3.151)$$

$$M_0 =(-56F_0^4 + 560F_0^3F_1r - 1016F_0^3F_2r^2 - 1776F_0^2F_1^2r^2 + 5784F_0^2F_1F_2r^3$$
$$- 4440F_0^2F_2^2r^4 - 630F_0^2H_1r + 1800F_0^2H_2r^2 - 3510F_0^2H_3r^3 + 5760F_0^2H_4r^4$$
$$+ 2096F_0F_1^3r^3 - 9384F_0F_1^2F_2r^4 + 13344F_0F_1F_2^2r^5 + 2340F_0F_1H_1r^2$$
$$- 5760F_0F_1H_2r^3 + 10260F_0F_1H_3r^4 - 15840F_0F_1H_4r^5 - 6056F_0F_2^3r^6$$
$$- 3420F_0F_2H_1r^3 + 7920F_0F_2H_2r^4 - 13500F_0F_2H_3r^5 + 20160F_0F_2H_4r^6$$
$$- 1800F_0L_2r^2 + 5400F_0L_3r^3 - 10800F_0L_4r^4 + 18000F_0L_5r^5 - 27000F_0L_6r^6$$

$$- 824F_1^4r^4 + 4616F_1^3F_2r^5 - 9336F_1^2F_2^2r^6 - 1710F_1^2H_1r^3 + 3960F_1^2H_2r^4$$
$$- 6750F_1^2H_3r^5 + 10080F_1^2H_4r^6 + 8120F_1F_2^3r^7 + 4500F_1F_2H_1r^4$$
$$- 10080F_1F_2H_2r^5 + 16740F_1F_2H_3r^6 - 24480F_1F_2H_4r^7 + 1800F_1L_2r^3$$
$$- 5400F_1L_3r^4 + 10800F_1L_4r^5 - 18000F_1L_5r^6 + 27000F_1L_6r^7 - 2576F_2^4r^8$$
$$- 2790F_2^2H_1r^5 + 6120F_2^2H_2r^6 - 9990F_2^2H_3r^7 + 14400F_2^2H_4r^8 - 1800F_2L_2r^4$$
$$+ 5400F_2L_3r^5 - 10800F_2L_4r^6 + 18000F_2L_5r^7 - 27000F_2L_6r^8 - 675H_1^2r^2$$
$$+ 2700H_1H_2r^3 - 4050H_1H_3r^4 + 5400H_1H_4r^5 - 2700H_2^2r^4 + 8100H_2H_3r^5$$
$$- 10800H_2H_4r^6 - 6075H_3^2r^6 + 16200H_3H_4r^7 - 10800H_4^2r^8 + 27000M_3r^3$$
$$- 81000M_4r^4 + 162000M_5r^5 - 270000M_6r^6 + 405000M_7r^7 - 567000M_8r^8$$
$$+ 756000M_9r^9)/27000, \tag{3.152}$$

respectively.

3.2.1.1.2 Case $r = 0, \lambda_3 \neq 0$

From equations (3.147), (3.62), (3.63), (3.64) and (3.65), one obtains the conditions

$$\lambda_4 = 0, \tag{3.153}$$
$$L_{2x} = (-3F_0L_3 - F_1L_2 - 75M_3)/5, \tag{3.154}$$
$$M_2 = (-40F_0^3F_2 - 144F_0^2F_1^2 + 180F_0^2H_2 + 360F_0F_1H_1 - 600F_0L_2 - 225H_1^2)/9000, \tag{3.155}$$
$$M_1 = 7F_0^2(8F_0F_1 - 15H_1)/4500, \tag{3.156}$$
$$M_0 = -7F_0^4/3375, \tag{3.157}$$

respectively.

3.2.1.1.3 Case $\lambda_3 = 0$ and $r = 0$

From equations (3.147), (3.62), (3.63), (3.64) and (3.65), one obtains the same conditions as in equations (3.153) - (3.157).

3.2.1.2 Case $\lambda_3 r \neq 0$

From equation (3.147), one obtains the derivative

$$\Delta_y = (-\lambda_4\Delta)/(10\lambda_3 r^2). \tag{3.158}$$

From the relations $(\Delta_y)_x = (\Delta_x)_y$ and $(\Delta_y)_y = \Delta_{yy}$, one obtains the conditions

$$\begin{aligned}\lambda_{4y} =&(240F_{2y}\lambda_3^2r^4 + 60\lambda_{3y}\lambda_4r^2 + 120r_y\lambda_3\lambda_4r + 68F_2^2\lambda_3^2r^4\\&- 4F_2\lambda_3\lambda_4r^2 - 180H_4\lambda_3^2r^4 - \lambda_4^2)/(60\lambda_3r^2), \qquad (3.159)\\ \lambda_{4x} =&(120F_{1y}\lambda_3^2r^3 + 60\lambda_{3x}\lambda_4r + 180r_y\lambda_3\lambda_4r - 8F_0\lambda_3\lambda_4\\&+ 48F_1F_2\lambda_3^2r^3 + 8F_1\lambda_3\lambda_4r - 28F_2^2\lambda_3^2r^4 - 12F_2\lambda_3\lambda_4r^2\\&- 60H_3\lambda_3^2r^3 + 60H_4\lambda_3^2r^4 - \lambda_4^2)/(60\lambda_3r), \qquad (3.160)\end{aligned}$$

respectively. From equations (3.62), (3.63), (3.64) and (3.65) one obtain the conditions

$$\begin{aligned}\lambda_{3y} =&(900L_{2x} - 900L_{2y}r - 1950L_{4x}r^2 + 1950L_{4y}r^3 + 3450L_{5x}r^3 - 3450L_{5y}r^4\\&- 690r_y\lambda_3r^2 + 51F_0\lambda_3r + 540F_0L_3 - 540F_0L_4r - 630F_0L_5r^2 + 2700F_0L_6r^3\\&- 127F_1\lambda_3r^2 + 180F_1L_2 - 540F_1L_3r + 150F_1L_4r^2 + 1320F_1L_5r^3\\&- 2700F_1L_6r^4 + 72F_2\lambda_3r^3 - 360F_2L_2r + 540F_2L_3r^2 + 240F_2L_4r^3\\&- 2010F_2L_5r^4 + 2700F_2L_6r^5 + 6\lambda_4r + 13500M_3 - 27000M_4r + 20700M_5r^2\\&+ 18900M_6r^3 - 89550M_7r^4 + 162000M_8r^5 - 162000M_9r^6)/(330r^3), \qquad (3.161)\\ M_2 =&(-40F_0^3F_2 - 144F_0^2F_1^2 + 696F_0^2F_1F_2r - 696F_0^2F_2^2r^2 + 180F_0^2H_2\\&- 540F_0^2H_3r + 1080F_0^2H_4r^2 + 288F_0F_1^3r - 1560F_0F_1^2F_2r^2\\&+ 2544F_0F_1F_2^2r^3 + 360F_0F_1H_1 - 1080F_0F_1H_2r + 2160F_0F_1H_3r^2\\&- 3600F_0F_1H_4r^3 - 1272F_0F_2^3r^4 - 720F_0F_2H_1r + 1800F_0F_2H_2r^2\\&- 3240F_0F_2H_3r^3 + 5040F_0F_2H_4r^4 - 600F_0L_2 + 1800F_0L_3r - 3600F_0L_4r^2\\&+ 6000F_0L_5r^3 - 9000F_0L_6r^4 - 144F_1^4r^2 + 904F_1^3F_2r^3 - 1992F_1^2F_2^2r^4\\&- 360F_1^2H_1r + 900F_1^2H_2r^2 - 1620F_1^2H_3r^3 + 2520F_1^2H_4r^4 + 1848F_1F_2^3r^5\\&+ 1080F_1F_2H_1r^2 - 2520F_1F_2H_2r^3 + 4320F_1F_2H_3r^4 - 6480F_1F_2H_4r^5\\&+ 600F_1L_2r - 1800F_1L_3r^2 + 3600F_1L_4r^3 - 6000F_1L_5r^4 + 9000F_1L_6r^5\\&- 616F_2^4r^6 - 720F_2^2H_1r^3 + 1620F_2^2H_2r^4 - 2700F_2^2H_3r^5 + 3960F_2^2H_4r^6\\&- 600F_2L_2r^2 + 1800F_2L_3r^3 - 3600F_2L_4r^4 + 6000F_2L_5r^5 - 9000F_2L_6r^6\\&- 225H_1^2 + 900H_1H_2r - 1350H_1H_3r^2 + 1800H_1H_4r^3 - 900H_2^2r^2\\&+ 2700H_2H_3r^3 - 3600H_2H_4r^4 - 2025H_3^2r^4 + 5400H_3H_4r^5 - 3600H_4^2r^6\\&+ 27000M_3r - 54000M_4r^2 + 90000M_5r^3 - 135000M_6r^4 + 189000M_7r^5\\&- 252000M_8r^6 + 324000M_9r^7)/9000, \qquad (3.162)\end{aligned}$$

$$
\begin{aligned}
M_1 =&(56F_0^3F_1 - 152F_0^3F_2r - 312F_0^2F_1^2r + 1200F_0^2F_1F_2r^2 - 1032F_0^2F_2^2r^3 \\
&- 105F_0^2H_1 + 390F_0^2H_2r - 855F_0^2H_3r^2 + 1500F_0^2H_4r^3 + 456F_0F_1^3r^2 \\
&- 2232F_0F_1^2F_2r^3 + 3384F_0F_1F_2^2r^4 + 570F_0F_1H_1r - 1500F_0F_1H_2r^2 \\
&+ 2790F_0F_1H_3r^3 - 4440F_0F_1H_4r^4 - 1608F_0F_2^3r^5 - 930F_0F_2H_1r^2 \\
&+ 2220F_0F_2H_2r^3 - 3870F_0F_2H_3r^4 + 5880F_0F_2H_4r^5 - 600F_0L_2r \\
&+ 1800F_0L_3r^2 - 3600F_0L_4r^3 + 6000F_0L_5r^4 - 9000F_0L_6r^5 - 200F_1^4r^3 \\
&+ 1184F_1^3F_2r^4 - 2496F_1^2F_2^2r^5 - 465F_1^2H_1r^2 + 1110F_1^2H_2r^3 - 1935F_1^2H_3r^4 \\
&+ 2940F_1^2H_4r^5 + 2240F_1F_2^3r^6 + 1290F_1F_2H_1r^3 - 2940F_1F_2H_2r^4 \\
&+ 4950F_1F_2H_3r^5 - 7320F_1F_2H_4r^6 + 600F_1L_2r^2 - 1800F_1L_3r^3 \\
&+ 3600F_1L_4r^4 - 6000F_1L_5r^5 + 9000F_1L_6r^6 - 728F_2^4r^7 - 825F_2^2H_1r^4 \\
&+ 1830F_2^2H_2r^5 - 3015F_2^2H_3r^6 + 4380F_2^2H_4r^7 - 600F_2L_2r^3 \\
&+ 1800F_2L_3r^4 - 3600F_2L_4r^5 + 6000F_2L_5r^6 - 9000F_2L_6r^7 - 225H_1^2r \\
&+ 900H_1H_2r^2 - 1350H_1H_3r^3 + 1800H_1H_4r^4 - 900H_2^2r^3 + 2700H_2H_3r^4 \\
&- 3600H_2H_4r^5 - 2025H_3^2r^5 + 5400H_3H_4r^6 - 3600H_4^2r^7 + 13500M_3r^2 \\
&- 36000M_4r^3 + 67500M_5r^4 - 108000M_6r^5 + 157500M_7r^6 - 216000M_8r^7 \\
&+ 283500M_9r^8)/4500, \qquad (3.163)
\end{aligned}
$$

$$
\begin{aligned}
M_0 =&(-56F_0^4 + 560F_0^3F_1r - 1016F_0^3F_2r^2 - 1776F_0^2F_1^2r^2 + 5784F_0^2F_1F_2r^3 \\
&- 4440F_0^2F_2^2r^4 - 630F_0^2H_1r + 1800F_0^2H_2r^2 - 3510F_0^2H_3r^3 \\
&+ 5760F_0^2H_4r^4 + 2096F_0F_1^3r^3 - 9384F_0F_1^2F_2r^4 + 13344F_0F_1F_2^2r^5 \\
&+ 2340F_0F_1H_1r^2 - 5760F_0F_1H_2r^3 + 10260F_0F_1H_3r^4 - 15840F_0F_1H_4r^5 \\
&- 6056F_0F_2^3r^6 - 3420F_0F_2H_1r^3 + 7920F_0F_2H_2r^4 - 13500F_0F_2H_3r^5 \\
&+ 20160F_0F_2H_4r^6 - 1800F_0L_2r^2 + 5400F_0L_3r^3 - 10800F_0L_4r^4 \\
&+ 18000F_0L_5r^5 - 27000F_0L_6r^6 - 824F_1^4r^4 + 4616F_1^3F_2r^5 - 9336F_1^2F_2^2r^6 \\
&- 1710F_1^2H_1r^3 + 3960F_1^2H_2r^4 - 6750F_1^2H_3r^5 + 10080F_1^2H_4r^6 \\
&+ 8120F_1F_2^3r^7 + 4500F_1F_2H_1r^4 - 10080F_1F_2H_2r^5 + 16740F_1F_2H_3r^6 \\
&- 24480F_1F_2H_4r^7 + 1800F_1L_2r^3 - 5400F_1L_3r^4 + 10800F_1L_4r^5 \\
&- 18000F_1L_5r^6 + 27000F_1L_6r^7 - 2576F_2^4r^8 - 2790F_2^2H_1r^5 + 6120F_2^2H_2r^6 \\
&- 9990F_2^2H_3r^7 + 14400F_2^2H_4r^8 - 1800F_2L_2r^4 + 5400F_2L_3r^5 - 10800F_2L_4r^6 \\
&+ 18000F_2L_5r^7 - 27000F_2L_6r^8 - 675H_1^2r^2 + 2700H_1H_2r^3 - 4050H_1H_3r^4
\end{aligned}
$$

$$+ 5400H_1H_4r^5 - 2700H_2^2r^4 + 8100H_2H_3r^5 - 10800H_2H_4r^6 - 6075H_3^2r^6$$
$$+ 16200H_3H_4r^7 - 10800H_4^2r^8 + 27000M_3r^3 - 81000M_4r^4 + 162000M_5r^5$$
$$- 270000M_6r^6 + 405000M_7r^7 - 567000M_8r^8 + 756000M_9r^9)/27000, \quad (3.164)$$

respectively.

3.2.2 Case $\lambda_1 \neq 0$

From equation (3.145), one obtains the derivative

$$\psi_{yy} = (2500\varphi_y\psi_y\Delta_y\lambda_1 + 50\varphi_y\psi_y\Delta(-15\lambda_{1y} + 2F_2\lambda_1) - \lambda_2\Delta^2)/(3750\varphi_y\lambda_1\Delta). \quad (3.165)$$

From equation (3.61), one gets equation

$$10\Delta_y\lambda_5r^2 + \lambda_6\Delta = 0 \quad (3.166)$$

where λ_5 and λ_6 are defined as equations (A.17) and (A.18).[8]

3.2.2.1 Case $\lambda_5 r = 0$

3.2.2.1.1 Case $\lambda_5 = 0, r \neq 0$

From equations (3.166), (3.62), (3.63), (3.64), (3.65) and the relation $(\psi_x)_{yy} = (\psi_{yy})_x$ one obtains the conditions

$$\lambda_6 = 0, \quad (3.167)$$

$$L_{2x} = (30L_{2y}r + 65L_{4x}r^2 - 65L_{4y}r^3 - 115L_{5x}r^3 + 115L_{5y}r^4 - 18F_0L_3$$
$$+ 18F_0L_4r + 21F_0L_5r^2 - 90F_0L_6r^3 - 6F_1L_2 + 18F_1L_3r - 5F_1L_4r^2$$
$$- 44F_1L_5r^3 + 90F_1L_6r^4 + 12F_2L_2r - 18F_2L_3r^2 - 8F_2L_4r^3$$
$$+ 67F_2L_5r^4 - 90F_2L_6r^5 + 33\lambda_1r^4 - 450M_3 + 900M_4r - 690M_5r^2$$
$$- 630M_6r^3 + 2985M_7r^4 - 5400M_8r^5 + 5400M_9r^6)/30, \quad (3.168)$$

$$M_2 = (-40F_0^3F_2 - 144F_0^2F_1^2 + 696F_0^2F_1F_2r - 696F_0^2F_2^2r^2 + 180F_0^2H_2$$
$$- 540F_0^2H_3r + 1080F_0^2H_4r^2 + 288F_0F_1^3r - 1560F_0F_1^2F_2r^2$$
$$+ 2544F_0F_1F_2^2r^3 + 360F_0F_1H_1 - 1080F_0F_1H_2r + 2160F_0F_1H_3r^2$$
$$- 3600F_0F_1H_4r^3 - 1272F_0F_2^3r^4 - 720F_0F_2H_1r + 1800F_0F_2H_2r^2$$
$$- 3240F_0F_2H_3r^3 + 5040F_0F_2H_4r^4 - 600F_0L_2 + 1800F_0L_3r$$
$$- 3600F_0L_4r^2 + 6000F_0L_5r^3 - 9000F_0L_6r^4 - 144F_1^4r^2 + 904F_1^3F_2r^3$$
$$- 1992F_1^2F_2^2r^4 - 360F_1^2H_1r + 900F_1^2H_2r^2 - 1620F_1^2H_3r^3$$
$$+ 2520F_1^2H_4r^4 + 1848F_1F_2^3r^5 + 1080F_1F_2H_1r^2 - 2520F_1F_2H_2r^3$$

[8] See Appendix

$$
\begin{aligned}
&+ 4320F_1F_2H_3r^4 - 6480F_1F_2H_4r^5 + 600F_1L_2r - 1800F_1L_3r^2 \\
&+ 3600F_1L_4r^3 - 6000F_1L_5r^4 + 9000F_1L_6r^5 - 616F_2^4r^6 - 720F_2^2H_1r^3 \\
&+ 1620F_2^2H_2r^4 - 2700F_2^2H_3r^5 + 3960F_2^2H_4r^6 - 600F_2L_2r^2 \\
&+ 1800F_2L_3r^3 - 3600F_2L_4r^4 + 6000F_2L_5r^5 - 9000F_2L_6r^6 - 225H_1^2 \\
&+ 900H_1H_2r - 1350H_1H_3r^2 + 1800H_1H_4r^3 - 900H_2^2r^2 + 2700H_2H_3r^3 \\
&- 3600H_2H_4r^4 - 2025H_3^2r^4 + 5400H_3H_4r^5 - 3600H_4^2r^6 + 27000M_3r \\
&- 54000M_4r^2 + 90000M_5r^3 - 135000M_6r^4 + 189000M_7r^5 - 252000M_8r^6 \\
&+ 324000M_9r^7)/9000, \qquad (3.169)
\end{aligned}
$$

$$
\begin{aligned}
M_1 =&(56F_0^3F_1 - 152F_0^3F_2r - 312F_0^2F_1^2r + 1200F_0^2F_1F_2r^2 - 1032F_0^2F_2^2r^3 \\
&- 105F_0^2H_1 + 390F_0^2H_2r - 855F_0^2H_3r^2 + 1500F_0^2H_4r^3 + 456F_0F_1^3r^2 \\
&- 2232F_0F_1^2F_2r^3 + 3384F_0F_1F_2^2r^4 + 570F_0F_1H_1r - 1500F_0F_1H_2r^2 \\
&+ 2790F_0F_1H_3r^3 - 4440F_0F_1H_4r^4 - 1608F_0F_2^3r^5 - 930F_0F_2H_1r^2 \\
&+ 2220F_0F_2H_2r^3 - 3870F_0F_2H_3r^4 + 5880F_0F_2H_4r^5 - 600F_0L_2r \\
&+ 1800F_0L_3r^2 - 3600F_0L_4r^3 + 6000F_0L_5r^4 - 9000F_0L_6r^5 - 200F_1^4r^3 \\
&+ 1184F_1^3F_2r^4 - 2496F_1^2F_2^2r^5 - 465F_1^2H_1r^2 + 1110F_1^2H_2r^3 - 1935F_1^2H_3r^4 \\
&+ 2940F_1^2H_4r^5 + 2240F_1F_2^3r^6 + 1290F_1F_2H_1r^3 - 2940F_1F_2H_2r^4 \\
&+ 4950F_1F_2H_3r^5 - 7320F_1F_2H_4r^6 + 600F_1L_2r^2 - 1800F_1L_3r^3 + 3600F_1L_4r^4 \\
&- 6000F_1L_5r^5 + 9000F_1L_6r^6 - 728F_2^4r^7 - 825F_2^2H_1r^4 + 1830F_2^2H_2r^5 \\
&- 3015F_2^2H_3r^6 + 4380F_2^2H_4r^7 - 600F_2L_2r^3 + 1800F_2L_3r^4 - 3600F_2L_4r^5 \\
&+ 6000F_2L_5r^6 - 9000F_2L_6r^7 - 225H_1^2r + 900H_1H_2r^2 - 1350H_1H_3r^3 \\
&+ 1800H_1H_4r^4 - 900H_2^2r^3 + 2700H_2H_3r^4 - 3600H_2H_4r^5 - 2025H_3^2r^5 \\
&+ 5400H_3H_4r^6 - 3600H_4^2r^7 + 13500M_3r^2 - 36000M_4r^3 + 67500M_5r^4 \\
&- 108000M_6r^5 + 157500M_7r^6 - 216000M_8r^7 + 283500M_9r^8)/4500, \qquad (3.170)
\end{aligned}
$$

$$
\begin{aligned}
M_0 =&(-56F_0^4 + 560F_0^3F_1r - 1016F_0^3F_2r^2 - 1776F_0^2F_1^2r^2 + 5784F_0^2F_1F_2r^3 \\
&- 4440F_0^2F_2^2r^4 - 630F_0^2H_1r + 1800F_0^2H_2r^2 - 3510F_0^2H_3r^3 + 5760F_0^2H_4r^4 \\
&+ 2096F_0F_1^3r^3 - 9384F_0F_1^2F_2r^4 + 13344F_0F_1F_2^2r^5 + 2340F_0F_1H_1r^2 \\
&- 5760F_0F_1H_2r^3 + 10260F_0F_1H_3r^4 - 15840F_0F_1H_4r^5 - 6056F_0F_2^3r^6 \\
&- 3420F_0F_2H_1r^3 + 7920F_0F_2H_2r^4 - 13500F_0F_2H_3r^5 + 20160F_0F_2H_4r^6 \\
&- 1800F_0L_2r^2 + 5400F_0L_3r^3 - 10800F_0L_4r^4 + 18000F_0L_5r^5 - 27000F_0L_6r^6
\end{aligned}
$$

$$- 824F_1^4r^4 + 4616F_1^3F_2r^5 - 9336F_1^2F_2^2r^6 - 1710F_1^2H_1r^3 + 3960F_1^2H_2r^4$$
$$- 6750F_1^2H_3r^5 + 10080F_1^2H_4r^6 + 8120F_1F_2^3r^7 + 4500F_1F_2H_1r^4$$
$$- 10080F_1F_2H_2r^5 + 16740F_1F_2H_3r^6 - 24480F_1F_2H_4r^7 + 1800F_1L_2r^3$$
$$- 5400F_1L_3r^4 + 10800F_1L_4r^5 - 18000F_1L_5r^6 + 27000F_1L_6r^7 - 2576F_2^4r^8$$
$$- 2790F_2^2H_1r^5 + 6120F_2^2H_2r^6 - 9990F_2^2H_3r^7 + 14400F_2^2H_4r^8 - 1800F_2L_2r^4$$
$$+ 5400F_2L_3r^5 - 10800F_2L_4r^6 + 18000F_2L_5r^7 - 27000F_2L_6r^8 - 675H_1^2r^2$$
$$+ 2700H_1H_2r^3 - 4050H_1H_3r^4 + 5400H_1H_4r^5 - 2700H_2^2r^4 + 8100H_2H_3r^5$$
$$- 10800H_2H_4r^6 - 6075H_3^2r^6 + 16200H_3H_4r^7 - 10800H_4^2r^8 + 27000M_3r^3$$
$$- 81000M_4r^4 + 162000M_5r^5 - 270000M_6r^6 + 405000M_7r^7 - 567000M_8r^8$$
$$+ 756000M_9r^9)/27000, \tag{3.171}$$

$$\begin{aligned}\lambda_{1xy} =&(75\varphi_y\psi_y\Delta(-100F_{1y}\lambda_1^2 + 200F_{2y}\lambda_1^2r + 300\lambda_{1x}\lambda_{1y} + 300\lambda_{1yy}\lambda_1r \\ &- 300\lambda_{1y}^2r + 300\lambda_{1y}r_y\lambda_1 - 140r_yF_2\lambda_1^2 - 68F_1F_2\lambda_1^2 + 136F_2^2\lambda_1^2r \\ &+ 85H_3\lambda_1^2 - 340H_4\lambda_1^2r) + 12500\Delta_y^2\lambda_1^2 + 1000\Delta_y\lambda_1\Delta(15\lambda_{1y} \\ &- 2F_2\lambda_1) + 2\Delta^2(-11250F_{2y}\lambda_1^2 + 15\lambda_{1x}\lambda_2 + 750\lambda_{1y}F_2\lambda_1 - 15\lambda_{1y}\lambda_2r \\ &- 15\lambda_{2x}\lambda_1 + 15\lambda_{2y}\lambda_1r - 45r_y\lambda_1\lambda_2 - 3F_1\lambda_1\lambda_2 - 4100F_2^2\lambda_1^2 \\ &+ 6F_2\lambda_1\lambda_2r + 11250H_4\lambda_1^2))/(22500\varphi_y\psi_y\lambda_1\Delta),\end{aligned} \tag{3.172}$$

respectively.

