สัญญาเลขที่ R2557B057

รายงานวิจัยฉบับสมบูรณ์

เงื่อนไขสำหรับการทำสมการเชิงอนุพันธ์สามัญอันดับสี่ให้เป็นเชิงเส้นโดยใช้การแปลงแบบซันด์มันทั่วไป

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สนับสนุนโดยสำนักงานคณะกรรมการวิจัยแห่งชาติ

และกองทุนวิจัยมหาวิทยาลัยนเรศวร

บทคัดย่อ

ในงานวิจัยนี้เราได้แบ่งผลการศึกษาออกเป็น 2 ส่วนใหญ่ๆ ซึ่งส่วนแรกเราได้ศึกษาการทำให้เป็นเชิงเส้นของสมการเชิงอนุพันธ์สามัญอันดับสี่โดยใช้การแปลงแบบซันด์มันทั่วไป ทำให้ได้มาซึ่งเงื่อนไขที่จำเป็นและเงื่อนไขที่เพียงพอสำหรับการทำให้เป็นสมการเชิงเส้นโดยใช้การแปลงแบบซันด์มันทั่วไป ตลอดจนได้มาซึ่งกระบวนการการได้มาของการแปลงเชิงเส้นในรูปแบบชัดแจ้ง ตัวอย่างประกอบการใช้ทฤษฎีบทเพื่อความเข้าใจ และได้ประยุกต์ใช้คลื่นจรกับสมการเชิงอนุพันธ์ย่อยบางสมการ

ส่วนที่สองเราได้ศึกษาการทำให้เป็นเชิงเส้นของสมการเชิงอนุพันธ์สามัญอันดับสามโดยใช้การแปลงเชิงเส้นทั่วไป ทำให้ได้มาซึ่งเงื่อนไขที่จำเป็นและเงื่อนไขที่เพียงพอสำหรับการทำให้เป็นสมการเชิงเส้น อีกทั้งได้มาซึ่งกระบวนการการได้มาของการแปลงเชิงเส้นในรูปแบบชัดแจ้งและตัวอย่างประกอบการใช้ทฤษฎีบท

คำสำคัญ : ปัญหาการทำให้เป็นเชิงเส้น/ การแปลงแบบซันด์มันทั่วไป/ การแปลงเชิงเส้นทั่วไป/ สมการเชิงอนุพันธ์สามัญไม่เชิงเส้น

Abstract

This research is divided into 2 main parts. In the first part we study of the linearization problem of fourth-order ordinary differential equation under the generalized Sundman transformation. We found the necessary and sufficient conditions which allow the fourth-order ordinary differential equation to be transformed to the simplest linear equation. Moreover, the procedure for obtaining the linearizing transformation are provided in explicit forms. The examples which is linearizable by our method is given and the linearization of traveling waves of partial differential equation are applied.

In the second part, we discuss the linearization problem of third-order ordinary differential equation under the generalized linearizing transformation. We identify the form of the linearizable equations and the conditions which allow the third-order ordinary differential equation to be transformed to the simplest linear equation. We also illustrate how to construct the generalized linearizing transformation. Some examples of linearizable equation are provided to demonstrate our procedure.

Keyword : Linearization problem/ generalized Sundman transformation/ generalized linearizing transformation/ nonlinear ordinary differential equations

Executive Summary

1. ความสำคัญและที่มาของปัญหา

ปัญหาการทำให้เป็นเชิงเส้น (linearization problem) นับเป็นแขนงที่สำคัญแขนงหนึ่งในสาขาของสมการเชิงอนุพันธ์ (differential equation) ในปัจจุบันนักคณิตศาสตร์ได้ศึกษาและวิจัยในแขนงดังกล่าวกันอย่างต่อเนื่อง ในการคิดค้นทฤษฎีเพื่อหาองค์ความรู้ใหม่ๆ นั้นนับว่ามีประโยชน์เป็นอย่างมากต่อทางวิชาการและการพัฒนาประเทศ เป็นที่ยอมรับว่าทฤษฎีและองค์ความรู้ใหม่ๆ ที่เกิดจากการวิจัยนั้น นอกจากจะมีประโยชน์อย่างมากในการพัฒนาความรู้เชิงวิชาการในสาขาและแขนงต่างๆ นั้นแล้ว บางครั้งยังสามารถนำไปประยุกต์ในสาขาอื่นๆ และเป็นพื้นฐานสำคัญในการพัฒนาทางวิทยาศาสตร์พื้นฐาน (basic science) ซึ่งเป็นการวิจัยพื้นฐาน (basic research) เพื่อสร้างองค์ความรู้ใหม่ อันถือเป็นพื้นฐานในการพัฒนาประเทศชาติต่อไป

ปัญหาการทำให้เป็นเชิงเส้นนับว่าเป็นแขนงหนึ่งที่สามารถประยุกต์ได้อย่างกว้างขวาง โดยเฉพาะอย่างยิ่งต่อการศึกษาเกี่ยวกับการหาผลเฉลยของสมการ ปัญหาที่มีความสำคัญทางกายภาพส่วนใหญ่มักจะอยู่ในรูปแบบของสมการเชิงอนุพันธ์ไม่เชิงเส้น (nonlinear differential equation) ซึ่งตามปกติการแก้ปัญหาที่อยู่ในรูปแบบไม่เชิงเส้นนั้นจะทำได้ยาก และไม่มีวิธีการหาผลเฉลยที่เป็นแบบแม่นตรง (exact solution) วิธีการเชิงตัวเลข (numerical method) นิยมนำมาแก้ปัญหา แต่ผลเฉลยที่ได้เป็นเพียงผลเฉลยประมาณค่า (approximate solution) อย่างไรก็ตามผลเฉลยแบบแม่นตรงมีความน่าสนใจกว่าเพราะว่าสามารถนำไปวิเคราะห์คุณสมบัติของสมการที่จะศึกษาได้ หนึ่งในวิธีการที่ใช้หาผลเฉลยแบบแม่นตรงนี้ คือ การทำสมการที่สนใจศึกษาให้มีความเป็นเชิงเส้น แล้วหาผลเฉลยโดยตรงจากวิธีการพื้นฐาน ซึ่งผลเฉลยที่ได้จากการแก้สมการเชิงเส้น ยังคงเป็นผลเฉลยของสมการที่มีมาแต่เดิมด้วย ซึ่งวิธีการดังกล่าวเราจำเป็นต้องหาการแปลง (transformation) เพื่อแปลงสมการตั้งต้นให้เป็นสมการเชิงเส้น

การแปลงที่น่าสนใจก็มีหลายรูปแบบด้วยกัน ตัวอย่างเช่น ถ้าการแปลงประกอบด้วยอนุพันธ์ เราจะเรียกการแปลงนี้ว่า การแปลงแบบแทนเจนต์ (tangent transformation) ถ้าการแปลงขึ้นอยู่กับตัวแปรอิสระและตัวแปรตามเท่านั้น เราจะเรียกการแปลงนี้ว่า การแปลงแบบจุด (point transformation) เราจะเรียกการแปลงแบบแทนเจนต์ที่ซึ่งนิยามด้วยการเปลี่ยนตัวแปรอิสระ ตัวแปรตาม และอนุพันธ์อันดับหนึ่งว่าการแปลงแบบคอนแทคท์ (contact transformation) และยังมีการแปลงอีกชนิดหนึ่งที่มีเซตของการแปลงจะแตกต่างจากการแปลงที่ได้กล่าวมาข้างต้น เนื่องจากประกอบด้วยพจน์ที่ไม่เฉพาะ (nonlocal term)

$$T = \int G\left(x(t),t\right)dt$$

การแปลงชนิดนี้จะเรียกว่า การแปลงแบบซันด์มันทั่วไป (generalized Sundman transformation) ซึ่งการแปลงที่ถูกเลือกมาใช้ในงานวิจัยนี้ก็คือ การแปลงแบบซันด์มันทั่วไป เนื่องจากการทำสมการเชิงอนุพันธ์สามัญอันดับสี่ให้เป็นเชิงเส้นโดยใช้การแปลงแบบซันด์มันทั่วไปยังไม่มีนักวิจัยท่านใดศึกษามาก่อน เนื่องด้วยตัวโครงสร้างของสมการที่เป็นอันดับสี่ค่อนข้างจะใหญ่และซับซ้อน และการประยุกต์การแปลงแบบซันด์มันทั่วไปเข้ากับสมการอันดับสี่ค่อนข้างจะทำได้ยากและใช้เวลานาน แต่อย่างไรก็ตามถ้าเราสามารถสร้างองค์ความรู้และทฤษฎีใหม่ๆขึ้นมาได้ ก็จะเป็นพื้นฐานที่สำคัญในการนำไปพัฒนาและประยุกต์ใช้ในสาขาต่างๆที่เกี่ยวข้องต่อไป อันจะเป็นพื้นฐานในการพัฒนาประเทศชาติต่อไป

2. วัตถุประสงค์

2.1 หาเงื่อนไขที่จำเป็นและเพียงพอที่ทำให้สมการเชิงอนุพันธ์สามัญอันดับสี่ใดๆ ในรูปแบบ

$$x^{(4)} = f\left(t, x, x', x'', x'''\right)$$

ซึ่งสามารถลดรูปไปสู่สมการเชิงเส้นในรูปแบบ

$$X^{(4)} = 0$$

โดยใช้การแปลงแบบซันด์มันทั่วไป ซึ่งอยู่ในรูปแบบ

$$X = F\left(t, x\right), \quad dT = G\left(t, x\right)dt, \quad \left(F_x G \neq 0\right)$$

2.2 หาการแปลงเชิงเส้น

2.3 หาตัวอย่าง และการนำไปประยุกต์ใช้

2.4 สร้างโปรแกรมสำเร็จรูปในการทดสอบความเป็นเชิงเส้น

3. ระเบียบวิธีวิจัย

3.1 ศึกษาโครงสร้างของสมการเชิงอนุพันธ์สามัญอันดับสี่ การแปลงในรูปแบบต่างๆ และผลงานวิจัยที่เกี่ยวข้องที่มีนักวิจัยทำมาก่อนหน้านี้ ด้วยการสืบค้นข้อมูลในอินเตอร์เน็ต เอกสาร และตำราจากห้องสมุด

3.2 ค้นคว้าหาเอกสาร ตำรา วารสาร และ เอกสารสิ่งพิมพ์ที่เกี่ยวข้องกับงานวิจัยที่กำลังดำเนินการวิจัยอยู่จากแหล่งข้อมูลต่างๆ

3.3 หาเงื่อนไขที่จำเป็นโดยวิธีการ change of derivatives

3.4 หาเงื่อนไขที่เพียงพอโดยใช้ compatibility theory

3.5 หาการแปลงเชิงเส้น

3.6 สร้างโปรแกรมสำเร็จรูปในการทดสอบความเป็นเชิงเส้นโดยใช้โปรแกรม Reduce

3.7 หาตัวอย่าง และการนำไปประยุกต์ใช้

3.8 เขียนและพิมพ์ผลงานวิจัยเพื่อส่งตีพิมพ์

3.9 รายงานสรุปผลโครงการ

4. แผนการดำเนินงานวิจัย

กิจกรรมและขั้นตอนดำเนินงาน	ปีงบประมาณ 2557 (6 เดือนแรก)					
	1	2	3	4	5	6
1. ศึกษาโครงสร้างของสมการเชิงอนุพันธ์สามัญอันดับสี่ การแปลงในรูปแบบต่างๆ และผลงานวิจัยที่เกี่ยวข้องที่มีนักวิจัยทำมาก่อนหน้านี้	▬					
2. ค้นคว้าหาเอกสาร ตำรา วารสาร และ เอกสาร		▬				

สิ่งพิมพ์ที่เกี่ยวข้องกับงานวิจัยที่กำลังดำเนินการวิจัยอยู่จากแหล่งข้อมูลต่างๆ						
3. คิดค้นและวิจัยเพื่อหาเงื่อนไขที่จำเป็นสำหรับการทำให้เป็นเชิงเส้น						
4. คิดค้นและวิจัยเพื่อหาเงื่อนไขที่เพียงพอสำหรับการทำให้เป็นเชิงเส้นในกรณี						

กิจกรรมและขั้นตอนดำเนินงาน	ปีงบประมาณ 2557 (6 เดือนหลัง)					
	7	8	9	10	11	12
1. คิดค้นและวิจัยเพื่อหาเงื่อนไขที่เพียงพอสำหรับการทำให้เป็นเชิงเส้น (ต่อ)						
2. คิดค้นและวิจัยเพื่อหาการแปลงเชิงเส้น						
3. สร้างโปรแกรมสำเร็จรูปในการทดสอบความเป็นเชิงเส้น						
4. คิดค้นและวิจัยเพื่อหาตัวอย่างและการประยุกต์ใช้						
5. เขียนและพิมพ์ผลงานวิจัยเพื่อส่งพิจารณาตีพิมพ์						
6. รายงานสรุปผลโครงการ						

5. ตัวชี้วัดเพื่อการเมินผลสำเร็จของโครงการ

คาดว่าจะตีพิมพ์ในวารสารระดับนานาชาติ (ไม่มีค่า Impact factor) 2 ผลงาน

Contents

Chapter 1

Basic concepts

1.1 The equivalence problem

Definition 1.1.1. Two equations are call *equivalent* if there is an invertible transformation which transforms one equation into another.

Definition 1.1.2. The problem of finding all equations, which are equivalent to a given equation is called an *equivalence problem.* If the given equation is a linear equation, then the equivalence problem is called a *linearization problem.*

1.2 Jacobian

Definition 1.2.1. If $F(u,v)$ and $G(u,v)$ are differentiable in a region, the *Jacobian determinant,* or briefly the *Jacobian*, of F and G with respect to u and v is the second-order functional determinant defined by

$$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}.$$

Similarly, the third-order determinant

$$\frac{\partial(F,G,H)}{\partial(u,v,w)} = \begin{vmatrix} F_u & F_v & F_w \\ G_u & G_v & G_w \\ H_u & H_v & H_w \end{vmatrix}$$

is called the Jacobian of F, G and H with respect to u, v and w.

1.3 The Inverse Function Theorem

Theorem 1.3.1. *(Inverse Function Theorem) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on some open set containing a, and suppose Jacobian of $f(a)$ not equal to zero. Then there is some open set V containing a and an open W containing $f(a)$ such that $f : V \to W$ has a continuous inverse $f^{-1} : W \to V$ which is differentiable for all $y \in W$.*

1.4 Point transformations

Definition 1.4.1. A transformation

$$\begin{aligned} t &= \varphi(x,y), \\ u &= \psi(x,y), \end{aligned} \tag{1.1}$$

where φ and ψ are sufficiently smooth functions is called a *point transformation.*

1.4.1 The mapping of a function by a point transformation

Assume that $y_0(x)$ is a given function. To obtain the transformed function $u_0(t)$, start with the equation

$$t = \varphi(x, y_0(x)).$$

Using Inverse Function Theorem, we can express x as $x = \alpha(t)$. Substituting x into the function $\psi(x, y_0(x))$, we get the transformed function

$$u_0(t) = \psi(\alpha(t), y_0(\alpha(t))).$$

Conversely, we have to change $u_0(t)$ to $y_0(x)$. Applying the Inverse Function Theorem to point transformations (2.1), we obtain

$$\begin{aligned} x &= \widetilde{\varphi}(t,u), \\ y &= \widetilde{\psi}(t,u). \end{aligned} \tag{1.2}$$

Let $u_0(t)$ be a given function of t. The first equation of (2.2) becomes

$$x = \widetilde{\varphi}(t, u_0(t)).$$

Using the Inverse Function Theorem, we find $t = H(x)$. Substituting t into the function $\widetilde{\psi}(t, u_0(t))$ the transformed function $y_0(x) = \widetilde{\psi}(H(x), u_0(H(x)))$ is obtained.

1.5 Generalized Sundman transformations

Definition 1.5.1. A non-point transformation

$$\begin{aligned} X &= F(t,x), \\ dT &= G(t,x)dt, \end{aligned} \tag{1.3}$$

where $F_x G \neq 0$ is called a *generalized Sundman transformation.*

1.5.1 The mapping of a function by a generalized Sundman transformation

Let us explain how a generalized Sundman transformation maps one function into another.

Assume that $x_0(t)$ is a given function. Integrating the second equation of (2.3), we obtain $T = Q(t)$, where

$$Q(t) = T_0 + \int_{t_0}^{t} G(s, x_0(s))ds$$

with some initial conditions T_0 and t_0. Using The Inverse Function Theorem, we find $t = Q^{-1}(T)$. Substituting t into the function $F(t, x_0(t))$, we get the transformed function

$$u_0(T) = F(Q^{-1}(T), x_0(Q^{-1}(T))).$$

Conversely, let $u_0(T)$ be a given function of T. Using The Inverse Function Theorem we solve the equation

$$u_0(T) = F(t, x)$$

with respect to $x : x = \phi(t, T)$. Solving the ordinary differential equation

$$\frac{dT}{dt} = G(t, \phi(t, T)),$$

we find $T = H(t)$. The function $H(t)$ can be written as an action of a functional $H = \mathcal{L}(u_0)$. Substituting $T = H(t)$ into the function $\phi(t, T)$, the transformed function $x_0(t) = \phi(t, H(t))$ is obtained.

Notice that for the case $G_x = 0$ the action of the functional $\mathcal{L}$ does not depend on the function $u_0(T)$. In this case the generalized Sundman transformation becomes a point transformation. Conversely, since for a point transformation the value dT in the generalized Sundman transformation is the total differential of T, then the compatibility condition for dT to be a total differential leads to the equation $G_x = 0$. Hence,the generalized Sundman transformation is a point transformation if and only if $G_x = 0$.

Formulae (2.3) also allows us to obtain the derivatives of $u_0(T)$ through the derivatives of the function $x_0(t)$, and vice versa.

Hence, using transformation (2.3), we can relate the solutions of two differential equations $Q(t, x, x', ..., x^{(n)}) = 0$ and $P(T, X, X', ..., X^{(n)}) = 0$. Therefore the knowledge of the general solution of one of them gives the general solution of the other equation, up to solving one ordinary differential equation of first-order and finding two inverse functions.

1.6 Clairaut's theorem on equality of mixed partials

1.6.1 Statement for second-order mixed partial of function of two variables

Suppose f is a real-valued function of two variables x, y and $f(x, y)$ is defined on an open subset U of $\mathbb{R}^2$. Suppose further that both the second-order mixed partial derivatives $f_{xy}(x, y)$ and $f_{yx}(x, y)$ exist and are continuous on U. Then, we have:

$$f_{xy} = f_{yx}$$

on all of U.

1.6.2 General statement

The statement can be generalized in two ways:

- We can generalize it to higher-order partial derivatives.
- We can generalize it to functions of more than two variables.

The general version states the following. Suppose f is a function of n variables defined on an open subset U of $\mathbb{R}^n$. Suppose all mixed partials with a certain number of differentiations in each input variable exist and are continuous on U. Then, all the mixed partials are continuous.

Some examples are given below:

- Suppose f is a function of two variables x and y, and the three mixed partials $f_{xxy}, f_{xyx}, f_{yxx}$ exist and are continuous on an open subset U of $\mathbb{R}^2$. Then, all three of them are equal on U. (Note that these mixed partials all involve differentiating twice with respect to x and once with respect to y).

- Suppose f is a function of three variables x, y, z, and the six mixed partials f_{xyz}, f_{xzy}, f_{yxz}, f_{yzx}, f_{zxy}, f_{zyx} exist and are continuous on an open subset U of $\mathbb{R}^3$. Then, all six of them are equal on U.

1.7 Completely integrable systems

One class of overdetermined systems, for which the problem of compatibility is solved, is the class of completely integrable systems. The theory of completely integrable systems is developed in the general case.

Definition 1.7.1. A system

$$\frac{\partial z^i}{\partial a^j} = f^i_j(a, z), \ (i = 1, 2, ..., N;\ j = 1, 2, ..., r) \tag{1.4}$$

is called *completely integrable* if it has a solution for any initial values a_0, z_0 in some open domain D.

Theorem 1.7.2. *A system of the type (1.26) is completely integrable if and only if all of the mixed derivatives equalities*

$$\frac{\partial f^i_j}{\partial a^\beta} + \sum_{\gamma=1}^{N} f^\gamma_\beta \frac{\partial f^i_j}{\partial z^\gamma} = \frac{\partial f^i_\beta}{\partial a^j} + \sum_{\gamma=1}^{N} f^\gamma_j \frac{\partial f^i_\beta}{\partial z^\gamma}, \ (i = 1, 2, ..., N;\ \beta, j = 1, 2, ..., r) \tag{1.5}$$

are identically satisfied with respect to the variable $(a, z) \in D$.

In practice, sometimes it is enough to use a particular case of the compatibility theorem:

Corollary 1.7.3. *If in an overdetermined system of partial differential equations all derivatives of order n are defined and comparison of all mixed derivatives of order $n+1$ does not produre new equations of order less or equal to n, then this system is compatible.*

1.8 Laguerre canonical form

Theorem 1.8.1. *(Laguerre-Forsyth canonical form) Any second-order linear ordinary differential equations*

$$y'' + a_1(x)y' + a_0(x)y = 0$$

can be transformed to the form

$$u'' = 0$$

by point transformation (1.1).

Theorem 1.8.2. *(Laguerre-Forsyth canonical form) Any kth-order linear ordinary differential equations*

$$y^{(k)} + \sum_{i=0}^{k-1} a_i(x) y^{(i)} = 0, \qquad k \geq 3$$

can be transformed to the form

$$u^{(k)} + \sum_{i=0}^{k-3} a_i(x) u^{(i)} = 0 \tag{1.6}$$

by point transformation (1.1).

Note that (1.6) is called the *Laguerre-Forsyth canonical form* of kth-order linear ordinary differential equations.

1.9 Linearization of second-order ordinary differential equations by point transformation

Theorem 1.9.1. ([1], S. Lie) *Any second-order ordinary differential equations $y'' = f(x, y, y')$ obtained from a linear equation*

$$u'' = 0 \tag{1.7}$$

by a point transformation (1.1) has to be either to the form

$$y'' + a(x,y)\, y'^3 + b(x,y)\, y'^2 + c(x,y)\, y' + d(x,y) = 0, \tag{1.8}$$

where

$$\begin{aligned} a &= \Delta^{-1}\left(\varphi_y \psi_{yy} - \varphi_{yy}\psi_y\right), \\ b &= \Delta^{-1}\left(\varphi_x \psi_{yy} - \varphi_{yy}\psi_x + 2\left(\varphi_y \psi_{xy} - \varphi_{xy}\psi_y\right)\right), \\ c &= \Delta^{-1}\left(\varphi_y \psi_{xx} - \varphi_{xx}\psi_y + 2\left(\varphi_x \psi_{xy} - \varphi_{xy}\psi_x\right)\right), \\ d &= \Delta^{-1}\left(\varphi_x \psi_{xx} - \varphi_{xx}\psi_x\right) \end{aligned} \tag{1.9}$$

and $\Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0$.

Proof.

Since $u = \psi(x, y)$, thus

$$\begin{aligned} u'(t) &= \tfrac{D_x\psi}{D_x\varphi} \\ &= \tfrac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y} \\ &= g(x, y, y'), \end{aligned}$$

$$\begin{aligned} u''(t) &= \tfrac{D_x g}{D_x\varphi} \\ &= \tfrac{g_x + y' g_y + y'' g_{y'}}{\varphi_x + y'\varphi_y} \\ &= P(x, y, y') \end{aligned} \tag{1.10}$$

where

$$\begin{aligned} g_x &= \frac{(\varphi_x + y'\varphi_y)\frac{\partial}{\partial x}(\psi_x + y'\psi_y) - (\psi_x + y'\psi_y)\frac{\partial}{\partial x}(\varphi_x + y'\varphi_y)}{(\varphi_x + y'\varphi_y)^2} \\ &= \frac{(\varphi_x + y'\varphi_y)\left(\psi_{xx} + \left(y'\frac{\partial}{\partial x}\psi_y + \psi_y\frac{\partial}{\partial x}y'\right)\right) - (\psi_x + y'\psi_y)\left(\varphi_{xx} + \left(y'\frac{\partial}{\partial x}\varphi_y + \varphi_y\frac{\partial}{\partial x}y'\right)\right)}{(\varphi_x + y'\varphi_y)^2} \\ &= \frac{(\varphi_x + y'\varphi_y)(\psi_{xx} + y'\psi_{xy} + \psi_y(0)) - (\psi_x + y'\psi_y)(\varphi_{xx} + y'\varphi_{xy} + \varphi_y(0))}{(\varphi_x + y'\varphi_y)^2} \\ &= \frac{(\varphi_x + y'\varphi_y)(\psi_{xx} + y'\psi_{xy}) - (\psi_x + y'\psi_y)(\varphi_{xx} + y'\varphi_{xy})}{(\varphi_x + y'\varphi_y)^2}, \end{aligned}$$

$$\begin{aligned} g_y &= \frac{(\varphi_x + y'\varphi_y)\frac{\partial}{\partial y}(\psi_x + y'\psi_y) - (\psi_x + y'\psi_y)\frac{\partial}{\partial y}(\varphi_x + y'\varphi_y)}{(\varphi_x + y'\varphi_y)^2} \\ &= \frac{(\varphi_x + y'\varphi_y)\left(\psi_{xy} + \left(y'\frac{\partial}{\partial y}\psi_y + \psi_y\frac{\partial}{\partial y}y'\right)\right) - (\psi_x + y'\psi_y)\left(\varphi_{xy} + \left(y'\frac{\partial}{\partial y}\varphi_y + \varphi_y\frac{\partial}{\partial y}y'\right)\right)}{(\varphi_x + y'\varphi_y)^2} \\ &= \frac{(\varphi_x + y'\varphi_y)(\psi_{xy} + y'\psi_{yy} + \psi_y(0)) - (\psi_x + y'\psi_y)(\varphi_{xy} + y'\varphi_{yy} + \varphi_y(0))}{(\varphi_x + y'\varphi_y)^2} \\ &= \frac{(\varphi_x + y'\varphi_y)(\psi_{xy} + y'\psi_{yy}) - (\psi_x + y'\psi_y)(\varphi_{xy} + y'\varphi_{yy})}{(\varphi_x + y'\varphi_y)^2}, \end{aligned}$$

$$\begin{aligned} g_{y'} &= \frac{(\varphi_x + y'\varphi_y)\frac{\partial}{\partial y'}(\psi_x + y'\psi_y) - (\psi_x + y'\psi_y)\frac{\partial}{\partial y'}(\varphi_x + y'\varphi_y)}{(\varphi_x + y'\varphi_y)^2} \\ &= \frac{(\varphi_x + y'\varphi_y)\left(0 + \left(y'\frac{\partial}{\partial y'}\psi_y + \psi_y\frac{\partial}{\partial y'}y'\right)\right) - (\psi_x + y'\psi_y)\left(0 + \left(y'\frac{\partial}{\partial y'}\varphi_y + \varphi_y\frac{\partial}{\partial y'}y'\right)\right)}{(\varphi_x + y'\varphi_y)^2} \\ &= \frac{(\varphi_x + y'\varphi_y)(0 + y'(0) + \psi_y(1)) - (\psi_x + y'\psi_y)(0 + y'(0) + \varphi_y(1))}{(\varphi_x + y'\varphi_y)^2} \\ &= \frac{(\varphi_x + y'\varphi_y)(\psi_y) - (\psi_x + y'\psi_y)(\varphi_y)}{(\varphi_x + y'\varphi_y)^2} \end{aligned}$$

and $D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + ...$ is total derivatives. Replacing $g_x, g_y, g_{y'}$ into equation (1.10) one gets

$$\begin{aligned} u'' = \; & (y''(\varphi_x\psi_y - \varphi_y\psi_x) + y'^3(\varphi_y\psi_{yy} - \varphi_{yy}\psi_y) + y'^2(\varphi_x\psi_{yy} - \varphi_{yy}\psi_x \\ & +2(\varphi_y\psi_{xy} - \varphi_{xy}\psi_y)) + y'(\varphi_y\psi_{xx} - \varphi_{xx}\psi_y \\ & +2(\varphi_x\psi_{xy} - \varphi_{xy}\psi_x)) + \varphi_x\psi_{xx} - \varphi_{xx}\psi_x)/(\varphi_x + y'\varphi_y)^3. \end{aligned} \tag{1.11}$$

Since the jacobian $\Delta \neq 0$, then after replacing u'' into equation (1.7) one gets equation (1.8).

Theorem 1.9.2. ([1], S. Lie) *Equation (1.8) is linearizable by point transformation (1.1) if and only if its coefficients satisfied the follows.*

(a) *If $\varphi_y = 0$ then the conditions are*

$$a = 0, \; c_y = 2b_x, \; d_{yy} - b_{xx} - b_x c + b_y d + d_y b = 0. \tag{1.12}$$

(b) *If $\varphi_y \neq 0$ then the conditions are*

$$\begin{aligned} &3a_{xx} - 2b_{xy} + c_{yy} - 3a_x c + 3a_y d + 2b_x b - 3c_x a - c_y b + 6d_y a = 0, \\ &b_{xx} - 2c_{xy} + 3d_{yy} - 6a_x d + b_x c + 3b_y d - 2c_y c - 3d_x a + 3d_y b = 0. \end{aligned} \tag{1.13}$$

Proof. We will find conditions from system of equation (1.9).
Case 1 : $\varphi_y = 0$.
From (1.9) one gets that

$$\begin{aligned} a &= 0, \\ b &= (\varphi_x\psi_y)^{-1}(\varphi_x\psi_{yy}) = \frac{\varphi_x\psi_{yy}}{\varphi_x\psi_y}, \\ c &= (\varphi_x\psi_y)^{-1}(-\varphi_{xx}\psi_y + 2\varphi_x\psi_{xy}) = \left(-\varphi_x{}^{-1}\varphi_{xx} + 2\psi_y{}^{-1}\psi_{xy}\right), \\ d &= (\varphi_x\psi_y)^{-1}(\varphi_x\psi_{xx} - \varphi_{xx}\psi_x) = \frac{\psi_{xx}}{\psi_y} - \frac{\varphi_{xx}\psi_x}{\varphi_x\psi_y} \end{aligned}$$

thus

$$a = 0, \; \psi_{yy} = \psi_y b, \; \psi_{xy} = \frac{1}{2}\left(\varphi_x{}^{-1}\psi_y\varphi_{xx} + \psi_y c\right), \; \psi_{xx} = \varphi_x{}^{-1}\psi_x\varphi_{xx} + \psi_y d. \tag{1.14}$$

Mixing the derivatives:

- $(\psi_{xy})_y = (\psi_{yy})_x$

$$\frac{1}{2}\left[\frac{\varphi_{xx}\psi_{yy}}{\varphi_x} + c\psi_{yy} + c_y\psi_y\right] = \psi_{xy}b + \psi_y b_x$$

$$\frac{\varphi_{xx}\psi_{yy}}{\varphi_x} + c\psi_{yy} + c_y\psi_y = 2\left(\psi_{xy}b + \psi_y b_x\right)$$

replacing (1.14), one gets

$$\begin{aligned}
\frac{\varphi_{xx}\psi_y b}{\varphi_x} + c\psi_y b + c_y\psi_y &= 2\left(\frac{1}{2}\left(\frac{\psi_y\varphi_{xx}}{\varphi_x} + \psi_y c\right)b + \psi_y b_x\right)\\
&= \frac{\psi_y\varphi_{xx}b}{\varphi_x} + c\psi_y b + 2\psi_y b_x\\
c_y\psi_y &= 2\psi_y b_x\\
c_y &= 2b_x
\end{aligned}$$

- $(\psi_{xy})_x = (\psi_{xx})_y$

$$\varphi_x^{-2}\left(2\varphi_x\varphi_{xxx} - 3\varphi_{xx}^2\right) = 4\left(d_y + bd\right) - \left(2c_x + c^2\right). \qquad (1.15)$$

Since $\varphi_y = 0$, then differentiating (1.15) respect to y one arrives at

$$d_{yy} - b_{xx} - b_x c + b_y d + d_y b = 0.$$

Hence a second-order ordinary differential equations in the form (1.8) can be linearizable by function $\varphi = \varphi(x)$ if and only if it's coefficients satisfied (1.12).