3.2.2.1.2 Case $r = 0, \lambda_5 \neq 0$

From equations (3.166), (3.62), (3.63), (3.64), (3.65) and the mixed derivative $(\psi_x)_{yy} = (\psi_{yy})_x$ one obtain the conditions

$$\lambda_6 = 0, \tag{3.173}$$

$$L_{2x} = (-3F_0L_3 - F_1L_2 - 75M_3)/5, \tag{3.174}$$

$$\begin{aligned}M_2 =&(-40F_0^3F_2 - 144F_0^2F_1^2 + 180F_0^2H_2 + 360F_0F_1H_1 - 600F_0L_2 \\ &- 225H_1^2)/9000,\end{aligned} \tag{3.175}$$

$$M_1 = (7F_0^2(8F_0F_1 - 15H_1))/4500, \tag{3.176}$$

$$M_0 = (-7F_0^4)/3375, \tag{3.177}$$

$$\begin{aligned}\lambda_{1xy} =&(75\varphi_y\psi_y\Delta(-100F_{1y}\lambda_1^2 + 300\lambda_{1x}\lambda_{1y} - 68F_1F_2\lambda_1^2 + 85H_3\lambda_1^2) \\ &+ 12500\Delta_y^2\lambda_1^2 + 1000\Delta_y\lambda_1\Delta(15\lambda_{1y} - 2F_2\lambda_1) + 2\Delta^2(-11250F_{2y}\lambda_1^2 \\ &+ 15\lambda_{1x}\lambda_2 + 750\lambda_{1y}F_2\lambda_1 - 15\lambda_{2x}\lambda_1 - 3F_1\lambda_1\lambda_2 - 4100F_2^2\lambda_1^2 \\ &+ 11250H_4\lambda_1^2))/(22500\varphi_y\psi_y\lambda_1\Delta),\end{aligned} \tag{3.178}$$

respectively.

3.2.2.1.3 Case $\lambda_5 = 0$ and $r = 0$

From equations (3.166), (3.62), (3.63), (3.64), (3.65) and the mixed derivative $(\psi_x)_{yy} = (\psi_{yy})_x$ one obtain the same conditions as in equations (3.173) - (3.178).

3.2.2.2 Case $\lambda_5 r \neq 0$

From equation(3.166), one obtains the derivative

$$\Delta_y = (-\lambda_6\Delta)/(10\lambda_5 r^2). \tag{3.179}$$

From the relations $(\Delta_y)_y = \Delta_{yy}$ and $(\Delta_y)_x = (\Delta_x)_y$, one obtains the conditions

$$\begin{aligned}\lambda_{6y} =&(240F_{2y}\lambda_5^2r^4 + 60\lambda_{5y}\lambda_6r^2 + 120r_y\lambda_5\lambda_6r + 68F_2^2\lambda_5^2r^4 - 4F_2\lambda_5\lambda_6r^2\\&- 180H_4\lambda_5^2r^4 - \lambda_6^2)/(60\lambda_5r^2), \qquad (3.180)\\ \lambda_{6x} =&(120F_{1y}\lambda_5^2r^3 + 60\lambda_{5x}\lambda_6r + 180r_y\lambda_5\lambda_6r - 8F_0\lambda_5\lambda_6 + 48F_1F_2\lambda_5^2r^3\\&+ 8F_1\lambda_5\lambda_6r - 28F_2^2\lambda_5^2r^4 - 12F_2\lambda_5\lambda_6r^2 - 60H_3\lambda_5^2r^3 + 60H_4\lambda_5^2r^4\\&- \lambda_6^2)/(60\lambda_5r), \qquad (3.181)\end{aligned}$$

respectively. From equations (3.62), (3.63), (3.64) and (3.65) one obtains the conditions

$$\begin{aligned}\lambda_{5y} =&(900L_{2x} - 900L_{2y}r - 1950L_{4x}r^2 + 1950L_{4y}r^3 + 3450L_{5x}r^3 - 3450L_{5y}r^4\\&- 690r_y\lambda_5r^2 + 51F_0\lambda_5r + 540F_0L_3 - 540F_0L_4r - 630F_0L_5r^2\\&+ 2700F_0L_6r^3 - 127F_1\lambda_5r^2 + 180F_1L_2 - 540F_1L_3r + 150F_1L_4r^2\\&+ 1320F_1L_5r^3 - 2700F_1L_6r^4 + 72F_2\lambda_5r^3 - 360F_2L_2r + 540F_2L_3r^2\\&+ 240F_2L_4r^3 - 2010F_2L_5r^4 + 2700F_2L_6r^5 - 990\lambda_1r^4 + 6\lambda_6r\\&+ 13500M_3 - 27000M_4r + 20700M_5r^2 + 18900M_6r^3 - 89550M_7r^4\\&+ 162000M_8r^5 - 162000M_9r^6)/(330r^3), \qquad (3.182)\\ M_2 =&(-40F_0^3F_2 - 144F_0^2F_1^2 + 696F_0^2F_1F_2r - 696F_0^2F_2^2r^2 + 180F_0^2H_2\\&- 540F_0^2H_3r + 1080F_0^2H_4r^2 + 288F_0F_1^3r - 1560F_0F_1^2F_2r^2\\&+ 2544F_0F_1F_2^2r^3 + 360F_0F_1H_1 - 1080F_0F_1H_2r + 2160F_0F_1H_3r^2\\&- 3600F_0F_1H_4r^3 - 1272F_0F_2^3r^4 - 720F_0F_2H_1r + 1800F_0F_2H_2r^2\\&- 3240F_0F_2H_3r^3 + 5040F_0F_2H_4r^4 - 600F_0L_2 + 1800F_0L_3r - 3600F_0L_4r^2\\&+ 6000F_0L_5r^3 - 9000F_0L_6r^4 - 144F_1^4r^2 + 904F_1^3F_2r^3 - 1992F_1^2F_2^2r^4\\&- 360F_1^2H_1r + 900F_1^2H_2r^2 - 1620F_1^2H_3r^3 + 2520F_1^2H_4r^4 + 1848F_1F_2^3r^5\\&+ 1080F_1F_2H_1r^2 - 2520F_1F_2H_2r^3 + 4320F_1F_2H_3r^4 - 6480F_1F_2H_4r^5\end{aligned}$$

$$
\begin{aligned}
&+600F_1L_2r-1800F_1L_3r^2+3600F_1L_4r^3-6000F_1L_5r^4+9000F_1L_6r^5\\
&-616F_2^4r^6-720F_2^2H_1r^3+1620F_2^2H_2r^4-2700F_2^2H_3r^5+3960F_2^2H_4r^6\\
&-600F_2L_2r^2+1800F_2L_3r^3-3600F_2L_4r^4+6000F_2L_5r^5-9000F_2L_6r^6\\
&-225H_1^2+900H_1H_2r-1350H_1H_3r^2+1800H_1H_4r^3-900H_2^2r^2\\
&+2700H_2H_3r^3-3600H_2H_4r^4-2025H_3^2r^4+5400H_3H_4r^5-3600H_4^2r^6\\
&+27000M_3r-54000M_4r^2+90000M_5r^3-135000M_6r^4+189000M_7r^5\\
&-252000M_8r^6+324000M_9r^7)/9000, \qquad (3.183)
\end{aligned}
$$

$$
\begin{aligned}
M_1=&(56F_0^3F_1-152F_0^3F_2r-312F_0^2F_1^2r+1200F_0^2F_1F_2r^2-1032F_0^2F_2^2r^3\\
&-105F_0^2H_1+390F_0^2H_2r-855F_0^2H_3r^2+1500F_0^2H_4r^3+456F_0F_1^3r^2\\
&-2232F_0F_1^2F_2r^3+3384F_0F_1F_2^2r^4+570F_0F_1H_1r-1500F_0F_1H_2r^2\\
&+2790F_0F_1H_3r^3-4440F_0F_1H_4r^4-1608F_0F_2^3r^5-930F_0F_2H_1r^2\\
&+2220F_0F_2H_2r^3-3870F_0F_2H_3r^4+5880F_0F_2H_4r^5-600F_0L_2r\\
&+1800F_0L_3r^2-3600F_0L_4r^3+6000F_0L_5r^4-9000F_0L_6r^5-200F_1^4r^3\\
&+1184F_1^3F_2r^4-2496F_1^2F_2^2r^5-465F_1^2H_1r^2+1110F_1^2H_2r^3-1935F_1^2H_3r^4\\
&+2940F_1^2H_4r^5+2240F_1F_2^3r^6+1290F_1F_2H_1r^3-2940F_1F_2H_2r^4\\
&+4950F_1F_2H_3r^5-7320F_1F_2H_4r^6+600F_1L_2r^2-1800F_1L_3r^3+3600F_1L_4r^4\\
&-6000F_1L_5r^5+9000F_1L_6r^6-728F_2^4r^7-825F_2^2H_1r^4+1830F_2^2H_2r^5\\
&-3015F_2^2H_3r^6+4380F_2^2H_4r^7-600F_2L_2r^3+1800F_2L_3r^4-3600F_2L_4r^5\\
&+6000F_2L_5r^6-9000F_2L_6r^7-225H_1^2r+900H_1H_2r^2-1350H_1H_3r^3\\
&+1800H_1H_4r^4-900H_2^2r^3+2700H_2H_3r^4-3600H_2H_4r^5-2025H_3^2r^5\\
&+5400H_3H_4r^6-3600H_4^2r^7+13500M_3r^2-36000M_4r^3+67500M_5r^4\\
&-108000M_6r^5+157500M_7r^6-216000M_8r^7+283500M_9r^8)/4500, \qquad (3.184)
\end{aligned}
$$

$$
\begin{aligned}
M_0=&(-56F_0^4+560F_0^3F_1r-1016F_0^3F_2r^2-1776F_0^2F_1^2r^2+5784F_0^2F_1F_2r^3\\
&-4440F_0^2F_2^2r^4-630F_0^2H_1r+1800F_0^2H_2r^2-3510F_0^2H_3r^3+5760F_0^2H_4r^4\\
&+2096F_0F_1^3r^3-9384F_0F_1^2F_2r^4+13344F_0F_1F_2^2r^5+2340F_0F_1H_1r^2\\
&-5760F_0F_1H_2r^3+10260F_0F_1H_3r^4-15840F_0F_1H_4r^5-6056F_0F_2^3r^6\\
&-3420F_0F_2H_1r^3+7920F_0F_2H_2r^4-13500F_0F_2H_3r^5+20160F_0F_2H_4r^6\\
&-1800F_0L_2r^2+5400F_0L_3r^3-10800F_0L_4r^4+18000F_0L_5r^5-27000F_0L_6r^6\\
&-824F_1^4r^4+4616F_1^3F_2r^5-9336F_1^2F_2^2r^6-1710F_1^2H_1r^3+3960F_1^2H_2r^4
\end{aligned}
$$

$$
\begin{aligned}
&- 6750F_1^2H_3r^5 + 10080F_1^2H_4r^6 + 8120F_1F_2^3r^7 + 4500F_1F_2H_1r^4 \\
&- 10080F_1F_2H_2r^5 + 16740F_1F_2H_3r^6 - 24480F_1F_2H_4r^7 + 1800F_1L_2r^3 \\
&- 5400F_1L_3r^4 + 10800F_1L_4r^5 - 18000F_1L_5r^6 + 27000F_1L_6r^7 - 2576F_2^4r^8 \\
&- 2790F_2^2H_1r^5 + 6120F_2^2H_2r^6 - 9990F_2^2H_3r^7 + 14400F_2^2H_4r^8 - 1800F_2L_2r^4 \\
&+ 5400F_2L_3r^5 - 10800F_2L_4r^6 + 18000F_2L_5r^7 - 27000F_2L_6r^8 - 675H_1^2r^2 \\
&+ 2700H_1H_2r^3 - 4050H_1H_3r^4 + 5400H_1H_4r^5 - 2700H_2^2r^4 + 8100H_2H_3r^5 \\
&- 10800H_2H_4r^6 - 6075H_3^2r^6 + 16200H_3H_4r^7 - 10800H_4^2r^8 + 27000M_3r^3 \\
&- 81000M_4r^4 + 162000M_5r^5 - 270000M_6r^6 + 405000M_7r^7 - 567000M_8r^8 \\
&+ 756000M_9r^9)/27000, \qquad (3.185)
\end{aligned}
$$

respectively. From the mixed derivative $(\psi_x)_{yy} = (\psi_{yy})_x$, one obtains

$$75\varphi_y\psi_y\lambda_5^2\lambda_7r^4 + \lambda_8\Delta = 0 \qquad (3.186)$$

where λ_7 and λ_8 are defined as equations (A.19) and (A.20).[9]

3.2.2.2.1 Case $\lambda_7 = 0$

From equation (3.186), one obtains the condition

$$\lambda_8 = 0. \qquad (3.187)$$

3.2.2.2.2 Case $\lambda_7 \neq 0$

From equation (3.186), one obtains the derivative

$$\psi_y = (-\lambda_8\Delta)/(75\varphi_y\lambda_5^2\lambda_7r^4). \qquad (3.188)$$

From the relations $(\psi_y)_y = \psi_{yy}$ and $(\psi_y)_x = (\psi_x)_y$, one obtain the conditions

$$
\begin{aligned}
\lambda_{8y} =&(-330\lambda_{1y}\lambda_5\lambda_7\lambda_8r^3 + 1650\lambda_{7y}\lambda_1\lambda_5\lambda_8r^3 + 9000L_{2x}\lambda_1\lambda_7\lambda_8 \\
&- 9000L_{2y}\lambda_1\lambda_7\lambda_8r - 19500L_{4x}\lambda_1\lambda_7\lambda_8r^2 + 19500L_{4y}\lambda_1\lambda_7\lambda_8r^3 \\
&+ 34500L_{5x}\lambda_1\lambda_7\lambda_8r^3 - 34500L_{5y}\lambda_1\lambda_7\lambda_8r^4 - 300r_y\lambda_1\lambda_5\lambda_7\lambda_8r^2 \\
&+ 510F_0\lambda_1\lambda_5\lambda_7\lambda_8r + 5400F_0\lambda_1\lambda_7\lambda_8L_3 - 5400F_0\lambda_1\lambda_7\lambda_8L_4r \\
&- 6300F_0\lambda_1\lambda_7\lambda_8L_5r^2 + 27000F_0\lambda_1\lambda_7\lambda_8L_6r^3 - 1270F_1\lambda_1\lambda_5\lambda_7\lambda_8r^2 \\
&+ 1800F_1\lambda_1\lambda_7\lambda_8L_2 - 5400F_1\lambda_1\lambda_7\lambda_8L_3r + 1500F_1\lambda_1\lambda_7\lambda_8L_4r^2 \\
&+ 13200F_1\lambda_1\lambda_7\lambda_8L_5r^3 - 27000F_1\lambda_1\lambda_7\lambda_8L_6r^4 + 654F_2\lambda_1\lambda_5\lambda_7\lambda_8r^3 \\
&- 3600F_2\lambda_1\lambda_7\lambda_8L_2r + 5400F_2\lambda_1\lambda_7\lambda_8L_3r^2 + 2400F_2\lambda_1\lambda_7\lambda_8L_4r^3
\end{aligned}
$$

[9]See Appendix

$$- 20100F_2\lambda_1\lambda_7\lambda_8L_5r^4 + 27000F_2\lambda_1\lambda_7\lambda_8L_6r^5 - 9900\lambda_1^2\lambda_7\lambda_8r^4$$
$$+ 60\lambda_1\lambda_6\lambda_7\lambda_8r + 135000\lambda_1\lambda_7\lambda_8M_3 - 270000\lambda_1\lambda_7\lambda_8M_4r$$
$$+ 207000\lambda_1\lambda_7\lambda_8M_5r^2 + 189000\lambda_1\lambda_7\lambda_8M_6r^3 - 895500\lambda_1\lambda_7\lambda_8M_7r^4$$
$$+ 1620000\lambda_1\lambda_7\lambda_8M_8r^5 - 1620000\lambda_1\lambda_7\lambda_8M_9r^6$$
$$+ 33\lambda_2\lambda_5^3\lambda_7^2r^7)/(1650\lambda_1\lambda_5\lambda_7r^3), \quad (3.189)$$

$$\lambda_{8x} =(-30\lambda_{1y}\lambda_5\lambda_7\lambda_8r^2 + 300\lambda_{5x}\lambda_1\lambda_7\lambda_8r + 150\lambda_{7x}\lambda_1\lambda_5\lambda_8r - 40F_0\lambda_1\lambda_5\lambda_7\lambda_8$$
$$+ 10F_1\lambda_1\lambda_5\lambda_7\lambda_8r + 750F_2\lambda_1\lambda_5^3\lambda_7^2r^5 + 14F_2\lambda_1\lambda_5\lambda_7\lambda_8r^2 - 750\lambda_1\lambda_5^2\lambda_6\lambda_7^2r^3$$
$$+ 3\lambda_2\lambda_5^3\lambda_7^2r^6)/(150\lambda_1\lambda_5\lambda_7r), \quad (3.190)$$

respectively. All obtained results can be summarized in the following theorem.

Theorem 3.2.2. *Sufficient conditions for equation (3.29) to be linearizable via the point transformation (3.2) are as follows*

(a) *If $\lambda_1 = 0, \lambda_3 = 0$ and $r \neq 0$ then the conditions are (3.114), (3.115), (3.116), (3.117), (3.118), (3.120), (3.121), (3.122), (3.123), (3.124), (3.125), (3.126), (3.127), (3.128), (3.129), (3.130), (3.131), (3.132), (3.133), (3.134), (3.135), (3.136), (3.137), (3.138), (3.139), (3.141), (3.142), (3.143), (3.144), (3.146), (3.148), (3.149), (3.150), (3.151) and (3.152).*

(b) *If $\lambda_1 = 0, r = 0$ and $\lambda_3 \neq 0$ then the conditions are (3.114), (3.115), (3.116), (3.117), (3.118), (3.120), (3.121), (3.122), (3.123), (3.124), (3.125), (3.126), (3.127), (3.128), (3.129), (3.130), (3.131), (3.132), (3.133), (3.134), (3.135), (3.136), (3.137), (3.138), (3.139), (3.141), (3.142), (3.143), (3.144), (3.146), (3.153), (3.154), (3.155), (3.156) and (3.157).*

(c) *If $\lambda_1 = 0$, $\lambda_3 = 0$ and $r = 0$, then the conditions are (3.114), (3.115), (3.116), (3.117), (3.118), (3.120), (3.121), (3.122), (3.123), (3.124), (3.125), (3.126), (3.127), (3.128), (3.129), (3.130), (3.131), (3.132), (3.133), (3.134), (3.135), (3.136), (3.137), (3.138), (3.139), (3.141), (3.142), (3.143), (3.144), (3.146), (3.153), (3.154), (3.155), (3.156) and (3.157).*

(d) *If $\lambda_1 = 0$ and $\lambda_3 r \neq 0$, then the conditions are (3.114), (3.115), (3.116), (3.117), (3.118), (3.120), (3.121), (3.122), (3.123), (3.124), (3.125), (3.126), (3.127), (3.128), (3.129), (3.130), (3.131), (3.132), (3.133), (3.134), (3.135), (3.136), (3.137), (3.138), (3.139), (3.141), (3.142), (3.143), (3.144), (3.146), (3.159), (3.160), (3.161), (3.162), (3.163) and (3.164).*

(e) *If $\lambda_1 \neq 0, \lambda_5 = 0$ and $r \neq 0$ then the conditions are (3.114), (3.115), (3.116), (3.117), (3.118), (3.120), (3.121), (3.122), (3.123), (3.124), (3.125), (3.126), (3.127), (3.128), (3.129), (3.130), (3.131), (3.132), (3.133), (3.134), (3.135), (3.136), (3.137), (3.138),* *(3.139), (3.141), (3.142), (3.143), (3.144), (3.167), (3.168), (3.169), (3.170), (3.171) and (3.172).*

(f) *If $\lambda_1 \neq 0, r = 0$ and $\lambda_5 \neq 0$ then the conditions are (3.114), (3.115), (3.116), (3.117), (3.118), (3.120), (3.121), (3.122), (3.123), (3.124), (3.125), (3.126), (3.127), (3.128), (3.129), (3.130), (3.131), (3.132), (3.133), (3.134), (3.135), (3.136), (3.137), (3.138), (3.139), (3.141), (3.142), (3.143), (3.144), (3.173), (3.174), (3.175), (3.176), (3.177) and (3.178).*

(g) *If $\lambda_1 \neq 0$, $\lambda_5 = 0$ and $r = 0$, then the conditions are (3.114), (3.115), (3.116), (3.117), (3.118), (3.120), (3.121), (3.122), (3.123), (3.124), (3.125), (3.126), (3.127), (3.128), (3.129), (3.130), (3.131), (3.132), (3.133), (3.134), (3.135), (3.136), (3.137), (3.138), (3.139), (3.141), (3.142), (3.143), (3.144), (3.173), (3.174), (3.175), (3.176), (3.177) and (3.178).*

(h) *If $\lambda_1 \neq 0$, $\lambda_5 r \neq 0$ and $\lambda_7 = 0$, then the conditions are (3.114), (3.115), (3.116), (3.117), (3.118), (3.120), (3.121), (3.122), (3.123), (3.124), (3.125), (3.126), (3.127), (3.128), (3.129), (3.130), (3.131), (3.132), (3.133), (3.134), (3.135), (3.136), (3.137), (3.138), (3.139), (3.141), (3.142), (3.143), (3.144), (3.180), (3.181), (3.182), (3.183), (3.184), (3.185) and (3.187).*

(i) *If $\lambda_1 \neq 0$, $\lambda_5 r \neq 0$ and $\lambda_7 \neq 0$, then the conditions are (3.114), (3.115), (3.116), (3.117), (3.118), (3.120), (3.121), (3.122), (3.123), (3.124), (3.125), (3.126), (3.127), (3.128), (3.129), (3.130), (3.131), (3.132), (3.133), (3.134), (3.135), (3.136), (3.137), (3.138), (3.139), (3.141), (3.142), (3.143), (3.144), (3.180), (3.181), (3.182), (3.183), (3.184), (3.185), (3.189) and (3.190).*

□

3.3 Linearizing transformation

By the prove of Theorem 3.2.1, we arrive at the following Corollary.

Corollary 3.3.1. *Provided that the sufficient conditions in Theorem 3.2.1 are satisfied, the transformation (3.2) mapping equation (3.10) to a linear equation (3.3) is obtained by solving the following compatible system of equations for the functions $\varphi(x)$ and $\psi(x, y)$:*

(a) *(3.67), (3.68), (3.86) and (3.88).*

(b) *(3.67), (3.68), (3.86) and (3.92).*

(c) *(3.67), (3.68), (3.92) and (3.96).*

(d) *(3.67), (3.68), (3.86) and (3.99).*

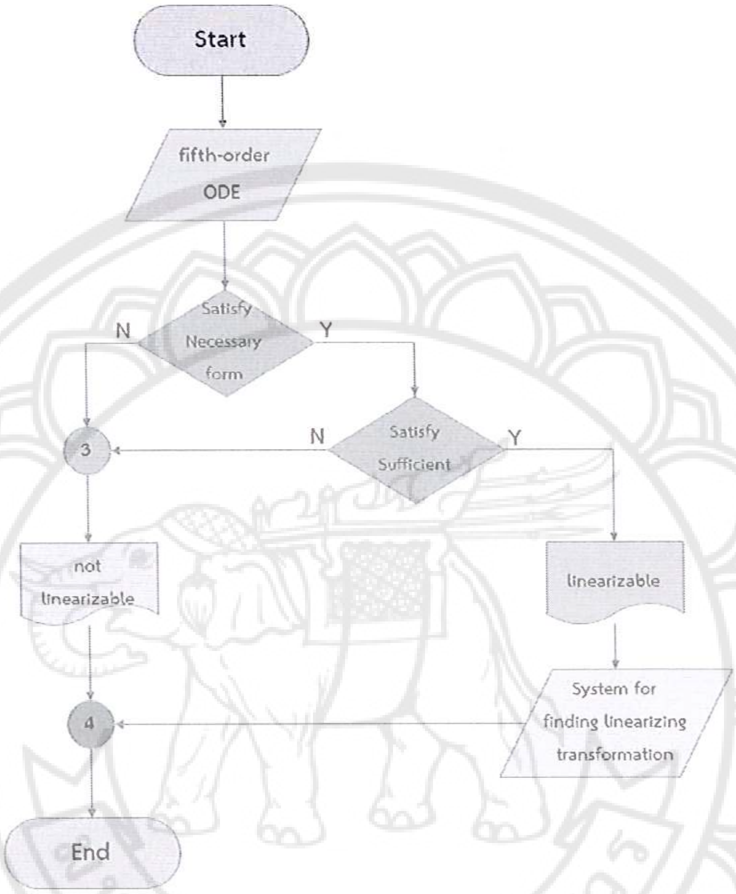
(e) *(3.67), (3.68), (3.86) and (3.102).*
(f) *(3.67), (3.68), (3.102) and (3.107).*

By the prove of Theorem 3.2.2, we arrive at the following Corollary.

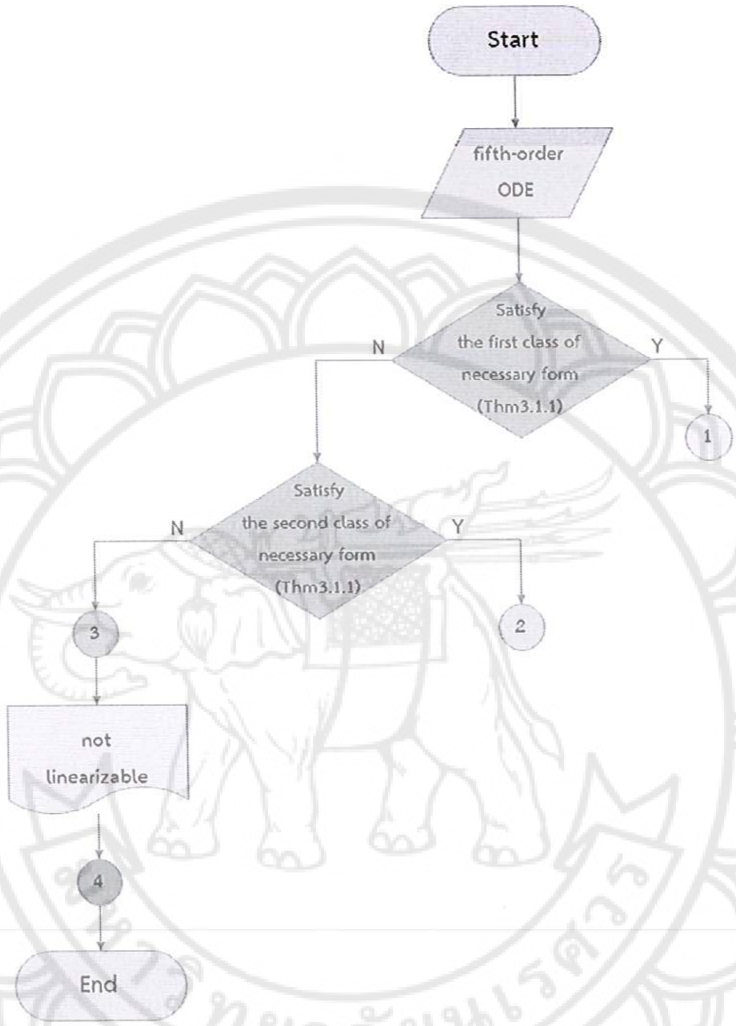
Corollary 3.3.2. *Provided that the sufficient conditions in Theorem 3.2.2 are satisfied, the transformation (3.66) mapping equation (3.29) to a linear equation (3.3) is obtained by solving the following compatible system of equations for the functions* $\varphi(x, y)$ *and* $\psi(x, y)$*:*
(a) *(3.110), (3.111), (3.112), (3.113), (3.119), and (3.140).*
(b) *(3.110), (3.111), (3.112), (3.113), (3.119), and (3.140).*
(c) *(3.110), (3.111), (3.112), (3.113), (3.119), and (3.140).*
(d) *(3.110), (3.111), (3.112), (3.113), (3.119), and (3.158).*
(e) *(3.110), (3.111), (3.112), (3.113), (3.119), and (3.165).*
(f) *(3.110), (3.111), (3.112), (3.113), (3.119), and (3.165).*
(g) *(3.110), (3.111), (3.112), (3.113), (3.119), and (3.165).*
(h) *(3.110), (3.111), (3.112), (3.113), (3.165), and (3.179).*
(i) *(3.110), (3.111), (3.112), (3.113), (3.179), and (3.188).*

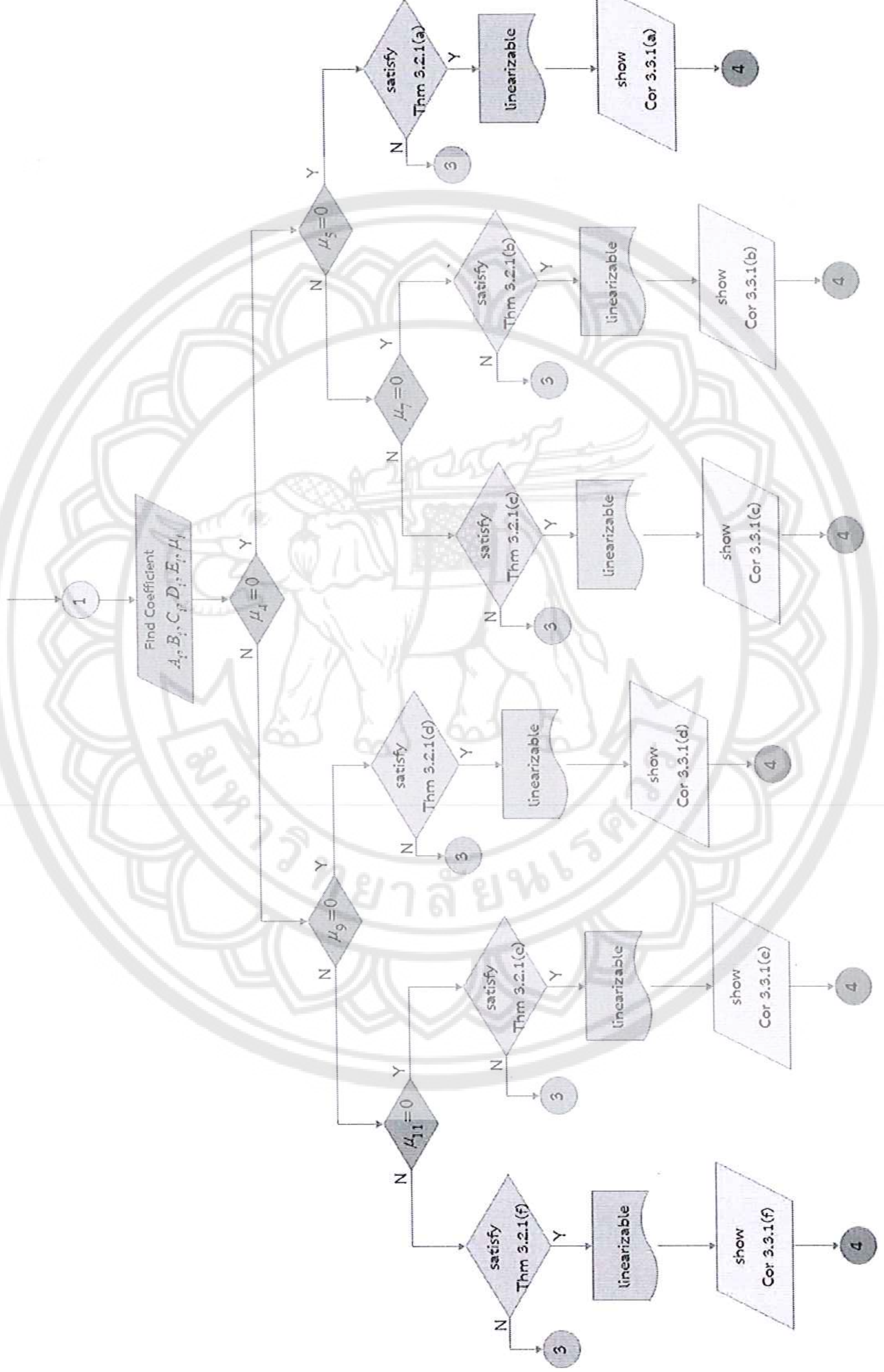
3.4 Flowchart of testing program

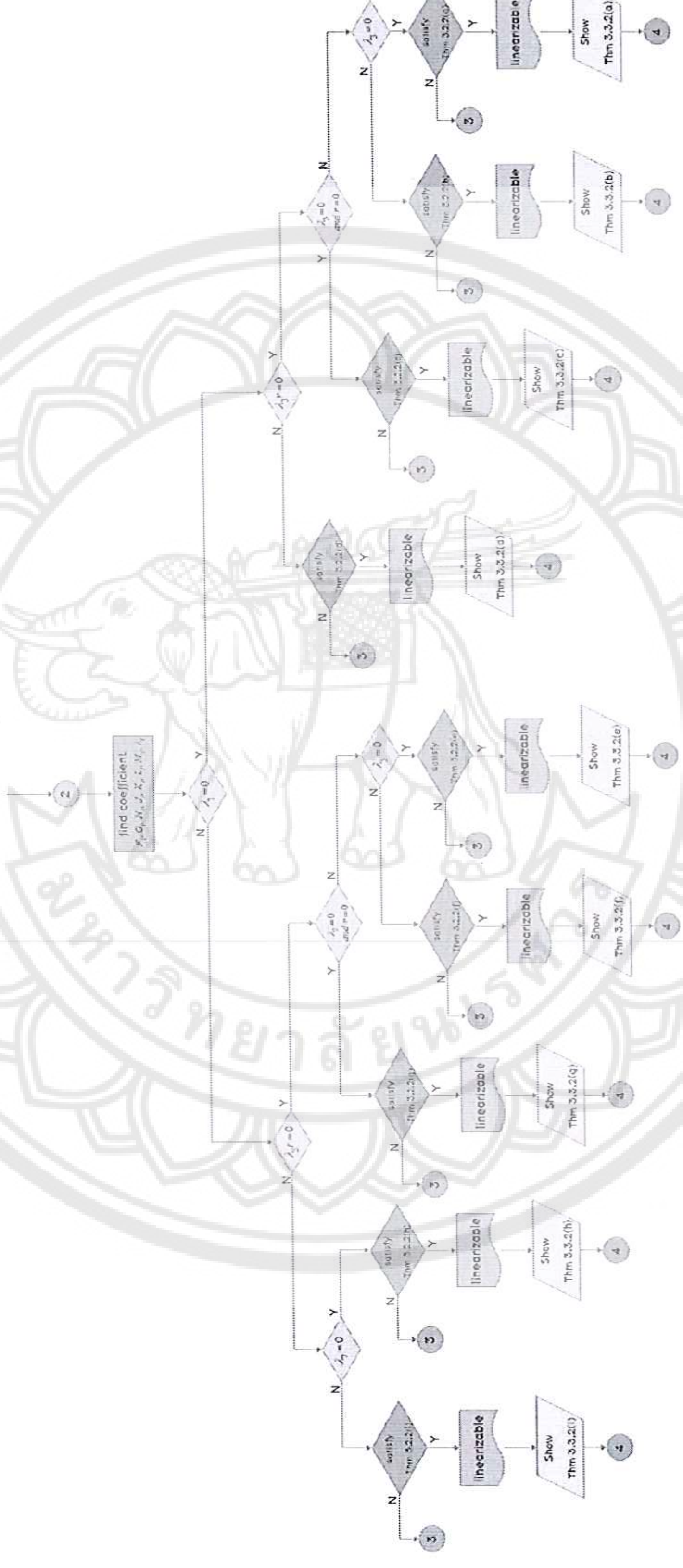
3.4.1 The flowchart shows demonstration of how to use the obtained main theorems.



3.4.2 The flowchart shows demonstration of how to use Theorem 3.1.1, Theorem 3.2.1, Theorem 3.2.2, Corollary 3.3.1 and Corollary 3.3.2




1
Find Coefficient
$A_i, B_i, C_i, D_i, E_i, \mu_i$
$\mu_1=0$
$\mu_5=0$
$\mu_7=0$
$\mu_9=0$
$\mu_{11}=0$
satisfy Thm 3.2.1(a)
satisfy Thm 3.2.1(b)
satisfy Thm 3.2.1(c)
satisfy Thm 3.2.1(d)
satisfy Thm 3.2.1(e)
satisfy Thm 3.2.1(f)
linearizable
show Cor 3.3.1(a)
show Cor 3.3.1(b)
show Cor 3.3.1(c)
show Cor 3.3.1(d)
show Cor 3.3.1(e)
show Cor 3.3.1(f)
Y
N
3
4


find coefficient
linearizable
Show
Thm 3.3.2(a)
Thm 3.3.2(b)
Thm 3.3.2(c)
Thm 3.3.2(d)
Thm 3.3.2(e)
Thm 3.3.2(f)
Thm 3.3.2(g)
Thm 3.3.2(h)
Thm 3.3.2(i)
Y
N

3.5 Examples

3.5.1 The examples for the first class

Example 3.5.1. Consider the nonlinear ordinary differential equation

$$x^2yy^{(5)} + 5(x^2y' + 2xy)y^{(4)} + 10(x^2y'' + 4xy' + 2y)y''' + 30xy''^2 + 60y'y'' = 0. \quad (3.191)$$

It is an equation of the form (3.10) with the coefficients

$$\begin{aligned}
&A_1 = \frac{5}{y},\ A_0 = \frac{10}{x},\ B_3 = \frac{10}{y},\ B_2 = 0,\ B_1 = \frac{40}{xy},\ B_0 = \frac{20}{x^2},\\
&C_1 = 0,\ C_0 = \frac{30}{xy},\ D_3 = 0,\ D_2 = 0,\ D_1 = \frac{60}{x^2y},\ D_0 = 0,\\
&E_5 = 0,\ E_4 = 0,\ E_3 = 0,\ E_2 = 0,\ E_1 = 0,\ E_0 = 0,\\
&\omega_1 = y,\ \omega_2 = \frac{y^2}{2},\ \mu_1 = 0,\ \mu_2 = \frac{-90000y^4}{x^4},\ \mu_3 = \frac{-3000y^3}{x^3},\ \mu_4 = 0,\\
&\mu_5 = 0,\ \mu_6 = 0,\ \mu_7 = 0,\ \mu_8 = 0,\ \mu_9 = 0,\ \mu_{10} = 0,\ \mu_{11} = 0,\ \mu_{12} = 0. \quad (3.192)
\end{aligned}$$

Since $\mu_4 = 0$ and $\mu_5 = 0$ then we apply Theorem 3.2.1 (a) for checking the linearity. The coefficients in equation (3.192) obey the conditions (3.69)-(3.84) and (3.91), so that one concludes that equation (3.191) is linearizable. Applying Corollary 3.3.1 (a), the linearizing transformation is found by solving the following equations.

$$\varphi_{xx} =(\varphi_x\psi_{1x}x - 2\psi_1)/(2\psi_1x), \quad (3.193)$$

$$\psi_{1xx} =(5\psi_{1x}^2x^2 - 4\psi_{1x}\psi_1x - 4\psi_1^2)/(4\psi_1x^2), \quad (3.194)$$

$$\begin{aligned}
\psi_{2xxxxx} =& 5(-3\psi_{1x}^4\psi_{2x}x^4 + 24\psi_{1x}^3\psi_{2xx}\psi_1x^4 + 24\psi_{1x}^3\psi_{2x}\psi_1x^3 - 48\psi_{1x}^2\psi_{2xxx}\psi_1^2x^4\\
&- 144\psi_{1x}^2\psi_{2xx}\psi_1^2x^3 - 72\psi_{1x}^2\psi_{2x}\psi_1^2x^2 + 32\psi_{1x}\psi_{2xxxx}\psi_1^3x^4\\
&+ 192\psi_{1x}\psi_{2xx}\psi_1^3x^3 + 288\psi_{1x}\psi_{2xx}\psi_1^3x^2 + 96\psi_{1x}\psi_{2x}\psi_1^3x - 64\psi_{2xxxx}\psi_1^4x^3\\
&- 192\psi_{2xxx}\psi_1^4x^2 - 192\psi_{2xx}\psi_1^4x - 48\psi_{2x}\psi_1^4)/(32\psi_1^4x^4), \quad (3.195)
\end{aligned}$$

$$\psi =(\psi_1y^2 + 2\psi_2)/2. \quad (3.196)$$

Since $\varphi_y = 0$, then one can take the simplest solution

$$\varphi = x.$$

Thus, equation (3.193) becomes

$$\begin{aligned}
\psi_{1x}x - 2\psi_1 &= 0\\
\frac{\psi_{1x}}{\psi_1} &= \frac{2}{x}\\
\int \frac{1}{\psi_1}d\psi_1 &= \int \frac{2}{x}dx\\
ln\psi_1 &= 2lnx + lnC\\
\psi_1 &= Cx^2.
\end{aligned}$$

Choosing C=2, we have

$$\psi_1 = 2x^2.$$

This solution satisfied equation (3.194). Equation (3.195) becomes

$$\psi_{2xxxxx} = 0.$$

Choosing the particular solution

$$\psi_2 = 0.$$

Hence equation (3.196) is in the form

$$\psi = x^2y^2.$$

So that one obtains the linearizing transformation

$$t = x, \quad u = x^2y^2. \tag{3.197}$$

Thus, the nonlinear equation (3.191) can be mapped by transformation (3.197) into the linear equation $u^{(5)}(t) = 0$. Next, to find the solution of (3.191). Since

$$u^{(5)}(t) = 0,$$

then we get the general solution

$$u(t) = C_0\frac{t^4}{24} + C_1\frac{t^3}{6} + C_2\frac{t^2}{2} + C_3t + C_4 \tag{3.198}$$

where C_0, C_1, C_2, C_3, C_4 are arbitrary constants. Substituting (3.197) into (3.198) we get,

$$x^2y^2 = C_0\frac{x^4}{24} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3x + C_4.$$

Example 3.5.2. Consider the nonlinear ordinary differential equation

$$\begin{aligned} &16y^4y^{(5)} - 40(y^3y' + 4y^4)y^{(4)} - 40(2y^3y'' - 3y^2y'^2 - 8y^3y' - 14y^4)y''' \\ &+60(3y'y^2 + 4y^3)y''^2 - 20(15yy'^3 + 36y^2y'^2 + 42y^3y' + 40y^4)y'' \\ &105y'^5 + 300yy'^4 + 420y^2y'^3 + 400y^3y'^2 + 384y^4y' = 0. \end{aligned} \tag{3.199}$$

It is an equation of the form equation (3.10) with the coefficients

$$\begin{aligned} &A_1 = \frac{-5}{2y},\ A_0 = -10,\ B_3 = \frac{-5}{y},\ B_2 = \frac{15}{2y^2},\ B_1 = \frac{20}{y}, \\ &B_0 = 35,\ C_1 = \frac{45}{4y^2},\ C_0 = \frac{15}{y},\ D_3 = \frac{-75}{4y^3},\ D_2 = \frac{-45}{y^2}, \\ &D_1 = \frac{-105}{2y},\ D_0 = -50,\ E_5 = \frac{105}{16y^4},\ E_4 = \frac{75}{4y^3},\ E_3 = \frac{105}{4y^2}, \\ &E_2 = \frac{25}{y},\ E_1 = 24,\ E_0 = 0,\ \omega_1 = \frac{1}{\sqrt{y}},\ \omega_2 = 2\sqrt{y}, \\ &\mu_1 = 0,\ \mu_2 = \frac{-288000}{y^2},\ \mu_3 = \frac{5000}{y\sqrt{y}},\ \mu_4 = 0,\ \mu_5 = 0,\ \mu_6 = 0, \\ &\mu_7 = 0,\ \mu_8 = 0,\ \mu_9 = 0,\ \mu_{10} = 0,\ \mu_{11} = 0,\ \mu_{12} = 0. \end{aligned} \tag{3.200}$$

Since $\mu_4 = 0$ and $\mu_5 = 0$ then we apply Theorem 3.2.1 (a) for checking the linearity. The coefficients in equation (3.200) obey the conditions (3.69)-(3.84) and (3.91), so that one concludes that equation (3.199) is linearizable. Applying Corollary 3.3.1 (a), the linearizing transformation is found by solving the following equations.

$$\varphi_{xx} = \varphi_x(\psi_{1x} + 2\psi_1)/(2\psi_1), \tag{3.201}$$

$$\psi_{1xx} = \psi_{1x}(5\psi_{1x} + 4\psi_1), \tag{3.202}$$

$$\begin{aligned}\psi_{2xxxxx} =& (-15\psi_{1x}^4\psi_{2x} + 120\psi_{1x}^3\psi_{2xx}\psi_1 - 120\psi_{1x}^3\psi_{2x}\psi_1 - 240\psi_{1x}^2\psi_{2xxx}\psi_1^2 \\ &+ 720\psi_{1x}^2\psi_{2xx}\psi_1^2 - 480\psi_{1x}^2\psi_{2x}\psi_1^2 + 160\psi_{1x}\psi_{2xxxx}\psi_1^3 - 960\psi_{1x}\psi_{2xxx}\psi_1^3 \\ &+ 1760\psi_{1x}\psi_{2xx}\psi_1^3 - 960\psi_{1x}\psi_{2x}\psi_1^3 + 320\psi_{2xxxx}\psi_1^4 - 1120\psi_{2xxx}\psi_1^4 \\ &+ 1600\psi_{2xx}\psi_1^4 - 768\psi_{2x}\psi_1^4)/(32\psi_1^4),\end{aligned} \tag{3.203}$$

$$\psi = (\psi_1 y^2 + 2\psi_2)/2. \tag{3.204}$$

From equation (3.201),

$$\frac{\varphi_{xx}}{\varphi_x} = \frac{\psi_{1x}}{2\psi_1} + 1. \tag{3.205}$$

Taking the particular solution $\varphi = e^x$, then equation (3.205) becomes

$$\begin{aligned} 1 &= \frac{\psi_{1x}}{2\psi_1} + 1 \\ \frac{\psi_{1x}}{2\psi_1} &= 0 \\ \psi_{1x} &= 0 \\ \psi_1 &= C. \end{aligned}$$

Choosing C=$\frac{1}{2}$, we have

$$\psi_1 = \frac{1}{2}.$$

This solution satisfied equation (3.202). Equation (3.203) becomes

$$\psi_{2xxxxx} = 10\psi_{2xxxx} - 35\psi_{2xxx} + 50\psi_{2xx} - 24\psi_{2x}.$$

Choosing the particular solution

$$\psi_2 = 0.$$

Hence equation (3.204) is in the form

$$\psi = \sqrt{y}.$$

So that one obtains the linearizing transformation

$$t = e^x, \quad u = \sqrt{y}. \tag{3.206}$$

Thus, the nonlinear equation (3.199) can be mapped by transformation (3.206) into the linear equation $u^{(5)}(t) = 0$. Next, we will find the solution of (3.199). Since

$$u^{(5)}(t) = 0,$$

then we get the general solution

$$u(t) = C_0\frac{t^4}{24} + C_1\frac{t^3}{6} + C_2\frac{t^2}{2} + C_3 t + C_4 \tag{3.207}$$

where C_0, C_1, C_2, C_3, C_4 are arbitrary constants. Substituting (3.206) into (3.207) we get,

$$\sqrt{y} = C_0\frac{e^{4x}}{24} + C_1\frac{e^{3x}}{6} + C_2\frac{e^{2x}}{2} + C_3 e^x + C_4.$$

Example 3.5.3. Consider the nonlinear ordinary differential equation

$$\begin{aligned} &x^4yy^{(5)} + 5(x^4y' - 2x^3y)y^{(4)} + 5(2x^4y'' - 8x^3y' + 9x^2y)y''' \\ &-30x^3y''^2 + 15(9x^2y' - 7xy)y'' - 105xy'^2 + 105yy' = 0. \end{aligned} \tag{3.208}$$

It is an equation of the form (3.10) with the coefficients

$$\begin{aligned} &A_1 = \frac{5}{y},\ A_0 = \frac{-10}{x},\ B_3 = \frac{10}{y},\ B_2 = 0,\ B_1 = \frac{-40}{xy}, \\ &B_0 = \frac{45}{x^2},\ C_1 = 0,\ C_0 = \frac{-30}{xy},\ D_3 = 0,\ D_2 = 0, \\ &D_1 = \frac{135}{x^2y},\ D_0 = \frac{-105}{x^3},\ E_5 = 0,\ E_4 = 0,\ E_3 = 0, \\ &E_2 = \frac{-105}{x^3y},\ E_1 = \frac{105}{x^4},\ E_0 = 0,\ \omega_1 = y,\ \omega_2 = \frac{y^2}{2}, \\ &\mu_1 = 0,\ \mu_2 = \frac{-1260000y^4}{x^4},\ \mu_3 = \frac{10500y^3}{x^3},\ \mu_4 = 0,\ \mu_5 = 0, \\ &\mu_6 = 0,\ \mu_7 = 0,\ \mu_8 = 0,\ \mu_9 = 0,\ \mu_{10} = 0,\ \mu_{11} = 0,\ \mu_{12} = 0. \end{aligned} \tag{3.209}$$

Since $\mu_4 = 0$ and $\mu_5 = 0$ then we apply Theorem 3.2.1 (a) for checking the linearity. The coefficients in equation (3.209) obey the conditions (3.69)-(3.84) and (3.91), so that one concludes that equation (3.208) is linearizable. Applying Corollary 3.3.1 (a), the linearizing transformation is found by solving the following equations.

$$\varphi_{xx} = \varphi_x(\psi_{1x}x + 2\psi_1)/(2\psi_1 x), \tag{3.210}$$

$$\psi_{1xx} = \psi_{1x}(5\psi_{1x}x + 4\psi_1)/(4\psi_1 x), \tag{3.211}$$

$$\begin{aligned} \psi_{2xxxxx} =& 5(-3\psi_{1x}^4\psi_{2x}x^4 + 24\psi_{1x}^3\psi_{2xx}\psi x^4 - 24\psi_{1x}^3\psi_{2x}\psi x^3 - 48\psi_{1x}^2\psi_{2xxx}\psi^2x^4 \\ &+ 144\psi_{1x}^2\psi_{2xx}\psi^2x^3 - 144\psi_{1x}^2\psi_{2x}\psi^2x^2 + 32\psi_{1x}\psi_{2xxxx}\psi^3x^4 - 192\psi_{1x}\psi_{2xxx}\psi^3x^3 \\ &+ 480\psi_{1x}\psi_{2xx}\psi^3x^2 - 480\psi_{1x}\psi_{2x}\psi^3x + 64\psi_{2xxxx}\psi^4x^3 - 288\psi_{2xxx}\psi^4x^2 \\ &+ 672\psi_{2xx}\psi^4x - 672\psi_{2x}\psi^4)/(32\psi^4x^4), \end{aligned} \tag{3.212}$$

$$\psi = (\psi_1 y^2 + 2\psi_2)/2. \tag{3.213}$$

From equation (3.210),

$$\frac{\varphi_{xx}}{\varphi_x} = \frac{\psi_{1x}}{2\psi_1} + \frac{1}{x}. \tag{3.214}$$

Taking the particular solution $\varphi = x^2$, then equation (3.214) becomes

$$\begin{aligned} \frac{1}{x} &= \frac{\psi_{1x}}{2\psi_1} + \frac{1}{x} \\ \frac{\psi_{1x}}{2\psi_1} &= 0 \\ \psi_{1x} &= 0 \\ \psi_1 &= C. \end{aligned}$$

Choosing C=2, we have

$$\psi_1 = 2.$$

This solution satisfied equation (3.211). Equation (3.212) becomes

$$\psi_{2xxxxx} = 5(2\psi_{2xxxx}x^3 - 9\psi_{2xxx}x^2 + 21\psi_{2xx}x - 21\psi_{2x})/x^4.$$

Choosing the particular solution

$$\psi_2 = 0.$$

Hence equation (3.213) is in the form

$$\psi = y^2.$$

So that one obtains the linearizing transformation

$$t = x^2, \quad u = y^2. \tag{3.215}$$

Thus, the nonlinear equation (3.208) can be mapped by transformation (3.215) into the linear equation $u^{(5)}(t) = 0$. Next, we find the solution of (3.208). Since

$$u^{(5)}(t) = 0,$$

then we get

$$u(t) = C_0\frac{t^4}{24} + C_1\frac{t^3}{6} + C_2\frac{t^2}{2} + C_3t + C_4. \tag{3.216}$$

Substituting (3.215) into (3.216) we get,

$$y^2 = C_0\frac{x^8}{24} + C_1\frac{x^6}{6} + C_2\frac{x^4}{2} + C_3x^2 + C_4.$$

Example 3.5.4. (New generalized fifth-order nonlinear integrable equation) [9]
Consider the nonlinear partial differential equation

$$u_{ttt} - u_{txxxx} - \alpha(u_x u_{xt})_{xx} - \beta(u_x u_{xt})_x = 0, \tag{3.217}$$

where α and β are arbitrary constants. Of particular interest among solutions of equation (3.217) are traveling wave solutions:

$$u(x,t) = H(x - Dt),$$

where D is a constant phase velocity and the argument $x - Dt$ is a phase of the wave. Substituting the representation of a solution into equation (3.217), one finds

$$DH^{(5)} + (2\alpha DH' + \beta DH' - D^3)H''' + (2\alpha D + \beta D)H''^2 = 0. \tag{3.218}$$

This is an equation of the form (3.10) with coefficients

$$\begin{aligned}
&A_1 = 0,\ A_0 = 0,\ B_3 = 0,\ B_2 = 0,\ B_1 = 2\alpha + \beta,\\
&B_0 = 0,\ C_1 = 0,\ C_0 = 2\alpha + \beta,\ D_3 = 0,\ D_2 = 0,\\
&D_1 = 0,\ D_0 = 0,\ E_5 = 0,\ E_4 = 0,\ E_3 = 0,\\
&E_2 = 0,\ E_1 = 0,\ E_0 = 0,\ \omega_1 = 0,\ \omega_2 = y,\\
&\mu_1 = 0,\ \mu_2 = 0,\ \mu_3 = 0,\ \mu_4 = 0,\ \mu_5 = 0,\ \mu_6 = 0,\\
&\mu_7 = 0,\ \mu_8 = 0,\ \mu_9 = 0,\ \mu_{10} = 0,\ \mu_{11} = 0,\ \mu_{12} = 0.
\end{aligned} \tag{3.219}$$

Since $\mu_4 = 0$ and $\mu_5 = 0$ then we apply Theorem 3.2.1 (a) for checking the linearity. The coefficients in equation (3.219) obey the conditions (3.69)-(3.84) and (3.91) if and only if

$$\beta = -2\alpha.$$

Hence equation (3.218) is linearizable with the condition $\beta = -2\alpha$.