Case 2 : $\varphi_y \neq 0$.
Consider (1.9).

From $a = \Delta^{-1}\left(\varphi_y\psi_{yy} - \varphi_{yy}\psi_y\right)$, one gets

$$\psi_{yy} = \frac{\left(\varphi_{yy}\psi_y + a\Delta\right)}{\varphi_y}.$$

From $b = \Delta^{-1}\left(\varphi_x\psi_{yy} - \varphi_{yy}\psi_x + 2\left(\varphi_y\psi_{xy} - \varphi_{xy}\psi_y\right)\right)$, one gets

$$\psi_{xy} = \frac{2\varphi_{xy}\varphi_y\psi_y - \varphi_{yy}\Delta - \left(a\varphi_x - b\varphi_y\right)\Delta}{2\varphi_y^2}.$$

From $c = \Delta^{-1}\left(\varphi_y\psi_{xx} - \varphi_{xx}\psi_y + 2\left(\varphi_x\psi_{xy} - \varphi_{xy}\psi_x\right)\right)$, one gets

$$\psi_{xx} = \frac{2\varphi_{xy}\varphi_y\psi_x - \varphi_x\varphi_{yy}\psi_x - \varphi_x^2\psi_x a + \varphi_x\varphi_y\psi_x b + \varphi_y^2\left(\psi_y d - \psi_x c\right)}{\varphi_y^2}.$$

From $d = \Delta^{-1}\left(\varphi_x\psi_{xx} - \varphi_{xx}\psi_x\right)$, one gets

$$\varphi_{xx} = \left(\frac{2\varphi_{xy}\varphi_x\varphi_y - \varphi_x^2\varphi_{yy} - \varphi_x^3 a + \varphi_x^2\varphi_y b - \varphi_x\varphi_y^2 c + \varphi_y^3 d}{\varphi_y^2}\right).$$

Therefore, by the relations (1.9) we have

$$\begin{aligned}
\psi_{yy} &= \frac{(\varphi_{yy}\psi_y + a\Delta)}{\varphi_y}, \\
\psi_{xy} &= \frac{2\varphi_{xy}\varphi_y\psi_y - \varphi_{yy}\Delta - (a\varphi_x - b\varphi_y)\Delta}{2\varphi_y^2}, \\
\psi_{xx} &= \frac{2\varphi_{xy}\varphi_y\psi_x - \varphi_x\varphi_{yy}\psi_x - \varphi_x^2\psi_x a + \varphi_x\varphi_y\psi_x b + \varphi_y^2(\psi_y d - \psi_x c)}{\varphi_y^2}, \\
\varphi_{xx} &= \left(\frac{2\varphi_{xy}\varphi_x\varphi_y - \varphi_x^2\varphi_{yy} - \varphi_x^3 a + \varphi_x^2\varphi_y b - \varphi_x\varphi_y^2 c + \varphi_y^3 d}{\varphi_y^2}\right).
\end{aligned} \tag{1.16}$$

Mixing the derivatives :

- $(\psi_{xy})_y = (\psi_{yy})_x$

$$\begin{aligned}
2\varphi_y\varphi_{yyy} &= 3\left(\varphi_{yy}^2 - 2\varphi_{xy}\varphi_y a + 2\varphi_x\varphi_{yy} a + \varphi_x^2 a^2\right) - 2\varphi_x\varphi_y\left(a_y + ab\right) \\
&\quad + \varphi_y^2\left(2b_y - 4a_x + 4ac - b^2\right) \\
\varphi_{yyy} &= (3\left(\varphi_{yy}^2 - 2\varphi_{xy}\varphi_y a + 2\varphi_x\varphi_{yy} a + \varphi_x^2 a^2\right) - 2\varphi_x\varphi_y\left(a_y + ab\right) \\
&\quad + \varphi_y^2\left(2b_y - 4a_x + 4ac - b^2\right))/(2\varphi_y)
\end{aligned}$$

- $(\psi_{xy})_x = (\psi_{xx})_y$

$$\begin{aligned}
6\varphi_y^2\varphi_{xyy} &= 3\left(4\varphi_{xy}\varphi_{yy}\varphi_y - \varphi_x\varphi_{yy}^2 + 2\varphi_x\varphi_{yy}\varphi_y b - 2\varphi_{xy}\varphi_y^2 b\right) \\
&\quad + 3\varphi_x^3 a^2 + 3\varphi_x\varphi_y^2\left(-2a_x + 2ac - b^2\right) + 2\varphi_y^3\left(-b_x + 2c_y + 3ad\right) \\
\varphi_{xyy} &= (3\left(4\varphi_{xy}\varphi_{yy}\varphi_y - \varphi_x\varphi_{yy}^2 + 2\varphi_x\varphi_{yy}\varphi_y b - 2\varphi_{xy}\varphi_y^2 b\right) + 3\varphi_x^3 a^2 \\
&\quad + 3\varphi_x\varphi_y^2\left(-2a_x + 2ac - b^2\right) + 2\varphi_y^3\left(-b_x + 2c_y + 3ad\right))/(6\varphi_y^2).
\end{aligned}$$

Mixing the derivative again :

- $(\varphi_{xyy})_y = (\varphi_{yyy})_x$ and $(\varphi_{xx})_{yy} = (\varphi_{xyy})_x$ one obtains (1.13). □

1.10 Linearization of second-order ordinary differential equations by a generalized Sundman transformation

Theorem 1.10.1. *Any second-order ordinary differential equations $x'' = f(t, x, x')$ obtained from a linear equation*

$$X'' = 0 \tag{1.17}$$

by a generalized Sundman transformation (1.3) has to be either to the form

$$x'' + A_2\,(t,x)\,x'^2 + A_1\,(t,x)\,x' + A_0\,(t,x) = 0, \tag{1.18}$$

where

$$\begin{aligned} A_2 &= (F_{xx}G - F_xG_x)\,/K, \\ A_1 &= (2F_{tx}G - F_tG_x - F_xG_t)\,/K, \\ A_0 &= (F_{tt}G - F_tG_t)\,/K \end{aligned} \tag{1.19}$$

with $K = GF_x \neq 0$.

Proof. Since $X = F(t,x)$, thus

$$\begin{aligned} X'(x) &= \frac{D_tF(t,x)}{D_t\int G(t,x)dt} \\ &= \frac{F_t + x'F_x}{G} \end{aligned}$$

$$\begin{aligned} X''(x) =& \frac{D_t\left(\frac{F_t+x'F_x}{G}\right)}{D_t\int G(t,x)dt} \\ =& \frac{2F_{tx}Gx' + F_{tt}G - F_tG_t - F_tG_xx' + F_{xx}Gx'^2 - F_xG_tx' - F_xG_xx'^2 + F_xGx''}{G^3} \end{aligned} \tag{1.20}$$

where $D_t = \frac{\partial}{\partial t} + x'\frac{\partial}{\partial x} + x''\frac{\partial}{\partial x'} + ...$ is total derivatives.

From equation (1.20) one gets

$$\begin{aligned} X''(x) =& \ [x''(F_xG) + x'^2(F_{xx}G - F_xG_x) + x'(2F_{tx}G - F_tG_x - F_xG_t) \\ & +(F_{tt}G - F_tG_t)]/G^3. \end{aligned} \tag{1.21}$$

Replacing X'' into equation (1.17) one gets equation.

$$x''(F_xG) + x'^2(F_{xx}G - F_xG_x) + x'(2F_{tx}G - F_tG_x - F_xG_t) + (F_{tt}G - F_tG_t) = 0.$$

Dividing this equation by F_xG, we get

$$x'' + \left(\frac{F_{xx}G - F_xG_x}{F_xG}\right)x'^2 + \left(\frac{2F_{tx}G - F_tG_x - F_xG_t}{F_xG}\right)x' + \left(\frac{F_{tt}G - F_tG_t}{F_xG}\right) = 0.$$

Setting $K = F_xG$, so that one obatins the necessary form as equation (1.18) where $A_i;$, $i = 0,1,2$ are as system (1.19).

□

Theorem 1.10.2. *Equation (1.18) is linearizable by a generalized Sundman transformation (1.3) if and only if its coefficients satisfied the follows.*

(i) *If* $S_1(t,x) = 0$ *then the conditions are*

$$\begin{aligned} &S_1(t,x) = A_{1x} - 2A_{2t} = 0,\\ &S_2(t,x) = -2A_{0xx} - 2A_{0x}A_2 + 2A_{2tt} + 2A_{2t}A_1 - 2A_{2x}A_0 + 2S_{1t} + A_1S_1 = 0. \end{aligned} \tag{1.22}$$

(ii) *If* $S_1(t,x) \neq 0$ *then the conditions are*

$$\begin{aligned} &S_1^3 + S_{1x}S_2 - S_{2x}S_1 = 0,\\ &4A_{0x}S_1^2 - 2A_{1t}S_1^2 + 2S_{1t}S_2 + 4A_0A_2S_1^2 - A_1^2S_1^2 + S_2^2 - 2S_1S_{2t} = 0. \end{aligned} \tag{1.23}$$

Proof. From system of equation (1.19), one obtains the derivatives

$$\begin{aligned} F_{xx} &= \tfrac{A_2F_xG+F_xG_x}{G},\\ F_{tx} &= \tfrac{A_1F_xG+F_tG_x+F_xG_t}{2G},\\ F_{tt} &= \tfrac{A_0F_xG+F_tG_t}{G}. \end{aligned} \tag{1.24}$$

Mixing the derivatives:

- $(F_{xx})_t = (F_{tx})_x$

$$\begin{aligned} G_{tx} = \;& (2A_{1x}F_xG^2 - 4A_{2t}F_xG^2 + 2F_tG_{xx}G - 3F_tG_x^2\\ & -2F_tG_xA_2G + 3F_xG_tG_x + F_xG_xA_1G)/(2F_xG). \end{aligned}$$

- $(F_{tx})_x = (F_{tt})_x$

$$\begin{aligned} G_{tt} = \;& (4A_{0x}F_x^2G^2 - 2A_{1t}F_x^2G^2 + 2A_{1x}F_tF_xG^2 - 4A_{2t}F_tF_xG^2\\ & +2F_t^2G_{xx}G - 3F_t^2G_x^2 - 2F_t^2G_xA_2G + 3F_x^2G_t^2 + 2F_x^2G_xA_0G\\ & +4F_x^2A_0A_2G^2 - F_x^2A_1^2G^2)/(2F_x^2G). \end{aligned}$$

Mixing the derivatives again:

- $(G_{tx})_t = (G_{tt})_x$

$$\begin{aligned} &F_tG_x(A_{1x} - 2A_{2t}) + F_xG_t(-A_{1x} + 2A_{2t})\\ &+F_xG(-2A_{0xx} - 2A_{0x}A_2 + 2A_{1tx} + A_{1x}A_1 - 2A_{2tt} - 2A_{2x}A_0) = 0. \end{aligned} \tag{1.25}$$

From equation (1.25) setting $S_1(t,x)$ and $S_2(t,x)$ as the following

$$S_1(t,x) = A_{1x} - 2A_{2t},$$
$$S_2(t,x) = -2A_{0xx} - 2A_{0x}A_2 + 2A_{2tt} + 2A_{2t}A_1 - 2A_{2x}A_0 + 2S_{1t} + A_1S_1.$$

Substituting S_1 and, S_2 into equation (1.25) one gets

$$F_tG_xS_1 - F_xG_tS_1 + F_xGS_2 = 0. \tag{1.26}$$

Case 1 : $S_1 = 0$.
From equation (1.26) one obtains that

$$F_xGS_2 = 0$$

so that one gets the condition

$$S_2 = 0.$$

Case 2 : $S_1 \neq 0$.
From equation (1.26) one finds

$$G_t = (F_tG_xS_1 + F_xGS_2)/(F_xS_1).$$

Substituting G_t into G_{tt} and G_{tx} one arrives at the conditions

$$4A_{0x}S_1^2 - 2A_{1t}S_1^2 + 2S_{1t}S_2 + 4A_0A_2S_1^2 - A_1^2S_1^2 + S_2^2 - 2S_1S_{2t} = 0,$$

$$S_1^3 + S_{1x}S_2 - S_{2x}S_1 = 0$$

respectively.

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Chapter 2

Linearizability of nonlinear fourth-order ordinary differential equations by a generalized Sundman transformation

2.1 Introduction

2.1.1 Introduction to the research problem and its significance

Nonlinear problems are of interest to engineers, physicists, mathematicians and many other scientists because most equations are inherently nonlinear in nature. Nonlinear equations are difficult to solve. Although there are a number of well defined methods for the solution of linear ordinary differential equations, the same however, cannot be said in the case of nonlinear ordinary differential equations. While solving problems related to nonlinear ordinary differential equations it is often expedient to simplify equations by a suitable change of variables. In particular the possibility that a given equation could be linearized, i.e. transformed to a linear equation.

Two given ordinary differential equations are called equivalent if one can be transformed into the other by a change of variables. The equivalence problem consists of two parts: deciding equivalence and determining a transformation that connects the ordinary differential equations. If the given equation is a linear equation, then the equivalence problem is a linearization problem. Our motivation for considering this problem is to translate a known solution of an ordinary differential equation to solutions of ordinary differential equations which are equivalent to it, thus allowing a systematic use of collections of solved ordinary differential equations. A linear equation was a most attractive proposition due to the special properties of linear differential equations. The reduction of an ordinary differential equation to a linear ordinary differential equation besides simplification allows constructing an exact solution of the original equation. Analytical (exact) solution has value, firstly, as an exact description of a real process in the framework of a given model; secondly, as a model to compare various numerical methods; thirdly, as a basis to improve the models used. Therefore, the linearization problem plays a significant role in the nonlinear problem.

Point transformation, contact transformation, reduction of order, differential substitution, Generalized Sundman transformation etc. are some of the tools commonly used for solving the linearization problem. Transformations used for solving the linearization problem considered in this project employ generalized Sundman transformations.

2.1.2 Historical review

In 1883, the problem on linearization of second-order ordinary differential equations by means of point transformations was solved by S.Lie [1]. He showed that any linearizable second-order equation can be at most cubic in the first-order derivative, and gave the linearization test in term of the coefficients of these equations. Lie's approach has also been applied to third-order and fourth-order ordinary differential equations.

Another approach was developed by Cartan [2]. He used differential geometry for solving the linearization problem.

In 1992, the first generalized Sundman transformations was proposed by Sundman.

In 1994, Duarte, Moreira and Santos [3] considered the problem of linearization of second-order ordinary differential equations by means of generalized Sundman transformations to the Laguerre form.

In 2001, Berkovich [4] considered some application of generalized Sundman transformations to ordinary differential equations and earlier paper, which are summarized in Berkovich [5].

In 2003, Euler, Wolf, Leach and Euler [6] solved the problem of linearization of third-order ordinary differential equations. They found necessary and sufficient conditions for a third-order ordinary differential equations to be equivalent to $X'''(T) = 0$ under generalized Sundman transformations.

In 2010, Muriel and Romero [7] studied the class of nonlinear second-order equations that are linearizable by means of generalized Sundman transformations is identified as the class of equations admitting first integrals that are polynomials of first degree in the first-order derivative. Nakpim and Meleshko [8] showed that solution of the linearization problem for a second-order ordinary differential equation via the generalized Sundman transformation was considered earlier by Duarte, Moreira and Santos using the Laguerre form is not complete.

In 2013, Mustafa, AL-Dweik and Mara'beh [9] considered the linearization problem for nonlinear second-order ordinary differential equations to the Laguerre form by means of generalized Sundman transformations. They gave a new characterization of S-linearizable equations in terms of the coefficients of ordinary differential equations and one auxiliary function. This new criterion is used to obtain the general solutions for the first integral explicitly, providing a direct alternative procedure for constructing the first integrals and Sundman transformations.

Sundman symmetries were first introduced by Euler, Wolf, Leach and Euler [6] in 2003. They discovered that all third-order ordinary differential equations that can be linearized to the equation $X'''(T) = 0$ by the generalized Sundman transformation

$$X(T) = F(t,x),\ dT = G(t,x)dt,\ (F_xG \neq 0)$$

admit the symmetry

$$F(\tilde{t},\tilde{x}) = F^{-1}(t,x),\ G(\tilde{t},\tilde{x})d\tilde{t} = F^{-3/2}(t,x)G(t,x)dt$$

called a Sundman symmetry transformation. In 2004 [10], Euler and Euler investigated the Sundman symmetries of second-order autonomous equations

$$X'' + a_2X'^2 + a_1X' + a_0 = 0$$

where a_0, a_1 and a_2 are differentiable functions. Moreover, they found the Sundman symmetries of third-order autonomous equations

$$X''' + a_5X''^2 + a_4X'X'' + a_3X'^3 + a_2X'^2 + a_1X' + a_0 = 0$$

where $a_j (j = 0, 1, ..., 5)$ are differentiable functions.

The main goal of the present paper is to demonstrate possibilities of applications generalized Sundman transformations for a linearization problem. In the paper generalized Sundman transformations are applied for linearizing fourth-order ordinary differential equations. Complete study of this transformations mapping equations to the trivial fourth-order ordinary differential equation $X^{(4)}(T) = 0$ is given in the paper.

The manuscript is organized as follows. In section 2, the necessary conditions of linearization of a fourth-order ordinary differential equation are presented. In section 3, we state the theorems that yield criteria for a fourth-order ordinary differential equation to be linearizable via generalized Sundman transformations. Relations between coefficients of a linearizable equation and generalized Sundman transformations reducing this equation into a linear equation are presented in this section. Examples which illustrate the procedure of using the linearization theorems and some applications are presented in section 4. For the sake of simplicity of reading, cumbersome formulae of this section are moved into Appendix.

2.2 Necessary conditions of linearization

We begin with investigating the necessary conditions for linearization. We consider the fourth-order ordinary differential equations

$$x^{(4)} = f(t, x, x', x'', x''') \tag{2.1}$$

which can be transformed to a linear equation

$$X^{(4)}(T) = 0 \tag{2.2}$$

under the generalized Sundman transformation

$$\begin{aligned} X &= F(t, x), \\ dT &= G(t, x)\,dt. \end{aligned} \tag{2.3}$$

So that we arrive at the following theorem.

Theorem 2.2.1. *Any system of fourth-order ordinary differential equations (3.3) obtained from a linear system (3.5) by a generalized Sundman transformation (3.4) has to be the form*

$$\begin{aligned} x^{(4)} + (A_1x' + A_0)x''' + Bx''^2 + (C_2x'^2 + C_1x' + C_0)x'' \\ + D_4x'^4 + D_3x'^3 + D_2x'^2 + D_1x' + D_0 = 0 \end{aligned} \tag{2.4}$$

where

$$A_1 =(4F_{xx}G - 7F_xG_x)/(F_xG), \tag{2.5}$$

$$A_0 =(4F_{tx}G - F_tG_x - 6F_xG_t)/(F_xG), \tag{2.6}$$

$$B =(3F_{xx}G - 4F_xG_x)/(F_xG), \tag{2.7}$$

$$C_2 =(6F_{xxx}G^2 - 22F_{xx}G_xG - 7F_xG_{xx}G + 25F_xG_x^2)/(F_xG^2), \tag{2.8}$$

$$C_1 =(12F_{txx}G^2 - 26F_{tx}G_xG - 3F_tG_{xx}G + 10F_tG_x^2 - 18F_{xx}G_tG - 11F_xG_{tx}G + 40F_xG_tG_x)/(F_xG^2), \tag{2.9}$$

$$C_0 =(-18F_{tx}G_tG + 6F_{ttx}G^2 - 4F_{tt}G_xG - 3F_tG_{tx}G + 10F_tG_tG_x - 4F_xG_{tt}G + 15F_xG_t^2)/(F_xG^2), \tag{2.10}$$

$$D_4 =(F_{xxxx}G^3 - 6F_{xxx}G_xG^2 - 4F_{xx}G_{xx}G^2 + 15F_{xx}G_x^2G - F_xG_{xxx}G^2 + 10F_xG_{xx}G_xG - 15F_xG_x^3)/(F_xG^3), \tag{2.11}$$

$$D_3 =(4F_{txxx}G^3 - 18F_{txx}G_xG^2 - 8F_{tx}G_{xx}G^2 + 30F_{tx}G_x^2G - F_tG_{xxx}G^2 + 10F_tG_{xx}G_xG - 15F_tG_x^3 - 6F_{xxx}G_tG^2 - 8F_{xx}G_{tx}G^2 + 30F_{xx}G_tG_xG - 3F_xG_{txx}G^2 + 20F_xG_{tx}G_xG + 10F_xG_tG_{xx}G - 45F_xG_tG_x^2)/(F_xG^3), \tag{2.12}$$

$$D_2 =(-18F_{txx}G_tG^2 - 16F_{tx}G_{tx}G^2 + 60F_{tx}G_tG_xG + 6F_{ttxx}G^3 - 18F_{ttx}G_xG^2 - 4F_{tt}G_{xx}G^2 + 15F_{tt}G_x^2G - 3F_tG_{txx}G^2 + 20F_tG_{tx}G_xG + 10F_tG_tG_{xx}G - 45F_tG_tG_x^2 - 4F_{xx}G_{tt}G^2 + 15F_{xx}G_t^2G + 20F_xG_{tx}G_tG - 3F_xG_{ttx}G^2 + 10F_xG_{tt}G_xG - 45F_xG_t^2G_x)/(F_xG^3), \tag{2.13}$$

$$D_1 =(-8F_{tx}G_{tt}G^2 + 30F_{tx}G_t^2G + 4F_{tttx}G^3 - 6F_{ttt}G_xG^2 - 18F_{ttx}G_tG^2 - 8F_{tt}G_{tx}G^2 + 30F_{tt}G_tG_xG + 20F_tG_{tx}G_tG - 3F_tG_{ttx}G^2 + 10F_tG_{tt}G_xG - 45F_tG_t^2G_x - F_xG_{ttt}G^2 + 10F_xG_{tt}G_tG - 15F_xG_t^3)/(F_xG^3), \tag{2.14}$$

$$D_0 =(F_{tttt}G^3 - 6F_{ttt}G_tG^2 - 4F_{tt}G_{tt}G^2 + 15F_{tt}G_t^2G - F_tG_{ttt}G^2 + 10F_tG_{tt}G_tG - 15F_tG_t^3)/(F_xG^3) \tag{2.15}$$

with

$$F_xG \neq 0.$$

Proof. Applying a generalized Sundman transformation (3.4), one obtains the following transformation of the fourth-order derivatives

$$X'(T) = \frac{D_tF(t,x)}{D_t\int G(t,x)dt} = \frac{F_t + x'F_x}{G} = P(t,x,x'), \tag{2.16}$$

$$
\begin{aligned}
X''(T) &= \frac{D_t P}{D_t \int G(t,x)dt} = \frac{P_t + x'P_x + x''P_{x'}}{G} \\
&= \frac{2F_{tx}Gx' + F_{tt}G - F_tG_t - F_tG_xx' + F_{xx}Gx'^2 - F_xG_tx' - F_xG_xx'^2 + F_xGx''}{G^3} \\
&= Q(t,x,x',x''),
\end{aligned}
$$

$$
\begin{aligned}
X'''(T) &= \frac{D_t Q}{D_t \int G(t,x)dt} = \frac{Q_t + x'Q_x + x''Q_{x'} + x'''Q_{x''}}{G} \\
&= \frac{1}{G^5}[(F_xG^2)x''' + G(3F_{xx}G - 4F_xG_x)x'x'' + G(3F_{tx}G - F_tG_x - 3F_xG_t)x'' + ...] \\
&= R(t,x,x',x'',x'''),
\end{aligned}
$$

$$
\begin{aligned}
X^{(4)}(T) &= \frac{D_t R}{D_t \int G(t,x)dt} = \frac{R_x + x'R_x + x''R_{x'} + x'''R_{x''} + x^{(4)}R_{x'''}}{G} \\
&= \frac{1}{G^7}[(F_xG^3)x^{(4)} + G^2(4F_{xx}G - 7F_xG_x)x'x''' + G^2(4F_{tx}G - F_tG_x - 6F_xG_t)x''' + ...] \\
&= S(t,x,x',x'',x''',x^{(4)})
\end{aligned}
$$

where

$D_t = \frac{\partial}{\partial t} + x'\frac{\partial}{\partial x} + x''\frac{\partial}{\partial x'} + x'''\frac{\partial}{\partial x''} + x^{(4)}\frac{\partial}{\partial x'''} + ...$ is a total derivatives.

Substituting $X^{(4)}(T)$ into equation (3.5), we have

$$
\begin{aligned}
&(F_xG^3)x^{(4)} + (G^2(4F_{xx}G - 7F_xG_x)x' + G^2(4F_{tx}G - F_tG_x - 6F_xG_t))x''' \\
&+G^2(3F_{xx}G - 4F_xG_x)x''^2 + (G(6F_{xxx}G^2 + ...)x'^2 + ...)x'' + ... = 0.
\end{aligned}
$$

Dividing this equation by F_xG^3, we get

$$
\begin{aligned}
&x^{(4)} + \left(\left(\tfrac{4F_{xx}G-7F_xG_x}{F_xG}\right)x' + \left(\tfrac{4F_{tx}G-F_tG_x-6F_xG_t}{F_xG}\right)\right)x''' \\
&+\left(\tfrac{3F_{xx}G-4F_xG_x}{F_xG}\right)x''^2 + \left(\left(\tfrac{6F_{xxx}G^2+...}{F_xG^2}\right)x'^2 + ...\right)x'' + ... = 0.
\end{aligned}
$$

Denoting A_i, B, C_i and D_i as equations (2.5) - (2.15), so that we obtain the necessary form as equation (3.6). These prove the theorem.

□

2.3 Formulation of the linearization theorem

We have shown in the previous section that every linearizable fourth-order ordinary differential equation belongs to the class of equations (3.6). In this section, we formulate the main theorems containing necessary and sufficient conditions for linearization as well as the methods for constructing the linearizing transformations.

For obtaining sufficient conditions, one has to solve the compatibility problem. Considering the representations of the coefficients A_i, B, C_i and D_i through the unknown functions F and G. From equations (2.5) and (2.6), one can find the derivatives

$$F_{xx} = F_x(7G_x + A_1G)/(4G), \tag{2.17}$$

$$F_{tx} = (F_tG_x + 6F_xG_t + F_xA_0G)/(4G). \tag{2.18}$$

From equation (2.7) one obtains that

$$G_x = -GS_1/5 \tag{2.19}$$

where

$$S_1 = 3A_1 - 4B.$$

Notice that for the case $G_x = 0$, the generalized Sundman transformations becomes a point transformation, so that we assume $G_x \neq 0$, that means $S_1 \neq 0$ too.

Equations (2.8) and (2.11) provide the conditions

$$\begin{aligned} S_{1x} =&(300A_{1x} + 75A_1^2 + 10A_1S_1 - 200C_2 - 17S_1^2)/140, \\ A_{1xx} =&(-5100A_{1x}A_1 - 300A_{1x}S_1 + 8400C_{2x} - 925A_1^3 - 175A_1^2S_1 \\ &+ 4100A_1C_2 - 15A_1S_1^2 + 900C_2S_1 - 39200D_4 + 3S_1^3)/2800. \end{aligned} \tag{2.20}$$

From equation (2.10) one obtains the derivative

$$G_{tt} = (-20F_{tt}G^2S_1 + 38F_tG_tGS_1 + 3F_tG^2S_2 + 300F_xG_t^2 + 5F_xG^2S_3)/(200F_xG) \tag{2.21}$$

where

$$\begin{aligned} S_2 &= -4S_{1t} + A_0S_1, \\ S_3 &= -12A_{0t} - 3A_0^2 + 8C_0. \end{aligned}$$

From equation (2.9) one obtains the derivative

$$G_t = (-F_tGS_1^2 - F_xGS_4)/(22F_xS_1) \tag{2.22}$$

where

$$S_4 = 60A_{0x} + 15A_0A_1 - 5A_0S_1 - 20C_1 + 7S_2.$$

Comparing the mixed derivative $(G_t)_x = (G_x)_t$ one obtains equation

$$5F_tGS_1^2S_5 + F_xGS_6 = 0 \tag{2.23}$$

where

$$S_5 = -660A_{1x} - 165A_1^2 + 55A_1S_1 + 440C_2 - 34S_1^2,$$
$$S_6 = -1540S_{4x}S_1 + 1309A_0S_1^3 + 385A_1S_1S_4 - 1694S_1^2S_2 - 252S_1^2S_4 - 5S_4S_5.$$

Substituting the relation A_{1x} into A_{1xx} one obtains the condition

$$S_{5x} = (-1016400C_{2x} + 63525A_1^2S_1 - 254100A_1C_2 - 40040A_1S_1^2 - 990A_1S_5 - 84700C_2S_1 + 6098400D_4 + 5663S_1^3 + 20S_1S_5)/660. \quad (2.24)$$

Further analysis of the compatibility depends on value of S_5 in equation (2.23): it is separated into two cases, i.e., $S_5 = 0$ and $S_5 \neq 0$.

3.1 Case $S_5 = 0$

From equation (2.23) one obtains the condition

$$S_6 = 0. \quad (2.25)$$

Comparing the mixed derivative $(F_{xx})_t = (F_{tx})_x$ one arrives at the condition

$$A_{1t} = (-33A_0A_1 + 11A_0S_1 + 44C_1 - 22S_2 + 4S_4)/132. \quad (2.26)$$

From equation (2.12) one obtains the derivative

$$F_t = (F_xS_7)/(252S_1^3) \quad (2.27)$$

where

$$S_7 = -145200C_{1x} - 14520S_{2x} + 18150A_0A_1S_1 - 2464A_0S_1^2 - 36300A_1C_1 - 9075A_1S_2 + 1815A_1S_4 - 12100C_1S_1 + 435600D_3 - 7381S_1S_2 + 727S_1S_4.$$

Substituting F_t into F_{tx} one arrives at the condition

$$S_{7x} = (6930A_0S_1^3 + 55A_1S_7 - 1890S_1^2S_4 - 51S_1S_7)/110. \quad (2.28)$$

Substituting G_t into G_{tt} one arrives at the condition

$$S_{7t} = (11642400S_{4t}S_1^3 - 2910600A_0S_1^3S_4 + 27720A_0S_1^2S_7 + 6403320S_1^4S_3 + 2910600S_1^2S_2S_4 + 264600S_1^2S_4^2 - 37884S_1S_2S_7 + 252S_1S_4S_7 + 17S_7^2)/(55440S_1^2). \quad (2.29)$$

Comparing the mixed derivative $(G_{tt})_x = (G_x)_{tt}$ one obtains the condition

$$508200S_1^4S_3 + S_7^2 - 435600D_3S_7 + 12100C_1S_1S_7 - 4065600C_0S_1^4 - 1663200S_{4t}S_1^3 - 6098400S_{3x}S_1^3 + 12196800S_{2t}S_1^3 + 145200C_{1x}S_7 - 11(11S_2 + 47S_4)S_1S_7 + 5040(77S_2 - 6S_4)S_1^2S_4 + 1815(7S_2 - S_4 + 20C_1)A_1S_7 - 110((165A_1 - 53S_1)S_7 - 252(77S_2 - 16S_4)S_1^2)A_0S_1 = 0. \quad (2.30)$$

From equations (2.13), (2.15) and (2.14) one obtains the conditions (2.57), (2.58) and (2.59)[1].