Example 3.5.5. (The generalized fifth order Korteweg-de Vries (fKdV) equation) [10]
Consider the nonlinear partial differential equation

$$u_t + \alpha u^2 u_x + \beta u_x u_{xx} + \gamma u u_{xxx} + u_{xxxxx} = 0, \tag{3.220}$$

Substituting the traveling wave representation of a solution into equation (3.220), one finds

$$H^{(5)} + \gamma H H''' + \beta H' H'' + \alpha H^2 H' - DH' = 0, \tag{3.221}$$

It is an equation of the form (3.10) with the coefficients

$$\begin{aligned}
&A_1 = 0,\ A_0 = 0,\ B_3 = 0,\ B_2 = 0,\ B_1 = 0,\ B_0 = \gamma y,\ C_1 = 0,\ C_0 = 0,\ D_3 = 0,\\
&D_2 = 0,\ D_1 = \beta,\ D_0 = 0\, E_5 = 0,\ E_4 = 0,\ E_3 = 0,\ E_2 = -\beta,\ E_1 = \alpha y^2,\ E_0 = 0,\\
&\omega_1 = 1,\ \omega_2 = y,\ \mu_1 = 0,\ \mu_2 = 2000y^2(4\alpha - \gamma^2),\ \mu_3 = 0,\ \mu_4 = 300\beta,\\
&\mu_5 = 2400y(40\alpha - \beta\gamma),\ \mu_6 = 360000\beta\gamma y^2,\ \mu_7 = 0,\\
&\mu_8 = 191102976000000000000000\beta\gamma y^8(64000000\alpha^5 + 6400000\alpha^4\beta\gamma\\
&-10240000\alpha^4\gamma^2 + 240000\alpha^3\beta^2\gamma^2 + 1024000\alpha^3\beta\gamma^3 - 4000\alpha^2\beta^3\gamma^3\\
&-38400\alpha^2\beta^2\gamma^4 + 25\alpha\beta^4\gamma^4 + 640\alpha\beta^3\gamma^5 - 4\beta^4\gamma^6),\\
&\mu_9 = 720000\beta(40\alpha - \beta\gamma),\ \mu_{10} = 216000000\beta^2\gamma y,\\
&\mu_{11} = 1492992000000000000\beta^2\gamma y^4(64000\gamma^3 y - 4800\alpha^2\beta\gamma y + 120\alpha\beta^2\gamma^2 y\\
&+1800\alpha\beta^2\gamma - \beta^3\gamma^3 y - 45\beta^3\gamma^2),\\
&\mu_{12} = 29023764480000000000000000\beta^7\gamma y^5(1000\alpha^2 - 25\alpha\beta\gamma - 160\alpha\gamma^2\\
&+4\beta\gamma^3).
\end{aligned} \tag{3.222}$$

Since $\mu_4 \neq 0$, $\mu_9 \neq 0$ and $\mu_{11} \neq 0$ then we apply Theorem 3.2.1 (f) for checking the linearity. Because of the coefficients (3.222) do not satisfy the linearization conditions (3.79), (3.80), (3.83), (3.84), (3.103), (3.104), (3.108) and (3.109), hence the equation (3.221) is not linearizable.

3.5.2 The examples for the second class

Example 3.5.6. Consider the nonlinear equation

$$y^{(5)} - \tfrac{5}{y'}(3y''y^{(4)} + 2xy'''^2) + \tfrac{105}{y'^2}y''^2y''' - \tfrac{105}{y'^3}xy''^4 = 0. \tag{3.223}$$

It has the form of equation (3.29) with the following coefficients

$$\begin{aligned}
&r = 0,\ F_2 = 0,\ F_1 = 0,\ F_0 = 0,\ G_2 = 0,\ G_1 = 0,\ G_0 = 0,\\
&H_4 = 0,\ H_3 = 0,\ H_2 = 0,\ H_1 = 0,\ H_0 = 0,\ J_2 = 0,\ J_1 = 0,\\
&J_0 = 0,\ K_4 = 0,\ K_3 = 0,\ K_2 = 0,\ K_1 = 0,\ K_0 = 0,\ L_6 = 0,\\
&L_5 = 0,\ L_4 = 0,\ L_3 = 0,\ L_2 = 0,\ L_1 = 0,\ L_0 = 0,\ M_9 = 0,\\
&M_8 = 0,\ M_7 = 0,\ M_6 = 0,\ M_5 = 0,\ M_4 = 0,\ M_3 = 0,\ M_2 = 0,\\
&M_1 = 0,\ M_0 = 0,\ \lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 0,\ \lambda_4 = 0.
\end{aligned} \tag{3.224}$$

Since $\lambda_1 = 0$, $\lambda_3 = 0$ and $r = 0$ then we apply Theorem 3.2.2 (c) for checking the linearity. The coefficients in equation (3.224) obey the conditions (3.114)-(3.156) and (3.157), so that one concludes that equation (3.223) is linearizable. Applying Corollary 3.3.2 (c), the linearizing transformation is found by solving the following equations. For the functions $\varphi(x,y)$ and $\psi(x,y)$ we have

$$\varphi_x = 0, \tag{3.225}$$
$$\varphi_{yy} = 0, \tag{3.226}$$
$$\Delta_x = 0, \tag{3.227}$$
$$\Delta_{yy} = 0, \tag{3.228}$$
$$\psi_x = -\Delta, \tag{3.229}$$
$$\psi_{yyyyy} = 5\Delta_y(36\psi_{yyyy}\Delta^3 - 36\psi_{yyy}\Delta_y\Delta^2 + 12\psi_{yy}\Delta_y^2\Delta - \psi_y\Delta_y^3)/(54\Delta_y^4), \tag{3.230}$$

From equation (3.225), one obtains

$$\varphi = f(y).$$

Let us take its simplest solution

$$\varphi = y$$

and the solution satisfies equation (3.226). From equation (3.227) and equation (3.228) one can take the particular solution

$$\Delta = C. \tag{3.231}$$

Choosing $C = -1$, we have

$$\Delta = -1.$$

So that we obtain

$$\psi_x = 1.$$

Thus, we have

$$\psi = x + C.$$

Choosing $C = 0$, we have

$$\psi = x$$

which satisfy equation (3.230). So that one obtains the linearizing transformation

$$t = x, \quad u = y. \tag{3.232}$$

Thus, the nonlinear equation (3.223) can be mapped by transformation (3.232) into the linear equation $u^{(5)}(t) = 0$. Next, we will find the solution of (3.223). Since

$$u^{(5)}(t) = 0,$$

then we get

$$u(t) = C_0\frac{t^4}{24} + C_1\frac{t^3}{6} + C_2\frac{t^2}{2} + C_3 t + C_4 \qquad (3.233)$$

where C_0, C_1, C_2, C_3, C_4 are arbitrary constants. Substituting equation (3.232) into equation (3.233) we get,

$$y = C_0\frac{x^4}{24} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4.$$

Example 3.5.7. Consider the nonlinear equation

$$xy^{(5)} - \tfrac{5}{y'}((3xy'' - y')y^{(4)} - 2xy'''^2) + \tfrac{15}{y'^2}((7y''^2 - 4y'y'')y''' + 7y''^3) - 105\tfrac{x}{y'^3}y''^4 = 0. \qquad (3.234)$$

It has the form of equation (3.29) with the following coefficients

$$\begin{aligned}
&r = 0,\ F_2 = 0,\ F_1 = 5/x,\ F_0 = 0,\ G_2 = 0,\ G_1 = (-60)/x,\ G_0 = 0,\\
&H_4 = 0,\ H_3 = 0,\ H_2 = 0,\ H_1 = 0,\ H_0 = 0,\ J_2 = 0,\ J_1 = 105/x,\\
&J_0 = 0,\ K_4 = 0,\ K_3 = 0,\ K_2 = 0,\ K_1 = 0,\ K_0 = 0,\ L_6 = 0,\\
&L_5 = 0,\ L_4 = 0,\ L_3 = 0,\ L_2 = 0,\ L_1 = 0,\ L_0 = 0,\ M_9 = 0,\\
&M_8 = 0,\ M_7 = 0,\ M_6 = 0,\ M_5 = 0,\ M_4 = 0,\ M_3 = 0,\ M_2 = 0,\\
&M_1 = 0,\ M_0 = 0,\ \lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 0,\ \lambda_4 = 0. \qquad (3.235)
\end{aligned}$$

Since $\lambda_1 = 0, \lambda_3 = 0$ and $r = 0$ then we apply Theorem 3.2.2 (c) for checking the linearity. The coefficients in equation (3.235) obey the conditions (3.114)-(3.156) and (3.157), so that one concludes that equation (3.234) is linearizable. Applying Corollary 3.3.2 (c), the linearizing transformation is found by solving the following equations. For the functions $\varphi(x, y)$ and $\psi(x, y)$ we have

$$\begin{aligned}
\varphi_x &= 0, && (3.236)\\
\varphi_{yy} &= 0, && (3.237)\\
\Delta_x &= \Delta/x, && (3.238)\\
\Delta_{yy} &= 0, && (3.239)\\
\psi_x &= -\Delta/\varphi_y, && (3.240)\\
\psi_{yyyyy} &= 5\Delta_y(36\psi_{yyyy}\Delta^3 - 36\psi_{yyy}\Delta_y\Delta^2 + 12\psi_{yy}\Delta_y^2\Delta - \psi_y\Delta_y^3)/(54\Delta_y^4), && (3.241)
\end{aligned}$$

From equation (3.236), one obtains

$$\varphi = f(y).$$

Let us take its simplest solution

$$\varphi = y$$

and the solution satisfies equation (3.237). Equation (3.240) becomes

$$\psi_x = -\Delta \tag{3.242}$$

From equation (3.238),

$$\begin{aligned} \frac{\Delta_x}{\Delta} &= \frac{1}{x} \\ ln\Delta &= lnx + lnC \\ \Delta &= Cx. \end{aligned}$$

Choosing $C = -1$, we have

$$\Delta = -x$$

this particular solution satisfies equation (3.239). From equation (3.242), we have

$$\psi_x = x$$

so that

$$\psi = \frac{x^2}{2} + C.$$

Choosing $C = 0$, we have

$$\psi = \frac{x^2}{2}$$

and this solution satisfies equation (3.241). So that one obtains the linearizing transformation

$$t = \frac{x^2}{2}, \quad u = y. \tag{3.243}$$

Thus, the nonlinear equation (3.234) can be mapped by transformation (3.243) into the linear equation $u^{(5)}(t) = 0$. Next, we will find the solution of (3.234). Since

$$u^{(5)}(t) = 0,$$

then we get

$$u(t) = C_0\frac{t^4}{24} + C_1\frac{t^3}{6} + C_2\frac{t^2}{2} + C_3t + C_4 \tag{3.244}$$

where C_0, C_1, C_2, C_3, C_4 are arbitrary constants. Substituting equation (3.243) into equation (3.244) we get,

$$y = C_0\frac{x^8}{8} + C_1\frac{3x^6}{4} + C_2\frac{x^4}{2} + C_3\frac{x^2}{2} + C_4.$$

Chapter 4

Conclusions

This project is devoted to the study the fifth-order ordinary differential equations to be reduced to a simplest linear equation by point transformations. The results obtained are separated into two parts.

- The main Theorem 3.2.1 for the first class, provides necessary and sufficient conditions for linearization. The explicit procedure for constructing the linearizing point transformations are summarized in Corollary 3.3.1. Examples demonstrating the procedure of using the linearization theorem are presented. Linearization of traveling waves of partial differential equation are applied.

- The main Theorem 3.2.2 for the second class, provides necessary and sufficient conditions for linearization. The procedure for obtaining the linearizing point transformations are summarized in Corollary 3.3.2.

We can conclude that the criteria for fifth-order ordinary differential equations to be linearizable via point transformations are not completed, it is a particular case. Program for checking the linearizable criteria have also been developed.

Appendix A

Equation in section 3.2

$$
\begin{aligned}
\mu_1 =& 700A_{0x}^2\omega_{2y}^5A_0\omega_2 - 1750A_{0x}B_{0x}\omega_{2y}^5\omega_2 + 280A_{0x}\omega_{2y}^5A_0^3\omega_2 - 1050A_{0x}\omega_{2y}^5A_0B_0\omega_2 \\
&+ 2750A_{0x}\omega_{2y}^5D_0\omega_2 - 350B_{0x}\omega_{2y}^5A_0^2\omega_2 + 875B_{0x}\omega_{2y}^5B_0\omega_2 + 1250D_{0xx}\omega_{2y}^5\omega_2 \\
&+ 500D_{0x}\omega_{2y}^5A_0\omega_2 - 6250E_{1x}\omega_{2y}^5\omega_2 + 2250000\omega_{2xy}^5\omega_2 - 2250000\omega_{2xy}^4\omega_{2x}\omega_{2y} \\
&- 450000\omega_{2xy}^4\omega_{2y}A_0\omega_2 - 4500000\omega_{2xy}^3\omega_{2xxy}\omega_{2y}\omega_2 + 1125000\omega_{2xy}^3\omega_{2xx}\omega_{2y}^2 \\
&+ 450000\omega_{2xy}^3\omega_{2x}\omega_{2y}^2A_0 + 112500\omega_{2xy}^3\omega_{2y}^2B_0\omega_2 + 1125000\omega_{2xy}^2\omega_{2xxxy}\omega_{2y}^2\omega_2 \\
&- 375000\omega_{2xy}^2\omega_{2xxx}\omega_{2y}^3 + 3375000\omega_{2xy}^2\omega_{2xxy}\omega_{2x}\omega_{2y}^2 + 675000\omega_{2xy}^2\omega_{2xxy}\omega_{2y}^2A_0\omega_2 \\
&- 225000\omega_{2xy}^2\omega_{2xx}\omega_{2y}^3A_0 - 112500\omega_{2xy}^2\omega_{2x}\omega_{2y}^3B_0 - 37500\omega_{2xy}^2\omega_{2y}^3D_0\omega_2 \\
&- 187500\omega_{2xy}\omega_{2xxxxy}\omega_{2y}^3\omega_2 + 93750\omega_{2xy}\omega_{2xxxx}\omega_{2y}^4 - 750000\omega_{2xy}\omega_{2xxxy}\omega_{2x}\omega_{2y}^3 \\
&- 150000\omega_{2xy}\omega_{2xxxy}\omega_{2y}^3A_0\omega_2 + 75000\omega_{2xy}\omega_{2xxx}\omega_{2y}^4A_0 + 1687500\omega_{2xy}\omega_{2xxy}^2\omega_{2y}^2\omega_2 \\
&- 1125000\omega_{2xy}\omega_{2xxy}\omega_{2xx}\omega_{2y}^3 - 450000\omega_{2xy}\omega_{2xxy}\omega_{2x}\omega_{2y}^3A_0 \\
&- 112500\omega_{2xy}\omega_{2xxy}\omega_{2y}^3B_0\omega_2 + 56250\omega_{2xy}\omega_{2xx}\omega_{2y}^4B_0 + 37500\omega_{2xy}\omega_{2x}\omega_{2y}^4D_0 \\
&+ 18750\omega_{2xy}\omega_{2y}^4E_1\omega_2 + 18750\omega_{2xxxxxy}\omega_{2y}^4\omega_2 - 18750\omega_{2xxxxx}\omega_{2y}^5 \\
&+ 93750\omega_{2xxxxy}\omega_{2x}\omega_{2y}^4 + 18750\omega_{2xxxxy}\omega_{2y}^4A_0\omega_2 - 18750\omega_{2xxxx}\omega_{2y}^5A_0 \\
&- 375000\omega_{2xxxy}\omega_{2xxy}\omega_{2y}^3\omega_2 + 187500\omega_{2xxxy}\omega_{2xx}\omega_{2y}^4 + 75000\omega_{2xxxy}\omega_{2x}\omega_{2y}^4A_0 \\
&+ 18750\omega_{2xxxy}\omega_{2y}^4B_0\omega_2 + 187500\omega_{2xxx}\omega_{2xxy}\omega_{2y}^4 - 18750\omega_{2xxx}\omega_{2y}^5B_0 \\
&- 562500\omega_{2xxy}^2\omega_{2x}\omega_{2y}^3 - 112500\omega_{2xxy}^2\omega_{2y}^3A_0\omega_2 + 112500\omega_{2xxy}\omega_{2xx}\omega_{2y}^4A_0 \\
&+ 56250\omega_{2xxy}\omega_{2x}\omega_{2y}^4B_0 + 18750\omega_{2xxy}\omega_{2y}^4D_0\omega_2 - 18750\omega_{2xx}\omega_{2y}^5D_0 \\
&- 18750\omega_{2x}\omega_{2y}^5E_1 + 18750\omega_{2y}^6E_0 + 28\omega_{2y}^5A_0^5\omega_2 - 210\omega_{2y}^5A_0^3B_0\omega_2 \\
&+ 550\omega_{2y}^5A_0^2D_0\omega_2 + 350\omega_{2y}^5A_0B_0^2\omega_2 - 1250\omega_{2y}^5A_0E_1\omega_2 - 1250\omega_{2y}^5B_0D_0\omega_2, \qquad \text{(A.1)}
\end{aligned}
$$

$$
\begin{aligned}
\mu_2 =& -3200A_{0x}^2\omega_{2y}^4 - 18000A_{0x}\omega_{2xy}^2\omega_{2y}^2 - 2400A_{0x}\omega_{2xy}\omega_{2y}^3A_0 - 2000A_{0x}\omega_{2y}^4A_0^2 \\
&+ 5600A_{0x}\omega_{2y}^4B_0 + 12000B_{0x}\omega_{2xy}\omega_{2y}^3 - 4000D_{0x}\omega_{2y}^4 - 5625\omega_{2xy}^4 \\
&+ 4500\omega_{2xy}^3\omega_{2y}A_0 - 4950\omega_{2xy}^2\omega_{2y}^2A_0^2 + 9000\omega_{2xy}^2\omega_{2y}^2B_0 - 300\omega_{2xy}\omega_{2y}^3A_0^3 \\
&+ 3600\omega_{2xy}\omega_{2y}^3A_0B_0 - 12000\omega_{2xy}\omega_{2y}^3D_0 - 281\omega_{2y}^4A_0^4 + 1480\omega_{2y}^4A_0^2B_0 \\
&- 800\omega_{2y}^4A_0D_0 - 2000\omega_{2y}^4B_0^2 + 8000\omega_{2y}^4E_1, \qquad \text{(A.2)}
\end{aligned}
$$

$$
\begin{aligned}
\mu_3 =& 400A_{0x}\omega_{2xy}\omega_{2y}^2 - 100B_{0x}\omega_{2y}^3 + 375\omega_{2xy}^3 - 225\omega_{2xy}^2\omega_{2y}A_0 + 125\omega_{2xy}\omega_{2y}^2A_0^2 \\
& - 200\omega_{2xy}\omega_{2y}^2B_0 - 3\omega_{2y}^3A_0^3 - 20\omega_{2y}^3A_0B_0 + 100\omega_{2y}^3D_0,
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
\mu_4 =& -1200A_{0x}\omega_{2xy}\omega_{2yy}\omega_{2y}^2 + 240A_{0x}\omega_{2yy}\omega_{2y}^3A_0 - 80A_{0x}\omega_{2y}^4C_0 + 100D_{1x}\omega_{2y}^4 \\
& - 1125\omega_{2xy}^3\omega_{2yy} + 675\omega_{2xy}^2\omega_{2yy}\omega_{2y}A_0 - 375\omega_{2xy}\omega_{2yy}\omega_{2y}^2A_0^2 \\
& + 600\omega_{2xy}\omega_{2yy}\omega_{2y}^2B_0 + 57\omega_{2yy}\omega_{2y}^3A_0^3 - 120\omega_{2yy}\omega_{2y}^3A_0B_0 + 3\omega_{2yy}\mu_3 \\
& - 16\omega_{2y}^4A_0^2C_0 + 20\omega_{2y}^4A_0D_1 + 40\omega_{2y}^4B_0C_0 - 300\omega_{2y}^4E_2,
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
\mu_5 =& -24000A_{0x}^2\omega_{2yy}\omega_{2y}^4 + 213000A_{0x}\omega_{2xy}^2\omega_{2yy}\omega_{2y}^2 - 66000A_{0x}\omega_{2xy}\omega_{2yy}\omega_{2y}^3A_0 \\
& + 19200A_{0x}\omega_{2xy}\omega_{2y}^4C_0 - 48000A_{0x}\omega_{2xxy}\omega_{2yy}\omega_{2y}^3 - 4920A_{0x}\omega_{2yy}\omega_{2y}^4A_0^2 \\
& + 28800A_{0x}\omega_{2yy}\omega_{2y}^4B_0 - 3840A_{0x}\omega_{2y}^5A_0C_0 + 4800A_{0x}\omega_{2y}^5D_1 + 48000E_{1y}\omega_{2y}^5 \\
& - 24000E_{2x}\omega_{2y}^5 + 120\mu_{3x}\omega_{2yy}\omega_{2y} - 40\mu_{4x}\omega_{2y} + 388125\omega_{2xy}^4\omega_{2yy} \\
& - 256500\omega_{2xy}^3\omega_{2yy}\omega_{2y}A_0 + 18000\omega_{2xy}^3\omega_{2y}^2C_0 - 135000\omega_{2xy}^2\omega_{2xxy}\omega_{2yy}\omega_{2y} \\
& + 103350\omega_{2xy}^2\omega_{2yy}\omega_{2y}^2A_0^2 - 93000\omega_{2xy}^2\omega_{2yy}\omega_{2y}^2B_0 - 10800\omega_{2xy}^2\omega_{2y}^3A_0C_0 \\
& + 54000\omega_{2xy}\omega_{2xxy}\omega_{2yy}\omega_{2y}^2A_0 - 19140\omega_{2xy}\omega_{2yy}\omega_{2y}^3A_0^3 + 27600\omega_{2xy}\omega_{2yy}\omega_{2y}^3A_0B_0 \\
& - 1200\omega_{2xy}\omega_{2yy}\mu_3 + 6000\omega_{2xy}\omega_{2y}^4A_0^2C_0 - 9600\omega_{2xy}\omega_{2y}^4B_0C_0 + 280\omega_{2xy}\mu_4 \\
& - 15000\omega_{2xxy}\omega_{2yy}\omega_{2y}^3A_0^2 + 24000\omega_{2xxy}\omega_{2yy}\omega_{2y}^3B_0 + 165\omega_{2yy}\omega_{2y}^4A_0^4 \\
& + 3960\omega_{2yy}\omega_{2y}^4A_0^2B_0 - 8400\omega_{2yy}\omega_{2y}^4B_0^2 + 48000\omega_{2yy}\omega_{2y}^4E_1 + 168\omega_{2yy}\omega_{2y}A_0\mu_3 \\
& - 11\omega_{2yy}\mu_2 - 912\omega_{2y}^5A_0^3C_0 + 960\omega_{2y}^5A_0^2D_1 + 1920\omega_{2y}^5A_0B_0C_0 - 4800\omega_{2y}^5A_0E_2 \\
& - 2400\omega_{2y}^5B_0D_1 - 48\omega_{2y}^2C_0\mu_3 - 8\omega_{2y}A_0\mu_4,
\end{aligned}
\tag{A.5}
$$

$$
\begin{aligned}
\mu_6 =& -486000A_{0x}^2\omega_{2xy}\omega_{2yy}\omega_{2y}^4\omega_2 - 7344000A_{0x}^2\omega_{2xy}\omega_{2y}^6 + 1080000A_{0x}^2\omega_{2x}\omega_{2yy}\omega_{2y}^5 \\
& - 1126800A_{0x}^2\omega_{2yy}\omega_{2y}^5A_0\omega_2 + 950400A_{0x}^2\omega_{2y}^7A_0 + 350400A_{0x}^2\omega_{2y}^6C_0\omega_2 \\
& + 5872500A_{0x}\omega_{2xy}^3\omega_{2yy}\omega_{2y}^2\omega_2 + 270000A_{0x}\omega_{2xy}^3\omega_{2y}^4 - 7425000A_{0x}\omega_{2xy}^2\omega_{2x}\omega_{2yy}\omega_{2y}^3 \\
& + 8779500A_{0x}\omega_{2xy}^2\omega_{2yy}\omega_{2y}^3A_0\omega_2 + 810000A_{0x}\omega_{2xy}^2\omega_{2y}^5A_0 - 2409000A_{0x}\omega_{2xy}^2\omega_{2y}^4C_0\omega_2 \\
& - 9720000A_{0x}\omega_{2xy}\omega_{2xxy}\omega_{2yy}\omega_{2y}^3\omega_2 + 10800000A_{0x}\omega_{2xy}\omega_{2xxy}\omega_{2y}^5 \\
& - 1350000A_{0x}\omega_{2xy}\omega_{2x}\omega_{2yy}\omega_{2y}^4A_0 + 1003500A_{0x}\omega_{2xy}\omega_{2yy}\omega_{2y}^4A_0^2\omega_2 \\
& + 810000A_{0x}\omega_{2xy}\omega_{2yy}\omega_{2y}^4B_0\omega_2 - 3985200A_{0x}\omega_{2xy}\omega_{2y}^6A_0^2 + 8856000A_{0x}\omega_{2xy}\omega_{2y}^6B_0 \\
& - 438000A_{0x}\omega_{2xy}\omega_{2y}^5A_0C_0\omega_2 + 2160000A_{0x}\omega_{2xxxy}\omega_{2yy}\omega_{2y}^4\omega_2 + 10800000A_{0x}\omega_{2xxy}\omega_{2x}\omega_{2yy}\omega_{2y}^4 \\
& - 11592000A_{0x}\omega_{2xxy}\omega_{2yy}\omega_{2y}^4A_0\omega_2 - 432000A_{0x}\omega_{2xxy}\omega_{2y}^6A_0 + 3504000A_{0x}\omega_{2xxy}\omega_{2y}^5C_0\omega_2 \\
& + 999000A_{0x}\omega_{2x}\omega_{2yy}\omega_{2y}^5A_0^2 - 2160000A_{0x}\omega_{2x}\omega_{2yy}\omega_{2y}^5B_0 - 1088460A_{0x}\omega_{2yy}\omega_{2y}^5A_0^3\omega_2 \\
& + 2415600A_{0x}\omega_{2yy}\omega_{2y}^5A_0B_0\omega_2 - 216000A_{0x}\omega_{2yy}\omega_{2y}^5D_0\omega_2 - 3240A_{0x}\omega_{2yy}\omega_{2y}^2\mu_3\omega_2 \\
& + 555120A_{0x}\omega_{2y}^7A_0^3 - 1425600A_{0x}\omega_{2y}^7A_0B_0 + 864000A_{0x}\omega_{2y}^7D_0 + 324120A_{0x}\omega_{2y}^6A_0^2C_0\omega_2 \\
& - 700800A_{0x}\omega_{2y}^6B_0C_0\omega_2 + 17280A_{0x}\omega_{2y}^4\mu_3 - 2040A_{0x}\omega_{2y}^2\mu_4\omega_2 + 10800000E_{0y}\omega_{2y}^7 \\
& - 2160000E_{1x}\omega_{2y}^7 + 225\mu_{2x}\omega_{2yy}\omega_{2y}\omega_2 - 180\mu_{2x}\omega_{2y}^3 - 5400\mu_{3xx}\omega_{2yy}\omega_{2y}^2\omega_2 \\
& + 64800\mu_{3x}\omega_{2xy}\omega_{2yy}\omega_{2y}\omega_2 - 21600\mu_{3x}\omega_{2xy}\omega_{2y}^3 - 27000\mu_{3x}\omega_{2x}\omega_{2yy}\omega_{2y}^2 \\
& + 24120\mu_{3x}\omega_{2yy}\omega_{2y}^2A_0\omega_2 - 8760\mu_{3x}\omega_{2y}^3C_0\omega_2 + 1800\mu_{4xx}\omega_{2y}^2\omega_2 - 14400\mu_{4x}\omega_{2xy}\omega_{2y}\omega_2 \\
& + 9000\mu_{4x}\omega_{2x}\omega_{2y}^2 + 720\mu_{4x}\omega_{2y}^2A_0\omega_2 - 75\mu_{5x}\omega_{2y}\omega_2 + 53662500\omega_{2xy}^5\omega_{2yy}\omega_2 \\
& - 8100000\omega_{2xy}^5\omega_{2y}^2 - 34171875\omega_{2xy}^4\omega_{2x}\omega_{2yy}\omega_{2y} + 12200625\omega_{2xy}^4\omega_{2yy}\omega_{2y}A_0\omega_2
\end{aligned}
$$

$$- 202500\omega_{2xy}^4\omega_{2y}^3A_0 - 11086875\omega_{2xy}^4\omega_{2y}^2C_0\omega_2 - 67837500\omega_{2xy}^3\omega_{2xxy}\omega_{2yy}\omega_{2y}\omega_2$$
$$+ 28350000\omega_{2xy}^3\omega_{2xxy}\omega_{2y}^3 + 15187500\omega_{2xy}^3\omega_{2x}\omega_{2yy}\omega_{2y}^2A_0 - 12879000\omega_{2xy}^3\omega_{2yy}\omega_{2y}^2A_0^2\omega_2$$
$$+ 2025000\omega_{2xy}^3\omega_{2yy}\omega_{2y}^2B_0\omega_2 + 1512000\omega_{2xy}^3\omega_{2y}^4A_0^2 - 540000\omega_{2xy}^3\omega_{2y}^4B_0$$
$$+ 4927500\omega_{2xy}^3\omega_{2y}^3A_0C_0\omega_2 + 6075000\omega_{2xy}^2\omega_{2xxxy}\omega_{2yy}\omega_{2y}^2\omega_2 + 30375000\omega_{2xy}^2\omega_{2xxy}\omega_{2x}\omega_{2yy}\omega_{2y}^2$$
$$- 7087500\omega_{2xy}^2\omega_{2xxy}\omega_{2yy}\omega_{2y}^2A_0\omega_2 - 12150000\omega_{2xy}^2\omega_{2xxy}\omega_{2y}^4A_0 + 9855000\omega_{2xy}^2\omega_{2xxy}\omega_{2y}^3C_0\omega_2$$
$$- 2396250\omega_{2xy}^2\omega_{2x}\omega_{2yy}\omega_{2y}^3A_0^2 + 675000\omega_{2xy}^2\omega_{2x}\omega_{2yy}\omega_{2y}^3B_0 + 2966850\omega_{2xy}^2\omega_{2yy}\omega_{2y}^3A_0^3\omega_2$$
$$- 2587500\omega_{2xy}^2\omega_{2yy}\omega_{2y}^3A_0B_0\omega_2 - 607500\omega_{2xy}^2\omega_{2yy}\omega_{2y}^3D_0\omega_2 - 173475\omega_{2xy}^2\omega_{2yy}\mu_3\omega_2$$
$$- 16200\omega_{2xy}^2\omega_{2y}^5A_0^3 - 648000\omega_{2xy}^2\omega_{2y}^5A_0B_0 - 777450\omega_{2xy}^2\omega_{2y}^4A_0^2C_0\omega_2 + 219000\omega_{2xy}^2\omega_{2y}^4B_0C_0\omega_2$$
$$+ 32400\omega_{2xy}^2\omega_{2y}^2\mu_3 + 36000\omega_{2xy}^2\mu_4\omega_2 - 2430000\omega_{2xy}\omega_{2xxxy}\omega_{2yy}\omega_{2y}^3A_0\omega_2$$
$$+ 12150000\omega_{2xy}\omega_{2xxy}^2\omega_{2yy}\omega_{2y}^2\omega_2 - 12150000\omega_{2xy}\omega_{2xxy}\omega_{2x}\omega_{2yy}\omega_{2y}^3A_0$$
$$+ 9031500\omega_{2xy}\omega_{2xxy}\omega_{2yy}\omega_{2y}^3A_0^2\omega_2 + 2430000\omega_{2xy}\omega_{2xxy}\omega_{2yy}\omega_{2y}^3B_0\omega_2$$
$$+ 3618000\omega_{2xy}\omega_{2xxy}\omega_{2y}^5A_0^2 - 5400000\omega_{2xy}\omega_{2xxy}\omega_{2y}^5B_0 - 3942000\omega_{2xy}\omega_{2xxy}\omega_{2y}^4A_0C_0\omega_2$$
$$- 634500\omega_{2xy}\omega_{2x}\omega_{2yy}\omega_{2y}^4A_0^3 + 1890000\omega_{2xy}\omega_{2x}\omega_{2yy}\omega_{2y}^4A_0B_0$$
$$+ 108000\omega_{2xy}\omega_{2x}\omega_{2yy}\omega_{2y}\mu_3 - 45000\omega_{2xy}\omega_{2x}\omega_{2y}\mu_4 + 602460\omega_{2xy}\omega_{2yy}\omega_{2y}^4A_0^4\omega_2$$
$$- 1753200\omega_{2xy}\omega_{2yy}\omega_{2y}^4A_0^2B_0\omega_2 + 243000\omega_{2xy}\omega_{2yy}\omega_{2y}^4A_0D_0\omega_2 - 162000\omega_{2xy}\omega_{2yy}\omega_{2y}^4B_0^2\omega_2$$
$$- 91890\omega_{2xy}\omega_{2yy}\omega_{2y}A_0\mu_3\omega_2 - 2025\omega_{2xy}\omega_{2yy}\mu_2\omega_2 - 535680\omega_{2xy}\omega_{2y}^6A_0^4$$
$$+ 2440800\omega_{2xy}\omega_{2y}^6A_0^2B_0 - 2592000\omega_{2xy}\omega_{2y}^6B_0^2 - 205860\omega_{2xy}\omega_{2y}^5A_0^3C_0\omega_2$$
$$+ 613200\omega_{2xy}\omega_{2y}^5A_0B_0C_0\omega_2 + 35040\omega_{2xy}\omega_{2y}^2C_0\mu_3\omega_2 + 720\omega_{2xy}\omega_{2y}^2\mu_2$$
$$- 2880\omega_{2xy}\omega_{2y}A_0\mu_4\omega_2 + 600\omega_{2xy}\mu_5\omega_2 + 675000\omega_{2xxxy}\omega_{2yy}\omega_{2y}^4A_0^2\omega_2$$
$$- 1080000\omega_{2xxxy}\omega_{2yy}\omega_{2y}^4B_0\omega_2 - 2430000\omega_{2xxy}^2\omega_{2yy}\omega_{2y}^3A_0\omega_2 + 3375000\omega_{2xxy}\omega_{2x}\omega_{2yy}\omega_{2y}^4A_0^2$$
$$- 5400000\omega_{2xxy}\omega_{2x}\omega_{2yy}\omega_{2y}^4B_0 - 3687300\omega_{2xxy}\omega_{2yy}\omega_{2y}^4A_0^3\omega_2 + 6282000\omega_{2xxy}\omega_{2yy}\omega_{2y}^4A_0B_0\omega_2$$
$$- 118800\omega_{2xxy}\omega_{2y}^6A_0^3 + 216000\omega_{2xxy}\omega_{2y}^6A_0B_0 + 1095000\omega_{2xxy}\omega_{2y}^5A_0^2C_0\omega_2$$
$$- 1752000\omega_{2xxy}\omega_{2y}^5B_0C_0\omega_2 - 21600\omega_{2xxy}\omega_{2y}^3\mu_3 - 16200\omega_{2xxy}\omega_{2y}\mu_4\omega_2 + 18000\omega_{2xx}\omega_{2y}^2\mu_4$$
$$+ 199125\omega_{2x}\omega_{2yy}\omega_{2y}^5A_0^4 - 837000\omega_{2x}\omega_{2yy}\omega_{2y}^5A_0^2B_0 + 810000\omega_{2x}\omega_{2yy}\omega_{2y}^5B_0^2$$
$$- 5400\omega_{2x}\omega_{2yy}\omega_{2y}^2A_0\mu_3 + 1125\omega_{2x}\omega_{2yy}\omega_{2y}\mu_2 + 1800\omega_{2x}\omega_{2y}^2A_0\mu_4 - 225\omega_{2x}\omega_{2y}\mu_5$$
$$+ 10800000\omega_{2yy}\omega_{2y}^6E_0 - 221787\omega_{2yy}\omega_{2y}^5A_0^5\omega_2 + 953460\omega_{2yy}\omega_{2y}^5A_0^3B_0\omega_2$$
$$- 67500\omega_{2yy}\omega_{2y}^5A_0^2D_0\omega_2 - 950400\omega_{2yy}\omega_{2y}^5A_0B_0^2\omega_2 + 108000\omega_{2yy}\omega_{2y}^5B_0D_0\omega_2$$
$$+ 3717\omega_{2yy}\omega_{2y}^2A_0^2\mu_3\omega_2 + 3240\omega_{2yy}\omega_{2y}^2B_0\mu_3\omega_2 - 1050\omega_{2yy}\omega_{2y}A_0\mu_2\omega_2 + 77868\omega_{2y}^7A_0^5$$
$$- 388800\omega_{2y}^7A_0^3B_0 + 172800\omega_{2y}^7A_0^2D_0 + 475200\omega_{2y}^7A_0B_0^2 - 432000\omega_{2y}^7A_0E_1$$
$$- 432000\omega_{2y}^7B_0D_0 + 64605\omega_{2y}^6A_0^4C_0\omega_2 - 271560\omega_{2y}^6A_0^2B_0C_0\omega_2 + 262800\omega_{2y}^6B_0^2C_0\omega_2$$
$$+ 4752\omega_{2y}^4A_0^2\mu_3 - 12960\omega_{2y}^4B_0\mu_3 - 1752\omega_{2y}^3A_0C_0\mu_3\omega_2 - 36\omega_{2y}^3A_0\mu_2 - 408\omega_{2y}^2A_0^2\mu_4\omega_2$$
$$+ 1200\omega_{2y}^2B_0\mu_4\omega_2 + 365\omega_{2y}^2C_0\mu_2\omega_2 - 15\omega_{2y}A_0\mu_5\omega_2, \qquad \text{(A.6)}$$

$$
\begin{aligned}
\mu_7 =& -200A_{0x}\mu_{5x}\omega_{2y}^3\mu_5^2\mu_6 + 200A_{0x}\mu_{6x}\omega_{2y}^3\mu_5^3 - 40A_{0x}\omega_{2y}^3A_0\mu_5^3\mu_6 - 1000\mu_{5xxx}\omega_{2y}^3\mu_5^2\mu_6 \\
& + 6000\mu_{5xx}\mu_{5x}\omega_{2y}^3\mu_5\mu_6 - 3000\mu_{5xx}\mu_{6x}\omega_{2y}^3\mu_5^2 + 6000\mu_{5xx}\omega_{2xy}\omega_{2y}^2\mu_5^2\mu_6 - 600\mu_{5xx}\omega_{2y}^3A_0\mu_5^2\mu_6 \\
& - 6000\mu_{5x}^3\omega_{2y}^3\mu_6 + 6000\mu_{5x}^2\mu_{6x}\omega_{2y}^3\mu_5 - 12000\mu_{5x}^2\omega_{2xy}\omega_{2y}^2\mu_5\mu_6 + 1200\mu_{5x}^2\omega_{2y}^3A_0\mu_5\mu_6 \\
& - 3000\mu_{5x}\mu_{6xx}\omega_{2y}^3\mu_5^2 + 12000\mu_{5x}\mu_{6x}\omega_{2xy}\omega_{2y}^2\mu_5^2 - 1200\mu_{5x}\mu_{6x}\omega_{2y}^3A_0\mu_5^2 \\
& - 18000\mu_{5x}\omega_{2xy}^2\omega_{2y}\mu_5^2\mu_6 + 2400\mu_{5x}\omega_{2xy}\omega_{2y}^2A_0\mu_5^2\mu_6 + 6000\mu_{5x}\omega_{2xxy}\omega_{2y}^2\mu_5^2\mu_6 \\
& - 40\mu_{5x}\omega_{2y}^3A_0^2\mu_5^2\mu_6 - 200\mu_{5x}\omega_{2y}^3B_0\mu_5^2\mu_6 + 1000\mu_{6xxx}\omega_{2y}^3\mu_5^3 - 6000\mu_{6xx}\omega_{2xy}\omega_{2y}^2\mu_5^3 \\
& + 600\mu_{6xx}\omega_{2y}^3A_0\mu_5^3 + 18000\mu_{6x}\omega_{2xy}^2\omega_{2y}\mu_5^3 - 2400\mu_{6x}\omega_{2xy}\omega_{2y}^2A_0\mu_5^3 - 6000\mu_{6x}\omega_{2xxy}\omega_{2y}^2\mu_5^3 \\
& + 40\mu_{6x}\omega_{2y}^3A_0^2\mu_5^3 + 200\mu_{6x}\omega_{2y}^3B_0\mu_5^3 - 23625\omega_{2xy}^3\mu_5^3\mu_6 + 3375\omega_{2xy}^2\omega_{2y}A_0\mu_5^3\mu_6 \\
& + 18000\omega_{2xy}\omega_{2xxy}\omega_{2y}\mu_5^3\mu_6 45\omega_{2xy}\omega_{2y}^2A_0^2\mu_5^3\mu_6 - 600\omega_{2xy}\omega_{2y}^2B_0\mu_5^3\mu_6 - 2000\omega_{2xxxy}\omega_{2y}^2\mu_5^3\mu_6 \\
& - 1200\omega_{2xxy}\omega_{2y}^2A_0\mu_5^3\mu_6 - 11\omega_{2y}^3A_0^3\mu_5^3\mu_6 + 20\omega_{2y}^3A_0B_0\mu_5^3\mu_6 + 100\omega_{2y}^3D_0\mu_5^3\mu_6 - \mu_3\mu_5^3\mu_6, \qquad \text{(A.7)}
\end{aligned}
$$

$$
\begin{aligned}
\mu_8 =& 4800A_{0x}^2\omega_{2y}^4\mu_5^4\mu_6 - 33000A_{0x}\omega_{2xy}^2\omega_{2y}^2\mu_5^4\mu_6 - 6000A_{0x}\omega_{2xy}\omega_{2y}^3A_0\mu_5^4\mu_6 \\
& + 48000A_{0x}\omega_{2xxy}\omega_{2y}^3\mu_5^4\mu_6 + 4440A_{0x}\omega_{2y}^4A_0^2\mu_5^4\mu_6 - 9600A_{0x}\omega_{2y}^4B_0\mu_5^4\mu_6 \\
& - 120\mu_{3x}\omega_{2y}\mu_5^4\mu_6 + 240\mu_{5x}\omega_{2y}\mu_7 - 60\mu_{7x}\omega_{2y}\mu_5 - 151875\omega_{2xy}^4\mu_5^4\mu_6 \\
& + 67500\omega_{2xy}^3\omega_{2y}A_0\mu_5^4\mu_6 + 135000\omega_{2xy}^2\omega_{2xxy}\omega_{2y}\mu_5^4\mu_6 - 10650\omega_{2xy}^2\omega_{2y}^2A_0^2\mu_5^4\mu_6 \\
& + 3000\omega_{2xy}^2\omega_{2y}^2B_0\mu_5^4\mu_6 - 54000\omega_{2xy}\omega_{2xxy}\omega_{2y}^2A_0\mu_5^4\mu_6 - 2820\omega_{2xy}\omega_{2y}^3A_0^3\mu_5^4\mu_6 \\
& + 8400\omega_{2xy}\omega_{2y}^3A_0B_0\mu_5^4\mu_6 + 480\omega_{2xy}\mu_3\mu_5^4\mu_6 + 360\omega_{2xy}\mu_5\mu_7 \\
& + 15000\omega_{2xxy}\omega_{2y}^3A_0^2\mu_5^4\mu_6 - 24000\omega_{2xxy}\omega_{2y}^3B_0\mu_5^4\mu_6 + 885\omega_{2y}^4A_0^4\mu_5^4\mu_6 \\
& - 3720\omega_{2y}^4A_0^2B_0\mu_5^4\mu_6 + 3600\omega_{2y}^4B_0^2\mu_5^4\mu_6 - 24\omega_{2y}A_0\mu_3\mu_5^4\mu_6 - 24\omega_{2y}A_0\mu_5\mu_7 \\
& - 480\mu_1\mu_5^5 + 5\mu_2\mu_5^4\mu_6, \qquad \text{(A.8)}
\end{aligned}
$$

$$\mu_9 = -\mu_{4y}\omega_{2y}\mu_5 + \mu_{5y}\omega_{2y}\mu_4 - \omega_{2yy}\mu_4\mu_5, \qquad \text{(A.9)}$$

$$\mu_{10} = -\mu_{4y}\omega_{2y}\mu_6 + \mu_{6y}\omega_{2y}\mu_4 - 2\omega_{2yy}\mu_4\mu_6, \qquad \text{(A.10)}$$

$$
\begin{aligned}
\mu_{11} =& -3888000A_{0x}^2\omega_{2xy}\omega_{2yy}\omega_{2y}^4\mu_{10}\mu_4\mu_5\omega_2 + 3888000A_{0x}^2\omega_{2xy}\omega_{2yy}\omega_{2y}^4\mu_4\mu_6\mu_9\omega_2 \\
& - 58752000A_{0x}^2\omega_{2xy}\omega_{2y}^6\mu_{10}\mu_4\mu_5 + 58752000A_{0x}^2\omega_{2xy}\omega_{2y}^6\mu_4\mu_6\mu_9 \\
& + 8640000A_{0x}^2\omega_{2x}\omega_{2yy}\omega_{2y}^5\mu_{10}\mu_4\mu_5 - 8640000A_{0x}^2\omega_{2x}\omega_{2yy}\omega_{2y}^5\mu_4\mu_6\mu_9 \\
& - 9014400A_{0x}^2\omega_{2yy}\omega_{2y}^5A_0\mu_{10}\mu_4\mu_5\omega_2 + 9014400A_{0x}^2\omega_{2yy}\omega_{2y}^5A_0\mu_4\mu_6\mu_9\omega_2 \\
& + 7603200A_{0x}^2\omega_{2y}^7A_0\mu_{10}\mu_4\mu_5 - 7603200A_{0x}^2\omega_{2y}^7A_0\mu_4\mu_6\mu_9 \\
& + 2803200A_{0x}^2\omega_{2y}^6C_0\mu_{10}\mu_4\mu_5\omega_2 - 2803200A_{0x}^2\omega_{2y}^6C_0\mu_4\mu_6\mu_9\omega_2 \\
& + 46980000A_{0x}\omega_{2xy}^3\omega_{2yy}\omega_{2y}^2\mu_{10}\mu_4\mu_5\omega_2 - 46980000A_{0x}\omega_{2xy}^3\omega_{2yy}\omega_{2y}^2\mu_4\mu_6\mu_9\omega_2 \\
& + 2160000A_{0x}\omega_{2xy}^3\omega_{2y}^4\mu_{10}\mu_4\mu_5 - 2160000A_{0x}\omega_{2xy}^3\omega_{2y}^4\mu_4\mu_6\mu_9 \\
& - 59400000A_{0x}\omega_{2xy}^2\omega_{2x}\omega_{2yy}\omega_{2y}^3\mu_{10}\mu_4\mu_5 + 59400000A_{0x}\omega_{2xy}^2\omega_{2x}\omega_{2yy}\omega_{2y}^3\mu_4\mu_6\mu_9 \\
& + 70236000A_{0x}\omega_{2xy}^2\omega_{2yy}\omega_{2y}^3A_0\mu_{10}\mu_4\mu_5\omega_2 - 70236000A_{0x}\omega_{2xy}^2\omega_{2yy}\omega_{2y}^3A_0\mu_4\mu_6\mu_9\omega_2 \\
& + 6480000A_{0x}\omega_{2xy}^2\omega_{2y}^5A_0\mu_{10}\mu_4\mu_5 - 6480000A_{0x}\omega_{2xy}^2\omega_{2y}^5A_0\mu_4\mu_6\mu_9 \\
& - 19272000A_{0x}\omega_{2xy}^2\omega_{2y}^4C_0\mu_{10}\mu_4\mu_5\omega_2 + 19272000A_{0x}\omega_{2xy}^2\omega_{2y}^4C_0\mu_4\mu_6\mu_9\omega_2 \\
& - 77760000A_{0x}\omega_{2xy}\omega_{2xxy}\omega_{2yy}\omega_{2y}^3\mu_{10}\mu_4\mu_5\omega_2 \\
& + 77760000A_{0x}\omega_{2xy}\omega_{2xxy}\omega_{2yy}\omega_{2y}^3\mu_4\mu_6\mu_9\omega_2 \\
& + 86400000A_{0x}\omega_{2xy}\omega_{2xxy}\omega_{2y}^5\mu_{10}\mu_4\mu_5 - 86400000A_{0x}\omega_{2xy}\omega_{2xxy}\omega_{2y}^5\mu_4\mu_6\mu_9
\end{aligned}
$$