3.2 Case $S_5 \neq 0$

From equation (2.23), one obtains the derivative

$$F_t = (-F_x S_6)/(5S_1^2 S_5). \tag{2.31}$$

The relations $(F_t)_x = (F_x)_t$ and $(F_{xx})_t = (F_{tx})_x$ provide the conditions

$$\begin{aligned} S_{6x} =& (-7114800C_{2x}S_1S_6 - 5775A_0S_1^3S_5^2 + 444675A_1^2S_1^2S\_1778700A_1C_2S_1S_6 \\ & - 280280A_1S_1^3S_6 - 5775A_1S_1S_5S_6 - 592900C_2S_1^2S_6 \\ & + 42688800D_4S_1S_6 + 39641S_1^4S_6 + 1575S_1^2S_4S_5^2 - 931S_1^2S_5S_6 \\ & - 30S_5^2S_6)/(4620S_1S_5), \end{aligned} \tag{2.32}$$

$$\begin{aligned} A_{1t} =& (-5775A_0A_1S_1^2 + 1925A_0S_1^3 + 7700C_1S_1^2 - 3850S_1^2S_2 \\ & + 700S_1^2S_4 - 3S_6)/(23100S_1^2). \end{aligned} \tag{2.33}$$

Substituting G_t into G_{tt} one arrives at the condition

$$\begin{aligned} S_{4t} =& (6600S_{5t}S_1S_6 - 6600S_{6t}S_1S_5 + 6875A_0S_1S_4S_5^2 \\ & + 1650A_0S_1S_5S_6 - 15125S_1^2S_3S_5^2 - 6875S_2S_4S_5^2 \\ & - 2860S_2S_5S_6 - 625S_4^2S_5^2 + 30S_4S_5S_6 \\ & - 102S_6^2)/(27500S_1S_5^2). \end{aligned} \tag{2.34}$$

Equation (2.12) provides the condition

$$\begin{aligned} C_{1x} =& (-508200S_{2x}S_1S_5 + 635250A_0A_1S_1^2S_5 - 86240A_0S_1^3S_5 \\ & - 5775A_0S_1S_5^2 - 1270500A_1C_1S_1S_5 - 317625A_1S_1S_2S_5 \\ & + 63525A_1S_1S_4S_5 - 423500C_1S_1^2S_5 + 15246000D_3S_1S_5 \\ & - 258335S_1^2S_2S_5 + 25445S_1^2S_4S_5 + 1764S_1^2S_6 \\ & - 75S_4S_5^2 - 150S_5S_6)/(5082000S_1S_5). \end{aligned} \tag{2.35}$$

[1]See Appendix

Equating the mixed derivative $(G_{tt})_x = (G_x)_{tt}$ one obtains the condition

$$\begin{aligned}
&5(3(20S_4S_5 + 7S_6 + 550S_2S_5)S_6 + 296450S_1^4S_3S_5 - 847000C_0S_1^4S_5\\
&- 38115A_1S_1S_2S_6 - 1270500S_{3x}S_1^3S_5 + 152460S_{2x}S_1S_6\\
&+ 2541000S_{2t}S_1^3S_5)S_5 + 21(39875S_2S_4S_5^2 + 27335S_2S_5S_6\\
&+ 375S_4^2S_5^2 - 615S_4S_5S_6 + 306S_6^2)S_1^2 - 3300(S_{5t}S_6\\
&- S_{6t}S_5)(126S_1^2 + 5S_5)S_1 + 165(13475S_1^2S_2S_5\\
&- 5425S_1^2S_4S_5 - 1701S_1^2S_6 - 25S_5S_6)A_0S_1S_5 = 0. \qquad (2.36)
\end{aligned}$$

From equations (2.13),(2.15) and (2.14) one obtains the conditions (2.60),(2.61) and (2.62)[2].

All obtained results can be summarized in the following theorems.

Theorem 2.3.1. *Sufficient conditions for equation (3.6) to be linearizable via the generalized Sundman transformation (3.4) are as follows.*

(a) *If $S_5 = 0$, then the conditions are (2.20), (2.24), (2.25), (2.26), (2.28), (2.30), (2.29), (2.57), (2.58) and (2.59).*

(b) *If $S_5 \neq 0$, then the conditions are (2.20), (2.24), (2.32), (2.33), (2.36), (2.34), (2.35), (2.60), (2.61) and (2.62).*

□

Theorem 2.3.2. *Provided that the sufficient conditions in Theorem 3.3.1 are satisfied, the transformation (3.4) mapping equation (3.6) to a linear equation (3.5) is obtained by solving the following compatible system of equations for the functions $F(t,x)$ and $G(t,x)$:*

(a) *(2.17), (2.19), (2.22) and (2.27).*

(b) *(2.17), (2.19), (2.22) and (2.31).*

2.4 Examples

Example 1. Consider the nonlinear fouth-order ordinary differential equation

$$x^{(4)} - \frac{7}{x}x'x''' - \frac{4}{x}x''^2 + \frac{25}{x^2}x'^2x'' - \frac{15}{x^3}x'^4 = 0. \qquad (2.37)$$

It is an equation of the form (3.6) in Theorem 3.2.1 with the coefficients

$$A_1 = \frac{-7}{x},\ A_0 = 0,\ B = \frac{-4}{x},\ C_2 = \frac{25}{x^2},\ C_1 = 0,\ C_0 = 0,$$

[2] See Appendix

$$D_4 = \frac{-15}{x^3},\ D_3 = 0,\ D_2 = 0,\ D_1 = 0,\ D_0 = 0,\ S_1 = \frac{-5}{x},$$

$$S_2 = 0,\ S_3 = 0,\ S_4 = 0,\ S_5 = \frac{-630}{x^2},\ S_6 = 0,\ S_7 = 0.$$

One can check that these coefficients obey the conditions in Theorem 3.3.1 (b). Thus equation (3.61) is linearizable via a generalized Sundman transformation. For finding the functions F and G we have to solve equations in Theorem 3.3.3 (b), which become

$$F_{xx} = 0, \tag{2.38}$$

$$G_x = G/x, \tag{2.39}$$

$$G_t = 0, \tag{2.40}$$

$$F_t = 0. \tag{2.41}$$

From equation (2.41), one can take the simplest solution

$$F = x$$

and this solution satisfies equation (3.77). From equation (3.64), one can take the simplest solution

$$G = x$$

and this solution satisfies equation (3.63). So that one obtains the linearizing transformation

$$X = x,\ dT = xdt. \tag{2.42}$$

Hence equation (3.61) is mapped by the transformation (3.66) into the linear equation

$$X^{(4)}(T) = 0. \tag{2.43}$$

The general solution of equation (2.43) is

$$X(T) = \frac{c_1}{6}T^3 + \frac{c_2}{2}T^2 + c_3T + c_4$$

where c_1, c_2, c_3 and c_4 are arbitrary constants. Applying the generalized Sundman transformation (3.66) to equation (3.61), we obtain that the general solution of equation (3.66) is

$$x(t) = \frac{c_1}{6}\phi(t)^3 + \frac{c_2}{2}\phi(t)^2 + c_3\phi(t) + c_4$$

where the function $T = \phi(t)$ is a solution of the equation

$$\frac{dT}{dt} = \frac{c_1}{6}T^3 + \frac{c_2}{2}T^2 + c_3T + c_4.$$

Example 2. Consider the nonlinear fouth-order ordinary differential equation

$$-80x'^4t + 24x'^3x + 86x'^2x''tx - 18x'x''x^2 - 14x'x'''tx^2 - 8x''^2tx^2 + 2x'''x^3 + x^{(4)}tx^3 = 0. \tag{2.44}$$

It is an equation of the form (3.6) in Theorem 3.2.1 with the coefficients

$$A_1 = \frac{-14}{x},\ A_0 = \frac{2}{t},\ B = \frac{-8}{x},\ C_2 = \frac{86}{x^2},\ C_1 = \frac{18}{tx},\ C_0 = 0,$$

$$D_4 = \frac{-80}{x^3},\ D_3 = \frac{24}{tx^2},\ D_2 = 0,\ D_1 = 0,\ D_0 = 0,\ S_1 = \frac{-10}{x},\ S_2 = \frac{-20}{tx},$$

$$S_3 = \frac{12}{t^2},\ S_4 = \frac{-100}{tx},\ S_5 = \frac{560}{x^2},\ S_6 = \frac{-280000}{tx^3},\ S_7 = \frac{72600(-7x-40)}{tx^2}.$$

One can check that these coefficients obey the conditions in Theorem 3.3.1 (b). Thus equation (3.67) is linearizable via a generalized Sundman transformation. For finding the functions F and G we have to solve equations in Theorem 3.3.3 (b), which become

$$F_{xx} = 0, \tag{2.45}$$

$$G_x = 2G/x, \tag{2.46}$$

$$G_t = 0, \tag{2.47}$$

$$F_t = F_x x/t. \tag{2.48}$$

From equation (3.69), we get $G = Cx^2$. Choosing $C = 1$, we have

$$G = x^2$$

and this satisfies equation (3.70). The equation (2.48), becomes

$$tF_t - xF_x = 0$$

by Cauchy method, one arrives at

$$F = tx$$

and this satisfies equation (3.68). So that one obtains the linearizing transformation

$$X = tx,\ dT = x^2dt. \tag{2.49}$$

Hence equation (3.67) is mapped by the transformation (3.72) into the linear equation

$$X^{(4)}(T) = 0. \tag{2.50}$$

The general solution of equation (3.73) has the form

$$X(T) = \frac{c_1}{6}T^3 + \frac{c_2}{2}T^2 + c_3T + c_4$$

where c_1, c_2, c_3 and c_4 are arbitrary constants. Applying the generalized Sundman transformation (3.72) to equation (3.67), we obtain that the general solution of equation (3.72) is

$$x(t) = \frac{\left(\frac{c_1}{6}\phi(t)^3 + \frac{c_2}{2}\phi(t)^2 + c_3\phi(t) + c_4\right)}{t}$$

where the function $T = \phi(t)$ is a solution of the equation

$$\frac{dT}{dt} = \left(\frac{\frac{c_1}{6}T^3 + \frac{c_2}{2}T^2 + c_3T + c_4}{t}\right)^2.$$

2.5 Applications

- **One Class of fourth-order Partial Differential Equations**

Let us consider the nonlinear fourth-order partial differential equation [11]

$$u_{tt} = (\kappa u + \gamma u^2)_{xx} + \nu u u_{xxxx} + \mu u_{xxtt} + \alpha u_x u_{xxx} + \beta u_{xx}^2, \tag{2.51}$$

where α, β, μ, ν, γ and κ are arbitrary constants. This equation may be thought of as a fourth-order analogue of a generalization of the Camassa-Holm equation, about which there has been considerable recent interest. Furthermore, equation (2.51) is a Boussinesq-type equation which arises as a model of vibrations of harmonic mass-spring chain and admits both compacton and conventional solitons.

Of particular interest among solutions of equation (2.51) are traveling wave solutions :

$$u(x,t) = H(x - Dt),$$

where D is a constant phase velocity and the argument $x - Dt$ is a phase of the wave. Substituting the representation of a solution into equation (2.51), one finds

$$(\nu H + \mu D^2)H^{(4)} + \alpha H'H''' + \beta H''^2 + (2\gamma H + \kappa - D^2)H'' + 2\gamma H'^2 = 0. \tag{2.52}$$

This is an equation of the form (3.6) in Theorem 3.2.1 with coefficients

$$A_1 = \frac{1}{\mu D^2 + \nu H},\ A_0 = 0,\ B = \frac{\beta}{\mu D^2 + \nu H},\ C_2 = 0,\ C_1 = 0,$$

$$C_0 = \frac{2\gamma H + \kappa - D^2}{\mu D^2 + \nu H},\ D_4 = 0,\ D_3 = 0,\ D_2 = \frac{2\gamma}{\mu D^2 + \nu H},\ D_1 = 0,\ D_0 = 0,$$

$$S_1 = \frac{3\alpha - 4\beta}{D^2\mu + \nu H},\ S_2 = 0,\ S_3 = \frac{8(-D^2 + 2\gamma H + \kappa)}{D^2\mu + \nu H},\ S_4 = 0,$$

$$S_5 = \frac{2(-153\alpha^2 + 298\alpha\beta + 330\alpha\nu - 272\beta^2)}{D^4\mu^2 + 2D^2\mu\nu H + \nu^2 H^2},\ S_6 = 0,\ S_7 = 0. \tag{2.53}$$

Since $S_1 \neq 0$ and $S_5 \neq 0$ then we apply Theorem 3.3.1 (b) for checking the linearity. The coefficients in equation (2.53) obey the conditions (2.20), (2.24), (2.36), (2.34) and (2.60) if and only if

$$\alpha = 1,\ \beta = 0,\ \nu = \frac{2}{5},\ \gamma = 0,\ \kappa = D^2.$$

Hence equation (2.52) is linearizable with the condition $\alpha = 1$, $\beta = 0$, $\nu = \frac{2}{5}$, $\gamma = 0$, and $\kappa = D^2$.

- **The Generalized Lubrication Equation**

Let us consider the generalized lubrication equation [12]

$$h_t + (h^n h_{xxx})_x = 0, \tag{2.54}$$

where the non-negative function $h(x,t)$ denotes the height of the free surface above the solid substrate, x is the horizontal coordinate and t the time. The constant n denotes the kind of flow. The generalized lubrication equation (2.54) models the spreading of a thin film driven by surface tension.

Substituting the traveling wave representation of a solution into equation (2.54), one finds

$$H^n H^{(4)} + nH^{(n-1)} H' H''' - DH' = 0. \tag{2.55}$$

This is an equation of the form (3.6) in Theorem 3.2.1 with coefficients

$$A_1 = \frac{n}{H},\ A_0 = 0,\ B = 0,\ C_2 = 0,\ C_1 = 0,\ C_0 = 0,\ D_4 = 0,$$

$$D_3 = 0,\ D_2 = 0,\ D_1 = \frac{-D}{H^n},\ D_0 = 0,\ S_1 = \frac{3n}{H},\ S_2 = 0,$$

$$S_3 = 0,\ S_4 = 0,\ S_5 = \frac{6n(-51n + 110)}{H^2},\ S_6 = 0,\ S_7 = 0. \tag{2.56}$$

Since $S_1 \neq 0$ and $S_5 \neq 0$ then we apply Theorem 3.3.1 (b) for checking the linearity. The coefficients in equation (2.56) obey the conditions (2.20), (2.24) and (2.61) if and only if

$$D = 0,\ n = \frac{5}{2}.$$

Hence equation (2.55) is linearizable with the condition $D = 0$ and $n = \frac{5}{2}$.

2.6 Appendix

For proving theorems one needs relations between $F(t,x)$ and $G(t,x)$ and coefficients of equation (3.6). These relations are presented here.

$$
\begin{aligned}
&-(319440(120(C_{1xx}-3D_{3x})+(7S_2-S_4)C_2)(1575S_1S_4+2S_7)\\
&-(13684809600C_{0x}S_1^3+265646304000C_{1tx}S_1^2-796938912000D_{3t}S_1^2\\
&+13282315200S_{3xx}S_1^2+3421202400S_{3x}S_1^3+6426493920C_0S_1^4\\
&+22137192000C_1^2S_1^2-4829932800D_2S_1^3-420942060S_1^4S_3\\
&-3045876372S_1^2S_2^2+984017727S_1^2S_2S_4-73652733S_1^2S_4^2\\
&-150513S_1S_2S_7+39000S_1S_4S_7+380S_7^2)S_1+87120(2286900S_1S_2\\
&-780255S_1S_4-2459S_7)D_3S_1-22137192000(3A_1+S_1)C_{1t}S_1^3\\
&-100623600(165A_1+7S_1)S_{2t}S_1^3+3659040(660A_1+61S_1)S_{4t}S_1^3\\
&-2744280(3465A_1-662S_1)A_0^2S_1^4-59895(9(7S_2-S_4)\\
&+220C_1)(1575S_1S_4+2S_7)A_1^2+7260(9(108416S_1S_2-11893S_1S_4\\
&+69S_7)S_1^2+880(1575S_1S_4+2S_7)C_2)C_1+87120((110880A_0S_1^2\\
&-762300S_1S_2+317835S_1S_4+893S_7)S_1-385(1575S_1S_4+2S_7)A_1)C_{1x}\\
&-33(220(630(3630S_2-S_4)S_1-1469S_7)C_1S_1-(567151200C_0S_1^4\\
&+6174630000D_3S_1S_4+7840800D_3S_7-75467700S_1^4S_3-176091300S_1^2S_2^2\\
&+79771230S_1^2S_2S_4-5955390S_1^2S_4^2+149589S_1S_2S_7\\
&-24222S_1S_4S_7-10S_7^2))A_1+66(3025(33A_1^2-16C_2)(1575S_1S_4\\
&+2S_7)+3(4065600C_1S_1-146361600D_3-34425930S_1S_2\\
&+9275175S_1S_4+5699S_7)S_1^2+55(5239080C_1S_1+1376298S_1S_2\\
&-1268064S_1S_4-2825S_7)A_1S_1)A_0S_1)=0,
\end{aligned}
\tag{2.57}
$$

$$
\begin{aligned}
&-(5(106500018240S_1^8S_3^2 - S_7^4 + 27883641139200D_0S_1^9 + 2178409464000A_0^3S_1^7S_4 \\
&- 387272793600S_{4t}^2S_1^6 - 23236367616000S_{4ttt}S_1^7 - 12780002188800S_{3tt}S_1^8 \\
&+ 3872727936000C_{0t}S_1^7S_4) - 6(50644440S_1^4S_3 + 41S_7^2)(11S_2 - 3S_4)S_1S_7 \\
&- 96818198400(374S_2 + 73S_4)S_{3t}S_1^7 - 252(1892S_2 + 309S_4)(11S_2 - 3S_4)S_1^2S_7^2 \\
&- 21168(28424S_2^2 + 11121S_2S_4 + 771S_4^2)(11S_2 - 3S_4)S_1^3S_7 - 66679200(33176S_2^2 \\
&+ 2079S_2S_4 - 21S_4^2)(11S_2 + S_4)S_1^4S_4 - 146694240(78166S_2^2 + 31229S_2S_4 \\
&+ 1569S_4^2)S_1^6S_3 - 18441561600(1575S_1S_4 + 2S_7)S_{2tt}S_1^5 + 5029516800(17325A_0S_1^2 \\
&- 30415S_1S_2 - 2730S_1S_4 + 2S_7)S_{4tt}S_1^5 + 558835200(105(517S_2 + 39S_4)S_1S_4 \\
&+ (11S_2 - 3S_4)S_7)C_0S_1^5 - 104781600(105(2816S_2 + 167S_4)S_1S_4 + 4(11S_2 \\
&- 3S_4)S_7)A_0^2S_1^5 - 498960(87318000A_0^2S_1^4 - 116424000C_0S_1^4 + 8925840S_1^4S_3 \\
&+ 195074880S_1^2S_2^2 + 27130320S_1^2S_2S_4 + 493920S_1^2S_4^2 - 1372S_1S_2S_7 - 5124S_1S_4S_7 \\
&- 11S_7^2 - 5040(43505S_1S_2 + 3115S_1S_4 - 2S_7)A_0S_1^2)S_{4t}S_1^3 - 166320(791683200S_{4t}S_1^3 \\
&+ 147276360S_1^4S_3 + 210672S_1S_2S_7 + 3024S_1S_4S_7 + 121S_7^2 + 264600(1617S_2 \\
&+ 101S_4)S_1^2S_4 - 55440(5145S_1S_4 + 2S_7)A_0S_1^2)S_{2t}S_1^3 + 41580(11((11S_2 - 3S_4)S_7^2 \\
&- 52920000C_0S_1^4S_4) + 84(228S_2 + 61S_4)(11S_2 - 3S_4)S_1S_7 + 582120(253S_2 \\
&+ 96S_4)S_1^4S_3 + 17640(57431S_2^2 + 6129S_2S_4 + 84S_4^2)S_1^2S_4)A_0S_1^3) = 0,
\end{aligned}
\tag{2.58}
$$

$$
\begin{aligned}
&-(5(11((1343188S_1^4S_3+S_7^2)S_7+36883123200D_1S_1^6-33809529600S_{4tt}S_1^5\\
&-4354257600S_{3t}S_1^6-67619059200S_{3tx}S_1^5+135238118400S_{2tt}S_1^5\\
&+20490624000C_{0x}S_1^4S_4-69668121600C_{0t}S_1^6-77616(31834S_2\\
&+18193S_4)S_1^5S_3)-(56529S_2-15737S_4)S_1S_7^2-14112(980342S_2^2\\
&+682473S_2S_4+325453S_4^2)S_1^3S_4)-14(27719648S_2^2-13837703S_2S_4\\
&-1031223S_4^2)S_1^2S_7-894432000(36D_{3t}-C_1^2-12C_{1tx})(1575S_1S_4+2S_7)S_1^2\\
&-33541200(35112S_1S_2+12264S_1S_4+17S_7)S_{3x}S_1^3+4743200(148104S_1S_2\\
&+29268S_1S_4+41S_7)C_0S_1^4+894432000(3(1575S_1S_4+2S_7)A_1+(945S_1S_4\\
&+2S_7)S_1)C_{1t}S_1^2-2032800((160083S_1S_2+395346S_1S_4+110S_7)S_1\\
&-165(8295S_1S_4+4S_7)A_1)A_0^2S_1^3+5691840(2(114345A_0S_1^2+206910S_1S_2\\
&+81135S_1S_4+32S_7)S_1+165(1575S_1S_4+2S_7)A_1)S_{2t}S_1^2\\
&+12100(81108720S_1^4S_3+83S_7^2+154(224S_2+123S_4)S_1S_7+352800(407S_2\\
&+62S_4)S_1^2S_4)C_1S_1-435600(81108720S_1^4S_3+83S_7^2+176400(1078S_2\\
&+97S_4)S_1^2S_4+14(6688S_2+921S_4)S_1S_7)D_3+18480((249018000C_1S_1\\
&-8964648000D_3-14848680S_1S_2-33007380S_1S_4+20899S_7)S_1\\
&+7260(102900C_1S_1+36015S_1S_2-6720S_1S_4-2S_7)A_1-9240(40425A_1
\end{aligned}
$$

$$
\begin{aligned}
&-13876S_1)A_0S_1^2)S_{4t}S_1^2+145200(81108720S_1^4S_3+83S_7^2+380318400S_{4t}S_1^3\\
&+176400(1078S_2+97S_4)S_1^2S_4+14(6688S_2+921S_4)S_1S_7-36960(3360S_1S_4\\
&+S_7)A_0S_1^2)C_{1x}-110(165(81108720S_1^4S_3+83S_7^2+14(7744S_2+393S_4)S_1S_7\\
&+17640(15213S_2+101S_4)S_1^2S_4+36960(8295S_1S_4+4S_7)C_1S_1)A_1\\
&+(8708515200C_0S_1^4-4234728960S_1^4S_3-3747222864S_1^2S_2^2-12347618976S_1^2S_2S_4\\
&-842158296S_1^2S_4^2-8500030S_1S_2S_7+1347990S_1S_4S_7-4735S_7^2\\
&-146361600(3360S_1S_4+S_7)D_3+4065600(3675S_1S_4+S_7)C_1S_1)S_1)A_0S_1\\
&+1815((83(7S_2-S_4)S_7+123200S_1^3S_3)S_7+176400(1078S_2+97S_4)(7S_2\\
&-S_4)S_1^2S_4+388080(1463S_2-9S_4)S_1^4S_3+14(6688S_2+921S_4)(7S_2\\
&-S_4)S_1S_7-985600(630S_1S_4+S_7)C_0S_1^3+20(81108720S_1^4S_3+83S_7^2\\
&+176400(1078S_2+97S_4)S_1^2S_4+14(6688S_2+921S_4)S_1S_7)C_1)A_1)=0, \qquad (2.59)
\end{aligned}
$$

$$
\begin{aligned}
&15(10(2((59290S_1^6S_3 + 9S_6^2)S_5^2 + 4268880D_4S_1S_6^2 - 5929000D_2S_1^5S_5^2 \\
&+ 592900C_0S_1^6S_5^2 + 148225S_{3x}S_1^5S_5^2 - 3557400S_{2xx}S_1^2S_5S_6 \\
&+ 426888000D_{4t}S_1^2S_5S_6 - 71148000C_{2tx}S_1^2S_5S_6 + 5929000C_{0x}S_1^5S_5^2) \\
&- 88935(35S_2S_5 - S_6)A_1^2S_1^2S_6 - 154(434S_1^2 + 5S_5)S_{6t}S_1^3S_5 \\
&- 1925(1456S_1^2 - 15S_5)A_0^2S_1^4S_5^2 + 1540(6545S_1^2S_2S_5 - 700S_1^2S_4S_5 \\
&- 77S_1^2S_6 + 3S_5S_6)C_2S_6 - 11858000(3A_1 + S_1)C_{2t}S_1^2S_5S_6 \\
&+ 1422960(25A_0S_1S_5 - S_6)C_{2x}S_1S_6) - 7(5220S_4S_5 - 1907S_6 \\
&- 31450S_2S_5)S_1^2S_5S_6) - 49(87375S_4^2S_5^2 + 489295S_4S_5S_6 - 22368S_6^2 \\
&+ 125(25197S_4S_5 + 20923S_6)S_2S_5)S_1^4 - 539000(367S_1^2 + 15S_5 \\
&+ 3300C_2)C_1S_1^2S_5S_6 + 69300(50((154A_1S_1 - S_5)S_6 + 385A_0S_1^3S_5) \\
&- 7(750S_4S_5 + 2161S_6)S_1^2)S_{2x}S_1S_5 + 300(2((16709S_1^4 + 75S_5^2 \\
&- 40425A_1S_1S_5)S_6 - 28875A_0S_1^3S_5^2) + 525(30S_4S_5 + 53S_6)S_1^2S_5)S_{5t}S_1 \\
&+ 5775(2(1540((50C_1S_1^2S_5 - 3C_2S_6)S_6 + 25C_0S_1^4S_5^2) + 3(25S_2S_5 \\
&- 17S_6)S_5S_6) + 7((2250S_4S_5 + 3703S_6)S_2S_5 + 2(955S_4S_5 \\
&- 104S_6)S_6)S_1^2)A_1S_1 - 5(75(10(3(2845920D_4S_1 + S_5^2) + 118580C_2S_1^2 \\
&+ 177870A_1^2S_1^2)S_6 + 21(150S_4S_5 + 647S_6)S_1^2S_5) - 49(353325S_4S_5 \\
&+ 106619S_6 + 2733225S_2S_5)S_1^4 - 404250(6(220C_2 + S_5)S_6 - 55(3S_2S_5 \\
&- 2S_6)S_1^2)A_1S_1)A_0S_1S_5 = 0,
\end{aligned}
\tag{2.60}
$$

$$
\begin{aligned}
&-(3(((2500S_4^2S_5^2+3235S_4S_5S_6-384S_6^2)S_6+113437500S_2^3S_5^3)S_6\\
&+1512500000D_0S_1^5S_5^4-28359375A_0^3S_1^3S_5^3S_6+12100000S_{6t}^2S_1^2S_5^2\\
&+302500000S_{6ttt}S_1^3S_5^3-1815000000S_{5t}^3S_1^3S_6-302500000S_{5ttt}S_1^3S_5^2S_6)\\
&+30250000S_{3t}S_1^3S_5^3S_6+453750000S_{2tt}S_1^2S_5^3S_6-151250000C_{0t}S_1^3S_5^3S_6\\
&+20625(1125S_4S_5+644S_6)S_2^2S_5^2S_6+330(3375S_4S_5+238S_6)S_2S_5S_6^2\\
&+1375(31625S_2S_5+1500S_4S_5+404S_6)S_1^2S_3S_5^2S_6-687500(15S_4S_5\\
&+8S_6+440S_2S_5)C_0S_1^2S_5^2S_6+103125(75S_4S_5+62S_6+3025S_2S_5)A_0^2S_1^2S_5^2S_6\\
&+4125000(15S_4S_5+8S_6+330S_2S_5-165A_0S_1S_5)S_{6tt}S_1^2S_5^2+1650000((75S_4S_5\\
&+62S_6+1650S_2S_5)S_6+825(4S_{6t}-A_0S_6)S_1S_5)S_{5t}^2S_1^2-4125000(165((4S_{6t}\\
&-A_0S_6)S_5-8S_{5t}S_6)S_1+(15S_4S_5+8S_6+330S_2S_5)S_6)S_{5tt}S_1^2S_5\\
&+82500(1375(((12S_{6t}-5A_0S_6)S_5-12S_{5t}S_6)S_1+3(125S_4S_5+52S_6\\
&+4125S_2S_5)S_6)S_{2t}S_1S_5^2-275(3((850S_4S_5+83S_6)S_6+618750S_2^2S_5^2\\
&+75(375S_4S_5+266S_6)S_2S_5)-13750(50C_0-7S_3)S_1^2S_5^2)A_0S_1S_5S_6\\
&-3300(125(55(20C_0-3S_3-15A_0^2)S_1S_5+3(25S_4S_5+28S_6\\
&+825S_2S_5)A_0)S_1S_5-((850S_4S_5+83S_6)S_6+309375S_2^2S_5^2+150(125S_4S_5\\
&+107S_6)S_2S_5))S_{6t}S_1S_5+3300(125(55((20C_0-3S_3-15A_0^2)S_6-120S_{6tt})S_1S_5\\
&+3(25S_4S_5+28S_6+825S_2S_5)A_0S_6-12(25S_4S_5+28S_6+550S_2S_5\\
&-275A_0S_1S_5)S_{6t})S_1S_5-((850S_4S_5+83S_6)S_6+309375S_2^2S_5^2\\
&+150(125S_4S_5+107S_6)S_2S_5)S_6)S_{5t}S_1)=0,
\end{aligned}
\tag{2.61}
$$

$$
\begin{aligned}
&5(((9(360S_4S_5 + 137S_6 + 8250S_2S_5)S_6 - 448525S_1^4S_3S_5)S_6 - 762300000D_1S_1^5S_5^2 \\
&+ 25410000C_1S_1^2S_2S_5S_6 - 76230000S_{6tt}S_1^4S_5 - 24139500S_{3x}S_1^3S_5S_6 \\
&+ 31762500S_{3t}S_1^5S_5^2 - 304920000S_{2tx}S_1^2S_5S_6 - 50820000C_{1t}S_1^3S_5S_6 \\
&+ 101640000C_{0x}S_1^3S_5S_6 + 508200000C_{0t}S_1^5S_5^2)S_5 + 990000(S_{5tt}S_5 - 2S_{5t}^2) \\
&(77S_1^2 + 4S_5)S_1^2S_6 - 66000(728S_1^2 - 15S_5)A_0^2S_1^2S_5^2S_6 + 44000(994S_1^2 \\
&- 15S_5)C_0S_1^2S_5^2S_6 + 115500(660A_1 - 1037S_1)S_{2t}S_1^2S_5^2S_6 + 1155(3(1500S_4S_5 \\
&+ 481S_6 + 33000S_2S_5)S_2 + 2750(8C_0 - S_3)S_1^2S_5)A_1S_1S_5S_6) + 3(137500S_3S_5^3 \\
&- 195125S_4^2S_5^2 - 335265S_4S_5S_6 + 50526S_6^2 - 318876250S_2^2S_5^2 - 35(477975S_4S_5 \\
&+ 115363S_6)S_2S_5)S_1^2S_6 + 9900(25(3080A_1S_1S_2 + 3S_6 + 2(1813S_1^2 - 20S_5)A_0S_1)S_5 \\
&- 28(3(75S_4S_5 + S_6) + 5650S_2S_5)S_1^2)S_{6t}S_1S_5 - 69300(5500((8S_{6t} - 3A_0S_6)S_5 \\
&- 8S_{5t}S_6)S_1 + (1500S_4S_5 + 481S_6 + 33000S_2S_5)S_6)S_{2x}S_1S_5 + 900(25(22(20((77S_1^2 \\
&+ 4S_5)S_{6t} - 77A_1S_2S_6) - (1813S_1^2 + 20S_5)A_0S_6)S_1 + (60S_4S_5 - S_6 \\
&+ 1320S_2S_5)S_6)S_5 + 308(3(75S_4S_5 + S_6) + 5650S_2S_5)S_1^2S_6)S_{5t}S_1 \\
&- 15(25(84700((2C_1S_6 - 5S_1^2S_3S_5 - 20C_0S_1^2S_5)S_1 + 12A_1S_2S_6)S_1 \\
&+ 3(300S_4S_5 + 347S_6 + 6600S_2S_5)S_6)S_5 - 7(216025S_4S_5 + 75573S_6 \\
&+ 8268700S_2S_5)S_1^2S_6)A_0S_1S_5 = 0. \qquad (2.62)
\end{aligned}
$$

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Chapter 3

Linearizability of nonlinear third-order ordinary differential equations by using a generalized linearizing transformation

3.1 Introduction

There has been major interest in the nonlinear problems, since most equations are inherently nonlinear in nature. In general the nonlinear problems are very difficult to solve explicitly. It is of interest to provide general criteria for the linearizability of nonlinear ordinary differential equations, as they can then be reduced to easily solvable equations. Therefore, the approach of investigating nonlinear ordinary differential equations via transforming to simpler ordinary differential equations becomes important and has been quite plentiful in analysis of physical problems. This includes the classical linearization problem of finding transformations that linearize a given ordinary differential equation. The linearization problem has been studied in many aspects. A short review can be found in [1, 2]. The tools commonly used for solving the linearization problem are the transformations such as point transformation, contact transformation, reduction of order, differential substitution, generalized Sundman transformation etc. For this paper, we employ the extension of the generalized Sundman transformations.