$$- 10800000A_{0x}\omega_{2xy}\omega_{2x}\omega_{2yy}\omega_{2y}^4A_0\mu_{10}\mu_4\mu_5 + 10800000A_{0x}\omega_{2xy}\omega_{2x}\omega_{2yy}\omega_{2y}^4A_0\mu_4\mu_6\mu_9$$
$$+ 8028000A_{0x}\omega_{2xy}\omega_{2yy}\omega_{2y}^4A_0^2\mu_{10}\mu_4\mu_5\omega_2 - 8028000A_{0x}\omega_{2xy}\omega_{2yy}\omega_{2y}^4A_0^2\mu_4\mu_6\mu_9\omega_2$$
$$+ 6480000A_{0x}\omega_{2xy}\omega_{2yy}\omega_{2y}^4B_0\mu_{10}\mu_4\mu_5\omega_2 - 6480000A_{0x}\omega_{2xy}\omega_{2yy}\omega_{2y}^4B_0\mu_4\mu_6\mu_9\omega_2$$
$$- 31881600A_{0x}\omega_{2xy}\omega_{2y}^6A_0^2\mu_{10}\mu_4\mu_5 + 31881600A_{0x}\omega_{2xy}\omega_{2y}^6A_0^2\mu_4\mu_6\mu_9$$
$$+ 70848000A_{0x}\omega_{2xy}\omega_{2y}^6B_0\mu_{10}\mu_4\mu_5 - 70848000A_{0x}\omega_{2xy}\omega_{2y}^6B_0\mu_4\mu_6\mu_9$$
$$- 3504000A_{0x}\omega_{2xy}\omega_{2y}^5A_0C_0\mu_{10}\mu_4\mu_5\omega_2 + 3504000A_{0x}\omega_{2xy}\omega_{2y}^5A_0C_0\mu_4\mu_6\mu_9\omega_2$$
$$+ 345600A_{0x}\omega_{2xy}\omega_{2y}^2\mu_{10}\mu_4^3\omega_2 + 17280000A_{0x}\omega_{2xxxy}\omega_{2yy}\omega_{2y}^4\mu_{10}\mu_4\mu_5\omega_2$$
$$- 17280000A_{0x}\omega_{2xxxy}\omega_{2yy}\omega_{2y}^4\mu_4\mu_6\mu_9\omega_2 + 86400000A_{0x}\omega_{2xxy}\omega_{2x}\omega_{2yy}\omega_{2y}^4\mu_{10}\mu_4\mu_5$$
$$- 86400000A_{0x}\omega_{2xxy}\omega_{2x}\omega_{2yy}\omega_{2y}^4\mu_4\mu_6\mu_9 - 92736000A_{0x}\omega_{2xxy}\omega_{2yy}\omega_{2y}^4A_0\mu_{10}\mu_4\mu_5\omega_2$$
$$+ 92736000A_{0x}\omega_{2xxy}\omega_{2yy}\omega_{2y}^4A_0\mu_4\mu_6\mu_9\omega_2 - 3456000A_{0x}\omega_{2xxy}\omega_{2y}^6A_0\mu_{10}\mu_4\mu_5$$
$$+ 3456000A_{0x}\omega_{2xxy}\omega_{2y}^6A_0\mu_4\mu_6\mu_9 + 28032000A_{0x}\omega_{2xxy}\omega_{2y}^5C_0\mu_{10}\mu_4\mu_5\omega_2$$
$$- 28032000A_{0x}\omega_{2xxy}\omega_{2y}^5C_0\mu_4\mu_6\mu_9\omega_2 + 7992000A_{0x}\omega_{2x}\omega_{2yy}\omega_{2y}^5A_0^2\mu_{10}\mu_4\mu_5$$
$$- 7992000A_{0x}\omega_{2x}\omega_{2yy}\omega_{2y}^5A_0^2\mu_4\mu_6\mu_9 - 17280000A_{0x}\omega_{2x}\omega_{2yy}\omega_{2y}^5B_0\mu_{10}\mu_4\mu_5$$
$$+ 17280000A_{0x}\omega_{2x}\omega_{2yy}\omega_{2y}^5B_0\mu_4\mu_6\mu_9 - 8707680A_{0x}\omega_{2yy}\omega_{2y}^5A_0^3\mu_{10}\mu_4\mu_5\omega_2$$
$$+ 8707680A_{0x}\omega_{2yy}\omega_{2y}^5A_0^3\mu_4\mu_6\mu_9\omega_2 + 19324800A_{0x}\omega_{2yy}\omega_{2y}^5A_0B_0\mu_{10}\mu_4\mu_5\omega_2$$
$$- 19324800A_{0x}\omega_{2yy}\omega_{2y}^5A_0B_0\mu_4\mu_6\mu_9\omega_2 - 1728000A_{0x}\omega_{2yy}\omega_{2y}^5D_0\mu_{10}\mu_4\mu_5\omega_2$$
$$+ 1728000A_{0x}\omega_{2yy}\omega_{2y}^5D_0\mu_4\mu_6\mu_9\omega_2 - 25920A_{0x}\omega_{2yy}\omega_{2y}^2\mu_{10}\mu_3\mu_4\mu_5\omega_2$$
$$+ 25920A_{0x}\omega_{2yy}\omega_{2y}^2\mu_3\mu_4\mu_6\mu_9\omega_2 + 4440960A_{0x}\omega_{2y}^7A_0^3\mu_{10}\mu_4\mu_5$$
$$- 4440960A_{0x}\omega_{2y}^7A_0^3\mu_4\mu_6\mu_9 - 11404800A_{0x}\omega_{2y}^7A_0B_0\mu_{10}\mu_4\mu_5$$
$$+ 11404800A_{0x}\omega_{2y}^7A_0B_0\mu_4\mu_6\mu_9 + 6912000A_{0x}\omega_{2y}^7D_0\mu_{10}\mu_4\mu_5$$
$$- 6912000A_{0x}\omega_{2y}^7D_0\mu_4\mu_6\mu_9 + 2592960A_{0x}\omega_{2y}^6A_0^2C_0\mu_{10}\mu_4\mu_5\omega_2$$
$$- 2592960A_{0x}\omega_{2y}^6A_0^2C_0\mu_4\mu_6\mu_9\omega_2 - 5606400A_{0x}\omega_{2y}^6B_0C_0\mu_{10}\mu_4\mu_5\omega_2$$
$$+ 5606400A_{0x}\omega_{2y}^6B_0C_0\mu_4\mu_6\mu_9\omega_2 + 138240A_{0x}\omega_{2y}^4\mu_{10}\mu_3\mu_4\mu_5$$
$$- 138240A_{0x}\omega_{2y}^4\mu_3\mu_4\mu_6\mu_9 - 69120A_{0x}\omega_{2y}^3A_0\mu_{10}\mu_4^3\omega_2$$
$$- 13440A_{0x}\omega_{2y}^2\mu_{10}\mu_4^2\mu_5\omega_2 + 13440A_{0x}\omega_{2y}^2\mu_4^2\mu_6\mu_9\omega_2$$
$$+ 86400000E_{0y}\omega_{2y}^7\mu_{10}\mu_4\mu_5 - 86400000E_{0y}\omega_{2y}^7\mu_4\mu_6\mu_9$$
$$- 17280000E_{1x}\omega_{2y}^7\mu_{10}\mu_4\mu_5 + 17280000E_{1x}\omega_{2y}^7\mu_4\mu_6\mu_9$$
$$+ 1800\mu_{2x}\omega_{2yy}\omega_{2y}\mu_{10}\mu_4\mu_5\omega_2 - 1800\mu_{2x}\omega_{2yy}\omega_{2y}\mu_4\mu_6\mu_9\omega_2$$
$$- 1440\mu_{2x}\omega_{2y}^3\mu_{10}\mu_4\mu_5 + 1440\mu_{2x}\omega_{2y}^3\mu_4\mu_6\mu_9$$
$$- 43200\mu_{3xx}\omega_{2yy}\omega_{2y}^2\mu_{10}\mu_4\mu_5\omega_2 + 43200\mu_{3xx}\omega_{2yy}\omega_{2y}^2\mu_4\mu_6\mu_9\omega_2$$
$$+ 518400\mu_{3x}\omega_{2xy}\omega_{2yy}\omega_{2y}\mu_{10}\mu_4\mu_5\omega_2 - 518400\mu_{3x}\omega_{2xy}\omega_{2yy}\omega_{2y}\mu_4\mu_6\mu_9\omega_2$$
$$- 172800\mu_{3x}\omega_{2xy}\omega_{2y}^3\mu_{10}\mu_4\mu_5 + 172800\mu_{3x}\omega_{2xy}\omega_{2y}^3\mu_4\mu_6\mu_9$$
$$- 216000\mu_{3x}\omega_{2x}\omega_{2yy}\omega_{2y}^2\mu_{10}\mu_4\mu_5 + 216000\mu_{3x}\omega_{2x}\omega_{2yy}\omega_{2y}^2\mu_4\mu_6\mu_9$$
$$+ 192960\mu_{3x}\omega_{2yy}\omega_{2y}^2A_0\mu_{10}\mu_4\mu_5\omega_2 - 192960\mu_{3x}\omega_{2yy}\omega_{2y}^2A_0\mu_4\mu_6\mu_9\omega_2$$
$$- 70080\mu_{3x}\omega_{2y}^3C_0\mu_{10}\mu_4\mu_5\omega_2 + 70080\mu_{3x}\omega_{2y}^3C_0\mu_4\mu_6\mu_9\omega_2$$
$$+ 28800\mu_{4x}^2\omega_{2y}^2\mu_{10}\mu_5\omega_2 - 28800\mu_{4x}^2\omega_{2y}^2\mu_6\mu_9\omega_2$$

$$
\begin{aligned}
&- 28800\mu_{4x}\mu_{5x}\omega_{2y}^2\mu_{10}\mu_4\omega_2 + 28800\mu_{4x}\mu_{6x}\omega_{2y}^2\mu_4\mu_9\omega_2 \\
&- 43200\mu_{4x}\omega_{2xy}\omega_{2y}\mu_{10}\mu_4\mu_5\omega_2 + 14400\mu_{4x}\omega_{2xy}\omega_{2y}\mu_4\mu_6\mu_9\omega_2 \\
&+ 72000\mu_{4x}\omega_{2x}\omega_{2y}^2\mu_{10}\mu_4\mu_5 - 72000\mu_{4x}\omega_{2x}\omega_{2y}^2\mu_4\mu_6\mu_9 \\
&- 2880\mu_{4x}\omega_{2y}^2A_0\mu_{10}\mu_4\mu_5\omega_2 + 2880\mu_{4x}\omega_{2y}^2A_0\mu_4\mu_6\mu_9\omega_2 \\
&- 360\mu_{4x}\omega_{2y}\mu_{10}\mu_5^2\omega_2 + 360\mu_{4x}\omega_{2y}\mu_5\mu_6\mu_9\omega_2 + 14400\mu_{5xx}\omega_{2y}^2\mu_{10}\mu_4^2\omega_2 \\
&- 72000\mu_{5x}\omega_{2xy}\omega_{2y}\mu_{10}\mu_4^2\omega_2 + 8640\mu_{5x}\omega_{2y}^2A_0\mu_{10}\mu_4^2\omega_2 - 240\mu_{5x}\omega_{2y}\mu_{10}\mu_4\mu_5\omega_2 \\
&+ 360\mu_{5x}\omega_{2y}\mu_4\mu_6\mu_9\omega_2 - 14400\mu_{6xx}\omega_{2y}^2\mu_4^2\mu_9\omega_2 + 100800\mu_{6x}\omega_{2xy}\omega_{2y}\mu_4^2\mu_9\omega_2 \\
&- 8640\mu_{6x}\omega_{2y}^2A_0\mu_4^2\mu_9\omega_2 - 120\mu_{6x}\omega_{2y}\mu_4\mu_5\mu_9\omega_2 + 429300000\omega_{2xy}^5\omega_{2yy}\mu_{10}\mu_4\mu_5\omega_2 \\
&- 429300000\omega_{2xy}^5\omega_{2yy}\mu_4\mu_6\mu_9\omega_2 - 64800000\omega_{2xy}^5\omega_{2y}^2\mu_{10}\mu_4\mu_5 \\
&+ 64800000\omega_{2xy}^5\omega_{2y}^2\mu_4\mu_6\mu_9 - 273375000\omega_{2xy}^4\omega_{2x}\omega_{2yy}\omega_{2y}\mu_{10}\mu_4\mu_5 \\
&+ 273375000\omega_{2xy}^4\omega_{2x}\omega_{2yy}\omega_{2y}\mu_4\mu_6\mu_9 + 97605000\omega_{2xy}^4\omega_{2yy}\omega_{2y}A_0\mu_{10}\mu_4\mu_5\omega_2 \\
&- 97605000\omega_{2xy}^4\omega_{2yy}\omega_{2y}A_0\mu_4\mu_6\mu_9\omega_2 - 1620000\omega_{2xy}^4\omega_{2y}^3A_0\mu_{10}\mu_4\mu_5 \\
&+ 1620000\omega_{2xy}^4\omega_{2y}^3A_0\mu_4\mu_6\mu_9 - 88695000\omega_{2xy}^4\omega_{2y}^2C_0\mu_{10}\mu_4\mu_5\omega_2 \\
&+ 88695000\omega_{2xy}^4\omega_{2y}^2C_0\mu_4\mu_6\mu_9\omega_2 - 542700000\omega_{2xy}^3\omega_{2xxy}\omega_{2yy}\omega_{2y}\mu_{10}\mu_4\mu_5\omega_2 \\
&+ 542700000\omega_{2xy}^3\omega_{2xxy}\omega_{2yy}\omega_{2y}\mu_4\mu_6\mu_9\omega_2 + 226800000\omega_{2xy}^3\omega_{2xxy}\omega_{2y}^3\mu_{10}\mu_4\mu_5 \\
&- 226800000\omega_{2xy}^3\omega_{2xxy}\omega_{2y}^3\mu_4\mu_6\mu_9 + 121500000\omega_{2xy}^3\omega_{2x}\omega_{2yy}\omega_{2y}^2A_0\mu_{10}\mu_4\mu_5 \\
&- 121500000\omega_{2xy}^3\omega_{2x}\omega_{2yy}\omega_{2y}^2A_0\mu_4\mu_6\mu_9 - 103032000\omega_{2xy}^3\omega_{2yy}\omega_{2y}^2A_0^2\mu_{10}\mu_4\mu_5\omega_2 \\
&+ 103032000\omega_{2xy}^3\omega_{2yy}\omega_{2y}^2A_0^2\mu_4\mu_6\mu_9\omega_2 + 16200000\omega_{2xy}^3\omega_{2yy}\omega_{2y}^2B_0\mu_{10}\mu_4\mu_5\omega_2 \\
&- 16200000\omega_{2xy}^3\omega_{2yy}\omega_{2y}^2B_0\mu_4\mu_6\mu_9\omega_2 + 12096000\omega_{2xy}^3\omega_{2y}^4A_0^2\mu_{10}\mu_4\mu_5 \\
&- 12096000\omega_{2xy}^3\omega_{2y}^4A_0^2\mu_4\mu_6\mu_9 - 4320000\omega_{2xy}^3\omega_{2y}^4B_0\mu_{10}\mu_4\mu_5 \\
&+ 4320000\omega_{2xy}^3\omega_{2y}^4B_0\mu_4\mu_6\mu_9 + 39420000\omega_{2xy}^3\omega_{2y}^3A_0C_0\mu_{10}\mu_4\mu_5\omega_2 \\
&- 39420000\omega_{2xy}^3\omega_{2y}^3A_0C_0\mu_4\mu_6\mu_9\omega_2 - 9720000\omega_{2xy}^3\mu_{10}\mu_4^3\omega_2 \\
&+ 48600000\omega_{2xy}^2\omega_{2xxxy}\omega_{2yy}\omega_{2y}^2\mu_{10}\mu_4\mu_5\omega_2 - 48600000\omega_{2xy}^2\omega_{2xxxy}\omega_{2yy}\omega_{2y}^2\mu_4\mu_6\mu_9\omega_2 \\
&+ 243000000\omega_{2xy}^2\omega_{2xxy}\omega_{2x}\omega_{2yy}\omega_{2y}^2\mu_{10}\mu_4\mu_5 - 243000000\omega_{2xy}^2\omega_{2xxy}\omega_{2x}\omega_{2yy}\omega_{2y}^2\mu_4\mu_6\mu_9 \\
&- 56700000\omega_{2xy}^2\omega_{2xxy}\omega_{2yy}\omega_{2y}^2A_0\mu_{10}\mu_4\mu_5\omega_2 + 56700000\omega_{2xy}^2\omega_{2xxy}\omega_{2yy}\omega_{2y}^2A_0\mu_4\mu_6\mu_9\omega_2 \\
&- 97200000\omega_{2xy}^2\omega_{2xxy}\omega_{2y}^4A_0\mu_{10}\mu_4\mu_5 + 97200000\omega_{2xy}^2\omega_{2xxy}\omega_{2y}^4A_0\mu_4\mu_6\mu_9 \\
&+ 78840000\omega_{2xy}^2\omega_{2xxy}\omega_{2y}^3C_0\mu_{10}\mu_4\mu_5\omega_2 - 78840000\omega_{2xy}^2\omega_{2xxy}\omega_{2y}^3C_0\mu_4\mu_6\mu_9\omega_2 \\
&- 19170000\omega_{2xy}^2\omega_{2x}\omega_{2yy}\omega_{2y}^3A_0^2\mu_{10}\mu_4\mu_5 + 19170000\omega_{2xy}^2\omega_{2x}\omega_{2yy}\omega_{2y}^3A_0^2\mu_4\mu_6\mu_9 \\
&+ 5400000\omega_{2xy}^2\omega_{2x}\omega_{2yy}\omega_{2y}^3B_0\mu_{10}\mu_4\mu_5 - 5400000\omega_{2xy}^2\omega_{2x}\omega_{2yy}\omega_{2y}^3B_0\mu_4\mu_6\mu_9 \\
&+ 23734800\omega_{2xy}^2\omega_{2yy}\omega_{2y}^3A_0^3\mu_{10}\mu_4\mu_5\omega_2 - 23734800\omega_{2xy}^2\omega_{2yy}\omega_{2y}^3A_0^3\mu_4\mu_6\mu_9\omega_2 \\
&- 20700000\omega_{2xy}^2\omega_{2yy}\omega_{2y}^3A_0B_0\mu_{10}\mu_4\mu_5\omega_2 + 20700000\omega_{2xy}^2\omega_{2yy}\omega_{2y}^3A_0B_0\mu_4\mu_6\mu_9\omega_2 \\
&- 4860000\omega_{2xy}^2\omega_{2yy}\omega_{2y}^3D_0\mu_{10}\mu_4\mu_5\omega_2 + 4860000\omega_{2xy}^2\omega_{2yy}\omega_{2y}^3D_0\mu_4\mu_6\mu_9\omega_2 \\
&- 1387800\omega_{2xy}^2\omega_{2yy}\mu_{10}\mu_3\mu_4\mu_5\omega_2 + 1387800\omega_{2xy}^2\omega_{2yy}\mu_3\mu_4\mu_6\mu_9\omega_2 \\
&- 129600\omega_{2xy}^2\omega_{2y}^5A_0^3\mu_{10}\mu_4\mu_5 + 129600\omega_{2xy}^2\omega_{2y}^5A_0^3\mu_4\mu_6\mu_9
\end{aligned}
$$

$$- 103680\omega_{2y}^4 B_0\mu_{10}\mu_3\mu_4\mu_5 + 103680\omega_{2y}^4 B_0\mu_3\mu_4\mu_6\mu_9 - 19008\omega_{2y}^3 A_0^3\mu_{10}\mu_4^3\omega_2$$
$$+ 34560\omega_{2y}^3 A_0 B_0\mu_{10}\mu_4^3\omega_2 - 14016\omega_{2y}^3 A_0 C_0\mu_{10}\mu_3\mu_4\mu_5\omega_2$$
$$+ 14016\omega_{2y}^3 A_0 C_0\mu_3\mu_4\mu_6\mu_9\omega_2 - 288\omega_{2y}^3 A_0\mu_{10}\mu_2\mu_4\mu_5 + 288\omega_{2y}^3 A_0\mu_2\mu_4\mu_6\mu_9$$
$$+ 172800\omega_{2y}^3 D_0\mu_{10}\mu_4^3\omega_2 - 2688\omega_{2y}^2 A_0^2\mu_{10}\mu_4^2\mu_5\omega_2 + 2688\omega_{2y}^2 A_0^2\mu_4^2\mu_6\mu_9\omega_2$$
$$+ 12480\omega_{2y}^2 B_0\mu_{10}\mu_4^2\mu_5\omega_2 - 12480\omega_{2y}^2 B_0\mu_4^2\mu_6\mu_9\omega_2 + 2920\omega_{2y}^2 C_0\mu_{10}\mu_2\mu_4\mu_5\omega_2$$
$$- 2920\omega_{2y}^2 C_0\mu_2\mu_4\mu_6\mu_9\omega_2 - 48\omega_{2y} A_0\mu_{10}\mu_4\mu_5^2\omega_2 + 48\omega_{2y} A_0\mu_4\mu_5\mu_6\mu_9\omega_2$$
$$- 1728\mu_{10}\mu_3\mu_4^3\omega_2 - 8\mu_{10}\mu_4\mu_5\mu_6 + \mu_{10}\mu_5^3\omega_2 + 8\mu_4\mu_6^2\mu_9 - \mu_5^2\mu_6\mu_9\omega_2, \quad \text{(A.11)}$$

$$\mu_{12} = 691200 A_{0x}^2\omega_{2y}^4\mu_{10}\mu_4^4\mu_9\omega_2^2 - 4752000 A_{0x}\omega_{2xy}^2\omega_{2y}^2\mu_{10}\mu_4^4\mu_9\omega_2^2$$
$$- 864000 A_{0x}\omega_{2xy}\omega_{2y}^3 A_0\mu_{10}\mu_4^4\mu_9\omega_2^2 + 6912000 A_{0x}\omega_{2xxy}\omega_{2y}^3\mu_{10}\mu_4^4\mu_9\omega_2^2$$
$$+ 639360 A_{0x}\omega_{2y}^4 A_0^2\mu_{10}\mu_4^4\mu_9\omega_2^2 - 1382400 A_{0x}\omega_{2y}^4 B_0\mu_{10}\mu_4^4\mu_9\omega_2^2$$
$$- 5\mu_{11x}\omega_{2y}\mu_4\mu_9\omega_2 - 17280\mu_{3x}\omega_{2y}\mu_{10}\mu_4^4\mu_9\omega_2^2 + 15\mu_{4x}\omega_{2y}\mu_1 1\mu_9\omega_2$$
$$+ 5\mu_{9x}\omega_{2y}\mu_1 1\mu_4\omega_2 - 21870000\omega_{2xy}^4\mu_{10}\mu_4^4\mu_9\omega_2^2 + 9720000\omega_{2xy}^3\omega_{2y} A_0\mu_{10}\mu_4^4\mu_9\omega_2^2$$
$$+ 19440000\omega_{2xy}^2\omega_{2xxy}\omega_{2y}\mu_{10}\mu_4^4\mu_9\omega_2^2 - 1533600\omega_{2xy}^2\omega_{2y}^2 A_0^2\mu_{10}\mu_4^4\mu_9\omega_2^2$$
$$+ 432000\omega_{2xy}^2\omega_{2y}^2 B_0\mu_{10}\mu_4^4\mu_9\omega_2^2 - 7776000\omega_{2xy}\omega_{2xxy}\omega_{2y}^2 A_0\mu_{10}\mu_4^4\mu_9\omega_2^2$$
$$- 406080\omega_{2xy}\omega_{2y}^3 A_0^3\mu_{10}\mu_4^4\mu_9\omega_2^2 + 1209600\omega_{2xy}\omega_{2y}^3 A_0 B_0\mu_{10}\mu_4^4\mu_9\omega_2^2$$
$$+ 69120\omega_{2xy}\mu_{10}\mu_3\mu_4^4\mu_9\omega_2^2 + 30\omega_{2xy}\mu_1 1\mu_4\mu_9\omega_2 + 2160000\omega_{2xxy}\omega_{2y}^3 A_0^2\mu_{10}\mu_4^4\mu_9\omega_2^2$$
$$- 3456000\omega_{2xxy}\omega_{2y}^3 B_0\mu_{10}\mu_4^4\mu_9\omega_2^2 + 5\omega_{2x}\omega_{2y}\mu_1 1\mu_4\mu_9 + 127440\omega_{2y}^4 A_0^4\mu_{10}\mu_4^4\mu_9\omega_2^2$$
$$- 535680\omega_{2y}^4 A_0^2 B_0\mu_{10}\mu_4^4\mu_9\omega_2^2 + 518400\omega_{2y}^4 B_0^2\mu_{10}\mu_4^4\mu_9\omega_2^2 - 3456\omega_{2y} A_0\mu_{10}\mu_3\mu_4^4\mu_9\omega_2^2$$
$$- 2\omega_{2y} A_0\mu_1 1\mu_4\mu_9\omega_2 - 69120\mu_1\mu_4^4\mu_9^2\omega_2^2 + 720\mu_{10}\mu_2\mu_4^4\mu_9\omega_2^2. \quad \text{(A.12)}$$

$$\lambda_1 = 5L_{6x} - 5L_{6y}r + F_1 L_6 - 2F_2 L_6 r + 5M_7 - 40M_8 r + 180M_9 r^2, \quad \text{(A.13)}$$

$$\lambda_2 = -700F_{2y}^2 F_2 + 1750F_{2y}H_{4y} - 280F_{2y}F_2^3 + 1050F_{2y}F_2H_4 - 2750F_{2y}L_6 + 350H_{4y}F_2^2 - 875H_{4y}H_4$$
$$- 1250L_{6yy} - 500L_{6y}F_2 - 6250M_{8y} + 18750M_{9x} + 37500M_{9y}r + 56250r_y M_9 + 3750F_1 M_9 - 28F_2^5$$
$$+ 210F_2^3 H_4 - 550F_2^2 L_6 - 350F_2 H_4^2 - 1250F_2 M_8 + 3750F_2 M_9 r + 1250H_4 L_6, \quad \text{(A.14)}$$

$$\lambda_3 = -15H_{3x} + 15H_{3y}r - 8F_0 H_4 - 3F_1 H_3 + 8F_1 H_4 r + 6F_2 H_3 r - 8F_2 H_4 r^2 + 10L_4 - 30L_5 r + 30L_6 r^2, \quad \text{(A.15)}$$

$$\lambda_4 = 60\lambda_{3y}r^2 - 450L_{3x} + 450L_{3y}r + 1200L_{4x}r - 1200L_{4y}r^2 - 1950L_{5x}r^2 + 1950L_{5y}r^3 + 120r_y\lambda_3 r$$
$$- 21F_0\lambda_3 - 270F_0 L_4 + 990F_0 L_5 r - 2160F_0 L_6 r^2 + 35F_1\lambda_3 r - 90F_1 L_3 + 510F_1 L_4 r - 1380F_1 L_5 r^2$$
$$+ 2160F_1 L_6 r^3 - 25F_2\lambda_3 r^2 + 180F_2 L_3 r - 750F_2 L_4 r^2 + 1770F_2 L_5 r^3 - 2160F_2 L_6 r^4 - 4500M_4$$
$$+ 18900M_5 r - 47250M_6 r^2 + 88650M_7 r^3 - 129600M_8 r^4 + 129600M_9 r^5, \quad \text{(A.16)}$$

$$\lambda_5 = -15H_{3x} + 15H_{3y}r - 8F_0 H_4 - 3F_1 H_3 + 8F_1 H_4 r + 6F_2 H_3 r - 8F_2 H_4 r^2 + 10L_4 - 30L_5 r + 30L_6 r^2, \quad \text{(A.17)}$$

$$\lambda_6 = 60\lambda_{5y}r^2 - 450L_{3x} + 450L_{3y}r + 1200L_{4x}r - 1200L_{4y}r^2 - 1950L_{5x}r^2 + 1950L_{5y}r^3 + 120r_y\lambda_5 r$$
$$- 21F_0\lambda_5 - 270F_0 L_4 + 990F_0 L_5 r - 2160F_0 L_6 r^2 + 35F_1\lambda_5 r - 90F_1 L_3 + 510F_1 L_4 r - 1380F_1 L_5 r^2$$
$$+ 2160F_1 L_6 r^3 - 25F_2\lambda_5 r^2 + 180F_2 L_3 r - 750F_2 L_4 r^2 + 1770F_2 L_5 r^3 - 2160F_2 L_6 r^4 + 540\lambda_1 r^3$$
$$- 4500M_4 + 18900M_5 r - 47250M_6 r^2 + 88650M_7 r^3 - 129600M_8 r^4 + 129600M_9 r^5, \quad \text{(A.18)}$$

$$\lambda_7 = 100F_{1y}\lambda_1^2 - 200F_{2y}\lambda_1^2 r + 300\lambda_{1xy}\lambda_1 - 300\lambda_{1x}\lambda_{1y} - 300\lambda_{1yy}\lambda_1 r + 300\lambda_{1y}^2 r - 300\lambda_{1y}r_y\lambda_1 + 140r_y F_2\lambda_1^2 + 68F_1F_2\lambda_1^2 - 136F_2^2\lambda_1^2 r - 85H_3\lambda_1^2 + 340H_4\lambda_1^2 r, \quad (A.19)$$

$$\lambda_8 = 22500F_{2y}\lambda_1^2\lambda_5^2 r^4 - 30\lambda_{1x}\lambda_2\lambda_5^2 r^4 - 1500\lambda_{1y}F_2\lambda_1\lambda_5^2 r^4 + 1500\lambda_{1y}\lambda_1\lambda_5\lambda_6 r^2 + 30\lambda_{1y}\lambda_2\lambda_5^2 r^5 + 30\lambda_{2x}\lambda_1\lambda_5^2 r^4 - 30\lambda_{2y}\lambda_1\lambda_5^2 r^5 + 90r_y\lambda_1\lambda_2\lambda_5^2 r^4 + 6F_1\lambda_1\lambda_2\lambda_5^2 r^4 + 8200F_2^2\lambda_1^2\lambda_5^2 r^4 - 200F_2\lambda_1^2\lambda_5\lambda_6 r^2 - 12F_2\lambda_1\lambda_2\lambda_5^2 r^5 - 22500H_4\lambda_1^2\lambda_5^2 r^4 - 125\lambda_1^2\lambda_6^2. \quad (A.20)$$

Bibliography

[1] S. Lie, (1883). *Klassifikation und Integration von gewöhnlichen Differentialgleichungen zwischen x, y, die eine Gruppe von Transformationen gestatten. III* Archiv for Matematik og Naturvidenskab, 8(4), 371-427. Reprinted in Lie's Gessammelte Abhandlundgen, 1924, 5, paper XIY, pp. 362-427.

[2] R. Liouville, (1889). *Sur les Invariantes de Certaines Equations* J. de lÉcole Polytechnique 59, 7-88.

[3] A. M. Tresse, (1896). *Détermination des Invariants Ponctuels de lÉquation Différentielle Ordinaire du Second Ordre $y'' = f(x, y, y')$.* Preisschriften der fürstlichen Jablonowskischen Geselschaft XXXII, Leipzig, S.Herzel.

[4] E. Cartan, (1924). *Sur les varietes a connexion projective.* Bull. Soc. Math. France 52, 205-241.

[5] Bocharov, A. V.,Sokolov, V. V. and Svinolupov, S. I. (1993). *On some equivalence problems for differential equations.* Preprint ESI. 54.

[6] Grebot, G. (1997). *The characterization of third order ordinary differential equations admitting a transtive fiber-preserving point symmetry group.* Journal of Mathematical Analysis and Applications. 206, 364-388.

[7] Ibragimov, N. H. and Meleshko, S. V. (2005). *Linearization of third-order ordinary differential equations by point and contact transformations.* Journal of Mathematical Analysis and Applications. 308, 266-289

[8] Ibragimov, N. H. , Meleshko, S. V. and Suksern, S. (2008). *Linearization of fourth-order ordinary differential equations by point transformations.* Journal of Physics A: Mathematical and Theoretical. 41,

[9] Abdul-Majid Wazwaz. (2013). *Kink solutions for three new fifth order nonlinear equations.* Journal of Applied Mathematical Modeling.

[10] Abdus, S. M. and Tania, A. (2012). *Some exact solutions of two fifth order KdV-type nonlinear partial differential equations.* Journal of Partial Differential Equation. Vol. 25, No. 4, pp. 357-367.

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1. ได้ผลงานตีพิมพ์ในวารสารวิชาการนานาชาติ

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ภาคผนวก

ประกอบด้วย

1. ผลงานตีพิมพ์ในวารสารวิชาการนานาชาติ เรื่อง

    On the Fiber Preserving Transformations for the Fifth-Order Ordinary Differential Equations

2. ตัวอย่างการใช้ได้โปรแกรมทดสอบความเป็นเชิงเส้นของสมการเชิงอนุพันธ์สามัญอันดับห้าโดยการแปลงแบบจุด

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Research Article

On the Fiber Preserving Transformations for the Fifth-Order Ordinary Differential Equations

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This paper is devoted to the study of the linearization problem of fifth-order ordinary differential equations by means of fiber preserving transformations. The necessary and sufficient conditions for linearization are obtained. The procedure for obtaining the linearizing transformations is provided in explicit form. Examples demonstrating the procedure of using the linearization theorems are presented.

1. Introduction

1.1. The Research Problem and Its Significance. In mathematics, a nonlinear equation is an equation which is not linear; that is, an equation which does not satisfy the superposition principle, or whose output is not directly proportional to its input. Less technically, a nonlinear equation is any problem where the variables to be solved for cannot be written as a linear combination of independent components.

Nonlinear problems are of interest to engineers, physicists, and mathematicians because most physical systems are inherently nonlinear in nature. Nonlinear equations are difficult to solve and give rise to interesting phenomena. While solving problems related to nonlinear ordinary differential equations, it is often expedient to simplify equations by a suitable change of variables. One of the fundamental methods to solve this relies upon the transformation of a given equation to another equation of standard form. The transformation may be to an equation of equal order or of greater or lesser order. In particular, the possibility that a given equation could be linearized, that is, transformed to a linear equation, was a most attractive proposition due to the special properties of linear differential equations. The reduction of an ordinary differential equation to a linear ordinary differential equation besides simplification allows us to construct an exact solution of the original equation.

One type of the classification problem is the equivalence problem. Two equations of differential equations are said to be equivalent if there exists an invertible transformation which transforms any solution of one equation to a solution of the other equation and vice versa. The linearization problem is a particular case of the equivalence problem, where one of the equations is a linear equation. It is one of the essential parts in the study of nonlinear equations.

The main difficulty in solving the linearization problem comes from the large number of complicated calculations. Because of this difficulty, no one attempts to solve this problem for nonlinear equations are higher than fourth. However if we can solve the linearization problem of fifth-order ordinary differential equations, then we should set a new process to solve the problems in Physics or Engineering.

1.2. Historical Review. The linearization, that is, mapping a nonlinear differential equation into a linear differential equation, is an important tool in the theory of differential equations. The problem of linearization of ordinary differential equations attracted attention of mathematicians such as Lie and Cartan. The first linearization problem for ordinary differential equations was solved by Lie [1, 2]. He found the general form of all ordinary differential equations of second-order that can be reduced to a linear equation by changing the independent and dependent variables. He showed that

$$B_0 = -5\left(2\varphi_{xxx}\varphi_x\psi_y - 9\varphi_{xx}^2\psi_y + 8\varphi_{xx}\varphi_x\psi_{xy} - 2\varphi_x^2\psi_{xxy}\right)\left(\varphi_x^2\psi_y\right)^{-1}, \tag{14}$$

$$C_1 = \frac{15\psi_{yyy}}{\psi_y}, \tag{15}$$

$$C_0 = \frac{-15\left(2\varphi_{xx}\psi_{yy} - \varphi_x\psi_{xyy}\right)}{\left(\varphi_x\psi_y\right)}, \tag{16}$$

$$D_3 = \frac{10\psi_{yyyy}}{\psi_y}, \tag{17}$$

$$D_2 = \frac{-30\left(2\varphi_{xx}\psi_{yyy} - \varphi_x\psi_{xyyy}\right)}{\left(\varphi_x\psi_y\right)}, \tag{18}$$

$$D_1 = (15\left(9\varphi_{xx}^2\psi_{yy} - 8\varphi_{xx}\varphi_x\psi_{xyy} + 2\varphi_x^2\psi_{xxyy} - 2\varphi_{xxx}\varphi_x\psi_{yy}\right))\left(\varphi_x^2\psi_y\right)^{-1}, \tag{19}$$

$$\begin{aligned} D_0 = -5\big(&\varphi_{xxxx}\varphi_x^2\psi_y - 12\varphi_{xxx}\varphi_{xx}\varphi_x\psi_y \\ &+ 6\varphi_{xxx}\varphi_x^2\psi_{xy} + 21\varphi_{xx}^3\psi_y - 27\varphi_{xx}^2\varphi_x\psi_{xy} \\ &+12\varphi_{xx}\varphi_x^2\psi_{xxy} - 2\varphi_x^3\psi_{xxxy}\big) \\ &\times\left(\varphi_x^3\psi_y\right)^{-1}, \end{aligned} \tag{20}$$

$$E_5 = \frac{\psi_{yyyyy}}{\psi_y}, \tag{21}$$

$$E_4 = \frac{-5\left(2\varphi_{xx}\psi_{yyyy} - \varphi_x\psi_{xyyyy}\right)}{\left(\varphi_x\psi_y\right)}, \tag{22}$$

$$\begin{aligned} E_3 = \big(-5\big(&\left(2\varphi_{xxx}\varphi_x - 9\varphi_{xx}^2\right)\psi_{yyy} \\ &+2\left(4\varphi_{xx}\psi_{xyyy} - \varphi_x\psi_{xxyyy}\right)\varphi_x\big) \\ &\times\left(\varphi_x^2\psi_y\right)^{-1}, \end{aligned} \tag{23}$$

$$\begin{aligned} E_2 = 5\big(&\left(6\left(2\varphi_{xx}\psi_{yy} - \varphi_x\psi_{xyy}\right)\varphi_{xxx} - \varphi_{xxxx}\varphi_x\psi_{yy}\right)\varphi_x \\ &- \big(21\varphi_{xx}^3\psi_{yy} - 27\varphi_{xx}^2\varphi_x\psi_{xyy} \\ &+12\varphi_{xx}\varphi_x^2\psi_{xxyy} - 2\varphi_x^3\psi_{xxxyy}\big)\big) \\ &\times\left(\varphi_x^3\psi_y\right)^{-1}, \end{aligned} \tag{24}$$

$$\begin{aligned} E_1 = \big(5\big(&21\varphi_{xx}^4\psi_y - 42\varphi_{xx}^3\varphi_x\psi_{xy} + 27\varphi_{xx}^2\varphi_x^2\psi_{xxy} \\ &-8\varphi_{xx}\varphi_x^3\psi_{xxxy} + \varphi_x^4\psi_{xxxxy} + 2\varphi_{xxx}^2\varphi_x^2\psi_y\big) \\ &- \psi_{xxxxx}\varphi_x^3\psi_y + 5\left(3\varphi_{xx}\psi_y - 2\varphi_x\psi_{xy}\right)\varphi_{xxxx}\varphi_x^2 \\ &- 15\left(7\varphi_{xx}^2\psi_y - 8\varphi_{xx}\varphi_x\psi_{xy} + 2\varphi_x^2\psi_{xxy}\right) \\ &\times\varphi_{xxx}\varphi_x\big)\left(\varphi_x^4\psi_y\right)^{-1}, \end{aligned} \tag{25}$$

$$\begin{aligned} E_0 = -\big(&\psi_{xxxxx}\varphi_x^3\psi_x - 15\varphi_{xxxx}\varphi_{xx}\varphi_x^2\psi_x \\ &+ 5\varphi_{xxxx}\varphi_x^3\psi_{xx} - 10\varphi_{xxx}^2\varphi_x^2\psi_x \\ &+ 105\varphi_{xxx}\varphi_{xx}^2\varphi_x\psi_x - 60\varphi_{xxx}\varphi_{xx}\varphi_x^2\psi_{xx} \\ &+ 10\varphi_{xxx}\varphi_x^3\psi_{xxx} \\ &- 105\varphi_{xx}^4\psi_x + 105\varphi_{xx}^3\varphi_x\psi_{xx} \\ &-45\varphi_{xx}^2\varphi_x^2\psi_{xxx} + 10\varphi_{xx}\varphi_x^3\psi_{xxxx} - \varphi_x^4\psi_{xxxxx}\big) \\ &\times\left(\varphi_x^4\psi_y\right)^{-1}. \end{aligned} \tag{26}$$

Proof. Applying a fiber preserving transformation (7), one obtains the following transformation of derivatives:

$$u'(t) = \frac{D_x\psi}{D_x\varphi} = \frac{\psi_x + y'\psi_y}{\varphi_x} = P\left(x, y, y'\right), \tag{27}$$

$$\begin{aligned} u''(t) &= \frac{D_xP}{D_x\varphi} = \frac{P_x + y'P_y + y''P_{y'}}{\varphi_x} \\ &= \frac{1}{\varphi_x^3}\big[\left(\varphi_x\psi_y\right)y'' + \left(\varphi_x\psi_{yy}\right)y'^2 \\ &\quad + \left(-\varphi_{xx}\psi_y + 2\varphi_x\psi_{xy}\right)y' - \varphi_{xx}\psi_x + \varphi_x\psi_{xx}\big] \\ &= Q\left(x, y, y', y''\right), \end{aligned} \tag{28}$$

$$\begin{aligned} u'''(t) &= \frac{D_xQ}{D_x\varphi} = \frac{Q_x + y'Q_y + y''Q_{y'} + y'''Q_{y''}}{\varphi_x} \\ &= \frac{1}{\varphi_x^5}\big[\left(\varphi_x^2\psi_y\right)y''' + \left(3\varphi_x^2\psi_{yy}\right)y'y'' \\ &\quad +3\varphi_x\left(-\varphi_{xx}\psi_y + \varphi_x\psi_{xy}\right)y'' + \cdots\big] \\ &= R\left(x, y, y', y'', y'''\right), \end{aligned} \tag{29}$$

$$\begin{aligned} u^{(4)}(t) &= \frac{D_xR}{D_x\varphi} \\ &= \frac{R_x + y'R_y + y''R_{y'} + y'''R_{y''} + y^{(4)}R_{y'''}}{\varphi_x} \\ &= \frac{1}{\varphi_x^7}\big[\left(\varphi_x^3\psi_y\right)y^{(4)} + \left(4\varphi_x^3\psi_{yy}\right)y'y''' \\ &\quad +2\varphi_x^2\left(-3\varphi_{xx}\psi_y + 2\varphi_x\psi_{xy}\right)y''' + \cdots\big] \\ &= S\left(x, y, y', y'', y''', y^{(4)}\right), \end{aligned} \tag{30}$$

$$\begin{aligned} u^{(5)}(t) &= \frac{D_xS}{D_x\varphi} \\ &= \frac{S_x + y'S_y + y''S_{y'} + y'''S_{y''} + y^{(4)}S_{y'''} + y^{(5)}S_{y^{(4)}}}{\varphi_x} \\ &= \frac{1}{\varphi_x^9}\big[\left(\varphi_x^4\psi_y\right)y^{(5)} + \left(5\varphi_x^4\psi_{yy}\right)y'y^{(4)} \\ &\quad +5\varphi_x^3\left(-2\varphi_{xx}\psi_y + \varphi_x\psi_{xy}\right)y^{(4)} + \cdots\big], \end{aligned} \tag{31}$$

where $D_x = (\partial/\partial x) + y'(\partial/\partial y) + y''(\partial/\partial y') + y'''(\partial/\partial y'') + y^{(4)}(\partial/\partial y''') + y^{(5)}(\partial/\partial y^{(4)}) + \cdots$ is a total derivative. Substituting $u^{(5)}(t)$ into the linear equation (5), we have

$$\begin{aligned} y^{(5)} &+ \left(\left(\frac{5\psi_{yy}}{\psi_y}\right) y' - 5\left(2\varphi_{xx}\psi_y - \varphi_x\psi_{xy}\right)\right) y^{(4)} \\ &+ \left(\left(\frac{10\psi_{yy}}{\psi_y}\right) y'' + \left(\frac{10\psi_{yyy}}{\psi_y}\right) y'^2 - \left(\frac{20\left(2\varphi_{xx}\psi_{yy} - \varphi_x\psi_{xyy}\right)}{\left(\varphi_x\psi_y\right)}\right) y' + \cdots\right) y''' \\ &+ \left(\left(\frac{15\psi_{yyy}}{\psi_y}\right) y' + \cdots\right) y''^2 + \cdots = 0. \end{aligned} \tag{32}$$

Denoting A_i, B_i, C_i, D_i, and E_i as (10)–(21), we obtain the necessary form (8). These prove the theorem. □

3. Formulation of the Linearization Theorem

We have shown in the previous section that every linearizable fifth-order ordinary differential equation belongs to the class of (8). In this section, we formulate the main theorems containing necessary and sufficient conditions for linearization as well as the methods for constructing the linearizing transformations.

Theorem 3. *Sufficient conditions for* (8) *to be linearizable via a fiber preserving transformation are as follows:*

$$A_{0y} = A_{1x}, \tag{33}$$

$$B_3 = 2A_1, \tag{34}$$

$$A_{1y} = \frac{-\left(2A_1^2 - 5B_2\right)}{10}, \tag{35}$$

$$A_{1x} = \frac{-\left(4A_0A_1 - 5B_1\right)}{20}, \tag{36}$$

$$B_2 = \frac{2C_1}{3}, \tag{37}$$

$$B_1 = \frac{4C_0}{3}, \tag{38}$$

$$C_{1y} = \frac{-\left(2A_1C_1 - 15D_3\right)}{10}, \tag{39}$$

$$C_{0y} = \frac{-\left(2A_1C_0 - 5D_2\right)}{10}, \tag{40}$$

$$D_{3y} = \frac{-\left(A_1D_3 - 50E_5\right)}{5}, \tag{41}$$

$$D_{2y} = \frac{-\left(A_1D_2 - 30E_4\right)}{5}, \tag{42}$$

$$B_{0y} = \frac{-2\left(15A_{0x}A_1 - 25C_{0x} + 3A_0^2A_1 - 5A_0C_0\right)}{75}, \tag{43}$$

$$C_{0x} = \frac{\left(30A_{0x}A_1 + 6A_0^2A_1 - 10A_0C_0 - 15A_1B_0 + 25D_1\right)}{50}, \tag{44}$$

$$A_{0xx} = \frac{-\left(60A_{0x}A_0 - 75B_{0x} + 8A_0^3 - 30A_0B_0 + 50D_0\right)}{50}, \tag{45}$$

$$D_{1y} = \frac{-\left(A_1D_1 - 15E_3\right)}{5}, \tag{46}$$

$$\begin{aligned} D_{0y} = \big(&60A_{0x}A_0A_1 - 100A_{0x}C_0 - 75B_{0x}A_1 \\ &+ 125D_{1x} + 12A_0^3A_1 - 20A_0^2C_0 - 45A_0A_1B_0 \\ &+ 25A_0D_1 - 75A_1D_0 + 50B_0C_0 + 375E_2\big)(750)^{-1}, \end{aligned} \tag{47}$$

$$\begin{aligned} B_{0xx} = \big(2\big(&175A_{0x}^2 + 70A_{0x}A_0^2 - 325A_{0x}B_0 \\ &- 75B_{0x}A_0 + 500D_{0x} + 7A_0^4 \\ &- 65A_0^2B_0 + 100A_0D_0 + 100B_0^2 - 625E_1\big)\big) \\ &\times (375)^{-1}. \end{aligned} \tag{48}$$

Proof. For obtaining sufficient conditions, one has to solve the compatibility problem. Considering the representations of the coefficients A_i, B_i, C_i, D_i, and E_i through the unknown functions φ and ψ. We first rewrite the expressions (9) and (10) for A_1 and A_0 in the following form:

$$\psi_{yy} = \frac{\psi_y A_1}{5}, \tag{49}$$

$$\varphi_{xx} = \frac{\left(5\psi_{xy} - \psi_y A_0\right)\varphi_x}{\left(10\psi_y\right)}. \tag{50}$$

Differentiating (50) with respect to y, one obtains the condition (33). Substituting the expressions of ψ_{yy} and φ_{xx} into (11), (12), (13), (15), (16), (17), (18), (21), and (22) one gets conditions (34)–(42), respectively. From (14) we have

$$\begin{aligned} \psi_{xxy} = -\big(&20A_{0x}\psi_y^2 - 125\psi_{xy}^2 + 10\psi_{xy}\psi_y A_0 \\ &+ 7\psi_y^2A_0^2 - 20\psi_y^2B_0\big)\left(100\psi_y\right)^{-1}. \end{aligned} \tag{51}$$

Comparing the mixed derivative $(\psi_{xxy})_y = (\psi_{yy})_{xx}$ one obtains the condition (43). Equations (19), (20), (23), (24), and (25) provide the conditions (44)–(48), respectively.

Consider the form of $\psi_{yy} : \psi_{yy} = (\psi_y A_1/5)$ one can solve that

$$\psi_y = \omega_1(x, y)\,\psi_1(x), \tag{52}$$

where $\omega_1(x, y) = e^{\int (A_1/5)dy}$ and $\psi_1(x) = e^{K_1(x)}$. Since $\psi_y \neq 0$ then ψ_1 and ω_1 cannot be zero. From $\omega_1 = e^{\int (A_1/5)dy}$, we found the relation

$$A_1 = 5\frac{\omega_{1y}}{\omega_1}. \tag{53}$$

Relations $(A_1)_x = A_{1x}$ and $(A_1)_y = A_{1y}$ provide the conditions

$$\omega_{1xy} = \frac{\left(15\omega_{1x}\omega_{1y} - 3A_0\omega_{1y}\omega_1 + C_0\omega_1^2\right)}{(15\omega_1)}, \qquad \omega_{1yy} = \frac{C_1\omega_1}{15}, \tag{54}$$

respectively, and the relation (52) satisfied the condition $(\psi_y)_y = \psi_{yy}$. cComposing the relation $(\psi_y)_{xx} = \psi_{xxy}$, one has the equation

$$\begin{aligned}\psi_{1xx} = \Big(&-20A_{0x}\omega_1^2\psi_1^2 - 100\omega_{1xx}\omega_1\psi_1^2 + 125\omega_{1x}^2\psi_1^2 \\ &+ 50\omega_{1x}\psi_{1x}\omega_1\psi_1 - 10\omega_{1x}A_0\omega_1\psi_1^2 \\ &+ 125\psi_{1x}^2\omega_1^2 - 10\psi_{1x}A_0\omega_1^2\psi_1 - 7A_0^2\omega_1^2\psi_1^2 \\ &+ 20B_0\omega_1^2\psi_1^2\Big)\left(100\omega_1^2\psi_1\right)^{-1}.\end{aligned} \tag{55}$$

Consider $\psi_y = \omega_1(x, y)\psi_1(x)$; one can solve that

$$\psi = \psi_1(x)\,\omega_2(x, y) + \psi_2(x), \tag{56}$$

where $\omega_2(x, y) = \int \omega_1(x, y)dy$. Because of $\int \omega_1(x, y)dy = \omega_2(x, y)$ then

$$\omega_1(x, y) = \omega_{2y}. \tag{57}$$

Since $\omega_1 \neq 0$ then $\omega_{2y} \neq 0$. Substituting ω_1 into ω_{1xy} and ω_{1yy} we obtain the additional conditions

$$\omega_{2xyy} = \frac{\left(\left(-3\omega_{2yy}A_0 + \omega_{2y}C_0\right)\omega_{2y} + 15\omega_{2xy}\omega_{2yy}\right)}{\left(15\omega_{2y}\right)}, \qquad \omega_{2yyy} = \frac{\omega_{2y}C_1}{15}, \tag{58}$$

respectively, and these satisfied the relations $(\psi)_y = \psi_y$, $(\psi)_{yy} = \psi_{yy}$, and $(\psi)_{xxy} = \psi_{xxy}$. From (26), setting $\mu_1(x, y)$, $\mu_2(x, y)$, and $\mu_3(x, y)$ as (A.1), (see Appendix) then we obtain

$$\begin{aligned}\psi_{2xxxxx} = \Big(&-140625\psi_{1x}^4\psi_{2x}\omega_{2y}^5 + 1125000\psi_{1x}^3\psi_{2xx}\omega_{2y}^5\psi_1 \\ &+ 112500\psi_{1x}^3\psi_{2x}\omega_{2y}^4\psi_1\left(-5\omega_{2xy} + \omega_{2y}A_0\right) \\ &- 2250000\psi_{1x}^2\psi_{2xxx}\omega_{2y}^5\psi_1^2 \\ &+ 675000\psi_{1x}^2\psi_{2xx}\omega_{2y}^4\psi_1^2\left(5\omega_{2xy} - \omega_{2y}A_0\right) \\ &+ 11250\psi_{1x}^2\psi_{2x}\omega_{2y}^3\psi_1^2 \\ &\times\left(-40A_{0x}\omega_{2y}^2 - 75\omega_{2xy}^2 + 30\omega_{2xy}\omega_{2y}A_0 \right. \\ &\qquad \left. - 11\omega_{2y}^2A_0^2 + 20\omega_{2y}^2B_0\right) \\ &+ 1500000\psi_{1x}\psi_{2xxxx}\omega_{2y}^5\psi_1^3 \\ &+ 900000\psi_{1x}\psi_{2xxx}\omega_{2y}^4\psi_1^3\left(-5\omega_{2xy} + \omega_{2y}A_0\right) \\ &+ 75000\psi_{1x}\psi_{2xx}\omega_{2y}^3\psi_1^3 \\ &\times\left(16A_{0x}\omega_{2y}^2 + 45\omega_{2xy}^2 - 18\omega_{2xy}\omega_{2y}A_0 \right. \\ &\qquad \left. + 5\omega_{2y}^2A_0^2 - 8\omega_{2y}^2B_0\right) + 1500\psi_{1x}\psi_{2x}\omega_{2y}^2\psi_1^3 \\ &\times\left(200A_{0x}\omega_{2xy}\omega_{2y}^2 - 40A_{0x}\omega_{2y}^3A_0 \right. \\ &\quad + 375\omega_{2xy}^3 - 225\omega_{2xy}^2\omega_{2y}A_0 + 85\omega_{2xy}\omega_{2y}^2A_0^2 \\ &\quad - 100\omega_{2xy}\omega_{2y}^2B_0 - 11\omega_{2y}^3A_0^3 + 20\omega_{2y}^3A_0B_0 \\ &\quad \left. - 2\mu_3\right) + 300000\psi_{2xxxx}\omega_{2y}^4\psi_1^4 \\ &\quad \times\left(5\omega_{2xy} - \omega_{2y}A_0\right) \\ &\quad + 30000\psi_{2xxx}\omega_{2y}^3\psi_1^4 \\ &\quad \times\left(-20A_{0x}\omega_{2y}^2 - 75\omega_{2xy}^2 + 30\omega_{2xy}\omega_{2y}A_0 \right. \\ &\qquad \left. -7\omega_{2y}^2A_0^2 + 10\omega_{2y}^2B_0\right) \\ &\quad + 3000\psi_{2xx}\omega_{2y}^2\mu_3\psi_1^4 \\ &\quad + 25\psi_{2x}\omega_{2y}\mu_2\psi_1^4 + 16\mu_1\psi_1^5\Big) \\ &\times\left(300000\omega_{2y}^5\psi_1^4\right)^{-1}.\end{aligned} \tag{59}$$