The linearization problem for a second-order ordinary differential equation was investigated with respect to a generalized Sundman transformation

$$X = F(t,x),\ dT = G(t,x)dt \tag{3.1}$$

by Duarte et al. [3] earlier. They obtained the form of the linearizable equations and the conditions which allow the second-order ordinary differential equation to be transformed to the free particle equation. A characterization of these equations that can be linearized by means of generalized Sundman transformations in terms of first integral and procedure for construction of linearizing transformations has been given by Muriel and Romero [4]. In [5], Mustafa et al. gave a new characterization of linearizable equations in terms of the coefficients of ordinary differential equation and one auxiliary function. In [6], Nakpim and Meleshko pointed out that the solution of the linearization problem for a second-order ordinary differential equation via the generalized Sundman transformation considered earlier by Duarte et al. using the Laguerre form is not complete.

The linearization problem for a third-order ordinary differential equation was also investigated with respect to a generalized Sundman transformation [7, 8]. Criteria for a third-order ordinary differential equation to be equivalent to the linear equation $X'''(T) = 0$ with respect to a Sundman transformation were presented in [8]. The generalized Sundman transformation was also applied for obtaining necessary and sufficient conditions for a third-order ordinary differential equation to be equivalent to a linear equation in the Laguerre form [6]. Some applications of the generalized Sundman transformation to ordinary differential equations were considered in [9] and earlier papers, summarized in the book [10].

The linearization problem of a fourth-order ordinary differential equation with respect to generalized Sundman transformations was studied in [11]. They found the necessary and sufficient conditions which allow the fourth-order ordinary differential equation to be transformed to the simplest linear equation.

In this work, we expose a more general transformation, i.e. the extension of the generalized Sundman transformation

$$X = F(t,x), \;\; dT = G(t,x,x')dt. \tag{3.2}$$

This transformation was studied in [12, 13, 14] where they designated the transformation as the *generalized linearizing transformation*. They showed that this transformation can be utilized to linearize a wider class of nonlinear ordinary differential equations and, in particular, certain equations which cannot be linearized by the non-point and invertible point transformations. If the function G in (3.2) is independent of the variable x' then it becomes a non-point transformation (vide equation (3.1)). On the other hand, if G is a differentiable function then it becomes an invertible point transformation. So, (3.2) is a unified transformation as it includes non-point and invertible point transformations as special cases. An example of an equation which can be linearized by a transformation of the form (3.2) is given in [13]. It is worth noting that any second-order equation $x'' = f(t,x,x')$ can be transformed by a transformation (3.2) to the free particle equation, and that this is not so for third-order ordinary differential equations. Hence, the linearization problem using generalized linearizing transformations become interesting for ordinary differential equations of order greater than 2. In [12], the authors applied a particular class of transformations (3.2), where the function $G(t, x, x')$ is linear with respect to x'.

We are now paying attention to the case where G is a polynomial function in x' and in particular where it is linear in x' with coefficients which are arbitrary functions of t and x. To be specific, we focus here on the case

$$X = F(t, x), \; dT = (G_1(t, x)\, x' + G_2(t, x))\, dt.$$

Notice that for the case $G_1 = 0$, the generalized linearizing transformation becomes a generalized Sundman transformation, so that we assume $G_1 \neq 0$.

3.2 Necessary conditions of linearization

Here we consider a nonlinear third-order ordinary differential equation

$$x''' = f(t, x, x', x''). \tag{3.3}$$

Our goal in this section is to describe all equations (3.3) which are equivalent with respect to generalized linearizing transformations

$$X = F(t, x), \; dT = (G_1(t, x)\, x' + G_2(t, x))\, dt \tag{3.4}$$

to a linear equation

$$X'''(T) = 0. \tag{3.5}$$

We begin with investigating the necessary conditions for linearization, i.e. the general form of third-order equation (3.3) that can be obtained from a linear equation (3.5) by any generalized linearizing transformation (3.4).

Applying a generalized linearizing transformation (3.4), one obtains the following transformation of the third-order derivatives

$$X'(T) = \frac{D_t F}{G_1 x' + G_2} = \frac{F_t + x' F_x}{G_1 x' + G_2} = P(t, x, x'),$$

where $A_i (i = 0, 1, 2)$ and $B_j (j = 0, 1, ..., 5)$ are functions of t and x determined as following

$$
\begin{aligned}
A_2 =& (3((F_{tx} - F_{xx}r)G_1 - F_tG_{1x}) + (2(2G_{1x}r - r_xG_1) - G_{1t})F_x)/(KG_1), && (3.7)\\
A_1 =& -((2G_{1x}r + 5r_xG_1 + 4G_{1t})F_t - 3(F_{tt} - F_{xx}r^2)G_1 + ((3r_t - 4r_xr)G_1 \\
& - 4G_{1x}r^2 - 2G_{1t}r)F_x)/(KG_1), && (3.8)\\
A_0 =& -(3F_{tx}G_1r^2 - 3F_{tt}G_1r + 4F_tG_{1t}r - F_tG_{1x}r^2 + 6F_tr_tG_1 \\
& - F_tr_xG_1r - 3F_xG_{1t}r^2 - 3F_xr_tG_1r)/(KG_1), && (3.9)\\
B_5 =& -((F_{xxx}G_1 - 3F_{xx}G_{1x})G_1 - (G_{1xx}G_1 - 3G_{1x}^2)F_x)/(KG_1^2), && (3.10)\\
B_4 =& (3(G_{1t} + 2G_{1x}r + r_xG_1)F_{xx}G_1 + (G_{1xx}G_1 - 3G_{1x}^2)F_t \\
& + (2G_{1tx}G_1 - 6G_{1t}G_{1x} + 2G_{1xx}G_1r - 6G_{1x}^2r - 4G_{1x}r_xG_1 \\
& + r_{xx}G_1^2)F_x + (2(3F_{tx}G_{1x} - F_{xxx}G_1r) - 3F_{txx}G_1)G_1)/(KG_1^2), && (3.11)\\
B_3 =& ((3F_{tt}G_{1x} - F_{xxx}G_1r^2 - 3F_{ttx}G_1 - 6F_{txx}G_1r + 6(G_{1t} + 2G_{1x}r \\
& + r_xG_1)F_{tx})G_1 + (2G_{1tx}G_1 - 6G_{1t}G_{1x} + 2G_{1xx}G_1r - 6G_{1x}^2r \\
& - 4G_{1x}r_xG_1 + r_{xx}G_1^2)F_t + 3((G_{1x}r + r_xG_1 + G_{1t})r + G_{1t}r \\
& + r_tG_1)F_{xx}G_1 + ((2G_{1tx}r + G_{1tt})G_1 - 3(G_{1x}r + r_xG_1)^2 - 6(G_{1t}r \\
& + r_tG_1)G_{1x} - 3(2(G_{1x}r + r_xG_1) + G_{1t})G_{1t} + (2G_{1x}r_x + r_{xx}G_1 \\
& + G_{1xx}r)G_1r + 2(G_{1x}r_t + r_{tx}G_1 + G_{1t}r_x + G_{1tx}r)G_1)F_x)/(KG_1^2), && (3.12)\\
B_2 =& (((2G_{1tx}r + G_{1tt})G_1 - 3(G_{1x}r + r_xG_1)^2 - 6(G_{1t}r + r_tG_1)G_{1x} \\
& - 3(2(G_{1x}r + r_xG_1) + G_{1t})G_{1t} + (2G_{1x}r_x + r_{xx}G_1 + G_{1xx}r)G_1r \\
& + 2(G_{1x}r_t + r_{tx}G_1 + G_{1t}r_x + G_{1tx}r)G_1)F_t - ((6(G_{1x}r + r_xG_1 \\
& + G_{1t})(G_{1t}r + r_tG_1) - G_{1tt}G_1r - (2G_{1t}r_t + r_{tt}G_1 + G_{1tt}r)G_1 \\
& - 2(G_{1x}r_t + r_{tx}G_1 + G_{1t}r_x + G_{1tx}r)G_1r)F_x + ((3F_{txx}r^2 + F_{ttt})G_1 \\
& - 3(G_{1t} + 2G_{1x}r + r_xG_1)F_{tt} - 3((G_{1t}r + r_tG_1)F_{xx} - 2F_{ttx}G_1)r \\
& - 6((G_{1x}r + r_xG_1 + G_{1t})r + G_{1t}r + r_tG_1)F_{tx})G_1))/(KG_1^2), && (3.13)\\
B_1 =& -((6(G_{1x}r + r_xG_1 + G_{1t})(G_{1t}r + r_tG_1) - G_{1tt}G_1r - (2G_{1t}r_t \\
& + r_{tt}G_1 + G_{1tt}r)G_1 - 2(G_{1x}r_t + r_{tx}G_1 + G_{1t}r_x + G_{1tx}r)G_1r)F_t \\
& - ((3((G_{1x}r + r_xG_1 + G_{1t})r + G_{1t}r + r_tG_1)F_{tt} - ((2F_{ttt} + 3F_{ttx}r)G_1 \\
& - 6(G_{1t}r + r_tG_1)F_{tx})r)G_1 + ((2G_{1t}r_t + r_{tt}G_1 + G_{1tt}r)G_1r \\
& - 3(G_{1t}r + r_tG_1)^2)F_x))/(KG_1^2), && (3.14)\\
B_0 =& (((2G_{1t}r_t + r_{tt}G_1 + G_{1tt}r)G_1r - 3(G_{1t}r + r_tG_1)^2)F_t \\
& + (3(G_{1t}r + r_tG_1)F_{tt} - F_{ttt}G_1r)G_1r)/(KG_1^2). && (3.15)
\end{aligned}
$$

Thus, we proved the theorem.

Theorem 3.2.1. *Any third-order ordinary differential equations (3.3) obtained from a linear equation (3.5) by a generalized linearizing transformation (3.4) has to be the form (3.6).*

3.3 Formulation of the linearization theorem

We have shown in the previous section that every linearizable third-order ordinary differential equation belongs to the class of equations (3.6). In this section, we formulate the main theorems containing necessary and sufficient conditions for linearization as well as the methods for constructing the linearizing transformations.

For obtaining sufficient conditions, one has to solve the compatibility problem. Consider the representations of the coefficients A_i and B_i through the unknown functions F and G_1. According to our notation $K = F_t - F_x r$, we define the derivative F_t as

$$F_t = F_x r + K. \tag{3.16}$$

From equations (3.7) and (3.8), one can find the derivatives

$$K_x = (F_x G_{1t} - F_x G_{1x} r - F_x r_x G_1 + 3G_{1x} K + A_2 G_1 K)/(3G_1), \tag{3.17}$$

$$\begin{aligned} K_t =&(F_x G_{1t} r - F_x G_{1x} r^2 - F_x r_x G_1 r + 4G_{1t} K - G_{1x} K r \\ &+ G_1 K(5r_x + A_1 - A_2 r))/(3G_1). \end{aligned} \tag{3.18}$$

From equation (3.9) one obtains the condition

$$r_t = (6r_x r - A_0 + A_1 r - A_2 r^2)/6. \tag{3.19}$$

Equation (3.10) defines the derivative

$$F_{xxx} = (3F_{xx} G_{1x} G_1 + F_x G_{1xx} G_1 - 3F_x G_{1x}^2 - B_5 G_1^2 K)/G_1^2. \tag{3.20}$$

So that equation (3.11) becomes

$$\begin{gathered} 6F_{xx} G_{1t} G_1 - 6F_{xx} G_{1x} G_1 r - 6F_{xx} r_x G_1^2 + 3F_x G_{1tx} G_1 - 12F_x G_{1t} G_{1x} \\ -F_x G_{1t} A_2 G_1 - 3F_x G_{1xx} G_1 r + 12F_x G_{1x}^2 r + F_x G_{1x} G_1 (6r_x + A_2 r) \\ +F_x G_1^2(-3r_{xx} + r_x A_2) - 6G_{1xx} G_1 K + 9G_{1x}^2 K \\ +G_1^2 K(-3A_{2x} - A_2^2 - 3B_4 + 15B_5 r) = 0. \end{gathered} \tag{3.21}$$

The compatibility analysis depends on the value of F_x. A complete study of all cases is given here.

3.3.1 Case $F_x = 0$

In this case, the form of derivatives F_t, K_x and K_t become

$$F_t = K, \tag{3.22}$$

$$K_x = (3G_{1x} + A_2G_1)K/(3G_1), \tag{3.23}$$

$$K_t = (4G_{1t} - G_{1x}r + G_1(5r_x + A_1 - A_2r))K/(3G_1). \tag{3.24}$$

Substituting F_x into F_{xxx} one arrives at the condition

$$B_5 = 0. \tag{3.25}$$

Comparing the mixed derivatives $(F_x)_t = (F_t)_x$, one gets the derivative

$$G_{1x} = -(A_2G_1)/3. \tag{3.26}$$

In this case, $(F_{xxx})_t = (F_t)_{xxx}$ is satisfied. Equations (3.11) and (3.12) give the conditions

$$A_{2x} = (-2A_2^2 - 9B_4)/3, \tag{3.27}$$

$$r_{xx} = (-9A_{1x} + 6A_{2t} + 3r_xA_2 - 3A_1A_2 - 2A_2^2r - 9B_3)/36.$$

Comparing the mixed derivative $(K_t)_x = (K_x)_t$ one obtains the condition

$$A_{1x} = (-6A_{2t} - 3r_xA_2 - 5A_1A_2 + 2A_2^2r - 15B_3 + 24B_4r)/3. \tag{3.28}$$

Equation (3.13) provides the derivative

$$G_{1tt} = (2250G_{1t}^2 + 150G_{1t}G_1h_1 + G_1^2h_2)/(1350G_1) \tag{3.29}$$

where

$$\begin{aligned} h_1 =& 15r_x + 3A_1 - 2A_2r, \\ h_2 =& -225A_{0x} - 1350A_{1t} - 1350A_{2t}r - 1050A_0A_2 - 477A_1^2 + 516A_1A_2r \\ &+ 33A_1h_1 - 432A_2^2r^2 - 57A_2h_1r - 4050B_2 + 4275B_3r - 4275B_4r^2 - 8h_1^2. \end{aligned}$$

The relation $(r_x)_x = r_{xx}$ gives the condition

$$h_{1x} = 4A_{2t}. \tag{3.30}$$

Comparing the mixed derivative $(G_{1tt})_x = (G_{1x})_{tt}$ one arrives at the condition

$$A_{2tt} = (50A_{2t}h_1 - h_{2x})/450. \tag{3.31}$$

Solving equations (3.14) and (3.15), one finds the conditions

$$A_{0t} =(15930A_{1t}r + 15930A_{2t}r^2 - 1260h_{1x}r - 1575A_0A_1 + 11970A_0A_2r + 5517A_1^2r - 5697A_1A_2r^2 - 558A_1h_1r + 4986A_2^2r^3 + 504A_2h_1r^2 - 8100B_1 + 48600B_2r - 48600B_3r^2 + 48600B_4r^3 + 148h_1^2r + 8h_2r)/1350, \quad (3.32)$$

$$B_0 =(-3240A_{1t}r^2 - 3240A_{2t}r^3 + 180h_{1x}r^2 - 135A_0^2 + 270A_0A_1r - 2430A_0A_2r^2 - 1107A_1^2r^2 + 1134A_1A_2r^3 + 108A_1h_1r^2 - 999A_2^2r^4 - 108A_2h_1r^3 + 1620B_1r - 9720B_2r^2 + 9720B_3r^3 - 9720B_4r^4 - 28h_1^2r^2 - 2h_2r^2)/1620. \quad (3.33)$$

3.3.2 Case $F_x \neq 0$

From equations (3.21) and (3.12) one obtains the derivatives

$$G_{1tx} =(-6F_{xx}G_{1t}G_1 + 6F_{xx}G_{1x}G_1r + 6F_{xx}r_xG_1^2 + 12F_xG_{1t}G_{1x} + F_xG_{1t}A_2G_1 + 3F_xG_{1xx}G_1r - 12F_xG_{1x}^2r + F_xG_{1x}G_1(-6r_x - A_2r) + F_xG_1^2(3r_{xx} - r_xA_2) + 6G_{1xx}G_1K - 9G_{1x}^2K + G_1^2K(3A_{2x} + A_2^2 + 3B_4 - 15B_5r))/(3F_xG_1), \quad (3.34)$$

$$G_{1tt} =(-24F_{xx}F_xG_{1t}G_1r + 24F_{xx}F_xG_{1x}G_1r^2 + 24F_{xx}F_xr_xG_1^2r - 24F_{xx}G_{1t}G_1K + 24F_{xx}G_{1x}G_1Kr + 24F_{xx}r_xG_1^2K + 14F_x^2G_{1t}^2 + 20F_x^2G_{1t}G_{1x}r + 2F_x^2G_{1t}G_1(r_x + A_1) + 6F_x^2G_{1xx}G_1r^2 - 34F_x^2G_{1x}^2r^2 + F_x^2G_{1x}G_1(-26r_xr - A_0 - A_1r - A_2r^2) + F_x^2G_1^2(-A_{0x} + A_{1x}r - A_{2x}r^2 + 12r_{xx}r - 4r_x^2 - r_xA_1 - 2r_xA_2r) + 24F_xG_{1t}G_{1x}K + 24F_xG_{1xx}G_1Kr - 60F_xG_{1x}^2Kr - 24F_xG_{1x}r_xG_1K + 2F_xG_1^2K(3A_{1x} + 18r_{xx} - 3r_xA_2 + A_1A_2 + 3B_3 - 6B_4r) + 24G_{1xx}G_1K^2 - 36G_{1x}^2K^2 + 4G_1^2K^2(3A_{2x} + A_2^2 + 3B_4 - 15B_5r))/(6F_x^2G_1). \quad (3.35)$$

Comparing the mixed derivative $(K_t)_x = (K_x)_t$ one obtains

$$G_{1xx} =(6F_{xx}G_{1t}G_1K - 6F_{xx}G_{1x}G_1Kr - 6F_{xx}r_xG_1^2K - F_x^2G_{1t}^2 + 2F_x^2G_{1t}G_{1x}r + 2F_x^2G_{1t}r_xG_1 - F_x^2G_{1x}^2r^2 - 2F_x^2G_{1x}r_xG_1r - F_x^2r_x^2G_1^2 - 6F_xG_{1t}G_{1x}K + 6F_xG_{1x}^2Kr + 6F_xG_{1x}r_xG_1K + F_xG_1^2K(-3A_{2t} + 3A_{2x}r - A_1A_2 + 2A_2^2r - 3B_3 + 12B_4r - 30B_5r^2) + 9G_{1x}^2K^2 + G_1^2K^2(-3A_{2x} - A_2^2 - 3B_4 + 15B_5r))/(6G_1K^2). \quad (3.36)$$

Equation (3.13) becomes

$$F_x s_1 + 2K s_2 = 0 \tag{3.37}$$

where

$$\begin{aligned} s_1 =& -6A_{1t} + 6A_{1x}r + 12A_{2t}r - 12A_{2x}r^2 - 5A_0A_2 - 2A_1^2 + 13A_1A_2r \\ & - 13A_2^2r^2 - 18B_2 + 54B_3r - 108B_4r^2 + 180B_5r^3, \\ s_2 =& -3A_{1x} + 6A_{2t} - 18r_{xx} + 3r_xA_2 + A_1A_2 - 2A_2^2r + 3B_3 \\ & - 12B_4r + 30B_5r^2. \end{aligned}$$

Further analysis of the compatibility depends on value of s_1 in equation (3.37): it is separated into two cases, i.e., $s_1 = 0$ and $s_1 \neq 0$.

Case $s_1 \neq 0$

From equation (3.37), one finds

$$F_x = -(2Ks_2)/s_1. \tag{3.38}$$

Since this case $F_x \neq 0$, then $s_2 \neq 0$ too. Comparing the mixed derivatives $(F_x)_t = (F_t)_x$, one gets the derivative

$$G_{1t} = (3G_{1x}s_1(2rs_2 - s_1) + G_1s_3)/(6s_1s_2) \tag{3.39}$$

where

$$\begin{aligned} s_3 =& -6r_xs_1s_2 + 6s_{1t}s_2 - 6s_{1x}rs_2 - 6s_{2t}s_1 + 6s_{2x}rs_1 \\ & - 2A_1s_1s_2 + 4A_2rs_1s_2 - A_2s_1^2. \end{aligned}$$

Substituting F_x into F_{xxx}, G_{1t} into G_{1tx} and G_{1tt} one arrives at the conditions

$$\begin{aligned} s_{2xx} =&(-12A_{2t}s_1^2s_2^2 + 12A_{2x}rs_1^2s_2^2 - 6A_{2x}s_1^3s_2 + 36r_xs_{1x}s_1s_2^2 - 36r_xs_{2x}s_1^2s_2 \\ &- 12r_xA_2s_1^2s_2^2 + 18s_{1xx}s_1^2s_2 - 36s_{1x}^2s_1s_2 + 36s_{1x}s_{2x}s_1^2 + 12s_{1x}A_2s_1^2s_2 \\ &- 6s_{1x}s_2s_3 - 12s_{2x}A_2s_1^3 + 6s_{2x}s_1s_3 - 4A_1A_2s_1^2s_2^2 + 8A_2^2rs_1^2s_2^2 \\ &- 2A_2^2s_1^3s_2 + 2A_2s_1s_2s_3 - 12B_3s_1^2s_2^2 + 48B_4rs_1^2s_2^2 - 120B_5r^2s_1^2s_2^2 \\ &+ 9B_5s_1^4)/(18s_1^3), \end{aligned} \tag{3.40}$$

$$\begin{aligned} s_{3x} =&(-6A_{1x}s_1^3s_2 - 6A_{2t}s_1^3s_2 + 18A_{2x}rs_1^3s_2 - 9A_{2x}s_1^4 + 36r_x^2s_1^2s_2^2 \\ &- 36r_xs_{1x}s_1^2s_2 + 36r_xs_{2x}s_1^3 + 6r_xA_2s_1^3s_2 - 12r_xs_1s_2s_3 + 12s_{1x}s_1s_3 \\ &- 4A_1A_2s_1^3s_2 + 8A_2^2rs_1^3s_2 - 3A_2^2s_1^4 - 12B_3s_1^3s_2 + 48B_4rs_1^3s_2 \\ &- 9B_4s_1^4 - 120B_5r^2s_1^3s_2 + 45B_5rs_1^4 - 2s_1^3s_2^2 + s_3^2)/(6s_1^2), \end{aligned} \tag{3.41}$$

$$\begin{aligned} s_{3t} =&(-6A_{0x}s_1^3s_2 - 3A_{1x}s_1^4 - 6A_{2t}rs_1^3s_2 - 12A_{2t}s_1^4 + 12A_{2x}r^2s_1^3s_2 \\ &+ 9A_{2x}rs_1^4 + 36r_x^2rs_1^2s_2^2 - 36r_x^2s_1^3s_2 - 36r_xs_{1x}rs_1^2s_2 + 36r_xs_{2x}rs_1^3 \\ &- 6r_xA_1s_1^3s_2 + 18r_xA_2rs_1^3s_2 - 3r_xA_2s_1^4 - 12r_xrs_1s_2s_3 + 12s_{1t}s_1s_3 \\ &- 4A_1A_2rs_1^3s_2 - 5A_1A_2s_1^4 + 8A_2^2r^2s_1^3s_2 + 7A_2^2rs_1^4 - 12B_3rs_1^3s_2 \\ &- 15B_3s_1^4 + 48B_4r^2s_1^3s_2 + 51B_4rs_1^4 - 120B_5r^3s_1^3s_2 - 105B_5r^2s_1^4 \\ &- 2rs_1^3s_2^2 + rs_3^2 + 5s_1^4s_2)/(6s_1^2). \end{aligned} \tag{3.42}$$

Equations (3.14) and (3.15) provide the conditions

$$\begin{aligned} A_{0t} =&(6A_{0x}r + 6A_{2t}r^2 - 6A_{2x}r^3 - 7A_0A_1 + 9A_0A_2r + 5A_1^2r - 8A_1A_2r^2 \\ &+ A_2^2r^3 - 36B_1 + 54B_2r - 54B_3r^2 + 36B_4r^3 - rs_1)/6, \end{aligned} \tag{3.43}$$

$$\begin{aligned} B_0 =&(-A_0^2 + 2A_0A_1r - 2A_0A_2r^2 - A_1^2r^2 + 2A_1A_2r^3 - A_2^2r^4 + 12B_1r \\ &- 12B_2r^2 + 12B_3r^3 - 12B_4r^4 + 12B_5r^5)/12. \end{aligned} \tag{3.44}$$

Comparing the mixed derivatives $(G_{1tt})_x = (G_{1tx})_t$, $(G_{1xx})_t = (G_{1tx})_x$ and $(F_{xxx})_t = (F_t)_{xxx}$ one gets the conditions

$$\begin{aligned} A_{1xx} =&(-33A_{1x}A_2s_1^2 - 18A_{2tx}s_1^2 - 108A_{2t}r_xs_1s_2 + 24A_{2t}A_2s_1^2 \\ &+ 18A_{2t}s_3 + 54A_{2xx}rs_1^2 + 108A_{2x}r_xrs_1s_2 + 18A_{2x}r_xs_1^2 \\ &- 30A_{2x}A_1s_1^2 + 102A_{2x}A_2rs_1^2 - 18A_{2x}rs_3 - 90B_{3x}s_1^2 + 54B_{4t}s_1^2 \\ &+ 306B_{4x}rs_1^2 - 270B_{5t}rs_1^2 - 630B_{5x}r^2s_1^2 - 36r_xA_1A_2s_1s_2 \\ &+ 72r_xA_2^2rs_1s_2 + 27r_xA_2^2s_1^2 - 108r_xB_3s_1s_2 + 432r_xB_4rs_1s_2 \\ &+ 252r_xB_4s_1^2 - 1080r_xB_5r^2s_1s_2 - 1260r_xB_5rs_1^2 + 36r_xs_1s_2^2 \\ &- 18s_{1x}s_1s_2 + 30s_{2x}s_1^2 + 45A_0B_5s_1^2 - 5A_1A_2^2s_1^2 + 6A_1A_2s_3 \\ &- 45A_1B_5rs_1^2 + 10A_2^3rs_1^2 - 12A_2^2rs_3 - 15A_2B_3s_1^2 + 60A_2B_4rs_1^2 \\ &- 105A_2B_5r^2s_1^2 + 5A_2s_1^2s_2 + 18B_3s_3 - 72B_4rs_3 + 180B_5r^2s_3 \\ &- 6s_2s_3)/(18s_1^2), \end{aligned} \tag{3.45}$$

$$\begin{aligned} A_{2tt} =&(36A_{2tx}rs_1 + 72A_{2t}r_xs_1 - 18A_{2xx}r^2s_1 - 72A_{2x}r_xrs_1 - 3A_{2x}A_0s_1 \\ &+ 3A_{2x}A_1rs_1 - 3A_{2x}A_2r^2s_1 - 18B_{3t}s_1 + 18B_{3x}rs_1 + 72B_{4t}rs_1 \\ &- 72B_{4x}r^2s_1 - 180B_{5t}r^2s_1 + 180B_{5x}r^3s_1 + 24r_xA_1A_2s_1 - 48r_xA_2^2rs_1 \\ &+ 72r_xB_3s_1 - 288r_xB_4rs_1 + 720r_xB_5r^2s_1 - 6r_xs_1s_2 - 3s_{1x}s_1 \\ &+ 3A_0A_2^2s_1 - 12A_0B_4s_1 + 60A_0B_5rs_1 + 4A_1^2A_2s_1 - 19A_1A_2^2rs_1 \\ &+ 6A_1B_3s_1 - 12A_1B_4rs_1 + 19A_2^3r^2s_1 + 18A_2B_2s_1 - 66A_2B_3rs_1 \\ &+ 144A_2B_4r^2s_1 - 240A_2B_5r^3s_1 + 2A_2s_1^2 + s_3)/(18s_1), \end{aligned} \tag{3.46}$$

$$
\begin{aligned}
A_{2tx} =& (-6A_{1x}A_2s_1^2s_2 - 72A_{2t}r_xs_1s_2^2 + 90A_{2t}s_{1x}s_1s_2 - 90A_{2t}s_{2x}s_1^2 \\
&- 24A_{2t}A_2s_1^2s_2 + 12A_{2t}s_2s_3 + 18A_{2xx}rs_1^2s_2 - 9A_{2xx}s_1^3 \\
&+ 72A_{2x}r_xrs_1s_2^2 + 18A_{2x}r_xs_1^2s_2 - 90A_{2x}s_{1x}rs_1s_2 + 90A_{2x}s_{2x}rs_1^2 \\
&- 6A_{2x}A_1s_1^2s_2 + 48A_{2x}A_2rs_1^2s_2 - 18A_{2x}A_2s_1^3 - 12A_{2x}rs_2s_3 \\
&- 18B_{3x}s_1^2s_2 + 72B_{4x}rs_1^2s_2 - 27B_{4x}s_1^3 + 54B_{5t}s_1^3 - 180B_{5x}r^2s_1^2s_2 \\
&+ 81B_{5x}rs_1^3 - 24r_xA_1A_2s_1s_2^2 + 48r_xA_2^2rs_1s_2^2 + 12r_xA_2^2s_1^2s_2 \\
&- 72r_xB_3s_1s_2^2 + 288r_xB_4rs_1s_2^2 + 72r_xB_4s_1^2s_2 - 720r_xB_5r^2s_1s_2^2 \\
&- 360r_xB_5rs_1^2s_2 + 135r_xB_5s_1^3 + 30s_{1x}A_1A_2s_1s_2 - 60s_{1x}A_2^2rs_1s_2 \\
&+ 90s_{1x}B_3s_1s_2 - 360s_{1x}B_4rs_1s_2 + 900s_{1x}B_5r^2s_1s_2 - 30s_{2x}A_1A_2s_1^2 \\
&+ 60s_{2x}A_2^2rs_1^2 - 90s_{2x}B_3s_1^2 + 360s_{2x}B_4rs_1^2 - 900s_{2x}B_5r^2s_1^2 \\
&- 8A_1A_2^2s_1^2s_2 + 4A_1A_2s_2s_3 + 18A_1B_5s_1^3 + 16A_2^3rs_1^2s_2 - 4A_2^3s_1^3 \\
&- 8A_2^2rs_2s_3 - 24A_2B_3s_1^2s_2 + 96A_2B_4rs_1^2s_2 - 18A_2B_4s_1^3 \\
&- 240A_2B_5r^2s_1^2s_2 + 54A_2B_5rs_1^3 + 12B_3s_2s_3 - 48B_4rs_2s_3 \\
&+ 120B_5r^2s_2s_3)/(18s_1^2s_2).
\end{aligned}
\tag{3.47}
$$

Case $s_1 = 0$

From equation (3.37), one finds the condition

$$s_2 = 0. \tag{3.48}$$

Equations (3.14) and (3.15) give the conditions

$$
\begin{aligned}
A_{0t} =& (6A_{0x}r + 6A_{2t}r^2 - 6A_{2x}r^3 - 7A_0A_1 + 9A_0A_2r + 5A_1^2r - 8A_1A_2r^2 \\
&+ A_2^2r^3 - 36B_1 + 54B_2r - 54B_3r^2 + 36B_4r^3)/6,
\end{aligned}
\tag{3.49}
$$

$$
\begin{aligned}
B_0 =& (-A_0^2 + 2A_0A_1r - 2A_0A_2r^2 - A_1^2r^2 + 2A_1A_2r^3 - A_2^2r^4 + 12B_1r \\
&- 12B_2r^2 + 12B_3r^3 - 12B_4r^4 + 12B_5r^5)/12.
\end{aligned}
\tag{3.50}
$$

From the mixed derivative $(G_{1xx})_t = (G_{1tx})_x$, one finds the condition

$$
\begin{aligned}
A_{2tt} =& (36A_{2tx}r + 72A_{2t}r_x - 18A_{2xx}r^2 - 72A_{2x}r_xr - 3A_{2x}A_0 + 3A_{2x}A_1r \\
&- 3A_{2x}A_2r^2 - 18B_{3t} + 18B_{3x}r + 72B_{4t}r - 72B_{4x}r^2 - 180B_{5t}r^2 \\
&+ 180B_{5x}r^3 + 24r_xA_1A_2 - 48r_xA_2^2r + 72r_xB_3 - 288r_xB_4r \\
&+ 720r_xB_5r^2 + 3A_0A_2^2 - 12A_0B_4 + 60A_0B_5r + 4A_1^2A_2 - 19A_1A_2^2r \\
&+ 6A_1B_3 - 12A_1B_4r + 19A_2^3r^2 + 18A_2B_2 - 66A_2B_3r + 144A_2B_4r^2 \\
&- 240A_2B_5r^3)/18.
\end{aligned}
$$

The relation $(G_{1tt})_x = (G_{1tx})_t$ becomes

$$18(F_xG_{1t} - F_xG_{1x}r - F_xr_xG_1 - G_{1x}K)s_4 + G_1Ks_5 = 0 \tag{3.51}$$

where

$$\begin{aligned}
s_4 =& 3A_{2t} - 3A_{2x}r + A_1A_2 - 2A_2^2r + 3B_3 - 12B_4r + 30B_5r^2,\\
s_5 =& 18A_{1xx} + 27A_{1x}A_2 - 36A_{2xx}r + 24A_{2x}A_1 - 102A_{2x}A_2r + 72B_{3x} - 54B_{4t}\\
&- 234B_{4x}r + 270B_{5t}r + 450B_{5x}r^2 - 15r_xA_2^2 - 180r_xB_4 + 900r_xB_5r\\
&+ 6s_{4x} - 45A_0B_5 + 13A_1A_2^2 + 45A_1B_5r - 26A_2^3r + 39A_2B_3\\
&- 156A_2B_4r + 345A_2B_5r^2 - 8A_2s_4.
\end{aligned}$$

The relation $(A_{2t})_t - A_{2tt} = 0$ provides the condition

$$s_{4t} = (12r_xs_4 + 3s_{4x}r + A_1s_4 - 2A_2rs_4)/3. \tag{3.52}$$

Further study depends on s_4.

- Case $s_4 \neq 0$

  From equation (3.51), one gets the derivative

$$g_{1t} = (18(F_xG_{1x}r + F_xr_xG_1 + G_{1x}K)s_4 - G_1Ks_5)/(18F_xs_4). \tag{3.53}$$

Differentiating g_{1t} with respect to x, one obtains the derivative

$$F_x = (Ks_6)/(108s_4^3) \tag{3.54}$$

where

$$s_6 = 324A_{2x}s_4^2 - 36s_{4x}s_5 + 36s_{5x}s_4 + 108A_2^2s_4^2 + 324B_4s_4^2 - 1620B_5rs_4^2 - s_5^2.$$

The relations $(F_x)_t = (F_t)_x$, $(G_{1t})_t = G_{1tt}$, $(F_{xxx})_t = (F_t)_{xxx}$, $(F_x)_{xx} = F_{xxx}$ provide the conditions

$$
\begin{aligned}
s_{6t} =&(30r_x s_6 + 3s_{6x} r + 2A_1 s_6 - 4A_2 r s_6 + 108A_2 s_4^3 + 18s_4^2 s_5)/3, &(3.55)\\
s_{5t} =&(-108A_{1x}s_4^2 - 108A_{2x} r s_4^2 + 108r_x A_2 s_4^2 + 180r_x s_4 s_5\\
&+ 36s_{4x} r s_5 - 36A_1 A_2 s_4^2 + 12A_1 s_4 s_5 - 36A_2^2 r s_4^2 - 24A_2 r s_4 s_5\\
&- 108B_3 s_4^2 + 108B_4 r s_4^2 + 540B_5 r^2 s_4^2 + r s_5^2 + r s_6\\
&- 144s_4^3)/(36s_4), &(3.56)\\
A_{2xx} =&(-5832A_{2x}A_2 s_4^3 - 8748B_{4x}s_4^3 + 17496B_{5t}s_4^3 + 26244B_{5x} r s_4^3\\
&+ 43740r_x B_5 s_4^3 - 126s_{4x}s_6 + 45s_{6x}s_4 + 5832A_1 B_5 s_4^3 - 1296A_2^3 s_4^3\\
&- 5832A_2 B_4 s_4^3 + 17496A_2 B_5 r s_4^3 + 12A_2 s_4 s_6 - s_5 s_6)/(2916s_4^3), &(3.57)\\
s_{6xx} =&(-324A_{2x}s_4^2 s_6 + 2916s_{4xx}s_4 s_6 - 11664s_{4x}^2 s_6 + 5832s_{4x}s_{6x}s_4\\
&+ 1944s_{4x}A_2 s_4 s_6 - 162s_{4x}s_5 s_6 - 648s_{6x}A_2 s_4^2 + 54s_{6x}s_4 s_5\\
&- 108A_2^2 s_4^2 s_6 + 18A_2 s_4 s_5 s_6 - 104976B_5 s_4^5 + s_6^2)/(972s_4^2). &(3.58)
\end{aligned}
$$

- Case $s_4 = 0$

From equation (3.51), one gets the condition

$$s_5 = 0. \tag{3.59}$$

Comparing the mixed derivative $(F_{xxx})_t = (F_t)_{xxx}$, one arrives at the condition

$$
\begin{aligned}
A_{2xx} =&(-18A_{2x}A_2 - 27B_{4x} + 54B_{5t} + 81B_{5x} r + 135r_x B_5 + 18A_1 B_5\\
&- 4A_2^3 - 18A_2 B_4 + 54A_2 B_5 r)/9. &(3.60)
\end{aligned}
$$

All obtained results can be summarized in the following theorems.

Theorem 3.3.1. *Sufficient conditions for equation (3.6) to be linearizable via the generalized linearizing transformation (3.4) with $F_x = 0$ are equations (3.19), (3.25), (3.27), (3.28), (3.30), (3.31), (3.32) and (3.33).*

Corollary 3.3.2. *Provided that the sufficient conditions in Theorem 3.3.1 are satisfied, the transformation (3.4) mapping equation (3.6) to a linear equation (3.5) is obtained by solving the compatible system of equations (3.22), (3.23), (3.24), (3.26) and (3.29) for the functions $F(t), G_1(t,x)$ and $G_2(t,x)$.*

Theorem 3.3.3. *Sufficient conditions for equation (3.6) to be linearizable via the generalized linearizing transformation (3.4) with $F_x \neq 0$ are as follows.*

(a) *If* $s_1 \neq 0$, *then the conditions are (3.19), (3.40), (3.41), (3.42), (3.43), (3.44), (3.45), (3.46) and (3.47).*

(b) *If* $s_1 = 0, s_4 \neq 0$, *then the conditions are (3.19), (3.48), (3.49), (3.50), (3.52), (3.55), (3.56), (3.57) and (3.58).*

(c) *If* $s_1 = 0, s_4 = 0$, *then the conditions are (3.19), (3.48), (3.49), (3.50), (3.59) and (3.60).*

Corollary 3.3.4. *Provided that the sufficient conditions in Theorem 3.3.3 are satisfied, the transformation (3.4) mapping equation (3.6) to a linear equation (3.5) is obtained by solving the following compatible system of equations for the functions* $F(t,x), G_1(t,x)$ *and* $G_2(t,x)$:

(a) *(3.16), (3.17), (3.18), (3.36), (3.38) and (3.39).*

(b) *(3.16), (3.17), (3.18), (3.36), (3.53) and (3.54).*

(c) *(3.16), (3.17), (3.18), (3.20), (3.34), (3.35) and (3.36).*

3.4 Examples

For understanding the procedure of using the linearization theorems we consider the following examples.

Example 1. Consider the nonlinear third-order ordinary differential equation

$$\begin{gathered} 3x'^4t^2 + 2x'^3t(3t+2x) + 3x'^2x''t^2x + x'^2(3t^2+8tx+3x^2) \\ +2x'x''tx(t+2x) - x'x'''t^2x^2 + 2x'x(2t+3x) + 3x''^2t^2x^2 \\ +x''tx(-t+4x) - x'''t^2x^2 + 3x^2 = 0. \end{gathered} \tag{3.61}$$

It is an equation of the form (3.6) in Theorem 3.2.1 with the coefficients

$$A_2 = -\frac{3}{x},\ A_1 = -\frac{2(t+2x)}{tx},\ A_0 = \frac{t-4x}{tx},\ B_5 = 0,\ B_4 = -\frac{3}{x^2},$$
$$B_3 = -\frac{2(3t+2x)}{tx^2},\ B_2 = -\frac{3t^2+8tx+3x^2}{t^2x^2},\ B_1 = -\frac{2(2t+3x)}{t^2x},$$
$$B_0 = -\frac{3}{t^2},\ r = 1,\ h_1 = -\frac{12}{t},\ h_2 = -\frac{450}{t^2}.$$

One can check that these coefficients obey the conditions in Theorem 3.3.1. Thus equation (3.61) is linearizable via a generalized linearizing transformation. For finding the functions F, G_1 and G_2 we have to solve equations in Corollary 3.3.2, which become

$$F_x = 0,\ \ F_t = K, \tag{3.62}$$

$$G_{1x} = G_1/x, \ \ G_{1tt} = (5G_{1t}^2 t^2 - 4G_{1t}G_1 t - G_1^2)/(3G_1 t^2), \tag{3.63}$$

$$K_x = 0, \ \ K_t = (4KG_{1t}t - G_1)/(3G_1 t). \tag{3.64}$$

From the first equation of system (3.63), we get $G_1 = xf(t)$, choosing $f(t) = t$ we have

$$G_1 = xt$$

and this solution satisfies the second equation. Since $r = 1$, then we obtain

$$G_2 = xt.$$

System (3.64) becomes

$$K_x = 0, \ \ K_t = 0$$

one can take the simplest solution

$$K = 1.$$

System (3.77) becomes

$$F_x = 0, \ \ F_t = 1 \tag{3.65}$$

so that we get the particular solution

$$F = t.$$

Thus one obtains the linearizing transformation

$$X = t, \ dT = tx(x' + 1)dt. \tag{3.66}$$

Hence equation (3.61) is mapped by the transformation (3.66) into the linear equation (3.5).

Example 2. Consider the nonlinear third-order ordinary differential equation

$$\begin{aligned} &3x'^5t^2 + x'^4t(3t + 4x) + x'^3x(4t + 3x) + x'^2x''tx(3t + x) + 3x'^2x^2 \\ &+4x'x''tx^2 - x'x'''t^2x^2 + 3x''^2t^2x^2 = 0. \end{aligned} \tag{3.67}$$

It is an equation of the form (3.6) in Theorem 3.2.1 with the coefficients

$$A_2 = -\frac{(3t + x)}{tx}, \ A_1 = -\frac{4}{t}, \ A_0 = 0, \ B_5 = -\frac{3}{x^2}, \ B_4 = -\frac{(3t + 4x)}{tx^2},$$

$$B_3 = -\frac{(4t + 3x)}{t^2x}, \ B_2 = -\frac{3}{t^2}, \ B_1 = 0, \ B_0 = 0, \ r = 0,$$

$$s_1 = -\frac{2}{t^2}, \ s_2 = \frac{1}{t^2}, \ s_3 = \frac{12(t - x)}{t^5x}.$$

One can check that these coefficients obey the conditions in Theorem 3.3.3 (a). Thus equation (3.67) is linearizable via a generalized linearizing transformation. For finding the functions F, G_1 and G_2 we have to solve equations in Corollary 3.3.4 (a), which become

$$F_x = K, \quad F_t = K, \tag{3.68}$$

$$\begin{aligned} G_{1t} &= (G_{1x}tx - G_1t + G_1x)/(tx), \\ G_{1xx} &= (G_{1x}^2tx^2 + 4G_{1x}G_1tx - 4G_{1x}G_1x^2 - 5G_1^2t + 4G_1^2x)/(3G_1tx^2), \end{aligned} \tag{3.69}$$

$$K_x = (4K(G_{1x}x - G_1))/(3G_1x), \quad K_t = (4K(G_{1x}x - G_1))/(3G_1x). \tag{3.70}$$

From the first equation of system (3.69), one can take the particular solution

$$G_1 = tx$$

and this solution satisfies the second equation. Since $r = 0$, then we obtain

$$G_2 = 0.$$

System (3.70) becomes

$$K_x = 0, \quad K_t = 0$$

one can take the simplest solution

$$K = 1.$$

System (3.68) becomes

$$F_x = 1, \quad F_t = 1 \tag{3.71}$$

so that we get the particular solution

$$F = t + x.$$

Thus one obtains the linearizing transformation

$$X = t + x, \ dT = txx'dt. \tag{3.72}$$

Hence equation (3.67) is mapped by the transformation (3.72) into the linear equation (3.5).

Example 3. Consider the nonlinear third-order ordinary differential equation

$$3x''^2x^2 - 3x'^4 - 3x'^2x''x - x'x'''x^2 = 0. \tag{3.73}$$

Note that this equation can be reduced to an autonomous equation by the substitution:

$$x = tv(s), \ s = ln(t)$$

and then to the second-order ordinary differential equation:

$$y''z^2y^2(z+y) = y'^2z^2y(-2z+y) - 3y'zy(z^2+y^2) - 3z^4 - 14z^3y - 20z^2y^2 - 15zy^3 - 3y^4$$

where $y = y(z)$. However, the latter equation is not linearizable by point transformations.

Equation (3.73) is an equation of the form (3.6) in Theorem 3.2.1 with the coefficients

$$A_2 = \frac{3}{x},\ A_1 = 0,\ A_0 = 0,\ B_5 = 0,\ B_4 = \frac{3}{x^2},\ B_3 = 0,\ B_2 = 0,$$
$$B_1 = 0,\ B_0 = 0,\ r = 0,\ s_1 = 0,\ s_2 = 0,\ s_4 = 0,\ s_5 = 0.$$

One can check that these coefficients obey the conditions in Theorem 3.3.3 (c). Thus equation (3.73) is linearizable via a generalized linearizing transformation. For finding the functions F, G_1 and G_2 we have to solve equations in Corollary 3.3.4 (c), which become

$$\begin{gathered} F_t = K, \\ F_{xxx} = (6F_{xx}F_xG_{1t}G_1Kx^2 + 18F_{xx}G_{1x}G_1K^2x^2 - F_x^3G_{1t}^2x^2 \\ -6F_x^2G_{1t}G_{1x}Kx^2 - 9F_xG_{1x}^2K^2x^2 - 9F_xG_1^2K^2)/(6G_1^2K^2x^2), \end{gathered} \tag{3.74}$$

$$\begin{gathered} G_{1tt} = (5G_{1t}^2)/(3G_1), \\ G_{1tx} = (G_{1t}(-F_xG_{1t}x + 6G_{1x}Kx + 3G_1K))/(3G_1Kx), \\ G_{1xx} = (6F_{xx}G_{1t}G_1Kx^2 - F_x^2G_{1t}^2x^2 - 6F_xG_{1t}G_{1x}Kx^2 \\ +9G_{1x}^2K^2x^2 - 9G_1^2K^2)/(6G_1K^2x^2), \end{gathered} \tag{3.75}$$

$$K_x = (F_xG_{1t}x + 3G_{1x}Kx + 3G_1K)/(3G_1x),\quad K_t = (4G_{1t}K)/(3G_1). \tag{3.76}$$

From the first equation of system (3.75), one can take the particular solution

$$G_1 = x$$

and this solution satisfies the second and third equations. Since $r = 0$, then we obtain

$$G_2 = 0.$$

System (3.76) becomes

$$K_x = (2K)/x,\quad K_t = 0$$

one can take the particular solution

$$K = x^2.$$

System (3.74) becomes

$$F_x = x^2, \quad F_{xxx} = (3(F_{xx}x - F_x))/x^2 \tag{3.77}$$

so that one obtains the particular solution of the first equation as

$$F = tx^2$$

and this solution satisfies the second equation. Then we get the linearizing transformation

$$X = tx^2, \; dT = xx'dt. \tag{3.78}$$

Hence equation (3.73) is mapped by the transformation (3.78) into the linear equation (3.5).

3.5 Conclusion

This paper is devoted to find the conditions which allow the third-order ordinary differential equation to be transformed to the simplest linear equation. Necessary conditions which guarantee that the third-order ordinary differential equation can be linearized are found in Theorem 3.2.1. Theorem 3.3.1 and Theorem 3.3.3 are sufficient conditions for the linearization problem. The linearizing transformation can be found by solving the compatible system in Corollary 3.3.2 and Corollary 3.3.4. Finally, some examples are provided to demonstrate our procedure.

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LINEARIZABILITY OF NONLINEAR FOURTH-ORDER ORDINARY DIFFERENTIAL EQUATIONS BY A GENERALIZED SUNDMAN TRANSFORMATION

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Abstract

We present the linearization problem of fourth-order ordinary differential equation under the generalized Sundman transformation $X = F(t, x)$, $dT = G(t, x)dt$. We found the necessary and sufficient conditions which allow the fourth-order ordinary differential equation $x^{(4)}(t) = f(t, x, x', x'', x''')$ to be transformed to $X^{(4)}(T) = 0$. The process of getting the linearizing transformation is constructed symmetrically and explicitly. Some examples of linearizable equation are provided to demonstrate our procedure and also the linearization of traveling waves of partial differential equation is applied.

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1. Introduction

1.1. Introduction to the research problem and its significance

Nonlinear problems are of interest to engineers, physicists, mathematicians and many other scientists because most equations are inherently nonlinear in nature. Nonlinear equations are difficult to solve. Although there are a number of well defined methods for the solution of linear ordinary differential equations, the same, however, cannot be said in the case of nonlinear ordinary differential equations. While solving problems related to nonlinear ordinary differential equations, it is often expedient to simplify equations by a suitable change of variables. In particular, the possibility that a given equation could be linearized, i.e., transformed to a linear equation.

Two given ordinary differential equations are called *equivalent* if one can be transformed into the other by a change of variables. The equivalence problem consists of two parts: deciding equivalence and determining a transformation that connects the ordinary differential equations. If the given equation is a linear equation, then the equivalence problem is a linearization problem. Our motivation for considering this problem is to translate a known solution of an ordinary differential equation to solutions of ordinary differential equations which are equivalent to it, thus allowing a systematic use of collections of solved ordinary differential equations. A linear equation was a most attractive proposition due to the special properties of linear differential equations. The reduction of an ordinary differential equation to a linear ordinary differential equation besides simplification allows constructing an exact solution of the original equation. Analytical (exact) solution has value, firstly, as an exact description of a real process in the framework of a given model; secondly, as a model to compare various numerical methods; thirdly, as a basis to improve the models used. Therefore, the linearization problem plays a significant role in the nonlinear problem.

Point transformation, contact transformation, reduction of order, differential substitution, generalized Sundman transformation, etc. are some of the tools commonly used for solving the linearization problem. Transformations used for solving the linearization problem considered in this project employ generalized Sundman transformations.

1.2. Historical review

In 1883, the problem on linearization of second-order ordinary differential equations by means of point transformations was solved by Lie [1]. He showed that any linearizable second-order equation can be at most cubic in the first-order derivative, and gave the linearization test in terms of the coefficients of these equations. Lie's approach has also been applied to third-order and fourth-order ordinary differential equations.

Another approach was developed by Cartan [2]. He used differential geometry for solving the linearization problem.

In 1992, the first generalized Sundman transformation was proposed by Sundman.

In 1994, Duarte et al. [3] considered the problem of linearization of second-order ordinary differential equations by means of generalized Sundman transformations to the Laguerre form.

In 2001, Berkovich [4] considered some application of generalized Sundman transformations to ordinary differential equations and earlier paper, which are summarized in Berkovich [5].

In 2003, Euler et al. [6] solved the problem of linearization of third-order ordinary differential equations. They found the necessary and sufficient conditions for a third-order ordinary differential equation to be equivalent to $X'''(T) = 0$ under generalized Sundman transformations.

In 2010, Muriel and Romero [7] studied the class of nonlinear second-order equations that are linearizable by means of generalized Sundman transformations is identified as the class of equations admitting first integrals that are polynomials of first degree in the first-order derivative. Nakpim and Meleshko [8] showed that the solution of the linearization problem for a second-order ordinary differential equation via the generalized Sundman transformation was considered earlier by Duarte et al. using the Laguerre form is not complete.

In 2013, Mustafa et al. [9] considered the linearization problem for nonlinear second-order ordinary differential equations to the Laguerre form by means of generalized Sundman transformations. They gave a new characterization of S-linearizable equations in terms of the coefficients of ordinary differential equations and one auxiliary function. This new criterion is used to obtain the general solutions for the first integral explicitly, providing a direct alternative procedure for constructing the first integrals and Sundman transformations.

Sundman symmetries were first introduced by Euler et al. [6] in 2003. They discovered that all third-order ordinary differential equations that can be linearized to the equation $X'''(T) = 0$ by the generalized Sundman transformation

$$X(T) = F(t, x), \; dT = G(t, x)dt, \; (F_x G \neq 0)$$

admit the symmetry

$$F(\tilde{t}, \tilde{x}) = F^{-1}(t, x), \; G(\tilde{t}, \tilde{x})d\tilde{t} = F^{-3/2}(t, x)G(t, x)dt$$

called a *Sundman symmetry transformation*. In 2004, Euler and Euler [10] investigated the Sundman symmetries of second-order autonomous equations

$$X'' + a_2 X'^2 + a_1 X' + a_0 = 0,$$

where a_0, a_1 and a_2 are differentiable functions. Moreover, they found the Sundman symmetries of third-order autonomous equations

$$X''' + a_5 X''^2 + a_4 X' X'' + a_3 X'^3 + a_2 X'^2 + a_1 X' + a_0 = 0,$$

where a_j $(j = 0, 1, \ldots, 5)$ are differentiable functions.

The main goal of the present paper is to demonstrate possibilities of applications generalized Sundman transformations for a linearization problem. In the paper, generalized Sundman transformations are applied for linearizing fourth-order ordinary differential equations. Complete study of generalized Sundman transformations mapping equations to the trivial fourth-order ordinary differential equation $X^{(4)}(T) = 0$ is given in the paper.

The manuscript is organized as follows: In Section 2, the necessary conditions of linearization of a fourth-order ordinary differential equation are presented. In Section 3, we state the theorems that yield criteria for a fourth-order ordinary differential equation to be linearizable via generalized Sundman transformations. Relations between coefficients of a linearizable equation and generalized Sundman transformations reducing this equation into a linear equation are presented in this section. Examples which illustrate the procedure of using the linearization theorems and some applications are presented in Section 4. For the sake of simplicity of reading, cumbersome formulae of this section are moved into Appendix.

2. Necessary Conditions of Linearization

We begin with investigating the necessary conditions for linearization. We consider the fourth-order ordinary differential equations

$$x^{(4)} = f(t, x, x', x'', x''') \tag{1}$$

which can be transformed to a linear equation

$$X^{(4)}(T) = 0 \tag{2}$$

under the generalized Sundman transformation

$$X = F(t, x),$$
$$dT = G(t, x)dt, \tag{3}$$

so that we arrive at the following theorem:

Theorem 1. *Any fourth-order ordinary differential equations* (1) *obtained from a linear equation* (2) *by a generalized Sundman* *transformation* (3) *has to be the form*

$$x^{(4)} + (A_1x' + A_0)x''' + Bx''^2 + (C_2x'^2 + C_1x' + C_0)x''$$
$$+ D_4x'^4 + D_3x'^3 + D_2x'^2 + D_1x' + D_0 = 0, \quad (4)$$

where

$$A_1 = (4F_{xx}G - 7F_xG_x)/(F_xG), \quad (5)$$

$$A_0 = (4F_{tx}G - F_tG_x - 6F_xG_t)/(F_xG), \quad (6)$$

$$B = (3F_{xx}G - 4F_xG_x)/(F_xG), \quad (7)$$

$$C_2 = (6F_{xxx}G^2 - 22F_{xx}G_xG - 7F_xG_{xx}G + 25F_xG_x^2)/(F_xG^2), \quad (8)$$

$$C_1 = (12F_{txx}G^2 - 26F_{tx}G_xG - 3F_tG_{xx}G + 10F_tG_x^2 - 18F_{xx}G_tG$$
$$- 11F_xG_{tx}G + 40F_xG_tG_x)/(F_xG^2), \quad (9)$$

$$C_0 = (-18F_{tx}G_tG + 6F_{ttx}G^2 - 4F_{tt}G_xG - 3F_tG_{tx}G + 10F_tG_tG_x$$
$$- 4F_xG_{tt}G + 15F_xG_t^2)/(F_xG^2), \quad (10)$$

$$D_4 = (F_{xxxx}G^3 - 6F_{xxx}G_xG^2 - 4F_{xx}G_{xx}G^2 + 15F_{xx}G_x^2G$$
$$- F_xG_{xxx}G^2 + 10F_xG_{xx}G_xG - 15F_xG_x^3)/(F_xG^3), \quad (11)$$

$$D_3 = (4F_{txxx}G^3 - 18F_{txx}G_xG^2 - 8F_{tx}G_{xx}G^2 + 30F_{tx}G_x^2G - F_tG_{xxx}G^2$$
$$+ 10F_tG_{xx}G_xG - 15F_tG_x^3 - 6F_{xxx}G_tG^2 - 8F_{xx}G_{tx}G^2$$
$$+ 30F_{xx}G_tG_xG - 3F_xG_{txx}G^2 + 20F_xG_{tx}G_xG$$
$$+ 10F_xG_tG_{xx}G - 45F_xG_tG_x^2)/(F_xG^3), \quad (12)$$

$$D_2 = (-18F_{txx}G_tG^2 - 16F_{tx}G_{tx}G^2 + 60F_{tx}G_tG_xG + 6F_{ttxx}G^3$$
$$- 18F_{ttx}G_xG^2 - 4F_{tt}G_{xx}G^2 + 15F_{tt}G_x^2G - 3F_tG_{txx}G^2$$

$$+ 20F_tG_{tx}G_xG + 10F_tG_tG_{xx}G - 45F_tG_tG_x^2 - 4F_{xx}G_{tt}G^2$$

$$+ 15F_{xx}G_t^2G + 20F_xG_{tx}G_tG - 3F_xG_{ttx}G^2$$

$$+ 10F_xG_{tt}G_xG - 45F_xG_t^2G_x)/(F_xG^3), \quad (13)$$

$$D_1 = (-8F_{tx}G_{tt}G^2 + 30F_{tx}G_t^2G + 4F_{tttx}G^3 - 6F_{ttt}G_xG^2$$

$$- 18F_{ttx}G_tG^2 - 8F_{tt}G_{tx}G^2 + 30F_{tt}G_tG_xG + 20F_tG_{tx}G_tG$$

$$- 3F_tG_{ttx}G^2 + 10F_tG_{tt}G_xG - 45F_tG_t^2G_x - F_xG_{ttt}G^2$$

$$+ 10F_xG_{tt}G_tG - 15F_xG_t^3)/(F_xG^3), \quad (14)$$

$$D_0 = (F_{tttt}G^3 - 6F_{ttt}G_tG^2 - 4F_{tt}G_{tt}G^2 + 15F_{tt}G_t^2G - F_tG_{ttt}G^2$$

$$+ 10F_tG_{tt}G_tG - 15F_tG_t^3)/(F_xG^3) \quad (15)$$

with

$$F_xG \neq 0.$$

Proof. Applying a generalized Sundman transformation (3), one obtains the following transformation of the fourth-order derivatives:

$$X'(T) = \frac{D_tF(t, x)}{D_t\int G(t, x)dt} = \frac{F_t + x'F_x}{G} = P(t, x, x'),$$

$$X''(T) = \frac{D_tP}{D_t\int G(t, x)dt} = \frac{P_t + x'P_x + x''P_{x'}}{G}$$

$$= \frac{2F_{tx}Gx' + F_{tt}G - F_tG_t - F_tG_xx' + F_{xx}Gx'^2 - F_xG_tx' - F_xG_xx'^2 + F_xGx''}{G^3}$$

$$= Q(t, x, x', x''),$$

$$X'''(T) = \frac{D_t Q}{D_t \int G(t, x)dt} = \frac{Q_t + x'Q_x + x''Q_{x'} + x'''Q_{x''}}{G}$$

$$= \frac{1}{G^5}[(F_x G^2)x''' + G(3F_{xx}G - 4F_x G_x)x'x''$$

$$+ G(3F_{tx}G - F_t G_x - 3F_x G_t)x'' + \cdots]$$

$$= R(t, x, x', x'', x'''),$$

$$X^{(4)}(T) = \frac{D_t R}{D_t \int G(t, x)dt} = \frac{R_t + x'R_x + x''R_{x'} + x'''R_{x''} + x^{(4)}R_{x'''}}{G}$$

$$= \frac{1}{G^7}[(F_x G^3)x^{(4)} + G^2(4F_{xx}G - 7F_x G_x)x'x'''$$

$$+ G^2(4F_{tx}G - F_t G_x - 6F_x G_t)x''' + \cdots]$$

$$= S(t, x, x', x'', x''', x^{(4)}), \tag{16}$$

where

$D_t = \frac{\partial}{\partial t} + x'\frac{\partial}{\partial x} + x''\frac{\partial}{\partial x'} + x'''\frac{\partial}{\partial x''} + x^{(4)}\frac{\partial}{\partial x'''} + \cdots$ is a total derivative.

Substituting $X^{(4)}(T)$ into equation (2), we have

$$(F_x G^3)x^{(4)} + (G^2(4F_{xx}G - 7F_x G_x)x' + G^2(4F_{tx}G - F_t G_x - 6F_x G_t))x'''$$

$$+ G^2(3F_{xx}G - 4F_x G_x)x''^2 + (G(6F_{xxx}G^2 + \cdots)x'^2 + \cdots)x'' + \cdots = 0.$$

Dividing this equation by $F_x G^3$, we get

$$x^{(4)} + \left(\left(\frac{4F_{xx}G - 7F_x G_x}{F_x G}\right)x' + \left(\frac{4F_{tx}G - F_t G_x - 6F_x G_t}{F_x G}\right)\right)x'''$$

$$+ \left(\frac{3F_{xx}G - 4F_x G_x}{F_x G}\right)x''^2 + \left(\left(\frac{6F_{xxx}G^2 + \cdots}{F_x G^2}\right)x'^2 + \cdots\right)x'' + \cdots = 0.$$

Denoting A_i, B, C_i and D_i as equations (5)-(15), so that we obtain the necessary form as equation (4). These prove the theorem. □

3. Formulation of the Linearization Theorem

We have shown in the previous section that every linearizable fourth-order ordinary differential equation belongs to the class of equations (4). In this section, we formulate the main theorems containing necessary and sufficient conditions for linearization as well as the methods for constructing the linearizing transformations.

For obtaining sufficient conditions, one has to solve the compatibility problem. Consider the representations of the coefficients A_i, B, C_i and D_i through the unknown functions F and G. From equations (5) and (6), one can find the derivatives

$$F_{xx} = F_x(7G_x + A_1G)/(4G), \quad (17)$$

$$F_{tx} = (F_tG_x + 6F_xG_t + F_xA_0G)/(4G). \quad (18)$$

From equation (7), one obtains that

$$G_x = -GS_1/5, \quad (19)$$

where

$$S_1 = 3A_1 - 4B.$$

Notice that for the case $G_x = 0$, the generalized Sundman transformation becomes a point transformation, so that we assume $G_x \neq 0$, that means $S_1 \neq 0$ too.

Equations (8) and (11) provide the conditions

$$S_{1x} = (300A_{1x} + 75A_1^2 + 10A_1S_1 - 200C_2 - 17S_1^2)/140,$$

$$A_{1xx} = (-5100A_{1x}A_1 - 300A_{1x}S_1 + 8400C_{2x} - 925A_1^3 - 175A_1^2S_1 + 4100A_1C_2 - 15A_1S_1^2 + 900C_2S_1 - 39200D_4 + 3S_1^3)/2800. \quad (20)$$

From equation (10), one obtains the derivative

$$G_{tt} = (-20F_{tt}G^2S_1 + 38F_tG_tGS_1 + 3F_tG^2S_2 + 300F_xG_t^2 + 5F_xG^2S_3)/(200F_xG), \tag{21}$$

where

$$S_2 = -4S_{1t} + A_0S_1,$$

$$S_3 = -12A_{0t} - 3A_0^2 + 8C_0.$$

From equation (9), one obtains the derivative

$$G_t = (-F_tGS_1^2 - F_xGS_4)/(22F_xS_1), \tag{22}$$

where

$$S_4 = 60A_{0x} + 15A_0A_1 - 5A_0S_1 - 20C_1 + 7S_2.$$

Comparing the mixed derivative $(G_t)_x = (G_x)_t$, one obtains the equation

$$5F_tGS_1^2S_5 + F_xGS_6 = 0, \tag{23}$$

where

$$S_5 = -660A_{1x} - 165A_1^2 + 55A_1S_1 + 440C_2 - 34S_1^2,$$

$$S_6 = -1540S_{4x}S_1 + 1309A_0S_1^3 + 385A_1S_1S_4 - 1694S_1^2S_2 - 252S_1^2S_4 - 5S_4S_5.$$

Substituting the relation A_{1x} into A_{1xx}, one obtains the condition

$$S_{5x} = (-1016400C_{2x} + 63525A_1^2S_1 - 254100A_1C_2 - 40040A_1S_1^2 - 990A_1S_5 - 84700C_2S_1 + 6098400D_4 + 5663S_1^3 + 20S_1S_5)/660. \tag{24}$$

Further analysis of the compatibility depends on the value of S_5 in equation (23): it is separated into two cases, i.e., $S_5 = 0$ and $S_5 \neq 0$.

3.1. Case $S_5 = 0$

From equation (23), one obtains the condition

$$S_6 = 0. \tag{25}$$

Comparing the mixed derivative $(F_{xx})_t = (F_{tx})_x$, one arrives at the condition

$$A_{1t} = (-33A_0A_1 + 11A_0S_1 + 44C_1 - 22S_2 + 4S_4)/132. \tag{26}$$

From equation (12), one obtains the derivative

$$F_t = (F_x S_7)/(252S_1^3), \tag{27}$$

where

$$\begin{aligned} S_7 = &-145200C_{1x} - 14520S_{2x} + 18150A_0A_1S_1 - 2464A_0S_1^2 \\ &- 36300A_1C_1 - 9075A_1S_2 + 1815A_1S_4 - 12100C_1S_1 \\ &+ 435600D_3 - 7381S_1S_2 + 727S_1S_4. \end{aligned}$$

Substituting F_t into F_{tx}, one arrives at the condition

$$S_{7x} = (6930A_0S_1^3 + 55A_1S_7 - 1890S_1^2S_4 - 51S_1S_7)/110. \tag{28}$$

Substituting G_t into G_{tt}, one arrives at the condition

$$\begin{aligned} S_{7t} = &(11642400S_{4t}S_1^3 - 2910600A_0S_1^3S_4 + 27720A_0S_1^2S_7 \\ &+ 6403320S_1^4S_3 + 2910600S_1^2S_2S_4 + 264600S_1^2S_4^2 \\ &- 37884S_1S_2S_7 + 252S_1S_4S_7 + 17S_7^2)/(55440S_1^2). \end{aligned} \tag{29}$$

Comparing the mixed derivative $(G_{tt})_x = (G_x)_{tt}$, one obtains the condition

$$\begin{aligned} &508200S_1^4S_3 + S_7^2 - 435600D_3S_7 + 12100C_1S_1S_7 - 4065600C_0S_1^4 \\ &- 1663200S_{4t}S_1^3 - 6098400S_{3x}S_1^3 + 12196800S_{2t}S_1^3 \\ &+ 145200C_{1x}S_7 - 11(11S_2 + 47S_4)S_1S_7 + 5040(77S_2 - 6S_4)S_1^2S_4 \\ &+ 1815(7S_2 - S_4 + 20C_1)A_1S_7 - 110((165A_1 - 53S_1)S_7 \\ &- 252(77S_2 - 16S_4)S_1^2)A_0S_1 = 0. \end{aligned} \tag{30}$$

From equations (13), (15) and (14), one obtains the conditions (A.1), (A.2) and (A.3)[1].

3.2. Case $S_5 \neq 0$

From equation (23), one obtains the derivative

$$F_t = (-F_x S_6)/(5S_1^2 S_5). \tag{31}$$

The relations $(F_t)_x = (F_x)_t$ and $(F_{xx})_t = (F_{tx})_x$ provide the conditions

$$\begin{aligned} S_{6x} = (&-7114800C_{2x}S_1S_6 - 5775A_0S_1^3S_5^2 + 444675A_1^2S_1^2S \\ &- 1778700A_1C_2S_1S_6 - 280280A_1S_1^3S_6 - 5775A_1S_1S_5S_6 \\ &- 592900C_2S_1^2S_6 + 42688800D_4S_1S_6 + 39641S_1^4S_6 \\ &+ 1575S_1^2S_4S_5^2 - 931S_1^2S_5S_6 - 30S_5^2S_6)/(4620S_1S_5), \end{aligned} \tag{32}$$

$$\begin{aligned} A_{1t} = (&-5775A_0A_1S_1^2 + 1925A_0S_1^3 + 7700C_1S_1^2 - 3850S_1^2S_2 \\ &+ 700S_1^2S_4 - 3S_6)/(23100S_1^2). \end{aligned} \tag{33}$$

Substituting G_t into G_{tt}, one arrives at the condition

$$\begin{aligned} S_{4t} = (&6600S_{5t}S_1S_6 - 6600S_{6t}S_1S_5 + 6875A_0S_1S_4S_5^2 + 1650A_0S_1S_5S_6 \\ &- 15125S_1^2S_3S_5^2 - 6875S_2S_4S_5^2 - 2860S_2S_5S_6 - 625S_4^2S_5^2 \\ &+ 30S_4S_5S_6 - 102S_6^2)/(27500S_1S_5^2). \end{aligned} \tag{34}$$

Equation (12) provides the condition

$$\begin{aligned} C_{1x} = (&-508200S_{2x}S_1S_5 + 635250A_0A_1S_1^2S_5 - 86240A_0S_1^3S_5 \\ &- 5775A_0S_1S_5^2 - 1270500A_1C_1S_1S_5 - 317625A_1S_1S_2S_5 \end{aligned}$$

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[1]See Appendix.

$$+ 63525A_1S_1S_4S_5 - 423500C_1S_1^2S_5 + 15246000D_3S_1S_5$$
$$- 258335S_1^2S_2S_5 + 25445S_1^2S_4S_5 + 1764S_1^2S_6$$
$$- 75S_4S_5^2 - 150S_5S_6)/(5082000S_1S_5). \qquad (35)$$

Equating the mixed derivative $(G_{tt})_x = (G_x)_{tt}$, one obtains the condition

$$5(3(20S_4S_5 + 7S_6 + 550S_2S_5)S_6 + 296450S_1^4S_3S_5 - 847000C_0S_1^4S_5$$
$$- 38115A_1S_1S_2S_6 - 1270500S_{3x}S_1^3S_5 + 152460S_{2x}S_1S_6$$
$$+ 2541000S_{2t}S_1^3S_5)S_5 + 21(39875S_2S_4S_5^2 + 27335S_2S_5S_6$$
$$+ 375S_4^2S_5^2 - 615S_4S_5S_6 + 306S_6^2)S_1^2 - 3300(S_{5t}S_6$$
$$- S_{6t}S_5)(126S_1^2 + 5S_5)S_1 + 165(13475S_1^2S_2S_5$$
$$- 5425S_1^2S_4S_5 - 1701S_1^2S_6 - 25S_5S_6)A_0S_1S_5 = 0. \qquad (36)$$

From equations (13), (15) and (14), one obtains the conditions (A.4), (A.5) and (A.6)[2].

All obtained results can be summarized in the following theorems.

Theorem 2. *Sufficient conditions for equation* (4) *to be linearizable via the generalized Sundman transformation* (3) *are as follows*:

(a) *If* $S_5 = 0$, *then the conditions are* (20), (24), (25), (26), (28), (30), (29), (A.1), (A.2) *and* (A.3).

(b) *If* $S_5 \neq 0$, *then the conditions are* (20), (24), (32), (33), (36), (34), (35), (A.4), (A.5) *and* (A.6).

Theorem 3. *Provided that the sufficient conditions in Theorem* 2 *are satisfied, the transformation* (3) *mapping equation* (4) *to a linear equation*

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[2]See Appendix.

(2) is obtained by solving the following compatible system of equations for the functions $F(t, x)$ and $G(t, x)$:

(a) (17), (19), (22) *and* (27).

(b) (17), (19), (22) *and* (31).

4. Examples

Example 1. Consider the nonlinear fourth-order ordinary differential equation

$$x^{(4)} - \frac{7}{x}x'x''' - \frac{4}{x}x''^2 + \frac{25}{x^2}x'^2x'' - \frac{15}{x^3}x'^4 = 0. \tag{37}$$

It is an equation of the form (4) in Theorem 1 with the coefficients

$$A_1 = \frac{-7}{x},\ A_0 = 0,\ B = \frac{-4}{x},\ C_2 = \frac{25}{x^2},\ C_1 = 0,\ C_0 = 0,$$

$$D_4 = \frac{-15}{x^3},\ D_3 = 0,\ D_2 = 0,\ D_1 = 0,\ D_0 = 0,\ S_1 = \frac{-5}{x},$$

$$S_2 = 0,\ S_3 = 0,\ S_4 = 0,\ S_5 = \frac{-630}{x^2},\ S_6 = 0,\ S_7 = 0.$$

One can check that these coefficients obey the conditions in Theorem 2(b). Thus equation (37) is linearizable via a generalized Sundman transformation. For finding the functions F and G, we have to solve equations in Theorem 3(b), which become

$$F_{xx} = 0, \tag{38}$$

$$G_x = G/x, \tag{39}$$

$$G_t = 0, \tag{40}$$

$$F_t = 0. \tag{41}$$

From equation (41), one can take the simplest solution

$$F = x$$

and this solution satisfies equation (38). From equation (40), one can take the simplest solution

$$G = x$$

and this solution satisfies equation (39), so that one obtains the linearizing transformation

$$X = x, \; dT = x dt. \tag{42}$$

Hence equation (37) is mapped by the transformation (42) into the linear equation

$$X^{(4)}(T) = 0. \tag{43}$$

The general solution of equation (43) is

$$X(T) = \frac{c_1}{6}T^3 + \frac{c_2}{2}T^2 + c_3 T + c_4,$$

where c_1, c_2, c_3 and c_4 are arbitrary constants. Applying the generalized Sundman transformation (42) to equation (37), we obtain that the general solution of equation (42) is

$$x(t) = \frac{c_1}{6}\phi(t)^3 + \frac{c_2}{2}\phi(t)^2 + c_3\phi(t) + c_4,$$

where the function $T = \phi(t)$ is a solution of the equation

$$\frac{dT}{dt} = \frac{c_1}{6}T^3 + \frac{c_2}{2}T^2 + c_3 T + c_4.$$

Example 2. Consider the nonlinear fourth-order ordinary differential equation

$$-80x'^4 t + 24x'^3 x + 86x'^2 x'' tx - 18x'x''x^2 - 14x'x''' tx^2 - 8x''^2 tx^2$$
$$+ 2x''' x^3 + x^{(4)} tx^3 = 0. \tag{44}$$

It is an equation of the form (4) in Theorem 1 with the coefficients

$$A_1 = \frac{-14}{x},\ A_0 = \frac{2}{t},\ B = \frac{-8}{x},\ C_2 = \frac{86}{x^2},\ C_1 = \frac{18}{tx},\ C_0 = 0,$$

$$D_4 = \frac{-80}{x^3},\ D_3 = \frac{24}{tx^2},\ D_2 = 0,\ D_1 = 0,\ D_0 = 0,\ S_1 = \frac{-10}{x},\ S_2 = \frac{-20}{tx},$$

$$S_3 = \frac{12}{t^2},\ S_4 = \frac{-100}{tx},\ S_5 = \frac{560}{x^2},\ S_6 = \frac{-280000}{tx^3},\ S_7 = \frac{72600(-7x-40)}{tx^2}.$$

One can check that these coefficients obey the conditions in Theorem 2(b). Thus equation (44) is linearizable via a generalized Sundman transformation. For finding the functions F and G, we have to solve equations in Theorem 3(b), which become

$$F_{xx} = 0, \tag{45}$$

$$G_x = 2G/x, \tag{46}$$

$$G_t = 0, \tag{47}$$

$$F_t = F_x x/t. \tag{48}$$

From equation (46), we get $G = Cx^2$. Choosing $C = 1$, we have

$$G = x^2$$

and this satisfies equation (47). Equation (48) becomes

$$tF_t - xF_x = 0$$

by Cauchy method, one arrives at

$$F = tx$$

and this satisfies equation (45), so that one obtains the linearizing transformation

$$X = tx,\ dT = x^2 dt. \tag{49}$$

Hence equation (44) is mapped by the transformation (49) into the linear equation

$$X^{(4)}(T) = 0. \tag{50}$$

The general solution of equation (50) has the form

$$X(T) = \frac{c_1}{6}T^3 + \frac{c_2}{2}T^2 + c_3T + c_4,$$

where c_1, c_2, c_3 and c_4 are arbitrary constants. Applying the generalized Sundman transformation (49) to equation (44), we obtain that the general solution of equation (49) is

$$x(t) = \frac{\left(\frac{c_1}{6}\phi(t)^3 + \frac{c_2}{2}\phi(t)^2 + c_3\phi(t) + c_4\right)}{t},$$

where the function $T = \phi(t)$ is a solution of the equation

$$\frac{dT}{dt} = \left(\frac{\frac{c_1}{6}T^3 + \frac{c_2}{2}T^2 + c_3T + c_4}{t}\right)^2.$$

5. Applications

- **One class of fourth-order partial differential equations**

Let us consider the nonlinear fourth-order partial differential equation [11]

$$u_{tt} = (\kappa u + \gamma u^2)_{xx} + \nu u u_{xxxx} + \mu u_{xxtt} + \alpha u_x u_{xxx} + \beta u_{xx}^2, \tag{51}$$

where α, β, μ, ν, γ and κ are arbitrary constants. This equation may be thought of as a fourth-order analogue of a generalization of the Camassa-Holm equation, about which there has been considerable recent interest. Furthermore, equation (51) is a Boussinesq-type equation which arises as a model of vibrations of harmonic mass-spring chain and admits both compacton and conventional solitons.

Of particular interest among solutions of equation (51) are traveling wave solutions:

$$u(x, t) = H(x - Dt),$$

where D is a constant phase velocity and the argument $x - Dt$ is a phase of the wave. Substituting the representation of a solution into equation (51), one finds

$$(\nu H + \mu D^2)H^{(4)} + \alpha H'H''' + \beta H''^2 + (2\gamma H + \kappa - D^2)H'' + 2\gamma H'^2 = 0. \quad (52)$$

This is an equation of the form (4) in Theorem 1 with coefficients

$$A_1 = \frac{1}{\mu D^2 + \nu H}, \; A_0 = 0, \; B = \frac{\beta}{\mu D^2 + \nu H}, \; C_2 = 0, \; C_1 = 0,$$

$$C_0 = \frac{2\gamma H + \kappa - D^2}{\mu D^2 + \nu H}, \; D_4 = 0, \; D_3 = 0, \; D_2 = \frac{2\gamma}{\mu D^2 + \nu H}, \; D_1 = 0,$$

$$D_0 = 0, \; S_1 = \frac{3\alpha - 4\beta}{D^2\mu + \nu H}, \; S_2 = 0, \; S_3 = \frac{8(-D^2 + 2\gamma H + \kappa)}{D^2\mu + \nu H}, \; S_4 = 0,$$

$$S_5 = \frac{2(-153\alpha^2 + 298\alpha\beta + 330\alpha\nu - 272\beta^2)}{D^4\mu^2 + 2D^2\mu\nu H + \nu^2 H^2}, \; S_6 = 0, \; S_7 = 0. \quad (53)$$

Since $S_5 \neq 0$, we apply Theorem 2(b) for checking the linearity. The coefficients in equation (53) obey the conditions (20), (24), (36), (34) and (A.4) if and only if

$$\alpha = 1, \; \beta = 0, \; \nu = \frac{2}{5}, \; \gamma = 0, \; \kappa = D^2.$$

Hence equation (52) is linearizable with the conditions $\alpha = 1$, $\beta = 0$, $\nu = \frac{2}{5}$, $\gamma = 0$ and $\kappa = D^2$.

- **The generalized lubrication equation**

Let us consider the generalized lubrication equation [21]

$$h_t + (h^n h_{xxx})_x = 0, \tag{54}$$

where the non-negative function $h(x, t)$ denotes the height of the free surface above the solid substrate, x is the horizontal coordinate and t is the time. The constant n denotes the kind of flow. The generalized lubrication equation (54) models the spreading of a thin film driven by surface tension.

Substituting the traveling wave representation of a solution into equation (54), one finds

$$H^n H^{(4)} + nH^{(n-1)} H' H''' - DH' = 0. \tag{55}$$

This is an equation of the form (4) in Theorem 1 with coefficients

$$A_1 = \frac{n}{H},\ A_0 = 0,\ B = 0,\ C_2 = 0,\ C_1 = 0,\ C_0 = 0,\ D_4 = 0,$$

$$D_3 = 0,\ D_2 = 0,\ D_1 = \frac{-D}{H^n},\ D_0 = 0,\ S_1 = \frac{3n}{H},\ S_2 = 0,$$

$$S_3 = 0,\ S_4 = 0,\ S_5 = \frac{6n(-51n + 110)}{H^2},\ S_6 = 0,\ S_7 = 0. \tag{56}$$

Since $S_5 \neq 0$, we apply Theorem 2(b) for checking the linearity. The coefficients in equation (56) obey the conditions (20), (24) and (A.5) if and only if

$$D = 0,\ n = \frac{5}{2}.$$

Hence equation (55) is linearizable with the condition $D = 0$ and $n = \frac{5}{2}$.

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A. Appendix

For proving theorems, one needs relations between $F(t, x)$ and $G(t, x)$ and coefficients of equation (4). These relations are presented here:

$$-(319440(120(C_{1xx}-3D_{3x})+(7S_2-S_4)C_2)(1575S_1S_4+2S_7)$$

$$-(13684809600C_{0x}S_1^3+265646304000C_{1tx}S_1^2-796938912000D_{3t}S_1^2$$

$$+13282315200S_{3xx}S_1^2+3421202400S_{3x}S_1^3+6426493920C_0S_1^4$$

$$+22137192000C_1^2S_1^2-4829932800D_2S_1^3-420942060S_1^4S_3$$

$$-3045876372S_1^2S_2^2+984017727S_1^2S_2S_4-73652733S_1^2S_4^2$$

$$-150513S_1S_2S_7+39000S_1S_4S_7+380S_7^2)S_1+87120(2286900S_1S_2$$

$$-780255S_1S_4-2459S_7)D_3S_1-22137192000(3A_1+S_1)C_{1t}S_1^3$$

$$-100623600(165A_1+7S_1)S_{2t}S_1^3+3659040(660A_1+61S_1)S_{4t}S_1^3$$

$$-2744280(3465A_1-662S_1)A_0^2S_1^4-59895(9(7S_2-S_4)$$

$$+220C_1)(1575S_1S_4+2S_7)A_1^2+7260(9(108416S_1S_2-11893S_1S_4$$

$$+69S_7)S_1^2+880(1575S_1S_4+2S_7)C_2)C_1+87120((110880A_0S_1^2$$

$$-762300S_1S_2+317835S_1S_4+893S_7)S_1-385(1575S_1S_4+2S_7)A_1)C_{1x}$$

$$-33(220(630(3630S_2-S_4)S_1-1469S_7)C_1S_1-(567151200C_0S_1^4$$

$$+6174630000D_3S_1S_4+7840800D_3S_7-75467700S_1^4S_3$$

$$
\begin{aligned}
& -176091300S_1^2S_2^2 + 79771230S_1^2S_2S_4 - 5955390S_1^2S_4^2 + 149589S_1S_2S_7 \\
& - 24222S_1S_4S_7 - 10S_7^2))A_1 + 66(3025(33A_1^2 - 16C_2)(1575S_1S_4 \\
& + 2S_7) + 3(4065600C_1S_1 - 146361600D_3 - 34425930S_1S_2 \\
& + 9275175S_1S_4 + 5699S_7)S_1^2 + 55(5239080C_1S_1 + 1376298S_1S_2 \\
& - 1268064S_1S_4 - 2825S_7)A_1S_1)A_0S_1) = 0, \qquad \text{(A.1)} \\
& -(5(106500018240S_1^8S_3^2 - S_7^4 + 27883641139200D_0S_1^9 \\
& + 2178409464000A_0^3S_1^7S_4 - 387272793600S_{4t}^2S_1^6 \\
& - 23236367616000S_{4ttt}S_1^7 - 12780002188800S_{3tt}S_1^8 \\
& + 3872727936000C_{0t}S_1^7S_4) - 6(50644440S_1^4S_3 + 41S_7^2)(11S_2 - 3S_4)S_1S_7 \\
& - 96818198400(374S_2 + 73S_4)S_{3t}S_1^7 \\
& - 252(1892S_2 + 309S_4)(11S_2 - 3S_4)S_1^2S_7^2 \\
& - 21168(28424S_2^2 + 11121S_2S_4 + 771S_4^2)(11S_2 - 3S_4)S_1^3S_7 \\
& - 66679200(33176S_2^2 + 2079S_2S_4 - 21S_4^2)(11S_2 + S_4)S_1^4S_4 \\
& - 146694240(78166S_2^2 + 31229S_2S_4 + 1569S_4^2)S_1^6S_3 \\
& - 18441561600(1575S_1S_4 + 2S_7)S_{2tt}S_1^5 + 5029516800(17325A_0S_1^2 \\
& - 30415S_1S_2 - 2730S_1S_4 + 2S_7)S_{4tt}S_1^5 \\
& + 558835200(105(517S_2 + 39S_4)S_1S_4 + (11S_2 - 3S_4)S_7)C_0S_1^5 \\
& - 104781600(105(2816S_2 + 167S_4)S_1S_4 + 4(11S_2 - 3S_4)S_7)A_0^2S_1^5 \\
& - 498960(87318000A_0^2S_1^4 - 116424000C_0S_1^4 + 8925840S_1^4S_3
\end{aligned}
$$

$$+ 195074880S_1^2S_2^2 + 27130320S_1^2S_2S_4 + 493920S_1^2S_4^2 - 1372S_1S_2S_7$$

$$- 5124S_1S_4S_7 - 11S_7^2 - 5040(43505S_1S_2 + 3115S_1S_4 - 2S_7)A_0S_1^2)S_{4t}S_1^3$$

$$- 166320(791683200S_{4t}S_1^3 + 147276360S_1^4S_3 + 210672S_1S_2S_7$$

$$+ 3024S_1S_4S_7 + 121S_7^2 + 264600(1617S_2 + 101S_4)S_1^2S_4$$

$$- 55440(5145S_1S_4 + 2S_7)A_0S_1^2)S_{2t}S_1^3 + 41580(11((11S_2 - 3S_4)S_7^2$$

$$- 52920000C_0S_1^4S_4) + 84(228S_2 + 61S_4)(11S_2 - 3S_4)S_1S_7$$

$$+ 582120(253S_2 + 96S_4)S_1^4S_3 + 17640(57431S_2^2$$

$$+ 6129S_2S_4 + 84S_4^2)S_1^2S_4)A_0S_1^3) = 0, \qquad \text{(A.2)}$$

$$-(5(11((1343188S_1^4S_3 + S_7^2)S_7 + 36883123200D_1S_1^6$$

$$- 33809529600S_{4tt}S_1^5 - 4354257600S_{3t}S_1^6 - 67619059200S_{3tx}S_1^5$$

$$+ 135238118400S_{2tt}S_1^5 + 20490624000C_{0x}S_1^4S_4 - 69668121600C_{0t}S_1^6$$

$$- 77616(31834S_2 + 18193S_4)S_1^5S_3) - (56529S_2 - 15737S_4)S_1S_7^2$$

$$- 14112(980342S_2^2 + 682473S_2S_4 + 325453S_4^2)S_1^3S_4) - 14(27719648S_2^2$$

$$- 13837703S_2S_4 - 1031223S_4^2)S_1^2S_7 - 894432000(36D_{3t} - C_1^2$$

$$- 12C_{1tx})(1575S_1S_4 + 2S_7)S_1^2 - 33541200(35112S_1S_2 + 12264S_1S_4$$

$$+ 17S_7)S_{3x}S_1^3 + 4743200(148104S_1S_2 + 29268S_1S_4 + 41S_7)C_0S_1^4$$

$$+ 894432000(3(1575S_1S_4 + 2S_7)A_1 + (945S_1S_4 + 2S_7)S_1)C_{1t}S_1^2$$

$$- 2032800((160083S_1S_2 + 395346S_1S_4 + 110S_7)S_1$$

$$-165(8295S_1S_4+4S_7)A_1)A_0^2S_1^3+5691840(2(114345A_0S_1^2$$

$$+206910S_1S_2+81135S_1S_4+32S_7)S_1+165(1575S_1S_4+2S_7)A_1)S_{2t}S_1^2$$

$$+12100(81108720S_1^4S_3+83S_7^2+154(224S_2+123S_4)S_1S_7$$

$$+352800(407S_2+62S_4)S_1^2S_4)C_1S_1-435600(81108720S_1^4S_3+83S_7^2$$

$$+176400(1078S_2+97S_4)S_1^2S_4+14(6688S_2+921S_4)S_1S_7)D_3$$

$$+18480((249018000C_1S_1-8964648000D_3-14848680S_1S_2$$

$$-33007380S_1S_4+20899S_7)S_1+7260(102900C_1S_1+36015S_1S_2$$

$$-6720S_1S_4-2S_7)A_1-9240(40425A_1-13876S_1)A_0S_1^2)S_{4t}S_1^2$$

$$+145200(81108720S_1^4S_3+83S_7^2+380318400S_{4t}S_1^3$$

$$+176400(1078S_2+97S_4)S_1^2S_4+14(6688S_2+921S_4)S_1S_7$$

$$-36960(3360S_1S_4+S_7)A_0S_1^2)C_{1x}-110(165(81108720S_1^4S_3$$

$$+83S_7^2+14(7744S_2+393S_4)S_1S_7+17640(15213S_2+101S_4)S_1^2S_4$$

$$+36960(8295S_1S_4+4S_7)C_1S_1)A_1+(8708515200C_0S_1^4$$

$$-4234728960S_1^4S_3-3747222864S_1^2S_2^2-12347618976S_1^2S_2S_4$$

$$-842158296S_1^2S_4^2-8500030S_1S_2S_7+1347990S_1S_4S_7-4735S_7^2$$

$$-146361600(3360S_1S_4+S_7)D_3+4065600(3675S_1S_4$$

$$+S_7)C_1S_1)S_1)A_0S_1+1815((83(7S_2-S_4)S_7+123200S_1^3S_3)S_7$$

$$+176400(1078S_2+97S_4)(7S_2-S_4)S_1^2S_4+388080(1463S_2-9S_4)S_1^4S_3$$

$$+14(6688S_2+921S_4)(7S_2-S_4)S_1S_7-985600(630S_1S_4+S_7)C_0S_1^3$$

$$+ 20(81108720S_1^4S_3 + 83S_7^2 + 176400(1078S_2 + 97S_4)S_1^2S_4$$

$$+ 14(6688S_2 + 921S_4)S_1S_7)C_1)A_1) = 0, \tag{A.3}$$

$$15(10(2((59290S_1^6S_3 + 9S_6^2)S_5^2 + 4268880D_4S_1S_6^2 - 5929000D_2S_1^5S_5^2$$

$$+ 592900C_0S_1^6S_5^2 + 148225S_{3x}S_1^5S_5^2 - 3557400S_{2xx}S_1^2S_5S_6$$

$$+ 426888000D_{4t}S_1^2S_5S_6 - 71148000C_{2tx}S_1^2S_5S_6$$

$$+ 5929000C_{0x}S_1^5S_5^2) - 88935(35S_2S_5 - S_6)A_1^2S_1^2S_6 - 154(434S_1^2$$

$$+ 5S_5)S_{6t}S_1^3S_5 - 1925(1456S_1^2 - 15S_5)A_0^2S_1^4S_5^2$$

$$+ 1540(6545S_1^2S_2S_5 - 700S_1^2S_4S_5 - 77S_1^2S_6 + 3S_5S_6)C_2S_6$$

$$- 11858000(3A_1 + S_1)C_{2t}S_1^2S_5S_6 + 1422960(25A_0S_1S_5 - S_6)C_{2x}S_1S_6)$$

$$- 7(5220S_4S_5 - 1907S_6 - 31450S_2S_5)S_1^2S_5S_6) - 49(87375S_4^2S_5^2$$

$$+ 489295S_4S_5S_6 - 22368S_6^2 + 125(25197S_4S_5 + 20923S_6)S_2S_5)S_1^4$$

$$- 539000(367S_1^2 + 15S_5 + 3300C_2)C_1S_1^2S_5S_6$$

$$+ 69300(50((154A_1S_1 - S_5)S_6 + 385A_0S_1^3S_5)$$

$$- 7(750S_4S_5 + 2161S_6)S_1^2)S_{2x}S_1S_5 + 300(2((16709S_1^4 + 75S_5^2$$

$$- 40425A_1S_1S_5)S_6 - 28875A_0S_1^3S_5^2) + 525(30S_4S_5 + 53S_6)S_1^2S_5)S_{5t}S_1$$

$$+ 5775(2(1540((50C_1S_1^2S_5 - 3C_2S_6)S_6 + 25C_0S_1^4S_5^2) + 3(25S_2S_5$$

$$- 17S_6)S_5S_6) + 7((2250S_4S_5 + 3703S_6)S_2S_5 + 2(955S_4S_5$$

$$- 104S_6)S_6)S_1^2)A_1S_1 - 5(75(10(3(2845920D_4S_1 + S_5^2) + 118580C_2S_1^2$$

$$+ 177870A_1^2S_1^2)S_6 + 21(150S_4S_5 + 647S_6)S_1^2S_5) - 49(353325S_4S_5$$

$$+106619S_6 + 2733225S_2S_5)S_1^4 - 404250(6(220C_2 + S_5)S_6$$

$$- 55(3S_2S_5 - 2S_6)S_1^2)A_1S_1)A_0S_1S_5 = 0, \quad (A.4)$$

$$-(3(((2500S_4^2S_5^2 + 3235S_4S_5S_6 - 384S_6^2)S_6 + 113437500S_2^3S_5^3)S_6$$

$$+ 1512500000D_0S_1^5S_5^4 - 28359375A_0^3S_1^3S_5^3S_6 + 12100000S_{6t}^2S_1^2S_5^2$$

$$+ 302500000S_{6ttt}S_1^3S_5^3 - 1815000000S_{5t}^3S_1^3S_6 - 302500000S_{5ttt}S_1^3S_5^2S_6)$$

$$+ 30250000S_{3t}S_1^3S_5^3S_6 + 453750000S_{2tt}S_1^2S_5^3S_6 - 151250000C_{0t}S_1^3S_5^3S_6$$

$$+ 20625(1125S_4S_5 + 644S_6)S_2^2S_5^2S_6 + 330(3375S_4S_5 + 238S_6)S_2S_5S_6^2$$

$$+ 1375(31625S_2S_5 + 1500S_4S_5 + 404S_6)S_1^2S_3S_5^2S_6 - 687500(15S_4S_5 + 8S_6$$

$$+ 440S_2S_5)C_0S_1^2S_5^2S_6 + 103125(75S_4S_5 + 62S_6 + 3025S_2S_5)A_0^2S_1^2S_5^2S_6$$

$$+ 4125000(15S_4S_5 + 8S_6 + 330S_2S_5 - 165A_0S_1S_5)S_{6tt}S_1^2S_5^2$$

$$+ 1650000((75S_4S_5 + 62S_6 + 1650S_2S_5)S_6 + 825(4S_{6t} - A_0S_6)S_1S_5)S_{5t}^2S_1^2$$

$$- 4125000(165((4S_{6t} - A_0S_6)S_5 - 8S_{5t}S_6)S_1$$

$$+ (15S_4S_5 + 8S_6 + 330S_2S_5)S_6)S_{5tt}S_1^2S_5$$

$$+ 82500(1375((12S_{6t} - 5A_0S_6)S_5 - 12S_{5t}S_6)S_1 + 3(125S_4S_5 + 52S_6$$

$$+ 4125S_2S_5)S_6)S_{2t}S_1S_5^2 - 275(3((850S_4S_5 + 83S_6)S_6 + 618750S_2^2S_5^2$$

$$+ 75(375S_4S_5 + 266S_6)S_2S_5) - 13750(50C_0 - 7S_3)S_1^2S_5^2)A_0S_1S_5S_6$$

$$- 3300(125(55(20C_0 - 3S_3 - 15A_0^2)S_1S_5 + 3(25S_4S_5 + 28S_6$$

$$+ 825S_2S_5)A_0)S_1S_5 - ((850S_4S_5 + 83S_6)S_6 + 309375S_2^2S_5^2$$

$$+ 150(125S_4S_5 + 107S_6)S_2S_5))S_{6t}S_1S_5 + 3300(125(55((20C_0$$

$$- 3S_3 - 15A_0^2)S_6 - 120S_{6tt})S_1S_5 + 3(25S_4S_5 + 28S_6 + 825S_2S_5)A_0S_6$$

$$- 12(25S_4S_5 + 28S_6 + 550S_2S_5 - 275A_0S_1S_5)S_{6t})S_1S_5$$

$$- ((850S_4S_5 + 83S_6)S_6 + 309375S_2^2S_5^2$$

$$+ 150(125S_4S_5 + 107S_6)S_2S_5)S_6)S_{5t}S_1) = 0, \qquad (A.5)$$

$$5(((9(360S_4S_5 + 137S_6 + 8250S_2S_5)S_6 - 448525S_1^4S_3S_5)S_6$$

$$- 762300000D_1S_1^5S_5^2 + 25410000C_1S_1^2S_2S_5S_6 - 76230000S_{6tt}S_1^4S_5$$

$$- 24139500S_{3x}S_1^3S_5S_6 + 31762500S_{3t}S_1^5S_5^2 - 304920000S_{2tx}S_1^2S_5S_6$$

$$- 50820000C_{1t}S_1^3S_5S_6 + 101640000C_{0x}S_1^3S_5S_6 + 508200000C_{0t}S_1^5S_5^2)S_5$$

$$+ 990000(S_{5tt}S_5 - 2S_{5t}^2)(77S_1^2 + 4S_5)S_1^2S_6$$

$$- 66000(728S_1^2 - 15S_5)A_0^2S_1^2S_5^2S_6 + 44000(994S_1^2 - 15S_5)C_0S_1^2S_5^2S_6$$

$$+ 115500(660A_1 - 1037S_1)S_{2t}S_1^2S_5^2S_6$$

$$+ 1155(3(1500S_4S_5 + 481S_6 + 33000S_2S_5)S_2$$

$$+ 2750(8C_0 - S_3)S_1^2S_5)A_1S_1S_5S_6) + 3(137500S_3S_5^3$$

$$- 195125S_4^2S_5^2 - 335265S_4S_5S_6 + 50526S_6^2 - 318876250S_2^2S_5^2$$

$$- 35(477975S_4S_5 + 115363S_6)S_2S_5)S_1^2S_6 + 9900(25(3080A_1S_1S_2$$

$$+ 3S_6 + 2(1813S_1^2 - 20S_5)A_0S_1)S_5 - 28(3(75S_4S_5 + S_6)$$

$$+ 5650S_2S_5)S_1^2)S_{6t}S_1S_5 - 69300(5500((8S_{6t} - 3A_0S_6)S_5$$

$$- 8S_{5t}S_6)S_1 + (1500S_4S_5 + 481S_6 + 33000S_2S_5)S_6)S_{2x}S_1S_5$$

$$+ 900(25(22(20((77S_1^2 + 4S_5)S_{6t} - 77A_1S_2S_6) - (1813S_1^2$$

$$+ 20S_5)A_0S_6)S_1 + (60S_4S_5 - S_6 + 1320S_2S_5)S_6)S_5$$

$$+ 308(3(75S_4S_5 + S_6) + 5650S_2S_5)S_1^2S_6)S_{5t}S_1$$

$$- 15(25(84700((2C_1S_6 - 5S_1^2S_3S_5 - 20C_0S_1^2S_5)S_1 + 12A_1S_2S_6)S_1$$

$$+ 3(300S_4S_5 + 347S_6 + 6600S_2S_5)S_6)S_5 - 7(216025S_4S_5$$

$$+ 75573S_6 + 8268700S_2S_5)S_1^2S_6)A_0S_1S_5 = 0. \tag{A.6}$$

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Research Article

Linearizability of Nonlinear Third-Order Ordinary Differential Equations by Using a Generalized Linearizing Transformation

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We discuss the linearization problem of third-order ordinary differential equation under the generalized linearizing transformation. We identify the form of the linearizable equations and the conditions which allow the third-order ordinary differential equation to be transformed into the simplest linear equation. We also illustrate how to construct the generalized linearizing transformation. Some examples of linearizable equation are provided to demonstrate our procedure.

1. Introduction

There has been major interest in the nonlinear problems, since most equations are inherently nonlinear in nature. In general, the nonlinear problems are very difficult to solve explicitly. It is of interest to provide general criteria for the linearizability of nonlinear ordinary differential equations, as they can then be reduced to easily solvable equations. Therefore, the approach of investigating nonlinear ordinary differential equations via transforming to simpler ordinary differential equations becomes important and has been quite plentiful in analysis of physical problems. This includes the classical linearization problem of finding transformations that linearize a given ordinary differential equation. The linearization problem has been studied in many aspects. A short review can be found in [1, 2]. The tools commonly used for solving the linearization problem are the transformations such as point transformation, contact transformation, reduction of order, differential substitution, and generalized Sundman transformation. For this paper, we employ the extension of the generalized Sundman transformations.

The linearization problem for a second-order ordinary differential equation was investigated with respect to a generalized Sundman transformation

$$X = F(t,x), \qquad dT = G(t,x)\,dt \tag{1}$$

by Duarte et al. [3] earlier. They obtained the form of the linearizable equations and the conditions which allow the second-order ordinary differential equation to be transformed into the free particle equation. A characterization of these equations that can be linearized by means of generalized Sundman transformations in terms of first integral and procedure for construction of linearizing transformations has been given by Muriel and Romero [4]. In [5], Mustafa et al. gave a new characterization of linearizable equations in terms of the coefficients of ordinary differential equations and one auxiliary function. In [6], Nakpim and Meleshko pointed out that the solution of the linearization problem for a second-order ordinary differential equation via the generalized Sundman transformation considered earlier by Duarte et al. [3] using the Laguerre form is not complete.

The linearization problem for a third-order ordinary differential equation was also investigated with respect to a generalized Sundman transformation [7, 8]. Criteria for a third-order ordinary differential equation to be equivalent to the linear equation $X'''(T) = 0$ with respect to a Sundman transformation were presented in [8]. The generalized Sundman transformation was also applied for obtaining necessary and sufficient conditions for a third-order ordinary differential equation to be equivalent to a linear equation in the Laguerre form [6]. Some applications of the generalized Sundman transformation to ordinary differential equations were considered in [9] and earlier papers, summarized in the book [10].

The linearization problem of a fourth-order ordinary differential equation with respect to generalized Sundman transformations was studied in [11]. They found the necessary and sufficient conditions which allow the fourth-order ordinary differential equation to be transformed into the simplest linear equation.

In this work, we expose a more general transformation, that is, the extension of the generalized Sundman transformation

$$X = F(t,x), \qquad dT = G(t,x,x')\,dt. \tag{2}$$

This transformation was studied in [12–14] where they designated the transformation as the *generalized linearizing transformation*. They showed that this transformation can be utilized to linearize a wider class of nonlinear ordinary differential equations and, in particular, certain equations which cannot be linearized by the nonpoint and invertible point transformations. If the function G in (2) is independent of the variable x', then it becomes a nonpoint transformation (vide (1)). On the other hand, if G is a differentiable function, then it becomes an invertible point transformation. So (2) is a unified transformation as it includes nonpoint and invertible point transformations as special cases. An example of an equation which can be linearized by a transformation of the form (2) is given in [13]. It is worth noting that any second-order equation $x'' = f(t,x,x')$ can be transformed by a transformation (2) into the free particle equation and that this is not so for third-order ordinary differential equations. Hence, the linearization problem using generalized linearizing transformations becomes interesting for ordinary differential equations of order greater than 2. In [12], the authors applied a particular class of transformations (2), where the function $G(t,x,x')$ is linear with respect to x'.

We are now paying attention to the case where G is a polynomial function in x' and in particular where it is linear in x' with coefficients which are arbitrary functions of t and x. To be specific, we focus here on the case

$$X = F(t,x), \qquad dT = (G_1(t,x)\,x' + G_2(t,x))\,dt. \tag{3}$$

Notice that for the case $G_1 = 0$, the generalized linearizing transformation becomes a generalized Sundman transformation, so that we assume $G_1 \neq 0$.

The paper is organized as follows. In Section 2, the necessary conditions of linearization of a third-order ordinary differential equation are presented. In Section 3, we get the theorems that yield criteria for a third-order ordinary differential equation to be linearizable via generalized linearizing transformations. Examples which illustrate the procedure of using the linearization theorems are presented in Section 4.

2. Necessary Conditions of Linearization

Here we consider a nonlinear third-order ordinary differential equation

$$x''' = f(t,x,x',x''). \tag{4}$$

Our goal in this section is to describe all equations (4) which are equivalent with respect to generalized linearizing transformations

$$X = F(t,x), \qquad dT = (G_1(t,x)\,x' + G_2(t,x))\,dt \tag{5}$$

to a linear equation

$$X'''(T) = 0. \tag{6}$$

We begin with investigating the necessary conditions for linearization, that is, the general form of third-order equation (4) that can be obtained from a linear equation (6) by any generalized linearizing transformation (5).

Applying a generalized linearizing transformation (5), one obtains the following transformation of the third-order derivatives:

$$X'(T) = \frac{D_tF}{G_1x' + G_2} = \frac{F_t + x'F_x}{G_1x' + G_2} = P(t,x,x'),$$

$$\begin{aligned}
X''(T) &= \frac{D_tP}{G_1x' + G_2} \\
&= \frac{P_t + x'P_x + x''P_{x'}}{G_1x' + G_2} \\
&= -\Big(\big(G_{2x}x' + G_1x'' + G_{2t} + G_{1x}x'^2 + G_{1t}x'\big)F_t \\
&\qquad - \big(F_{tt} + F_{xx}x'^2 + 2F_{tx}x'\big)\big(G_1x' + G_2\big) \\
&\qquad + \big(G_{2x}x'^2 - G_2x'' + G_{2t}x' + G_{1x}x'^3 + G_{1t}x'^2\big)F_x\Big) \\
&\quad \times \big(G_1x' + G_2\big)^{-3} \\
&= Q(t,x,x',x''),
\end{aligned}$$

$$\begin{aligned}
X'''(T) &= \Big(\big(3G_{2x}^2x'^3 + G_2^2x''' + 3G_{2t}^2x' + 3G_{1x}^2x'^5 \\
&\qquad + 3G_{1t}^2x'^3 + \big(x'x''' - 3x''^2\big)G_1G_2 \\
&\qquad + 2\big(G_1x' - 2G_2\big)G_{2x}x'x'' \\
&\qquad - \big(G_{2tt} + G_{2xx}x'^2 + 2G_{2tx}x' + G_{1xx}x'^3 \\
&\qquad\quad + G_{1tt}x' + 2G_{1tx}x'^2\big)\big(G_1x' + G_2\big)x' \\
&\qquad + 3\big(\big(G_1x' - G_2\big)x'' + 2G_{2x}x'^2\big)G_{2t} \\
&\qquad + 6\big(G_{2x}x'^2 - G_2x'' + G_{2t}x'\big)G_{1x}x'^2
\end{aligned}$$

$$
\begin{aligned}
&+\Big(\big(\left(G_1x'-5G_2\right)x''+6G_{2x}x'^2+6G_{2t}x'\\
&\qquad+6G_{1x}x'^3\big)G_{1t}x'\Big)F_x\\
&-\Big(3\left(G_{2t}+G_{2x}x'+G_{1x}x'^2+G_{1t}x'+G_1x''\right)F_{tt}\\
&\quad-\left(\left(3F_{ttx}+F_{xxx}x'^2\right)x'+F_{ttt}+3F_{txx}x'^2\right)\\
&\quad\times\left(G_1x'+G_2\right)\\
&\quad+3\left(\left(G_{2t}+G_{2x}x'+G_{1x}x'^2+G_{1t}x'\right)x'-G_2x''\right)\\
&\quad\times F_{xx}x'+3\big(2\left(G_{2t}+G_{2x}x'+G_{1x}x'^2+G_{1t}x'\right)x'\\
&\qquad+\left(G_1x'-G_2\right)x''\big)F_{tx}\Big)\left(G_1x'+G_2\right)\\
&-\Big(\big(\left(G_{2tt}+G_{2xx}x'^2+2G_{2tx}x'+G_{1xx}x'^3\right.\\
&\quad\left.+G_{1tt}x'+2G_{1tx}x'^2\right)\\
&\quad\times\left(G_1x'+G_2\right)\\
&\quad-\left(3G_{2x}^2x'^2-G_1G_2x'''+3G_{2t}^2+3G_{1x}^2x'^4\right.\\
&\qquad\left.+3G_{1t}^2x'^2-\left(x'x'''-3x''^2\right)G_1^2\right)\\
&\quad-\left(5G_1x'-G_2\right)G_{2x}x''\\
&\quad-6\left(G_{2x}x'+G_1x''\right)G_{2t}\\
&\quad-3\left(\left(G_1x'-G_2\right)x''+2G_{2x}x'^2+2G_{2t}x'\right)\\
&\quad\times G_{1x}x'-2\Big(\left(2G_1x'-G_2\right)x''+3G_{2x}x'^2\\
&\qquad+3G_{2t}x'+3G_{1x}x'^3\Big)G_{1t}\big)F_t\Big)\\
&\times\left(G_1x'+G_2\right)^{-5}\\
&=R\left(t,x,x',x'',x'''\right),
\end{aligned}
\tag{7}
$$

where $D_t=\partial/\partial t+x'(\partial/\partial x)+x''(\partial/\partial x')+x'''(\partial/\partial x'')+\cdots$ is a total of derivatives. Substituting the resulting expression in linear equation (6) and setting $r=G_2/G_1$, $K=F_t-F_xr$, we arrive at the following equation:

$$
\begin{aligned}
&x'''+\frac{1}{x'+r}\\
&\quad\times\left[-3x''^2+\left(A_2x'^2+A_1x'+A_0\right)x''\right.\\
&\qquad\left.+B_5x'^5+B_4x'^4+B_3x'^3+B_2x'^2+B_1x'+B_0\right]\\
&=0,
\end{aligned}
\tag{8}
$$

where $A_i\,(i=0,1,2)$ and $B_j\,(j=0,1,\ldots,5)$ are functions of t and x determined as follows:

$$
\begin{aligned}
A_2&=\left(3\left(\left(F_{tx}-F_{xx}r\right)G_1-F_tG_{1x}\right)\right.\\
&\qquad\left.+\left(2\left(2G_{1x}r-r_xG_1\right)-G_{1t}\right)F_x\right)/\left(KG_1\right),\\
A_1&=-\Big(\left(2G_{1x}r+5r_xG_1+4G_{1t}\right)F_t\\
&\qquad-3\left(F_{tt}-F_{xx}r^2\right)G_1\\
&\qquad+\Big(\left(3r_t-4r_xr\right)G_1\\
&\qquad-4G_{1x}r^2-2G_{1t}r\Big)F_x\Big)/\left(KG_1\right),
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
A_0=&-\left(3F_{tx}G_1r^2-3F_{tt}G_1r+4F_tG_{1t}r\right.\\
&-F_tG_{1x}r^2+6F_tr_tG_1-F_tr_xG_1r\\
&\left.-3F_xG_{1t}r^2-3F_xr_tG_1r\right)/\left(KG_1\right),
\end{aligned}
\tag{10}
$$

$$
B_5=-\frac{\left(\left(F_{xxx}G_1-3F_{xx}G_{1x}\right)G_1-\left(G_{1xx}G_1-3G_{1x}^2\right)F_x\right)}{\left(KG_1^2\right)},
\tag{11}
$$

$$
\begin{aligned}
B_4=&\left(3\left(G_{1t}+2G_{1x}r+r_xG_1\right)F_{xx}G_1\right.\\
&+\left(G_{1xx}G_1-3G_{1x}^2\right)F_t\\
&+\Big(2G_{1tx}G_1-6G_{1t}G_{1x}+2G_{1xx}G_1r-6G_{1x}^2r\\
&\qquad-4G_{1x}r_xG_1+r_{xx}G_1^2\Big)F_x\\
&\left.+\left(2\left(3F_{tx}G_{1x}-F_{xxx}G_1r\right)-3F_{txx}G_1\right)G_1\right)\\
&\times\left(KG_1^2\right)^{-1},
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
B_3=&\Big(\Big(3F_{tt}G_{1x}-F_{xxx}G_1r^2-3F_{ttx}G_1-6F_{txx}G_1r\\
&\quad+6\left(G_{1t}+2G_{1x}r+r_xG_1\right)F_{tx}\Big)G_1\\
&+\Big(2G_{1tx}G_1-6G_{1t}G_{1x}+2G_{1xx}G_1r-6G_{1x}^2r\\
&\qquad-4G_{1x}r_xG_1+r_{xx}G_1^2\Big)F_t\\
&+3\left(\left(G_{1x}r+r_xG_1+G_{1t}\right)r+G_{1t}r+r_tG_1\right)F_{xx}G_1\\
&+\Big(\left(2G_{1tx}r+G_{1tt}\right)G_1-3\left(G_{1x}r+r_xG_1\right)^2\\
&\quad-6\left(G_{1t}r+r_tG_1\right)G_{1x}\\
&\quad-3\left(2\left(G_{1x}r+r_xG_1\right)+G_{1t}\right)G_{1t}\\
&\quad+\left(2G_{1x}r_x+r_{xx}G_1+G_{1xx}r\right)G_1r
\end{aligned}
$$

$$\begin{aligned}&+\ 2\left(G_{1x}r_t + r_{tx}G_1 + G_{1t}r_x + G_{1tx}r\right)G_1\Big)F_x\Big)\\ &\times\left(KG_1^2\right)^{-1},\end{aligned}\tag{13}$$

$$\begin{aligned}B_2 = &\Big(\Big(\left(2G_{1tx}r + G_{1tt}\right)G_1 - 3(G_{1x}r + r_xG_1)^2\\ &- 6\left(G_{1t}r + r_tG_1\right)G_{1x}\\ &- 3\left(2\left(G_{1x}r + r_xG_1\right) + G_{1t}\right)G_{1t}\\ &+ \left(2G_{1x}r_x + r_{xx}G_1 + G_{1xx}r\right)G_1r\\ &+ 2\left(G_{1x}r_t + r_{tx}G_1 + G_{1t}r_x + G_{1tx}r\right)G_1\Big)F_t\\ &-\Big(\Big(6\left(G_{1x}r + r_xG_1 + G_{1t}\right)\\ &\quad\times\left(G_{1t}r + r_tG_1\right) - G_{1tt}G_1r\\ &\quad-\left(2G_{1t}r_t + r_{tt}G_1 + G_{1tt}r\right)G_1\\ &\quad- 2\left(G_{1x}r_t + r_{tx}G_1 + G_{1t}r_x + G_{1tx}r\right)G_1r\Big)F_x\\ &\quad+\Big(\left(3F_{txx}r^2 + F_{ttt}\right)G_1\\ &\quad- 3\left(G_{1t} + 2G_{1x}r + r_xG_1\right)F_{tt}\\ &\quad- 3\left(\left(G_{1t}r + r_tG_1\right)F_{xx} - 2F_{ttx}G_1\right)r\\ &\quad- 6\left(\left(G_{1x}r + r_xG_1 + G_{1t}\right)r + G_{1t}r + r_tG_1\right)\\ &\quad\times F_{tx}\Big)G_1\Big)\Big)\\ &\times\left(KG_1^2\right)^{-1},\end{aligned}\tag{14}$$

$$\begin{aligned}B_1 = &-\Big(\left(6\left(G_{1x}r + r_xG_1 + G_{1t}\right)\left(G_{1t}r + r_tG_1\right)\right.\\ &- G_{1tt}G_1r - \left(2G_{1t}r_t + r_{tt}G_1 + G_{1tt}r\right)G_1\\ &- 2\left(G_{1x}r_t + r_{tx}G_1 + G_{1t}r_x + G_{1tx}r\right)G_1r\big)F_t\\ &-\Big(\left(3\left(\left(G_{1x}r + r_xG_1 + G_{1t}\right)r + G_{1t}r + r_tG_1\right)F_{tt}\right.\\ &\quad-\left(\left(2F_{ttt} + 3F_{ttx}r\right)G_1\right.\\ &\quad-6\left(G_{1t}r + r_tG_1\right)F_{tx}\big)r\big)G_1\\ &\quad+\Big(\left(2G_{1t}r_t + r_{tt}G_1 + G_{1tt}r\right)G_1r\\ &\quad-3(G_{1t}r + r_tG_1)^2\Big)F_x\Big)\Big)\\ &\times\left(KG_1^2\right)^{-1},\end{aligned}$$

$$\begin{aligned}B_0 = &\Big(\Big(\left(2G_{1t}r_t + r_{tt}G_1 + G_{1tt}r\right)G_1r\\ &-3(G_{1t}r + r_tG_1)^2\Big)F_t\\ &+\left(3\left(G_{1t}r + r_tG_1\right)F_{tt} - F_{ttt}G_1r\right)G_1r\Big)/\left(KG_1^2\right).\end{aligned}\tag{15}$$

Thus, we proved the theorem.

Theorem 1. *Any third-order ordinary differential equation* (4) *obtained from a linear equation* (6) *by a generalized linearizing transformation* (5) *has to be in the form* (8).

3. Formulation of the Linearization Theorem

We have shown in the previous section that every linearizable third-order ordinary differential equation belongs to the class of equations (8). In this section, we formulate the main theorems containing necessary and sufficient conditions for linearization as well as the methods for constructing the linearizing transformations.

For obtaining sufficient conditions, one has to solve the compatibility problem. Consider the representations of the coefficients A_i and B_i through the unknown functions F and G_1. According to our notation $K = F_t - F_xr$, we define the derivative F_t as

$$F_t = F_xr + K.\tag{16}$$

From (9), one can find the derivatives

$$\begin{aligned}K_x &= \frac{\left(F_xG_{1t} - F_xG_{1x}r - F_xr_xG_1 + 3G_{1x}K + A_2G_1K\right)}{\left(3G_1\right)},\\ K_t &= \Big(F_xG_{1t}r - F_xG_{1x}r^2 - F_xr_xG_1r + 4G_{1t}K - G_{1x}Kr\\ &\quad + G_1K\left(5r_x + A_1 - A_2r\right)\Big)/\left(3G_1\right).\end{aligned}\tag{17}$$

From (10), one obtains the condition

$$r_t = \frac{\left(6r_xr - A_0 + A_1r - A_2r^2\right)}{6}.\tag{18}$$

Equation (11) defines the derivative

$$F_{xxx} = \frac{\left(3F_{xx}G_{1x}G_1 + F_xG_{1xx}G_1 - 3F_xG_{1x}^2 - B_5G_1^2K\right)}{G_1^2}.\tag{19}$$

So that equation (12) becomes

$$\begin{aligned}&6F_{xx}G_{1t}G_1 - 6F_{xx}G_{1x}G_1r - 6F_{xx}r_xG_1^2\\ &\quad + 3F_xG_{1tx}G_1 - 12F_xG_{1t}G_{1x}\\ &\quad - F_xG_{1t}A_2G_1 - 3F_xG_{1xx}G_1r + 12F_xG_{1x}^2r\end{aligned}$$

$$+ F_x G_{1x} G_1 (6r_x + A_2 r)$$
$$+ F_x G_1^2 (-3r_{xx} + r_x A_2) - 6G_{1xx} G_1 K + 9G_{1x}^2 K$$
$$+ G_1^2 K \left(-3A_{2x} - A_2^2 - 3B_4 + 15B_5 r\right) = 0. \quad (20)$$

The compatibility analysis depends on the value of F_x. A complete study of all cases is given here.

3.1. Case $F_x=0$. In this case, the forms of derivatives F_t, K_x, and K_t become

$$F_t = K,$$
$$K_x = \frac{(3G_{1x} + A_2 G_1) K}{(3G_1)}, \quad (21)$$
$$K_t = \frac{(4G_{1t} - G_{1x} r + G_1 (5r_x + A_1 - A_2 r)) K}{(3G_1)}.$$

Substituting F_x into F_{xxx}, one arrives at the condition

$$B_5 = 0. \quad (22)$$

Comparing the mixed derivatives $(F_x)_t = (F_t)_x$, one gets the derivative

$$G_{1x} = \frac{-(A_2 G_1)}{3}. \quad (23)$$

In this case, $(F_{xxx})_t = (F_t)_{xxx}$ is satisfied. Equations (12) and (13) give the conditions

$$A_{2x} = \frac{\left(-2A_2^2 - 9B_4\right)}{3},$$
$$r_{xx} = \frac{\left(-9A_{1x} + 6A_{2t} + 3r_x A_2 - 3A_1 A_2 - 2A_2^2 r - 9B_3\right)}{36}. \quad (24)$$

Comparing the mixed derivative $(K_t)_x = (K_x)_t$, one obtains the condition

$$A_{1x} = \frac{\left(-6A_{2t} - 3r_x A_2 - 5A_1 A_2 + 2A_2^2 r - 15B_3 + 24B_4 r\right)}{3}. \quad (25)$$

Equation (14) provides the derivative

$$G_{1tt} = \frac{\left(2250G_{1t}^2 + 150G_{1t} G_1 h_1 + G_1^2 h_2\right)}{(1350G_1)}, \quad (26)$$

where

$$h_1 = 15r_x + 3A_1 - 2A_2 r,$$
$$h_2 = -225A_{0x} - 1350A_{1t} - 1350A_{2t} r - 1050A_0 A_2 - 477A_1^2 + 516A_1 A_2 r + 33A_1 h_1 - 432A_2^2 r^2 - 57A_2 h_1 r - 4050B_2 + 4275B_3 r - 4275B_4 r^2 - 8h_1^2. \quad (27)$$

The relation $(r_x)_x = r_{xx}$ gives the condition

$$h_{1x} = 4A_{2t}. \quad (28)$$

Comparing the mixed derivative $(G_{1tt})_x = (G_{1x})_{tt}$, one arrives at the condition

$$A_{2tt} = \frac{\left(50A_{2t} h_1 - h_{2x}\right)}{450}. \quad (29)$$

Solving (15), one finds the conditions

$$A_{0t} = \left(15930A_{1t} r + 15930A_{2t} r^2 - 1260h_{1x} r - 1575A_0 A_1 + 11970A_0 A_2 r + 5517A_1^2 r - 5697A_1 A_2 r^2 - 558A_1 h_1 r + 4986A_2^2 r^3 + 504A_2 h_1 r^2 - 8100B_1 + 48600B_2 r - 48600B_3 r^2 + 48600B_4 r^3 + 148h_1^2 r + 8h_2 r\right)/1350, \quad (30)$$

$$B_0 = \left(-3240A_{1t} r^2 - 3240A_{2t} r^3 + 180h_{1x} r^2 - 135A_0^2 + 270A_0 A_1 r - 2430A_0 A_2 r^2 - 1107A_1^2 r^2 + 1134A_1 A_2 r^3 + 108A_1 h_1 r^2 - 999A_2^2 r^4 - 108A_2 h_1 r^3 + 1620B_1 r - 9720B_2 r^2 + 9720B_3 r^3 - 9720B_4 r^4 - 28h_1^2 r^2 - 2h_2 r^2\right)/1620. \quad (31)$$

3.2. Case $F_x \neq 0$. From (20) and (13), one obtains the derivatives

$$G_{1tx} = \left(-6F_{xx} G_{1t} G_1 + 6F_{xx} G_{1x} G_1 r + 6F_{xx} r_x G_1^2 + 12F_x G_{1t} G_{1x} + F_x G_{1t} A_2 G_1 + 3F_x G_{1xx} G_1 r - 12F_x G_{1x}^2 r + F_x G_{1x} G_1 (-6r_x - A_2 r) + F_x G_1^2 (3r_{xx} - r_x A_2) + 6G_{1xx} G_1 K - 9G_{1x}^2 K + G_1^2 K \left(3A_{2x} + A_2^2 + 3B_4 - 15B_5 r\right)\right) \times (3F_x G_1)^{-1}, \quad (32)$$

$$G_{1tt} = \left(-24F_{xx} F_x G_{1t} G_1 r + 24F_{xx} F_x G_{1x} G_1 r^2 + 24F_{xx} F_x r_x G_1^2 r - 24F_{xx} G_{1t} G_1 K + 24F_{xx} G_{1x} G_1 K r + 24F_{xx} r_x G_1^2 K + 14F_x^2 G_{1t}^2 + 20F_x^2 G_{1t} G_{1x} r\right.$$

$$
\begin{aligned}
&+ 2F_x^2 G_{1t} G_1 (r_x + A_1) + 6F_x^2 G_{1xx} G_1 r^2 - 34 F_x^2 G_{1x}^2 r^2 \\
&+ F_x^2 G_{1x} G_1 \left(-26 r_x r - A_0 - A_1 r - A_2 r^2\right) \\
&+ F_x^2 G_1^2 \left(-A_{0x} + A_{1x} r - A_{2x} r^2 + 12 r_{xx} r \right. \\
&\qquad \left. - 4r_x^2 - r_x A_1 - 2 r_x A_2 r\right) \\
&+ 24 F_x G_{1t} G_{1x} K + 24 F_x G_{1xx} G_1 K r - 60 F_x G_{1x}^2 K r \\
&- 24 F_x G_{1x} r_x G_1 K \\
&+ 2F_x G_1^2 K (3A_{1x} + 18 r_{xx} - 3 r_x A_2 \\
&\qquad + A_1 A_2 + 3B_3 - 6B_4 r) + 24 G_{1xx} G_1 K^2 \\
&- 36 G_{1x}^2 K^2 + 4 G_1^2 K^2 \left(3A_{2x} + A_2^2 + 3B_4 - 15 B_5 r\right)\Big) \\
&\times \left(6F_x^2 G_1\right)^{-1}.
\end{aligned} \tag{33}
$$

Comparing the mixed derivative $(K_t)_x = (K_x)_t$, one obtains

$$
\begin{aligned}
G_{1xx} = &\Big(6F_{xx} G_{1t} G_1 K - 6F_{xx} G_{1x} G_1 K r \\
&- 6F_{xx} r_x G_1^2 K - F_x^2 G_{1t}^2 + 2F_x^2 G_{1t} G_{1x} r \\
&+ 2F_x^2 G_{1t} r_x G_1 - F_x^2 G_{1x}^2 r^2 - 2F_x^2 G_{1x} r_x G_1 r \\
&- F_x^2 r_x^2 G_1^2 - 6F_x G_{1t} G_{1x} K + 6F_x G_{1x}^2 K r \\
&+ 6F_x G_{1x} r_x G_1 K \\
&+ F_x G_1^2 K \Big(-3A_{2t} + 3A_{2x} r - A_1 A_2 + 2A_2^2 r \\
&\qquad - 3B_3 + 12 B_4 r - 30 B_5 r^2 \Big) \\
&+ 9 G_{1x}^2 K^2 + G_1^2 K^2 \left(-3A_{2x} - A_2^2 - 3B_4 + 15 B_5 r\right)\Big) \\
&\times \left(6 G_1 K^2\right)^{-1}.
\end{aligned} \tag{34}
$$

Equation (14) becomes

$$
F_x s_1 + 2K s_2 = 0, \tag{35}
$$

where

$$
\begin{aligned}
s_1 = &- 6A_{1t} + 6A_{1x} r + 12 A_{2t} r - 12 A_{2x} r^2 \\
&- 5A_0 A_2 - 2A_1^2 + 13 A_1 A_2 r - 13 A_2^2 r^2 \\
&- 18 B_2 + 54 B_3 r - 108 B_4 r^2 + 180 B_5 r^3,
\end{aligned} \tag{36}
$$
$$
\begin{aligned}
s_2 = &- 3A_{1x} + 6A_{2t} - 18 r_{xx} + 3 r_x A_2 \\
&+ A_1 A_2 - 2A_2^2 r + 3B_3 - 12 B_4 r + 30 B_5 r^2.
\end{aligned}
$$

Further analysis of the compatibility depends on value of s_1 in (35): it is separated into two cases; that is, $s_1 = 0$ and $s_1 \neq 0$.

3.2.1. Case $s_1 \neq 0$. From (35), one finds

$$
F_x = -\frac{(2K s_2)}{s_1}. \tag{37}
$$

Since this case $F_x \neq 0$, then $s_2 \neq 0$ too. Comparing the mixed derivatives $(F_x)_t = (F_t)_x$, one gets the derivative

$$
G_{1t} = \frac{\left(3 G_{1x} s_1 (2 r s_2 - s_1) + G_1 s_3\right)}{(6 s_1 s_2)}, \tag{38}
$$

where

$$
\begin{aligned}
s_3 = &- 6 r_x s_1 s_2 + 6 s_{1t} s_2 - 6 s_{1x} r s_2 - 6 s_{2t} s_1 \\
&+ 6 s_{2x} r s_1 - 2A_1 s_1 s_2 + 4 A_2 r s_1 s_2 - A_2 s_1^2.
\end{aligned} \tag{39}
$$

Substituting F_x into F_{xxx}, G_{1t} into G_{1tx} and G_{1tt}, one arrives at the conditions

$$
\begin{aligned}
s_{2xx} = &\Big(-12 A_{2t} s_1^2 s_2^2 + 12 A_{2x} r s_1^2 s_2^2 - 6 A_{2x} s_1^3 s_2 \\
&+ 36 r_x s_{1x} s_1 s_2^2 - 36 r_x s_{2x} s_1^2 s_2 - 12 r_x A_2 s_1^2 s_2^2 \\
&+ 18 s_{1xx} s_1^2 s_2 - 36 s_{1x}^2 s_1 s_2 + 36 s_{1x} s_{2x} s_1^2 \\
&+ 12 s_{1x} A_2 s_1^2 s_2 - 6 s_{1x} s_2 s_3 - 12 s_{2x} A_2 s_1^3 \\
&+ 6 s_{2x} s_1 s_3 - 4 A_1 A_2 s_1^2 s_2^2 + 8 A_2^2 r s_1^2 s_2^2 \\
&- 2A_2^2 s_1^3 s_2 + 2 A_2 s_1 s_2 s_3 - 12 B_3 s_1^2 s_2^2 \\
&+ 48 B_4 r s_1^2 s_2^2 - 120 B_5 r^2 s_1^2 s_2^2 + 9 B_5 s_1^4\Big) \\
&\times \left(18 s_1^3\right)^{-1},
\end{aligned}
$$
$$
\begin{aligned}
s_{3x} = &\Big(-6 A_{1x} s_1^3 s_2 - 6 A_{2t} s_1^3 s_2 + 18 A_{2x} r s_1^3 s_2 \\
&- 9 A_{2x} s_1^4 + 36 r_x^2 s_1^2 s_2^2 - 36 r_x s_{1x} s_1^2 s_2 + 36 r_x s_{2x} s_1^3 \\
&+ 6 r_x A_2 s_1^3 s_2 - 12 r_x s_1 s_2 s_3 + 12 s_{1x} s_1 s_3 \\
&- 4 A_1 A_2 s_1^3 s_2 + 8 A_2^2 r s_1^3 s_2 - 3 A_2^2 s_1^4 - 12 B_3 s_1^3 s_2 \\
&+ 48 B_4 r s_1^3 s_2 - 9 B_4 s_1^4 - 120 B_5 r^2 s_1^3 s_2 \\
&+ 45 B_5 r s_1^4 - 2 s_1^3 s_2^2 + s_3^2\Big) / \left(6 s_1^2\right),
\end{aligned}
$$
$$
\begin{aligned}
s_{3t} = &\Big(-6 A_{0x} s_1^3 s_2 - 3 A_{1x} s_1^4 - 6 A_{2t} r s_1^3 s_2 - 12 A_{2t} s_1^4 \\
&+ 12 A_{2x} r^2 s_1^3 s_2 + 9 A_{2x} r s_1^4 + 36 r_x^2 r s_1^2 s_2^2
\end{aligned}
$$

$$
\begin{aligned}
&- 36r_x^2 s_1^3 s_2 - 36r_x s_{1x} r s_1^2 s_2 + 36r_x s_{2x} r s_1^3 \\
&- 6r_x A_1 s_1^3 s_2 + 18r_x A_2 r s_1^3 s_2 - 3r_x A_2 s_1^4 \\
&- 12r_x r s_1 s_2 s_3 + 12 s_{1t} s_1 s_3 - 4A_1 A_2 r s_1^3 s_2 \\
&- 5A_1 A_2 s_1^4 + 8A_2^2 r^2 s_1^3 s_2 + 7A_2^2 r s_1^4 - 12B_3 r s_1^3 s_2 \\
&- 15B_3 s_1^4 + 48B_4 r^2 s_1^3 s_2 + 51B_4 r s_1^4 \\
&- 120B_5 r^3 s_1^3 s_2 - 105B_5 r^2 s_1^4 - 2r s_1^3 s_2^2 \\
&+ r s_3^2 + 5 s_1^4 s_2\big) / \left(6 s_1^2\right).
\end{aligned}
\tag{40}
$$

Equation (15) provides the conditions

$$
\begin{aligned}
A_{0t} = {} & \big(6A_{0x} r + 6A_{2t} r^2 - 6A_{2x} r^3 - 7A_0 A_1 \\
& + 9A_0 A_2 r + 5A_1^2 r - 8A_1 A_2 r^2 + A_2^2 r^3 - 36B_1 \\
& + 54B_2 r - 54B_3 r^2 + 36B_4 r^3 - r s_1\big) / 6, \\
B_0 = {} & \big(-A_0^2 + 2A_0 A_1 r - 2A_0 A_2 r^2 - A_1^2 r^2 \\
& + 2A_1 A_2 r^3 - A_2^2 r^4 + 12B_1 r - 12B_2 r^2 + 12B_3 r^3 \\
& - 12B_4 r^4 + 12B_5 r^5\big) / 12.
\end{aligned}
\tag{41}
$$

Comparing the mixed derivatives $(G_{1tt})_x = (G_{1tx})_t$, $(G_{1xx})_t = (G_{1tx})_x$, and $(F_{xxx})_t = (F_t)_{xxx}$, one gets the conditions

$$
\begin{aligned}
A_{1xx} = {} & \big(-33A_{1x} A_2 s_1^2 - 18A_{2tx} s_1^2 - 108A_{2t} r_x s_1 s_2 \\
& + 24A_{2t} A_2 s_1^2 + 18A_{2t} s_3 + 54A_{2xx} r s_1^2 \\
& + 108A_{2x} r_x r s_1 s_2 + 18A_{2x} r_x s_1^2 - 30A_{2x} A_1 s_1^2 \\
& + 102A_{2x} A_2 r s_1^2 - 18A_{2x} r s_3 - 90B_{3x} s_1^2 \\
& + 54B_{4t} s_1^2 + 306B_{4x} r s_1^2 - 270B_{5t} r s_1^2 - 630B_{5x} r^2 s_1^2 \\
& - 36r_x A_1 A_2 s_1 s_2 + 72r_x A_2^2 r s_1 s_2 + 27r_x A_2^2 s_1^2 \\
& - 108r_x B_3 s_1 s_2 + 432r_x B_4 r s_1 s_2 + 252r_x B_4 s_1^2 \\
& - 1080r_x B_5 r^2 s_1 s_2 - 1260r_x B_5 r s_1^2 + 36r_x s_1 s_2^2 \\
& - 18s_{1x} s_1 s_2 + 30s_{2x} s_1^2 + 45A_0 B_5 s_1^2 - 5A_1 A_2^2 s_1^2 \\
& + 6A_1 A_2 s_3 - 45A_1 B_5 r s_1^2 + 10A_2^3 r s_1^2 - 12A_2^2 r s_3 \\
& - 15A_2 B_3 s_1^2 + 60A_2 B_4 r s_1^2 - 105A_2 B_5 r^2 s_1^2 \\
& + 5A_2 s_1^2 s_2 + 18B_3 s_3 - 72B_4 r s_3 + 180B_5 r^2 s_3 \\
& - 6s_2 s_3\big) / \left(18 s_1^2\right), \\
A_{2tt} = {} & \big(36A_{2tx} r s_1 + 72A_{2t} r_x s_1 - 18A_{2xx} r^2 s_1 \\
& - 72A_{2x} r_x r s_1 - 3A_{2x} A_0 s_1 + 3A_{2x} A_1 r s_1 \\
& - 3A_{2x} A_2 r^2 s_1 - 18B_{3t} s_1 + 18B_{3x} r s_1 \\
& + 72B_{4t} r s_1 - 72B_{4x} r^2 s_1 - 180B_{5t} r^2 s_1 \\
& + 180B_{5x} r^3 s_1 + 24r_x A_1 A_2 s_1 - 48r_x A_2^2 r s_1 \\
& + 72r_x B_3 s_1 - 288r_x B_4 r s_1 + 720r_x B_5 r^2 s_1 - 6r_x s_1 s_2 \\
& - 3s_{1x} s_1 + 3A_0 A_2^2 s_1 - 12A_0 B_4 s_1 + 60A_0 B_5 r s_1 \\
& + 4A_1^2 A_2 s_1 - 19A_1 A_2^2 r s_1 + 6A_1 B_3 s_1 - 12A_1 B_4 r s_1 \\
& + 19A_2^3 r^2 s_1 + 18A_2 B_2 s_1 \\
& - 66A_2 B_3 r s_1 + 144A_2 B_4 r^2 s_1 \\
& - 240A_2 B_5 r^3 s_1 + 2A_2 s_1^2 + s_3\big) / \left(18 s_1\right), \\
A_{2tx} = {} & \big(-6A_{1x} A_2 s_1^2 s_2 - 72A_{2t} r_x s_1 s_2^2 + 90A_{2t} s_{1x} s_1 s_2 \\
& - 90A_{2t} s_{2x} s_1^2 - 24A_{2t} A_2 s_1^2 s_2 + 12A_{2t} s_2 s_3 \\
& + 18A_{2xx} r s_1^2 s_2 - 9A_{2xx} s_1^3 + 72A_{2x} r_x r s_1 s_2^2 \\
& + 18A_{2x} r_x s_1^2 s_2 - 90A_{2x} s_{1x} r s_1 s_2 + 90A_{2x} s_{2x} r s_1^2 \\
& - 6A_{2x} A_1 s_1^2 s_2 + 48A_{2x} A_2 r s_1^2 s_2 - 18A_{2x} A_2 s_1^3 \\
& - 12A_{2x} r s_2 s_3 - 18B_{3x} s_1^2 s_2 + 72B_{4x} r s_1^2 s_2 \\
& - 27B_{4x} s_1^3 + 54B_{5t} s_1^3 - 180B_{5x} r^2 s_1^2 s_2 \\
& + 81B_{5x} r s_1^3 - 24r_x A_1 A_2 s_1 s_2^2 + 48r_x A_2^2 r s_1 s_2^2 \\
& + 12r_x A_2^2 s_1^2 s_2 - 72r_x B_3 s_1 s_2^2 + 288r_x B_4 r s_1 s_2^2 \\
& + 72r_x B_4 s_1^2 s_2 - 720r_x B_5 r^2 s_1 s_2^2 - 360r_x B_5 r s_1^2 s_2 \\
& + 135r_x B_5 s_1^3 + 30s_{1x} A_1 A_2 s_1 s_2 - 60s_{1x} A_2^2 r s_1 s_2 \\
& + 90s_{1x} B_3 s_1 s_2 - 360s_{1x} B_4 r s_1 s_2 \\
& + 900s_{1x} B_5 r^2 s_1 s_2 - 30s_{2x} A_1 A_2 s_1^2 + 60s_{2x} A_2^2 r s_1^2 \\
& - 90s_{2x} B_3 s_1^2 + 360s_{2x} B_4 r s_1^2 - 900s_{2x} B_5 r^2 s_1^2 \\
& - 8A_1 A_2^2 s_1^2 s_2 + 4A_1 A_2 s_2 s_3 + 18A_1 B_5 s_1^3 \\
& + 16A_2^3 r s_1^2 s_2 - 4A_2^3 s_1^3 - 8A_2^2 r s_2 s_3 \\
& - 24A_2 B_3 s_1^2 s_2 + 96A_2 B_4 r s_1^2 s_2 - 18A_2 B_4 s_1^3 \\
& - 240A_2 B_5 r^2 s_1^2 s_2 + 54A_2 B_5 r s_1^3 + 12B_3 s_2 s_3 \\
& - 48B_4 r s_2 s_3 + 120B_5 r^2 s_2 s_3\big) / \left(18 s_1^2 s_2\right).
\end{aligned}
\tag{42}
$$

3.2.2. Case s_1=0. From (35), one finds the condition

$$s_2 = 0. \tag{43}$$

Equation (15) gives the conditions

$$\begin{aligned}A_{0t} = {} & \left(6A_{0x}r + 6A_{2t}r^2 - 6A_{2x}r^3 - 7A_0A_1\right.\\ & + 9A_0A_2r + 5A_1^2r - 8A_1A_2r^2 + A_2^2r^3\\ & \left. - 36B_1 + 54B_2r - 54B_3r^2 + 36B_4r^3\right)/6,\\ B_0 = {} & \left(-A_0^2 + 2A_0A_1r - 2A_0A_2r^2 - A_1^2r^2\right.\\ & + 2A_1A_2r^3 - A_2^2r^4 + 12B_1r - 12B_2r^2\\ & \left. + 12B_3r^3 - 12B_4r^4 + 12B_5r^5\right)/12.\end{aligned} \tag{44}$$

From the mixed derivative $(G_{1xx})_t = (G_{1tx})_x$, one finds the condition

$$\begin{aligned}A_{2tt} = {} & \left(36A_{2tx}r + 72A_{2t}r_x - 18A_{2xx}r^2\right.\\ & - 72A_{2x}r_xr - 3A_{2x}A_0 + 3A_{2x}A_1r\\ & - 3A_{2x}A_2r^2 - 18B_{3t} + 18B_{3x}r + 72B_{4t}r\\ & - 72B_{4x}r^2 - 180B_{5t}r^2 + 180B_{5x}r^3\\ & + 24r_xA_1A_2 - 48r_xA_2^2r + 72r_xB_3 - 288r_xB_4r\\ & + 720r_xB_5r^2 + 3A_0A_2^2 - 12A_0B_4 + 60A_0B_5r\\ & + 4A_1^2A_2 - 19A_1A_2^2r + 6A_1B_3 - 12A_1B_4r\\ & + 19A_2^3r^2 + 18A_2B_2 - 66A_2B_3r\\ & \left. + 144A_2B_4r^2 - 240A_2B_5r^3\right)/18.\end{aligned} \tag{45}$$

The relation $(G_{1tt})_x = (G_{1tx})_t$ becomes

$$18\left(F_xG_{1t} - F_xG_{1x}r - F_xr_xG_1 - G_{1x}K\right)s_4 + G_1Ks_5 = 0, \tag{46}$$

where

$$\begin{aligned}s_4 = {} & 3A_{2t} - 3A_{2x}r + A_1A_2 - 2A_2^2r + 3B_3\\ & - 12B_4r + 30B_5r^2,\\ s_5 = {} & 18A_{1xx} + 27A_{1x}A_2 - 36A_{2xx}r + 24A_{2x}A_1\\ & - 102A_{2x}A_2r + 72B_{3x} - 54B_{4t} - 234B_{4x}r\\ & + 270B_{5t}r + 450B_{5x}r^2 - 15r_xA_2^2 - 180r_xB_4\\ & + 900r_xB_5r + 6s_{4x} - 45A_0B_5 + 13A_1A_2^2\\ & + 45A_1B_5r - 26A_2^3r + 39A_2B_3 - 156A_2B_4r\\ & + 345A_2B_5r^2 - 8A_2s_4.\end{aligned} \tag{47}$$

The relation $(A_{2t})_t - A_{2tt} = 0$ provides the condition

$$s_{4t} = \frac{\left(12r_xs_4 + 3s_{4x}r + A_1s_4 - 2A_2rs_4\right)}{3}. \tag{48}$$

Further study depends on s_4.

(i) Case $s_4 \neq 0$

From (46), one gets the derivative

$$g_{1t} = \frac{\left(18\left(F_xG_{1x}r + F_xr_xG_1 + G_{1x}K\right)s_4 - G_1Ks_5\right)}{\left(18F_xs_4\right)}. \tag{49}$$

Differentiating g_{1t} with respect to x, one obtains the derivative

$$F_x = \frac{\left(Ks_6\right)}{\left(108s_4^3\right)}, \tag{50}$$

where

$$\begin{aligned}s_6 = {} & 324A_{2x}s_4^2 - 36s_{4x}s_5 + 36s_{5x}s_4 + 108A_2^2s_4^2\\ & + 324B_4s_4^2 - 1620B_5rs_4^2 - s_5^2.\end{aligned} \tag{51}$$

The relations $(F_x)_t = (F_t)_x, (G_{1t})_t = G_{1tt}, (F_{xxx})_t = (F_t)_{xxx}$, and $(F_x)_{xx} = F_{xxx}$ provide the conditions

$$\begin{aligned}s_{6t} = {} & \left(30r_xs_6 + 3s_{6x}r + 2A_1s_6 - 4A_2rs_6 + 108A_2s_4^3\right.\\ & \left. + 18s_4^2s_5\right)/3,\\ s_{5t} = {} & \left(-108A_{1x}s_4^2 - 108A_{2x}rs_4^2 + 108r_xA_2s_4^2\right.\\ & + 180r_xs_4s_5 + 36s_{4x}rs_5 - 36A_1A_2s_4^2\\ & + 12A_1s_4s_5 - 36A_2^2rs_4^2 - 24A_2rs_4s_5\\ & - 108B_3s_4^2 + 108B_4rs_4^2 + 540B_5r^2s_4^2 + rs_5^2\\ & \left. + rs_6 - 144s_4^3\right)/\left(36s_4\right),\\ A_{2xx} = {} & \left(-5832A_{2x}A_2s_4^3 - 8748B_{4x}s_4^3 + 17496B_{5t}s_4^3\right.\\ & + 26244B_{5x}rs_4^3 + 43740r_xB_5s_4^3 - 126s_{4x}s_6\\ & + 45s_{6x}s_4 + 5832A_1B_5s_4^3 - 1296A_2^3s_4^3\\ & - 5832A_2B_4s_4^3 + 17496A_2B_5rs_4^3\\ & \left. + 12A_2s_4s_6 - s_5s_6\right)/\left(2916s_4^3\right),\\ s_{6xx} = {} & \left(-324A_{2x}s_4^2s_6 + 2916s_{4xx}s_4s_6 - 11664s_{4x}^2s_6\right.\\ & + 5832s_{4x}s_{6x}s_4 + 1944s_{4x}A_2s_4s_6 - 162s_{4x}s_5s_6\\ & - 648s_{6x}A_2s_4^2 + 54s_{6x}s_4s_5 - 108A_2^2s_4^2s_6\\ & \left. + 18A_2s_4s_5s_6 - 104976B_5s_4^5 + s_6^2\right)/\left(972s_4^2\right).\end{aligned} \tag{52}$$

(ii) Case $s_4 = 0$

From (46), one gets the condition

$$s_5 = 0. \tag{53}$$

Comparing the mixed derivative $(F_{xxx})_t = (F_t)_{xxx}$, one arrives at the condition

$$A_{2xx} = (-18A_{2x}A_2 - 27B_{4x} + 54B_{5t} + 81B_{5x}r + 135r_xB_5 + 18A_1B_5 - 4A_2^3 - 18A_2B_4 + 54A_2B_5r)/9. \tag{54}$$

All obtained results can be summarized in the following theorems.

Theorem 2. *Sufficient conditions for* (8) *to be linearizable via the generalized linearizing transformation* (5) *with $F_x = 0$ are equations* (18), (22), (24), (25), (28), (29), (30), *and* (31).

Corollary 3. *Provided that the sufficient conditions in Theorem 2 are satisfied, the transformation* (5) *mapping equation* (8) *to a linear equation* (6) *is obtained by solving the compatible system of equations* (21), (23), *and* (26) *for the functions $F(t), G_1(t,x)$, and $G_2(t,x)$.*

Theorem 4. *Sufficient conditions for equation* (8) *to be linearizable via the generalized linearizing transformation* (5) *with $F_x \neq 0$ are as follows.*

(a) *If $s_1 \neq 0$, then the conditions are* (18), (40), (41), *and* (42).

(b) *If $s_1 = 0, s_4 \neq 0$, then the conditions are* (18), (43), (44), (48), *and* (52).

(c) *If $s_1 = 0, s_4 = 0$, then the conditions are* (18), (43), (44), (53), *and* (54).

Corollary 5. *Provided that the sufficient conditions in Theorem 4 are satisfied, the transformation* (5) *mapping equation* (8) *to a linear equation* (6) *is obtained by solving the following compatible system of equations for the functions $F(t,x), G_1(t,x)$, and $G_2(t,x)$:*

(a) (16), (17), (34), (37), *and* (38);

(b) (16), (17), (34), (49), *and* (50);

(c) (16), (17), (19), (32), (33), *and* (34).

4. Examples

For understanding the procedure of using the linearization theorems, we consider the following examples.

Example 1. Consider the nonlinear third-order ordinary differential equation

$$\begin{aligned} &3x'^4t^2 + 2x'^3t(3t+2x) + 3x'^2x''t^2x \\ &+ x'^2(3t^2 + 8tx + 3x^2) + 2x'x''tx(t+2x) \\ &- x'x'''t^2x^2 + 2x'x(2t+3x) + 3x''^2t^2x^2 \\ &+ x''tx(-t+4x) - x'''t^2x^2 + 3x^2 = 0. \end{aligned} \tag{55}$$

It is an equation of the form (8) in Theorem 1 with the coefficients

$$\begin{aligned} &A_2 = -\frac{3}{x}, \quad A_1 = -\frac{2(t+2x)}{tx}, \\ &A_0 = \frac{t-4x}{tx}, \quad B_5 = 0, \quad B_4 = -\frac{3}{x^2}, \\ &B_3 = -\frac{2(3t+2x)}{tx^2}, \quad B_2 = -\frac{3t^2+8tx+3x^2}{t^2x^2}, \\ &B_1 = -\frac{2(2t+3x)}{t^2x}, \quad B_0 = -\frac{3}{t^2}, \\ &r = 1, \quad h_1 = -\frac{12}{t}, \quad h_2 = -\frac{450}{t^2}. \end{aligned} \tag{56}$$

One can check that these coefficients obey the conditions in Theorem 2. Thus, (55) is linearizable via a generalized linearizing transformation. For finding the functions F, G_1, and G_2, we have to solve equations in Corollary 3, which become

$$F_x = 0, \quad F_t = K, \tag{57}$$

$$G_{1x} = \frac{G_1}{x}, \quad G_{1tt} = \frac{(5G_{1t}^2t^2 - 4G_{1t}G_1t - G_1^2)}{(3G_1t^2)}, \tag{58}$$

$$K_x = 0, \quad K_t = \frac{(4KG_{1t}t - G_1)}{(3G_1t)}. \tag{59}$$

From the first equation of system (58), we get $G_1 = xf(t)$, and choosing $f(t) = t$, we have

$$G_1 = xt \tag{60}$$

and this solution satisfies the second equation. Since $r = 1$, then we obtain

$$G_2 = xt. \tag{61}$$

System (59) becomes

$$K_x = 0, \quad K_t = 0, \tag{62}$$

and one can take the simplest solution

$$K = 1. \tag{63}$$

System (90) becomes

$$F_x = 0, \quad F_t = 1, \tag{64}$$

so that we get the particular solution

$$F = t. \quad (65)$$

Thus, one obtains the linearizing transformation

$$X = t, \qquad dT = tx\left(x' + 1\right) dt. \quad (66)$$

Hence, (55) is mapped by the transformation of (66) into the linear equation (6).

Example 2. Consider the nonlinear third-order ordinary differential equation

$$\begin{aligned} &3x'^5t^2 + x'^4t(3t+4x) + x'^3x(4t+3x) \\ &\quad + x'^2x''tx(3t+x) + 3x'^2x^2 + 4x'x''tx^2 \\ &\quad - x'x'''t^2x^2 + 3x''^2t^2x^2 = 0. \end{aligned} \quad (67)$$

It is an equation of the form (8) in Theorem 1 with the coefficients

$$\begin{gathered} A_2 = -\frac{(3t+x)}{tx}, \qquad A_1 = -\frac{4}{t}, \qquad A_0 = 0, \\ B_5 = -\frac{3}{x^2}, \qquad B_4 = -\frac{(3t+4x)}{tx^2}, \\ B_3 = -\frac{(4t+3x)}{t^2x}, \qquad B_2 = -\frac{3}{t^2}, \\ B_1 = 0, \qquad B_0 = 0, \qquad r = 0, \\ s_1 = -\frac{2}{t^2}, \qquad s_2 = \frac{1}{t^2}, \qquad s_3 = \frac{12(t-x)}{t^5x}. \end{gathered} \quad (68)$$

One can check that these coefficients obey the conditions in Theorem 4(a). Thus, (67) is linearizable via a generalized linearizing transformation. For finding the functions F, G_1, and G_2, we have to solve equations in Corollary 5(a), which become

$$F_x = K, \qquad F_t = K, \quad (69)$$

$$\begin{aligned} G_{1t} &= \frac{(G_{1x}tx - G_1t + G_1x)}{(tx)}, \\ G_{1xx} &= \left(G_{1x}^2tx^2 + 4G_{1x}G_1tx - 4G_{1x}G_1x^2 \right. \\ &\quad \left. - 5G_1^2t + 4G_1^2x\right) / \left(3G_1tx^2\right), \end{aligned} \quad (70)$$

$$\begin{aligned} K_x &= \frac{(4K(G_{1x}x - G_1))}{(3G_1x)}, \\ K_t &= \frac{(4K(G_{1x}x - G_1))}{(3G_1x)}. \end{aligned} \quad (71)$$

From the first equation of system (70), one can take the particular solution

$$G_1 = tx \quad (72)$$

and this solution satisfies the second equation. Since $r = 0$, then we obtain

$$G_2 = 0. \quad (73)$$

System (71) becomes

$$K_x = 0, \qquad K_t = 0, \quad (74)$$

and one can take the simplest solution

$$K = 1. \quad (75)$$

System (69) becomes

$$F_x = 1, \qquad F_t = 1, \quad (76)$$

so that we get the particular solution

$$F = t + x. \quad (77)$$

Thus, one obtains the linearizing transformation

$$X = t + x, \qquad dT = txx'dt. \quad (78)$$

Hence, (67) is mapped by the transformation of (78) into the linear equation (6).

Example 3. Consider the nonlinear third-order ordinary differential equation

$$3x''^2x^2 - 3x'^4 - 3x'^2x''x - x'x'''x^2 = 0. \quad (79)$$

Note that this equation can be reduced to an autonomous equation by the substitution

$$x = tv(s), \qquad s = \ln(t), \quad (80)$$

and then to the second-order ordinary differential equation

$$\begin{aligned} &y''z^2y^2(z+y) \\ &\quad = y'^2z^2y(-2z+y) - 3y'zy\left(z^2+y^2\right) \\ &\qquad - 3z^4 - 14z^3y - 20z^2y^2 - 15zy^3 - 3y^4, \end{aligned} \quad (81)$$

where $y = y(z)$. However, the latter equation is not linearizable by point transformations.

Equation (79) is an equation of the form (8) in Theorem 1 with the coefficients

$$\begin{gathered} A_2 = \frac{3}{x}, \qquad A_1 = 0, \qquad A_0 = 0, \\ B_5 = 0, \qquad B_4 = \frac{3}{x^2}, \qquad B_3 = 0, \qquad B_2 = 0, \\ B_1 = 0, \qquad B_0 = 0, \qquad r = 0, \\ s_1 = 0, \qquad s_2 = 0, \qquad s_4 = 0, \qquad s_5 = 0. \end{gathered} \quad (82)$$

One can check that these coefficients obey the conditions in Theorem 4(c). Thus, (79) is linearizable via a generalized linearizing transformation. For finding the functions F, G_1, and G_2, we have to solve equations in Corollary 5(c), which become

$$F_t = K,$$

$$F_{xxx} = \left(6F_{xx}F_xG_{1t}G_1Kx^2 + 18F_{xx}G_{1x}G_1K^2x^2 - F_x^3G_{1t}^2x^2 - 6F_x^2G_{1t}G_{1x}Kx^2 - 9F_xG_{1x}^2K^2x^2 - 9F_xG_1^2K^2\right) / \left(6G_1^2K^2x^2\right), \quad (83)$$

$$G_{1tt} = \frac{\left(5G_{1t}^2\right)}{\left(3G_1\right)},$$

$$G_{1tx} = \frac{\left(G_{1t}\left(-F_xG_{1t}x + 6G_{1x}Kx + 3G_1K\right)\right)}{\left(3G_1Kx\right)}, \quad (84)$$

$$G_{1xx} = \left(6F_{xx}G_{1t}G_1Kx^2 - F_x^2G_{1t}^2x^2 - 6F_xG_{1t}G_{1x}Kx^2 + 9G_{1x}^2K^2x^2 - 9G_1^2K^2\right) / \left(6G_1K^2x^2\right),$$

$$K_x = \frac{\left(F_xG_{1t}x + 3G_{1x}Kx + 3G_1K\right)}{\left(3G_1x\right)},$$

$$K_t = \frac{\left(4G_{1t}K\right)}{\left(3G_1\right)}. \quad (85)$$

From the first equation of system (84), one can take the particular solution

$$G_1 = x \quad (86)$$

and this solution satisfies the second and third equations. Since $r = 0$, then we obtain

$$G_2 = 0. \quad (87)$$

System (85) becomes

$$K_x = \frac{(2K)}{x}, \qquad K_t = 0, \quad (88)$$

and one can take the particular solution

$$K = x^2. \quad (89)$$

System (83) becomes

$$F_x = x^2, \qquad F_{xxx} = \frac{\left(3\left(F_{xx}x - F_x\right)\right)}{x^2}, \quad (90)$$

so that one obtains the particular solution of the first equation as

$$F = tx^2 \quad (91)$$

and this solution satisfies the second equation. Then we get the linearizing transformation

$$X = tx^2, \qquad dT = xx' dt. \quad (92)$$

Hence, equation (79) is mapped by the transformation of (92) into the linear equation (6).

5. Conclusion

This paper is devoted to find the conditions which allow the third-order ordinary differential equation to be transformed into the simplest linear equation. Necessary conditions which guarantee that the third-order ordinary differential equation can be linearized are found in Theorem 1. Theorems 2 and 4 are sufficient conditions for the linearization problem. The linearizing transformation can be found by solving the compatible system in Corollaries 3 and 5. Finally, some examples are provided to demonstrate our procedure.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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