By the proof of Theorem 3, we arrive at the following Corollary. □

Corollary 4. *Provided that the sufficient conditions in Theorem 3 are satisfied, the transformation* (7) *of mapping equation* (8) *to a linear equation $u^{(5)}(t)$ is obtained by solving the compatible system of equations* (49), (50), (56), (55), *and* (59) *for the functions $\varphi(x)$ and $\psi(x, y)$.*

4. Examples

Example 1. Consider the nonlinear ordinary differential equation

$$\begin{aligned}&x^2yy^{(5)} + 5\left(x^2y' + 2xy\right)y^{(4)} \\ &+ 10\left(x^2y'' + 4xy' + 2y\right)y''' + 30xy''^2 \\ &+ 60y'y'' = 0.\end{aligned} \tag{60}$$

It is an equation of the form (8) with the coefficients

$$\begin{gathered}A_1 = \frac{5}{y}, \quad A_0 = \frac{10}{x}, \quad B_3 = \frac{10}{y}, \\ B_2 = 0, \quad B_1 = \frac{40}{xy}, \quad B_0 = \frac{20}{x^2}, \\ C_1 = 0, \quad C_0 = \frac{30}{xy}, \quad D_3 = 0, \\ D_2 = 0, \quad D_1 = \frac{60}{x^2y}, \quad D_0 = 0, \\ E_5 = 0, \quad E_4 = 0, \quad E_3 = 0, \\ E_2 = 0, \quad E_1 = 0, \quad E_0 = 0, \\ \omega_1 = y, \quad \omega_2 = \frac{y^2}{2}, \quad \mu_1 = 0, \\ \mu_2 = \frac{-90000y^4}{x^4}, \quad \mu_3 = \frac{-3000y^3}{x^3}.\end{gathered} \tag{61}$$

Applying Theorem 3 for checking the linearity, the coefficients in (61) obey all the conditions in Theorem 3, so that one concludes that equation (60) is linearizable. Applying Corollary 4, the linearizing transformation is found by solving the following equations:

$$\varphi_{xx} = \frac{(\varphi_x \psi_{1x} x - 2\psi_1)}{(2\psi_1 x)}, \tag{62}$$

$$\psi_{1xx} = \frac{\left(5\psi_{1x}^2 x^2 - 4\psi_{1x}\psi_1 x - 4\psi_1^2\right)}{(4\psi_1 x^2)}, \tag{63}$$

$$\begin{aligned}\psi_{2xxxxx} = {} & \left(5\left(-3\psi_{1x}^4\psi_{2x}x^4 + 24\psi_{1x}^3\psi_{2xx}\psi_1 x^4\right.\right.\\ & + 24\psi_{1x}^3\psi_{2x}\psi_1 x^3 - 48\psi_{1x}^2\psi_{2xxx}\psi_1^2 x^4\\ & - 144\psi_{1x}^2\psi_{2xx}\psi_1^2 x^3 - 72\psi_{1x}^2\psi_{2x}\psi_1^2 x^2\\ & + 32\psi_{1x}\psi_{2xxxx}\psi_1^3 x^4 + 192\psi_{1x}\psi_{2xx}\psi_1^3 x^3\\ & + 288\psi_{1x}\psi_{2xx}\psi_1^3 x^2 + 96\psi_{1x}\psi_{2x}\psi_1^3 x\\ & - 64\psi_{2xxxx}\psi_1^4 x^3 - 192\psi_{2xxx}\psi_1^4 x^2\\ & \left.\left. -192\psi_{2xx}\psi_1^4 x - 48\psi_{2x}\psi_1^4\right)\right)\\ & \times \left(32\psi_1^4 x^4\right)^{-1}\end{aligned} \tag{64}$$

$$\psi = \frac{\left(\psi_1 y^2 + 2\psi_2\right)}{2}. \tag{65}$$

Since $\varphi_y = 0$, then one can take the simplest solution

$$\varphi = x. \tag{66}$$

Thus, (62) becomes $\psi_{1x}x - 2\psi_1 = 0$; the solution for this equation is

$$\psi_1 = Cx^2. \tag{67}$$

Choosing $C = 2$, we have

$$\psi_1 = 2x^2. \tag{68}$$

This solution satisfied (63). Equation (64) becomes

$$\psi_{2xxxxxx} = 0. \tag{69}$$

Choosing the particular solution

$$\psi_2 = 0. \tag{70}$$

Hence (65) is in the form

$$\psi = x^2 y^2. \tag{71}$$

So one obtains the linearizing transformation

$$t = x, \qquad u = x^2 y^2. \tag{72}$$

Thus, the nonlinear equation (60) can be mapped by transformation of (72) into the linear equation $u^{(5)}(t) = 0$. Next, we will find the solution of (60). Since

$$u^{(5)}(t) = 0, \tag{73}$$

then we get the general solution

$$u(t) = C_0\frac{t^4}{24} + C_1\frac{t^3}{6} + C_2\frac{t^2}{2} + C_3 t + C_4, \tag{74}$$

where C_0, C_1, C_2, C_3, and C_4 are arbitrary constants. Substituting (72) into (74) we get

$$x^2 y^2 = C_0\frac{x^4}{24} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4. \tag{75}$$

Example 2. Consider the nonlinear ordinary differential equation

$$\begin{aligned} & 16y^4 y^{(5)} - 40\left(y^3 y' + 4y^4\right) y^{(4)}\\ & - 40\left(2y^3 y'' - 3y^2 y'^2 - 8y^3 y' - 14y^4\right) y'''\\ & + 60\left(3y' y^2 + 4y^3\right) y''^2\\ & - 20\left(15yy'^3 + 36y^2 y'^2 + 42y^3 y' + 40y^4\right) y''\\ & \times 105y'^5 + 300yy'^4 + 420y^2 y'^3\\ & + 400y^3 y'^2 + 384y^4 y' = 0.\end{aligned} \tag{76}$$

It is an equation of the form equation (8) with the coefficients

$$\begin{gathered} A_1 = \frac{-5}{2y}, \qquad A_0 = -10, \qquad B_3 = \frac{-5}{y},\\ B_2 = \frac{15}{2y^2}, \qquad B_1 = \frac{20}{y}, \qquad B_0 = 35,\\ C_1 = \frac{45}{4y^2}, \qquad C_0 = \frac{15}{y}, \qquad D_3 = \frac{-75}{4y^3},\\ D_2 = \frac{-45}{y^2}, \qquad D_1 = \frac{-105}{2y}, \qquad D_0 = -50,\\ E_5 = \frac{105}{16y^4}, \qquad E_4 = \frac{75}{4y^3}, \qquad E_3 = \frac{105}{4y^2},\\ E_2 = \frac{25}{y}, \qquad E_1 = 24, \qquad E_0 = 0,\\ \omega_1 = \frac{1}{\sqrt{y}}, \qquad \omega_2 = 2\sqrt{y},\\ \mu_1 = 0, \qquad \mu_2 = \frac{-288000}{y^2}, \qquad \mu_3 = \frac{5000}{y\sqrt{y}}.\end{gathered} \tag{77}$$

Applying Theorem 3 for checking the linearity, the coefficients in (77) obey all conditions in Theorem 3, so that one concludes that (76) is linearizable. Applying Corollary 4, the linearizing transformation is found by solving the following equations:

$$\varphi_{xx} = \frac{\varphi_x(\psi_{1x} + 2\psi_1)}{(2\psi_1)}, \tag{78}$$

$$\psi_{1xx} = \psi_{1x}(5\psi_{1x} + 4\psi_1), \tag{79}$$

$$\begin{aligned}\psi_{2xxxxx} = \big(&-15\psi_{1x}^4\psi_{2x} + 120\psi_{1x}^3\psi_{2xx}\psi_1 \\ &- 120\psi_{1x}^3\psi_{2x}\psi_1 - 240\psi_{1x}^2\psi_{2xxx}\psi_1^2 \\ &+ 720\psi_{1x}^2\psi_{2xx}\psi_1^2 - 480\psi_{1x}^2\psi_{2x}\psi_1^2 \\ &+ 160\psi_{1x}\psi_{2xxxx}\psi_1^3 - 960\psi_{1x}\psi_{2xxx}\psi_1^3 \\ &+ 1760\psi_{1x}\psi_{2xx}\psi_1^3 - 960\psi_{1x}\psi_{2x}\psi_1^3 \\ &+ 320\psi_{2xxxx}\psi_1^4 - 1120\psi_{2xxx}\psi_1^4 \\ &+ 1600\psi_{2xx}\psi_1^4 - 768\psi_{2x}\psi_1^4\big) \\ &\times\left(32\psi_1^4\right)^{-1},\end{aligned} \tag{80}$$

$$\psi = \frac{(\psi_1 y^2 + 2\psi_2)}{2}. \tag{81}$$

From (78),

$$\frac{\varphi_{xx}}{\varphi_x} = \frac{\psi_{1x}}{2\psi_1} + 1. \tag{82}$$

Taking the particular solution $\varphi = e^x$, then the solution of (82) is

$$\psi_1 = C. \tag{83}$$

Choosing $C = (1/2)$, we have

$$\psi_1 = \frac{1}{2}. \tag{84}$$

This solution satisfied (79). Equation (80) becomes

$$\psi_{2xxxxx} = 10\psi_{2xxxx} - 35\psi_{2xxx} + 50\psi_{2xx} - 24\psi_{2x}. \tag{85}$$

Choosing the particular solution

$$\psi_2 = 0. \tag{86}$$

Hence (81) is in the form

$$\psi = \sqrt{y}. \tag{87}$$

So one obtains the linearizing transformation

$$t = e^x, \qquad u = \sqrt{y}. \tag{88}$$

Thus, the nonlinear equation (76) can be mapped by transformation of (88) into the linear equation $u^{(5)}(t) = 0$. Next, we will find the solution of (76). Since

$$u^{(5)}(t) = 0, \tag{89}$$

then we get the general solution

$$u(t) = C_0\frac{t^4}{24} + C_1\frac{t^3}{6} + C_2\frac{t^2}{2} + C_3 t + C_4, \tag{90}$$

where C_0, C_1, C_2, C_3, and C_4 are arbitrary constants. Substituting (88) into (90) we get

$$\sqrt{y} = C_0\frac{e^{4x}}{24} + C_1\frac{e^{3x}}{6} + C_2\frac{e^{2x}}{2} + C_3 e^x + C_4. \tag{91}$$

Appendix

Equations in Section 3

Consider

$$\begin{aligned}\mu_1 = {} & 700A_{0x}^2\omega_{2y}^5 A_0\omega_2 - 1750A_{0x}B_{0x}\omega_{2y}^5\omega_2 \\ &+ 280A_{0x}\omega_{2y}^5 A_0^3\omega_2 - 1050A_{0x}\omega_{2y}^5 A_0 B_0\omega_2 \\ &+ 2750A_{0x}\omega_{2y}^5 D_0\omega_2 - 350B_{0x}\omega_{2y}^5 A_0^2\omega_2 \\ &+ 875B_{0x}\omega_{2y}^5 B_0\omega_2 + 1250D_{0xx}\omega_{2y}^5\omega_2 \\ &+ 500D_{0x}\omega_{2y}^5 A_0\omega_2 - 6250E_{1x}\omega_{2y}^5\omega_2 \\ &+ 2250000\omega_{2xy}^5\omega_2 - 2250000\omega_{2xy}^4\omega_{2x}\omega_{2y} \\ &- 450000\omega_{2xy}^4\omega_{2y}A_0\omega_2 - 4500000\omega_{2xy}^3\omega_{2xxy}\omega_{2y}\omega_2 \\ &+ 1125000\omega_{2xy}^3\omega_{2xx}\omega_{2y}^2 \\ &+ 450000\omega_{2xy}^3\omega_{2x}\omega_{2y}^2 A_0 + 112500\omega_{2xy}^3\omega_{2y}^2 B_0\omega_2 \\ &+ 1125000\omega_{2xy}^2\omega_{2xxxy}\omega_{2y}^2\omega_2 \\ &- 375000\omega_{2xy}^2\omega_{2xxx}\omega_{2y}^3 \\ &+ 3375000\omega_{2xy}^2\omega_{2xxy}\omega_{2x}\omega_{2y}^2 \\ &+ 675000\omega_{2xy}^2\omega_{2xxy}\omega_{2y}^2 A_0\omega_2 \\ &- 225000\omega_{2xy}^2\omega_{2xx}\omega_{2y}^3 A_0 \\ &- 112500\omega_{2xy}^2\omega_{2x}\omega_{2y}^3 B_0 \\ &- 37500\omega_{2xy}^2\omega_{2y}^3 D_0\omega_2 \\ &- 187500\omega_{2xy}\omega_{2xxxxy}\omega_{2y}^3\omega_2 \\ &+ 93750\omega_{2xy}\omega_{2xxxx}\omega_{2y}^4 \\ &- 750000\omega_{2xy}\omega_{2xxxy}\omega_{2x}\omega_{2y}^3 \\ &- 150000\omega_{2xy}\omega_{2xxxy}\omega_{2y}^3 A_0\omega_2 \\ &+ 75000\omega_{2xy}\omega_{2xxx}\omega_{2y}^4 A_0 \\ &+ 1687500\omega_{2xy}\omega_{2xxy}^2\omega_{2y}^2\omega_2 \\ &- 1125000\omega_{2xy}\omega_{2xxy}\omega_{2xx}\omega_{2y}^3 \\ &- 450000\omega_{2xy}\omega_{2xxy}\omega_{2x}\omega_{2y}^3 A_0 \\ &- 112500\omega_{2xy}\omega_{2xxy}\omega_{2y}^3 B_0\omega_2 \\ &+ 56250\omega_{2xy}\omega_{2xx}\omega_{2y}^4 B_0\end{aligned}$$

$$
\begin{aligned}
&+ 37500\omega_{2xy}\omega_{2x}\omega_{2y}^{4}D_0 \\
&+ 18750\omega_{2xy}\omega_{2y}^{4}E_1\omega_2 \\
&+ 18750\omega_{2xxxxxy}\omega_{2y}^{4}\omega_2 \\
&- 18750\omega_{2xxxxx}\omega_{2y}^{5} \\
&+ 93750\omega_{2xxxxy}\omega_{2x}\omega_{2y}^{4} \\
&+ 18750\omega_{2xxxxy}\omega_{2y}^{4}A_0\omega_2 \\
&- 18750\omega_{2xxxx}\omega_{2y}^{5}A_0 \\
&- 375000\omega_{2xxxy}\omega_{2xxy}\omega_{2y}^{3}\omega_2 \\
&+ 187500\omega_{2xxxy}\omega_{2xx}\omega_{2y}^{4} \\
&+ 75000\omega_{2xxxy}\omega_{2x}\omega_{2y}^{4}A_0 \\
&+ 18750\omega_{2xxxy}\omega_{2y}^{4}B_0\omega_2 \\
&+ 187500\omega_{2xxx}\omega_{2xxy}\omega_{2y}^{4} \\
&- 18750\omega_{2xxx}\omega_{2y}^{5}B_0 \\
&- 562500\omega_{2xxy}^{2}\omega_{2x}\omega_{2y}^{3} \\
&- 112500\omega_{2xxy}^{2}\omega_{2y}^{3}A_0\omega_2 \\
&+ 112500\omega_{2xxy}\omega_{2xx}\omega_{2y}^{4}A_0 \\
&+ 56250\omega_{2xxy}\omega_{2x}\omega_{2y}^{4}B_0 \\
&+ 18750\omega_{2xxy}\omega_{2y}^{4}D_0\omega_2 \\
&- 18750\omega_{2xx}\omega_{2y}^{5}D_0 \\
&- 18750\omega_{2x}\omega_{2y}^{5}E_1 + 18750\omega_{2y}^{6}E_0 \\
&+ 28\omega_{2y}^{5}A_0^{5}\omega_2 - 210\omega_{2y}^{5}A_0^{3}B_0\omega_2 \\
&+ 550\omega_{2y}^{5}A_0^{2}D_0\omega_2 + 350\omega_{2y}^{5}A_0B_0^{2}\omega_2 \\
&- 1250\omega_{2y}^{5}A_0E_1\omega_2 - 1250\omega_{2y}^{5}B_0D_0\omega_2, \\
\mu_2 = &-3200A_{0x}^{2}\omega_{2y}^{4} - 18000A_{0x}\omega_{2xy}^{2}\omega_{2y}^{2} \\
&- 2400A_{0x}\omega_{2xy}\omega_{2y}^{3}A_0 - 2000A_{0x}\omega_{2y}^{4}A_0^{2} \\
&+ 5600A_{0x}\omega_{2y}^{4}B_0 + 12000B_{0x}\omega_{2xy}\omega_{2y}^{3} \\
&- 4000D_{0x}\omega_{2y}^{4} - 5625\omega_{2xy}^{4} \\
&+ 4500\omega_{2xy}^{3}\omega_{2y}A_0 - 4950\omega_{2xy}^{2}\omega_{2y}^{2}A_0^{2} \\
&+ 9000\omega_{2xy}^{2}\omega_{2y}^{2}B_0 - 300\omega_{2xy}\omega_{2y}^{3}A_0^{3} \\
&+ 3600\omega_{2xy}\omega_{2y}^{3}A_0B_0 - 12000\omega_{2xy}\omega_{2y}^{3}D_0 \\
&- 281\omega_{2y}^{4}A_0^{4} + 1480\omega_{2y}^{4}A_0^{2}B_0 \\
&- 800\omega_{2y}^{4}A_0D_0 - 2000\omega_{2y}^{4}B_0^{2} + 8000\omega_{2y}^{4}E_1, \\
\mu_3 = &\,400A_{0x}\omega_{2xy}\omega_{2y}^{2} - 100B_{0x}\omega_{2y}^{3} \\
&+ 375\omega_{2xy}^{3} - 225\omega_{2xy}^{2}\omega_{2y}A_0 \\
&+ 125\omega_{2xy}\omega_{2y}^{2}A_0^{2} - 200\omega_{2xy}\omega_{2y}^{2}B_0 \\
&- 3\omega_{2y}^{3}A_0^{3} - 20\omega_{2y}^{3}A_0B_0 + 100\omega_{2y}^{3}D_0.
\end{aligned}
\tag{A.1}
$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] S. Lie, "Klassifikation und integration von gewöhnlichendifferentialgleichungen zwischen *x*, *y*, die eine Gruppe von Transformationen gestatten. III," *Archiv for Matematik og Naturvidenskab*, vol. 8, no. 4, pp. 371–427, 1883.

[2] S. Lie, "Klassifikation und integration von gewöhnlichen differentialgleichungen zwischen *x*,*y*, die eine Gruppe von Transformationen gestatten. III," in *Gessammelte Abhandlungen*, S. Lie, Ed., vol. 5, paper XIY, pp. 362–427, 1924.

[3] R. Liouville, "Sur les invariantes de certaines equations," *Journal de l'École Polytechnique*, vol. 59, pp. 7–88, 1889.

[4] A. M. Tresse, *Détermination des Invariants Ponctuels de L'Équation Différentielle Ordinaire du Second Ordre $y'' = f(x, y, y')$*, Preisschriften der Fürstlichen Jablonowski'schen Gesellschaft XXXII, S.Herzel, Leipzig, Germany, 1896.

[5] E. Cartan, "Sur les variétés à connexion projective," *Bulletin de la Société Mathématique de France*, vol. 52, pp. 205–241, 1924.

[6] A. V. Bocharov, V. V. Sokolov, and S. I. Svinolupov, *On Some Equivalence Problems for Differential Equations*, Preprint ESI 54, 1993.

[7] G. Grebot, "The characterization of third order ordinary differential equations admitting a transitive fiber-preserving point symmetry group," *Journal of Mathematical Analysis and Applications*, vol. 206, no. 2, pp. 364–388, 1997.

[8] N. H. Ibragimov and S. V. Meleshko, "Linearization of third-order ordinary differential equations by point and contact transformations," *Journal of Mathematical Analysis and Applications*, vol. 308, no. 1, pp. 266–289, 2005.

[9] N. H. Ibragimov, S. V. Meleshko, and S. Suksern, "Linearization of fourth-order ordinary differential equations by point transformations," *Journal of Physics A: Mathematical and Theoretical*, vol. 41, no. 23, Article ID 235206, 19 pages, 2008.

โปรแกรมทดสอบความเป็นเชิงเส้นของสมการเชิงอนุพันธ์สามัญอันดับห้า

โดยการแปลงแบบจุด

โดย ดร. สุภาพร สุขเสริญ

```
0.00+0.15 secs          reduce
File  Edit  Font  Break  Load Package  Switch                    Help

1: in test;
                 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
                 %          Linearization test      %
                 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


off nat;


in a1;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                                                %
%   Linearization of fifth-order ODEs by point transformation  %
%                                                                %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%Case mu4 = 0, mu5 = 0

off echo;


fifthorder


point


vyd:=1;


vyd := 1



equation:=60*p*pp + 40*p*ppp*x + 5*p*pppp*x**2 + 30*pp**2*x + 10*pp*ppp*x**2 + 2
0*ppp*y + 10*pppp*x*y + ppppp*x**2*y $
```

ตัวอย่างผลจากการใช้โปรแกรม

```
1: in test;

        %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
        %              Linearization test                %
        %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

off nat;

in a1;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%       Linearization of fifth-order ODEs by point transformation          %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%Case mu4 = 0, mu5 = 0

off echo;

fifthorder

point

vyd:=1;

vyd := 1

equation:=60*p*pp + 40*p*ppp*x + 5*p*pppp*x**2 + 30*pp**2*x + 10*pp*ppp*x**2 +
20*ppp*y + 10*pppp*x*y + ppppp*x**2*y $


fifthorder(equation,vyd);


Application of the POINT procedure


equation = 60*p*pp + 40*p*ppp*x + 5*p*pppp*x**2 + 30*pp**2*x + 10*pp*ppp*x**2 +

20*ppp*y + 10*pppp*x*y + ppppp*x**2*y
```

```
aw1 = 5/y

aw0 = 10/x

bw3 = 10/y

bw2 = 0

bw1 = 40/(x*y)

bw0 = 20/x**2

cw1 = 0

cw0 = 30/(x*y)

dw3 = 0

dw2 = 0

dw1 = 60/(x**2*y)

dw0 = 0

ew5 = 0

ew4 = 0

ew3 = 0

ew2 = 0

ew1 = 0

ew0 = 0


om1 = y

om2 = y**2/2

mu1 = 0

mu2 = ( - 90000*y**4)/x**4

mu3 = ( - 3000*y**3)/x**3

mu4 = 0

mu5 = 0

mu6 = 0
```

mu7 = 0

mu8 = 0

mu9 = 0

mu10 = 0

mu11 = 0

mu12 = 0

Linearization by POINT transformation is possible

Linearization is to the equation u^(5)=0

df(fi,x,2) = (df(fi,x)*(df(psi1,x)*x - 2*psi1))/(2*psi1*x)

df(psi1,x,2) = (5*df(psi1,x)**2*x**2 - 4*df(psi1,x)*psi1*x - 4*psi1**2)/(4*psi1*x**2)

df(psi2,x,5) = (5*(- 3*df(psi1,x)**4*df(psi2,x)*x**4 + 24*df(psi1,x)**3*df(psi2
,x,2)*psi1*x**4 + 24*df(psi1,x)**3*df(psi2,x)*psi1*x**3 - 48*df(psi1,x)**2*df(
psi2,x,3)*psi1**2*x**4 - 144*df(psi1,x)**2*df(psi2,x,2)*psi1**2*x**3 - 72*df(
psi1,x)**2*df(psi2,x)*psi1**2*x**2 + 32*df(psi1,x)*df(psi2,x,4)*psi1**3*x**4 +
192*df(psi1,x)*df(psi2,x,3)*psi1**3*x**3 + 288*df(psi1,x)*df(psi2,x,2)*psi1**3*x
2 + 96*df(psi1,x)*df(psi2,x)*psi13*x - 64*df(psi2,x,4)*psi1**4*x**3 - 192*df
(psi2,x,3)*psi1**4*x**2 - 192*df(psi2,x,2)*psi1**4*x - 48*df(psi2,x)*psi1**4))/(
32*psi1**4*x**4)

psi = (psi1*y**2 + 2*psi2)/2

end;

2: