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รายงานวิจัยฉบับสมบูรณ์

โครงการ : การมีอยู่จริงของผลเฉลยสำหรับปัญหากึ่งดุลภาพบนปริภูมิเมตริกบริบูรณ์กับการประยุกต์สู่ทฤษฎีบทมินิแมกซ์

Existence of solutions for quasi-equilibrium problems on complete metric spaces with applications to minimax theorem

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สารบัญ

หน้า

บทคัดย่อ

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ชื่อโครงการ : การมีอยู่จริงของผลเฉลยสำหรับปัญหากึ่งดุลภาพบนปริภูมิเมตริกบริบูรณ์กับการประยุกต์สู่ทฤษฎีบทมินิแมกซ์

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โครงการวิจัยนี้เราได้สร้างและศึกษาเซตผลเฉลยของอสมการแปรผันซึ่งเราเรียกว่า อสมการแปรผันอีต้า และแนะนำแนวคิดวีคชาร์ปเซตของผลเฉลยไปสู่ปัญหาอสมการแปรผันอีต้าซึ่งมีความเกี่ยวข้องกับปัญหาสมดุลในปริภูมิบานาคสะท้อนปรับเรียบแบบเข้ม เรายังได้นำเสนอเงื่อนไขที่เพียงพอสำหรับการส่งที่เกี่ยวข้องกันนำไปสู่การส่งค่าคงที่บนเซตผลเฉลย เรายังได้แสดงให้เห็นถึงลักษณะเฉพาะของเซตผลเฉลยโดยฟังก์ชันช่องว่าง

เราได้แนะนำแนวคิดของการหดตัวเอฟไซพ์ชนิดอัตราส่วนและแก้ปัญหาจุดตรึงสำหรับการส่งดังกล่าวในปริภูมิเมตริกบริบูรณ์ที่มีสองอันดับบางส่วน ตัวอย่างที่สนับสนุนแนวความคิดได้ถูกสร้างขึ้นเพื่ออธิบายประโยชน์ของแนวคิดที่ได้สร้างขึ้นนี้ บทประยุกต์สามประการสู่โปรแกรมไดนามิก สมการเชิงอนุพันธ์เศษส่วน และสมการเชิงปริพันธ์ และบทประยุกต์อื่นๆ เราได้สร้างตัวอย่างที่เป็นพื้นผิวเพื่ออธิบายเกี่ยวกับการประมาณผลเฉลย

คำสำคัญ: อสมการแปรผันอีต้า ฟังก์ชันช่องว่าง ผลเฉลยวีคชาร์ป อสมการข้อจำกัดอันดับบางส่วน การหดตัวแบบเอฟ จุดตรึง

ABSTRACT

Project Code: R2560B078

Project Title: Existence of solutions for quasi-equilibrium problems on complete metric spaces with applications to minimax theorem

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We study the solution set of variational like-inequalities (in this sense we are called η-variational inequalities) and introduce the notion of a weak sharp set of solutions to η-variational inequality problem which related to equilibrium problems in reflexive, strictly convex and smooth Banach space. We also present sufficient conditions for the relevant mapping to be constant on the solutions. Moreover, we characterize the weak sharpness of the solutions of η-variational inequality by primal gap function.

We introduce the concept of (F,ψ)-rational type contraction and solve a fixed point problem for such mappings in a complete metric space endowed with two partial orders. Some examples are given to illustrate the usability of the established concept. Three applications to dynamic programming, fractional differential equation and integral equation are included here to highlight the usability of the obtained results. Using the derived results application to the system of dynamic programming along with an example is discussed. We also explain an illustrative example with graphical representation to validate the application of our result to integral equation, which includes some surfaces demonstrating the justification of approximate solution of the integral equation along with error function. Along this implementation, we give an entrance to the theory of fixed point with some relevant and innovative applications.

Keywords: η-variational inequality, Gap function, Weakly sharp solution; Constraint inequalities; partial order; F-contraction; Fixed point.

CHAPTER I

EXECUTIVE SUMMARY

Variational inequality has shown to be an important mathematical model in the study of many real problems, in particular equilibrium problems. It provides us with a tool for formulating and qualitatively analyzing the equilibrium problems in terms of existence and uniqueness of solutions, stability, and sensitivity analysis. The subject of variational inequalities has its origin in the calculus of variation associated with the minimization of infinite dimensional functionals.

Given a Banach space E, a subset K of E, and a functional $F : K \to E^*$ where E^* is the dual space of E, the Stampacchia Variational Inequality problem is the problem of solving for the variable $x \in K$ such that $\langle F(x), y - x\rangle \geq 0, \quad \text{for all } y \in K \quad (SVI)$ where $\langle \cdot, \cdot \rangle : E^* \times E \to \mathbb{R}$ is the duality paring while in the case of the Minty Variational Inequality it is to find the variable $x \in K$ such that $\langle F(y), x - y\rangle \leq 0, \quad \text{for all } y \in K \quad (MVI)$. Historically, (SVI) was introduced by Hartman and Stampacchia, and was subsequently expanded by Stampacchia in several papers. The study of the other problem (MVI) goes back to Minty, who studied the relationships of (SVI) and (MVI) in the case when F is a monotone operator. New impetus has been given to the field by the recent paper of Giannessi.

Burke and Ferris introduced the concept of a weak sharp minimum to present sufficient conditions for the finite identification, by iterative algorithm, of local minima associated with mathematical programming in space $\mathbb{R}^n$. Patriksson has generalized the concept of the weak sharpness of the solution set of a variational inequality problem (in short, VIP). Their concepts have been extended by Marcotte and Zhu to introduce another the notion of weak sharp solutions for variational inequalities. They also characterized the weak sharp solutions in terms of a dual gap function for variational inequalities. The relevant results have been obtained by Zhang et al.. It is further study by Wu and Wu. Hu and Song have extended the results of weak sharpness for the solutions of VIP under some continuity and monotonicity assumptions in Banach space. They also introduce the notion of weak sharp set of solutions to a variational inequality problem in a reflexive, strictly convex and smooth Banach space and present its several equivalent conditions. Liu and Wu studied weak sharp solutions for the variational inequality in terms of its primal gap function. They also characterized the weak sharpness of the solution set of VIP in terms of primal gap function. Recently, AL-Hamidan et al. give some characterization of weak sharp solutions for the VIP without considering the primal or dual gap function.

Motivated and inspired by the research mentioned above, we provide some general two concepts of Liu and Wu and Hu and Song to study the weak sharpness of solution set of η-variational inequality problem in Banach space. We also give some characterizations of weak sharp solutions for the η-VIP and also present its several equivalent conditions.

In recent times, many results developed related to metric fixed point theory endowed with a partial order. An early result in this direction was established by Ran and Reurings, where they presented a fixed point result, which can be considered as a junction of two fixed point theorems: Banach contraction principle and Knaster-Tarski fixed point theorem. Moreover, the result achieved by Ran and Reurings was extended and generalized by many researchers. On the other hand, Wardowski introduced the notion of F-contraction. This kind of contractions generalizes the Banach contraction. Newly, Piri and Kumam enhanced the results of Wardowski by launching the concept of an F-Suzuki contraction and obtained some curious fixed point results. Several extensions of this result have appeared in the reference therein. Very recently, Jleli and Samet presented a fixed point problem under two constraints inequalities.

Following this direction of research in this project, we introduce the concept of (F, ψ)-rational type contraction in the setup of metric space and examine the existence of fixed points for such type of contraction. Some examples and applications are given to illustrate the realized improvement.

CHAPTER II

CONTENTS OF RESEARCH

In this project, we obtain two papers. One is published in the international journal, and the other is submitted in the international journal as the following:

1

Natthaphon Artsawang, Ali Farajzadeh and Kasamsuk Ungchittrakool, Characterization of weak sharp solutions for generalized variational inequalities in Banach spaces, Journal of Computational Analysis and Applications, 2018, VOL. 25, NO.4 : 738-750, Impact Factor 2016=0.609

Lemma 1.1. *Assume that E is a reflexive, strictly convex and smooth Banach space. Let C be a closed convex subset of E and $\hat{x} \in C$. Then the following are equivalent:*

(i) $\hat{x}$ is a best approximation to x :$\|\eta(x,\hat{x})\| = \inf_{y\in C} \|\eta(x,y)\|$.

(ii) the inequality $\langle J(\eta(x,\hat{x})), \eta(y,\hat{x})\rangle \le 0, \quad \forall y \in C$ holds.

Proposition 1.2. *Consider a set $C \subseteq E$ and $\overline{x} \in C$. Then the following hold:*

(i) $T_C^{\eta}(\overline{x})$ is closed;

(ii) If C is convex, $T_C^{\eta}(\overline{x})$ is the closure of the cone generated by $\eta(C \times \{\overline{x}\})$, that is, $T_C^{\eta}(\overline{x}) = \overline{cone(\eta(C \times \{\overline{x}\}))}$

Let C be a nonempty closed convex subset of reflexive, strictly convex and smooth Banach space E. For a mapping F from E into E^*, the *η-variational inequality problem* [η-VIP] is to find a vector $x^* \in C$ such that

$$\langle F(x^*), \eta(x, x^*)\rangle \ge 0 \text{ for all } x \in C. \tag{1.1}$$

We denote the solution set of the η-VIP by C^{η}

The *η-dual variational inequality problem* [η-DVIP] is to find a vector $x_* \in C$ such that

$$\langle F(x), \eta(x, x_*)\rangle \ge 0 \text{ for all } x \in C. \tag{1.2}$$

We denote the solution set of the η-DVIP by C_{η}.

Now, we define the *primal gap function* $g(x)$ associated with η-VIP (1.1) as

$$g(x) := \sup_{y\in C}\{\langle F(x), \eta(x,y)\rangle\}, \text{ for all } x \in E,$$

and we setting

$$\Gamma(x) := \{y \in C : \langle F(x, \eta(x,y))\rangle = g(x)\}.$$

Similarly, we define the *dual gap function* $G(x)$ associated with η-DVIP (1.2) as

$$G(x) := \sup_{y\in C}\{\langle F(y), \eta(x,y)\rangle\}, \text{ for all } x \in E,$$

and we setting

$$\Lambda(x) := \{y \in C : \langle F(y, \eta(x,y))\rangle = G(x)\}.$$

Proposition 1.3. *Let $\hat{x} \in C$. Then* *(i)* $\hat{x} \in C^\eta \Leftrightarrow g(\hat{x}) = 0 \Leftrightarrow \hat{x} \in \Gamma(\hat{x})$;

(ii) $\hat{x} \in C_\eta \Leftrightarrow G(\hat{x}) = 0 \Leftrightarrow \hat{x} \in \Lambda(\hat{x})$.

Proposition 1.4. *If F is η-pseudomonotone on C, $C^\eta \subseteq C_\eta$.*

Proposition 1.5. *Let F be η-pseudomonotone$^+$ on C^η. Then F is constant on C^η.*

Proposition 1.6. *Let F be η-pseudomonotone$^+$ on C and $x^* \in C^\eta$. Then $C^\eta = \Lambda(x^*)$ and F is constant on $\Lambda(x^*)$.*

Proposition 1.7. *Suppose that F be η-pseudomonotone on C and $x^* \in C^\eta$. If F is constant on $\Gamma(x^*)$ then F is constant on C^η. And hence*

$$C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*).$$

Proposition 1.8. *Let F be η-pseudomonotone$^+$ on C. Then, for $x^* \in C^\eta$, F is constant on $\Gamma(x^*)$ if and only if*

$$C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*).$$

Proposition 1.9. *Let F be η-pseudomonotone$^+$ on C. Then the following are equivalent:*

(i) F *is constant on* $\Gamma(x^*)$ *for each* $x^* \in C^\eta$.

(ii) $C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*)$ *for each* $x^* \in C^\eta$.

(iii) $C^\eta = \Gamma(x^*) = \Lambda(x^*)$ *for each* $x^* \in C^\eta$.

(iv) $C^\eta = \Gamma(x^*)$ *for each* $x^* \in C^\eta$.

Lemma 1.10. *Let C be compact. If F is η-locally Lipschitz on C^η, then g is also η-locally Lipschitz on C^η.*

Proposition 1.11. *Let F be η-monotone on X and $x^* \in C^\eta$. Suppose that g is finite on X and η-Gateaux differentiable at x^*. Then $\partial_\eta g(x^*) = \{F(x^*)\}$.*

Theorem 1.12. *Let F be η-monotone on E and constant on $\Gamma(x^*)$ for some $x^* \in C^\eta$. Suppose that g is η-Gateaux differentiable, η-locally Lipschitz on C^η, and $g(x) < +\infty$ for all $x \in E$. Then C^η is weakly sharp if and only if there exists a positive number α such that*

$$\alpha d^\eta_{C^\eta}(x) \le g(x) \quad \text{for all } \; x \in C, \tag{1.3}$$

where $d^\eta_{C^\eta}(x) := \inf_{y \in C^\eta} \|\eta(x,y)\|$.

Corollary 1.13 ([1]). *Let F be monotone on R^n and constant on $\Gamma(x^*)$ for some $x^* \in C^*$. Suppose that g is Gateaux differentiable, locally Lipschitz on C^*, and $g(x) < +\infty$ for all $x \in R^n$. Then C^* is weakly sharp if and only if there exists a positive number α such that*

$$\alpha d_{C^*}(x) \le g(x) \quad \text{for all } \; x \in C.$$

2

Warut Saksirikun, Deepak Singh, Varsha Chauhan, and **Narin Petrot**, (Submitted). Computational and applicative approach through (F, ψ)-rational type contraction for existence of non-linear problems with two partial ordering.

Fixed point problem under two constraint inequalities for (F, ψ)-rational type contraction. The following problem will be discussed: find $x \in X$ such that

$$\begin{cases} x = Tx, \\ Ax \preceq_1 Bx, \\ Cx \preceq_2 Dx. \end{cases} \tag{2.1}$$

The dynamic processing gives fruitful tools for mathematical optimization and computer programming. We suppose that $W \subseteq U$ is a state space, $D \subseteq V$ is a decision space, where U and V are Banach spaces. In aforesaid system, the problem of dynamic programming associated to multistage process reduces to the problem of solving the functional equations:

$$\begin{aligned} h(x) &= \sup_{y \in D}\{f(x,y) + G(x,y,h(\rho(x,y)))\}, \quad x \in W; \\ g_i(x) &= \sup_{y \in D}\{f(x,y) + G_i(x,y,g_i(\rho(x,y)))\}, \quad x \in W,\ i = 1,2,3,4, \end{aligned} \tag{2.2}$$

where $\rho : W \times D \to W$, $f : W \times D \to \mathbb{R}$ and $G, G_1, G_2, G_3, G_4 : W \times D \times \mathbb{R} \to \mathbb{R}$.

Let $B(W)$ denote the set of all bounded real valued functions on W and for an arbitrary $h \in B(W)$, define

$$||h|| = \sup_{x \in W} |h(x)|.$$

Clearly, the pair $(B(W), ||.||)$ with the metric d defined by

$$d(h,k) = \sup_{x \in W} |h(x) - k(x)|,$$

for all $h, k \in B(W)$, is a Banach space. Precisely, the convergence in the space $B(W)$ with respect to $||.||$ is uniform and thus, if we consider a Cauchy sequence $\{h_n\}$ in $B(W)$, then $\{h_n\}$ converges uniformly to a function, say h^*, that is bounded and so $h^* \in B(W)$.

The mappings $T, H_1, H_2, H_3, H_4 : B(W) \to B(W)$ which are defined as follows:

$$\begin{aligned} T(h)(x) &= \sup_{y \in D}\{f(x,y) + G(x,y,h(\rho(x,y)))\}; \\ H_i(k)(x) &= \sup_{y \in D}\{f(x,y) + G_i(x,y,k(\rho(x,y)))\}, \end{aligned} \tag{2.3}$$

for all $h, k \in B(W)$, $x \in W$ and $i = 1,2,3,4$. Also consider $B(W)$ is equipped with two partial orders $\preceq_1$ and $\preceq_2$ in the following sense:

$$h(x) \preceq_1 k(x) \quad \text{implies} \quad h(x) \leq k(x);$$

$$h(x) \preceq_2 k(x) \quad \text{implies} \quad h(x) \geq k(x),$$

for all $h, k \in B(W)$.

Theorem 2.1. *Let (X, d) be a complete metric space endowed with two partial orders $\preceq_1$ and $\preceq_2$. Let $T, A, B, C, D : X \to X$ are given operators. Assume that the following assumptions are true:*

1. *$\preceq_i$ $(i = 1,2)$ is d-regular ;*
2. *T, A, B, C, D are continuous;*
3. *there exists $x_0 \in X$ such that $Ax_0 \preceq_1 Bx_0$;*
4. *T is $((A, B, \preceq_1), (C, D, \preceq_2))$-stable;*
5. *T is $((C, D, \preceq_2), (A, B, \preceq_1))$-stable;*
6. *T is (F, ψ)-rational type contraction on X.*

Then, the sequence $\{T^n x_0\}$ converges to some $u \in X$ and such $u \in X$ is a solution of the problem (2.1).

From Theorem 2.1, if $A = D = I_X$ and $B = C = T$ then we deduce the following corollary

Corollary 2.2. *Let (X, d) be a complete metric space endowed with two partial orders $\preceq_1$ and $\preceq_2$. Let $T : X \to X$ be a given operators. Assume that the following assumptions are true:*

1. *$\preceq_i$ $(i = 1, 2)$ is d-regular;*
2. *T is continuous;*
3. *there exists $x_0 \in X$ such that $x_0 \preceq_1 Tx_0$;*
4. *for all $x \in X$, we have $x \preceq_1 Tx \Longrightarrow T^2x \preceq_2 Tx$;*
5. *for all $x \in X$, we have $Tx \preceq_2 x \Longrightarrow Tx \preceq_1 T^2x$;*
6. *if there exist $F \in \Delta_F$, $\tau > 0$ and $\psi \in \Psi$ such that for all $x, y \in X$ with $Tx \neq Ty$, we have*

$$x \preceq_1 Tx,\ Ty \preceq_2 y \Longrightarrow F\Big(d(Tx, Ty)\Big) \leq F\Big(\psi\Big(M(x, y)\Big)\Big) - \tau \tag{2.4}$$

where $M(x, y) = \max\left\{d(x, y), \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty)]}{1+d(x,y)}\right\}$.

Then, T has a fixed point.

Remark 2.3. *Theorem 2.1 generalizes, improves and extends the Theorem 2.1 of H. Piri and P. Kumam [2] for two partial orders $\preceq_1$ and $\preceq_2$ along with rational type F-contraction.*

Remark 2.4. *By introducing Theorem 2.1, we generalized the results of Jleli et al. [3] and obtained the F-contraction version of [3].*

Theorem 2.5. *Suppose that the following conditions are satisfied:*
(1) $G(.,.,0), G_1(.,.,0), G_2(.,.,0), G_3(.,.,0), G_4(.,.,0) : W \times D \to \mathbb{R}$ and $f : W \times D \to \mathbb{R}$ are continuous and bounded functions;
(2) $H_1(h)(x) \leq H_2(h)(x), H_3(k)(x) \geq H_4(k)(x) \Longrightarrow$

$$|G(x, y, h(x)) - G(x, y, k(x))| \leq e^{-\tau}\psi(|h(x) - k(x)|)$$

for all $h(x), k(x) \in B(W)$, $x \in W$ and $y \in D$. Where $\psi \in \Psi$ is defined as in Theorem 2.1 and

$$\tau = \{|h(x) - k(x)|, |h(x) - Th(x)|, |k(x) - Tk(x)|, |k(x) - Th(x)|\} > 0;$$

(3) for every sequences $\{h_n\}, \{k_n\} \subset B(W)$ and $h, k \in B(W)$, if $\lim_{n\to\infty} \sup_{x\in W} |h_n(x) - h(x)| = \lim_{n\to\infty} \sup_{x\in W} |k_n(x) - k(x)| = 0$ and $h_n(x) \leq k_n(x)$ for all $n \in \mathbb{N}$, we have $h \leq l$.
(4) there exists $x_0 \in W$ such that

$$\sup_{y\in D}\{f(x, y) + G_1(x, y, h(\rho(x_0, y)))\} \leq \sup_{y\in D}\{f(x, y) + G_2(x, y, h(\rho(x_0, y)))\}$$

(5) there exists $h, k \in B(W)$ such that

$$H_1(h)(x) \leq H_2(h)(x) \Longrightarrow H_3T(h)(x) \geq H_4T(h)(x);$$

$$H_3(k)(x) \geq H_4(k)(x) \Longrightarrow H_1T(k)(x) \leq H_2T(k)(x).$$

Then the functional equation (2.2) has a bounded solution.

Consider the following fractional boundary value problem

$$
\begin{aligned}
{}^cD^\alpha u(t) + f(t, u(t)) = 0, \qquad 0 \geq t \geq 1, 1 < \alpha \geq 2; \\
u(0) = u(1) = 0,
\end{aligned} \tag{2.5}
$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous function and ${}^cD^\alpha$ represents the Caputo fractional derivative of order α and it is defined by

$$
{}^cD^\alpha = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds.
$$

Consider the following integral equation:

$$
u(t) = p(t) + \int_0^\Omega \lambda(t,s) f(s, u(s)) ds. \tag{2.6}
$$

We consider the space $X = C([0,\Omega], \mathbb{R})$ of all continuous functions defined on $[0,\Omega]$. Obviously, the space with the metric given by

$$
d(u,v) = \sup_{t \in [0,\Omega]} |u(t) - v(t)|, \quad u, v \in X
$$

is a complete metric space. Consider on $X = C([0,\Omega], \mathbb{R})$ equipped with the natural partial order relation, that is,

$$
u, v \in X, \quad u \leq v \iff u(t) \leq v(t), \quad t \in [0, \Omega].
$$

Theorem 2.6. *Consider, the nonlinear fractional differential equation (2.5). Assume that the following assertions hold:*

(i) there exist $\psi \in \Psi$ and $\tau > 0$ such that for all $u, v \in \mathbb{R}$, $u \leq v$

$$
f(t,u) - f(t,v) \geq 0 \text{ and } |f(t,u) - f(t,v)| \leq e^{-\tau} \psi(|v-u|), \quad \text{for all } t \in [0,1];
$$

(ii) there exists $u_0 \in X$ with $X = C([0,1], \mathbb{R})$ such that

$$
u_0(t) \leq \int_0^1 G(t,s) f(s, u_0(s))\, ds;
$$

(iii) $\sup_{t \in [0,1]} \int_0^1 G(t,s) ds \leq 1.$

Then the problem (2.5) has at least one solution in X.

Theorem 2.7. *Consider the problem (2.6) and assume that the following conditions are satisfied:*

(i) $f : [0,\Omega] \times \mathbb{R} \to \mathbb{R}$ is continuous;

(ii) $p : [0,\Omega] \to \mathbb{R}$ is continuous;

(iii) $\lambda : [0,\Omega] \times \mathbb{R} \to [0,\infty)$ is continuous;

(iv) there are $\psi \in \Psi$ and $\tau > 0$ such that for all $u, v \in \mathbb{R}$, $u \leq v$,

$$
f(s,u) - f(s,v) \geq 0 \text{ and } |f(s,u) - f(s,v)| \leq e^{-\tau} \psi(|v-u|);
$$

(v) assume that

$$
\sup_{t \in [0,\Omega]} \int_0^\Omega \lambda(t,s) ds \leq 1;
$$

(vi) there exists a $x_0 \in X$ with $(X = C([0,\Omega], \mathbb{R}))$ such that

$$
x_0(t) \leq p(t) + \int_0^\Omega \lambda(t,s) f(s, x_0(s)) ds.
$$

Then the integral equation (2.6) has a solution in X with $(X = C([0,\Omega], \mathbb{R}))$.

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CHAPTER III

OUTPUT

ผลลัพธ์จากโครงการวิจัยที่ได้รับทุนจากงบประมาณแผ่นดิน ประจำปี 2560

1. ผลงานวิจัยตีพิมพ์ในวารสารวิชาการนานาชาติ

- Natthaphon Artsawang, Ali Farajzadeh and **Kasamsuk Ungchittrakool,** Characterization of weak sharp solutions for generalized variational inequalities in Banach spaces, Journal of Computational Analysis and Applications, 2018, VOL. 25, NO.4 : 738–750, Impact Factor 2016=0.609
- Warut Saksirikun, Deepak Singh, Varsha Chauhan, and **Narin Petrot,** (Submitted). Computational and applicative approach through (F,\psi)-rational type contraction for existence of non-linear problems with two partial ordering.

2. การนำผลงานวิจัยไปใช้ประโยชน์

มีการนำไปใช้ประโยชน์ทั้งเชิงวิชาการและเชิงสาธารณะโดยทำให้มีการพัฒนาการเรียนการสอน รวมทั้งมีการสร้างเครือข่ายความร่วมมือในการทำวิจัย

ภาคผนวก

Characterization of weak sharp solutions for generalized variational inequalities in Banach spaces

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Characterization of weak sharp solutions for generalized variational inequalities in Banach spaces

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Abstract

In this paper, we study the solution set of variational like-inequalities (in this sense we are called η-variational inequalities) and introduce the notion of a weak sharp set of solutions to η-variational inequality problem in reflexive, strictly convex and smooth Banach space. We also present sufficient conditions for the relevant mapping to be constant on the solutions. Moreover, we characterize the weak sharpness of the solutions of η-variational inequality by primal gap function.

Keywords: η-variational inequality, Gap function, Weakly sharp solution

1. Introduction

Burke and Ferris [2] introduced the concept of a weak sharp minimum to present sufficient conditions for the finite identification, by iterative algorithm, of local minima associated with mathematical programming in space $\mathbb{R}^n$. Patriksson [7] has generalized the concept of the weak sharpness of the solution set of a variational inequality problem (in short, VIP). Their concepts have been extended by Marcotte and Zhu [6] to introduce another the notion of weak sharp solutions for variational inequalities. They also characterized the weak sharp solutions in terms of a dual gap function for variational inequalities. The relevant results have been obtained by Zhang et al. [12]. It is further study by Wu and Wu [9–11]. Hu and Song [4] have extended the results of weak sharpness for the solutions of VIP under some continuity and monotonicity assumptions in Banach space. They also introduce the notion of weak sharp set of solutions to a variational inequality problem in a reflexive, strictly convex and smooth Banach space and present its several equivalent conditions. Liu and Wu [5] studied weak sharp solutions for the variational inequality in terms of its primal gap function. They also characterized the weak sharpness of the solution set of VIP in terms of primal gap function. Recently, AL-Hamidan et al. [1] give some characterization of weak sharp solutions for the VIP without considering the primal or dual gap function.

In this paper, we provide some general two concepts of Liu and Wu [5] and Hu and Song [4] to study the weak sharpness of solution set of η-variational inequality problem in Banach space. We also give some characterizations of weak sharp solutions for the η-VIP and also present its several equivalent conditions. Our purpose in this paper is to develop the weak sharpness result in space $\mathbb{R}^n$.

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The paper is organized as follows. Section 2 discuss the new concepts of the Gateaux differentiable and Lipschitz continuity of the primal gap function and we also introduce the main definitions. Several equivalent conditions for F to be constant are discuss and present some relationship among C^η, C_η, $\Gamma(x^*)$, and $\Lambda(x^*)$ in Section 3. Finally, section 4 addresses the weak sharpness of C^η in terms of the primal gap function is characterized.

2. Preliminaries and formulations

Let E be a real Banach space with is topological dual space E^* and $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^* respectively. For a mapping from $\eta : E \times E$ to E . Let g be a mapping from E into Banach space Y. The mapping g is called directionally differentiable at a point $x \in E$ in a direction $v \in E$ if the limit

$$g'(x, v) := \lim_{t \downarrow 0} \frac{g(x + tv) - g(x)}{t}$$

exists. We say that g is directionally differentiable at x, if g is directionally differentiable at x in every direction $v \in E$.

The mapping g is called Gateaux differentiable at x if g is directionally differentiable at x and the directional derivative $g'(x, v)$ is linear and continuous in v and we denote this operator by $\nabla g(x)$, i.e. $\langle \nabla g(x), v \rangle = g'(x, v)$.

Definition 2.1. Let g be a mapping from E into Banach space Y. The mapping g is called η-Gateaux differentiable at x if g is Gateaux differentiable at x and there exists a unique $\xi \in E^*$ such that $\langle \xi, \eta(v, 0) \rangle = \langle \nabla g(x), v \rangle$, $\forall v \in E$. We denote this operator by $\nabla_\eta g(x)$ i.e. $\langle \nabla_\eta g(x), \eta(v, 0) \rangle = g'(x, v)$.

We defined η-subdifferential of a proper convex function f at $x \in E$ is given by

$$\partial_\eta f(x) := \{x^* \in E^* : \langle x^*, \eta(y, x) \rangle \le f(y) - f(x), \forall y \in E\}.$$

Let C be a closed convex subset of E. The mapping $P_C : E \to 2^C$ defined by

$$P_C(x) := \{y \in C : \|x - y\| = d(x, C)\},$$

is called *the metric projection operator.*

We known that if E is a reflexive and strictly convex Banach space, P_C is a single-valued mapping.

The duality mappings $J : E \to 2^{E^*}$ and $J^* : E^* \to 2^E$ are defined by

$$J(x) = \{x^* \in E^* : \langle x^*, x \rangle = \|x^*\|_*^2 = \|x\|^2\}, \forall x \in E$$

and

$$J^*(x^*) = \{x \in E : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|_*^2\}, \forall x^* \in E^*.$$

We know the following (see [8])

(i) if E is smooth, then J is single-valued;
(ii) if E is reflexive, then J is onto;
(iii) if E is strictly convex, then J is one-to-one;
(iv) if E is strictly convex, then J is strictly monotone.

Thought out this paper, we let $\eta : E \times E$ to E be satisfy the following condition;

(i) η is continuous on $E \times E$;
(ii) for any $x, y \in E$, $\eta(x, y) = -\eta(y, x)$;

(iii) for any $x, y \in E$ and α, β are scalars, $\eta(\alpha x + \beta y, 0) = \alpha\eta(x, 0) + \beta\eta(y, 0)$;
(iv) there exists $k > 0$ such that $\|\eta(x, y)\| = k\|x - y\|$ for all $x, y \in E$;
(v) $\eta(E \times \{0\}) = E$.

For a mapping g from a Banach space E into Banach space Y, we say that g is η-locally Lipschitz on E if for any $\overline{x} \in E$ there exist $\delta > 0$ and $L \geq 0$ such that

$$\|g(x) - g(y)\| \leq L\|\eta(x, y)\|, \quad \text{for all } \; x, y \in B(\overline{x}, \delta).$$

The following results are importance:

Lemma 2.2 ([3]). *Let E be a Banach space, $J : E \to 2^{E^*}$ a duality mapping and $\Phi(\|x\|) = \int_0^{\|x\|} ds$, $0 \neq x \in X$. Then $J(x) = \partial\Phi(\|x\|)$.*

Lemma 2.3. *Assume that E is a reflexive, strictly convex and smooth Banach space. Let C be a closed convex subset of E and $\hat{x} \in C$. Then the following are equivalent:*

(i) *$\hat{x}$ is a best approximation to x :$\|\eta(x, \hat{x})\| = \inf_{y \in C} \|\eta(x, y)\|$.*
(ii) *the inequality $\langle J(\eta(x, \hat{x})), \eta(y, \hat{x})\rangle \leq 0, \quad \forall y \in C$ holds.*

Proof. $(ii) \Rightarrow (i)$ For each $x \in E$. Let $\hat{x} \in C$ such that

$$\langle J(\eta(x, \hat{x})), \eta(y, \hat{x})\rangle \leq 0 \;\; \forall y \in C.$$

Then

$$\begin{aligned} \|\eta(x, \hat{x})\| \|J(\eta(x, \hat{x}))\|_* &= \langle J(\eta(x, \hat{x})), \eta(x, \hat{x})\rangle \\ &\leq \langle J(\eta(x, \hat{x})), \eta(x, \hat{x})\rangle + \langle J(\eta(x, \hat{x})), \eta(\hat{x}, y)\rangle, \quad \forall y \in C \\ &= \langle J(\eta(x, \hat{x})), \eta(x, y)\rangle, \quad \forall y \in C \\ &\leq \|J(\eta(x, \hat{x}))\|_* \|\eta(x, y)\|, \quad \forall y \in C. \end{aligned}$$

Hence, $\|\eta(x, \hat{x})\| = \inf_{y \in C} \|\eta(x, y)\|$.

$(i) \Rightarrow (ii)$ For each $x \in E$. Suppose that $\hat{x} \in C$ such that

$$\|\eta(x, \hat{x})\| = \inf_{y \in C} \|\eta(x, y)\|.$$

Since C is convex, we obtain that

$$\|\eta(x, \hat{x})\| \leq \|\eta(x, (1 - t)\hat{x} + ty)\|, \quad \forall y \in C \;\; \text{and} \;\; t \in [0, 1],$$

which implies that

$$\Phi(\|\eta(x, \hat{x})\|) - \Phi(\|\eta(x, (1 - t)\hat{x} + ty)\|), \quad \forall y \in C \;\; \text{and} \;\; t \in [0, 1],$$

where $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ give by $\Phi(x) = \int_0^x ds$, for all $x \in \mathbb{R}_+$.

By Lemma 2.2, $J(z) = \partial\Phi(\|z\|)$. It follows that

$$\begin{aligned} \langle J(\eta(x, (1 - t)\hat{x} + ty)), \eta(x, \hat{x}) - \eta(x, (1 - t)\hat{x} + ty)\rangle &\leq \Phi(\|\eta(x, \hat{x})\|) - \Phi(\|\eta(x, (1 - t)\hat{x} + ty)\|) \\ &\leq 0, \quad \forall y \in C \;\; \text{and} \;\; t \in [0, 1], \end{aligned}$$

that is,

$$t\langle J(\eta(x, (1 - t)\hat{x} + ty)), \eta(y, \hat{x})\rangle \leq 0, \quad \forall y \in C \;\; \text{and} \;\; t \in [0, 1]$$

Therefore,

$$\langle J(\eta(x,(1-t)\hat{x}+ty)), \eta(y,\hat{x})\rangle \leq 0, \quad \forall y \in C \quad \text{and} \quad t \in [0,1].$$

Taking $t \to 0$, we have

$$\langle J(\eta(x,\hat{x})), \eta(y,\hat{x})\rangle \leq 0, \quad \forall y \in C.$$

□

Remark 2.4. By definition of η for each $x \in E$ if $\hat{x} = P_C(x)$ then $\|\eta(x,\hat{x})\| = \inf_{y \in C} \|\eta(x,y)\|$.

If C is a closed convex subset of E and $\overline{x} \in C$, then the η-tangent cone to C at $\overline{x}$ has the form

$$\begin{aligned} T_C^\eta(\overline{x}) \quad = \quad & \{d \in E : \text{there exists a bounded sequence } \{d_k\} \subseteq X \text{ with } \eta(d_k,0) \to d, t_k \downarrow 0 \\ & \text{such that } \overline{x} + t_k d_k \in C, \quad \forall k \in \mathbb{N}\}. \end{aligned}$$

In the above, denote $x_k = \overline{x} + t_k d_k \in C$. Taking the limit as $k \to +\infty$, $t_k \to 0$, which implies that $t_k d_k \to 0$, thereby leading to $x_k \to \overline{x}$. Also from construction,

$$\frac{\eta(x_k,\overline{x})}{t_k} = \eta(d_k,0) \to d.$$

Thus, the η-tangent cone can be equivalently expressed as

$$\begin{aligned} T_C^\eta(\overline{x}) \quad = \quad & \{d \in E : \text{there exists sequence } \{x_k\} \subseteq C \text{ with } x_k \to \overline{x}, t_k \downarrow 0 \\ & \text{such that } \frac{\eta(x_k,\overline{x})}{t_k} \to d\}. \end{aligned}$$

Proposition 2.5. *Consider a set $C \subseteq E$ and $\overline{x} \in C$. Then the following hold:*

(i) *$T_C^\eta(\overline{x})$ is closed;*

(ii) *If C is convex, $T_C^\eta(\overline{x})$ is the closure of the cone generated by $\eta(C \times \{\overline{x}\})$, that is, $T_C^\eta(\overline{x}) = \overline{cone(\eta(C \times \{\overline{x}\}))}$*

Proof. (i) Suppose that $\{d_k\} \subseteq T_C^\eta(\overline{x})$ such that $d_k \to d$. Since $d_k \in T_C^\eta(\overline{x})$, there exist $\{x_k^r\} \subseteq C$ with $x_k^r \to \overline{x}$ and $\{t_k^r\} \subseteq \mathbb{R}_+$ with $t_k^r \to 0$ such that

$$\frac{\eta(x_k^r,\overline{x})}{t_k^r} \to d_k, \quad \forall k \in \mathbb{N}.$$

For a fixed k, there exists $\overline{r}$ such that

$$\left\|\frac{\eta(x_k^r,\overline{x})}{t_k^r} - d_k\right\| < \frac{1}{k}, \quad \forall r \geq \overline{r}.$$

Taking $k \to +\infty$, one can generate a sequence $\{x_k\} \subseteq C$ with $x_k \to \overline{x}$ and $t_k \downarrow 0$ such that

$$\frac{\eta(x_k,\overline{x})}{t_k} \to d.$$

Thus, $d \in T_C^\eta(\overline{x})$. Hence, $T_C^\eta(\overline{x})$ is closed.

(ii) Suppose that $d \in T_C^\eta(\overline{x})$, which implies that there exist $\{x_k\} \subseteq C$ with $x_k \to \overline{x}$ and $\{t_k\} \subseteq \mathbb{R}_+$ with $t_k \to 0$ such that

$$\frac{\eta(x_k,\overline{x})}{t_k} \to d.$$

Observe that $\eta(x_k,\overline{x}) \in \eta(C \times \{\overline{x}\})$. Since $t_k > 0$, $\frac{1}{t_k} > 0$. Therefore,

$$\frac{\eta(x_k,\overline{x})}{t_k} \in cone\ \eta(C \times \{\overline{x}\}).$$

Thus, $d \in \overline{cone(\eta(C \times \{\overline{x}\}))}$. Hence, $T_C^\eta(\overline{x}) \subseteq \overline{cone(\eta(C \times \{\overline{x}\}))}$.

Conversely, for each $x \in C$. Define a sequence

$$x_k = \overline{x} + \frac{1}{k}(x - \overline{x}), \quad \forall k \in \mathbb{N}.$$

By the convexity of C, it is obvious that $\{x_k\} \subseteq C$. Taking $k \to +\infty$, $x_k \to \overline{x}$, by construction, we obtain that

$$k\eta(x_k, \overline{x}) = \eta(x, \overline{x}).$$

Set $t_k = \frac{1}{k} > 0$, $t_k \to 0$ such that $\frac{\eta(x_k,\overline{x})}{t_k} \to \eta(x, \overline{x})$, which implies that $\eta(x, \overline{x}) \in T_C^\eta(\overline{x})$.

Since $x \in C$ is arbitrary, $\eta(C \times \{\overline{x}\}) \subseteq T_C^\eta(\overline{x})$. Because $T_C^\eta(\overline{x})$ is cone, we have $cone(\eta(C \times \{\overline{x}\})) \subseteq T_C^\eta(\overline{x})$. By (i), $T_C^\eta(\overline{x})$ is closed, which implies that $\overline{cone(\eta(C \times \{\overline{x}\}))} \subseteq T_C^\eta(\overline{x})$. □

The η-normal cone to C at $\overline{x}$ is defined by $N_C^\eta(\overline{x}) := [T_C^\eta(\overline{x})]^\circ$, where

$$A^\circ := \{x^* \in E^* : \langle x^*, x\rangle \leq 0, \forall x \in A\}.$$

If C is convex, then

$$N_C^\eta(\overline{x}) := \begin{cases} \{x^* \in E^* : \langle x^*, \eta(c, \overline{x})\rangle \leq 0 \text{ for all } c \in C\} & \text{if } \overline{x} \in C, \\ \emptyset, & \text{if } \overline{x} \notin C. \end{cases}$$

Let C be a nonempty closed convex subset of reflexive, strictly convex and smooth Banach space E. For a mapping F from E into E^*, the *η-variational inequality problem* [η-VIP] is to find a vector $x^* \in C$ such that

$$\langle F(x^*), \eta(x, x^*)\rangle \geq 0 \text{ for all } x \in C. \tag{2.1}$$

We denote the solution set of the η-VIP by C^η

The *η-dual variational inequality problem* [η-DVIP] is to find a vector $x_* \in C$ such that

$$\langle F(x), \eta(x, x_*)\rangle \geq 0 \text{ for all } x \in C. \tag{2.2}$$

We denote the solution set of the η-DVIP by C_η

Definition 2.6. The mapping $F : E \to E^*$ is said to be:

(i) *η-monotone* on C if $\langle F(x) - F(y), \eta(y, x)\rangle \geq 0$ for all $x, y \in C$;

(ii) *η-pseudomonotone* at $x \in C$ if for each $y \in C$ there holds

$$\langle F(x), \eta(y, x)\rangle \geq 0 \Rightarrow \langle F(y), \eta(y, x)\rangle \geq 0;$$

(iii) *η-pseudomonotone$^+$* on C if it is η-pseudomonotone at each point in C and, for all $x, y \in C$,

$$\left.\begin{array}{l} \langle F(y), \eta(x, y)\rangle \geq 0 \\ \langle F(x), \eta(x, y)\rangle = 0 \end{array}\right\} \Rightarrow F(x) = F(y).$$

Now, we define the *primal gap function* $g(x)$ associated with η-VIP (2.1) as

$$g(x) := \sup_{y \in C}\{\langle F(x), \eta(x, y)\rangle\}, \text{ for all } x \in E,$$

and we setting

$$\Gamma(x) := \{y \in C : \langle F(x, \eta(x, y))\rangle = g(x)\}.$$

Similarly, we define the *dual gap function* $G(x)$ associated with η-DVIP (2.2) as

$$G(x) := \sup_{y \in C}\{\langle F(y), \eta(x, y)\rangle\}, \text{ for all } x \in E,$$

and we setting

$$\Lambda(x) := \{y \in C : \langle F(y, \eta(x, y))\rangle = G(x)\}.$$

3. Sufficient condition for constancy of F on C^η and some properties of the primal gap function

In this section, we discuss about relations among C^η, C_η, $\Gamma(x^*)$, and $\Lambda(x^*)$. We study sufficient condition for F to be constant on C^η and also study the η-Lipschitz continuity and η-subdifferentiability of the primal gap function g in terms of the mapping F.

Proposition 3.1. *Let $\hat{x} \in C$. Then*

(i) $\hat{x} \in C^\eta \Leftrightarrow g(\hat{x}) = 0 \Leftrightarrow \hat{x} \in \Gamma(\hat{x})$;

(ii) $\hat{x} \in C_\eta \Leftrightarrow G(\hat{x}) = 0 \Leftrightarrow \hat{x} \in \Lambda(\hat{x})$.

Proof. (i) Consider

$$\begin{aligned} \hat{x} \in C^\eta \quad &\Leftrightarrow \quad \langle F(\hat{x}), \eta(y, \hat{x})\rangle \geq 0, \quad \forall y \in C \\ &\Leftrightarrow \quad \langle F(\hat{x}), \eta(\hat{x}, y)\rangle \leq 0, \quad \forall y \in C \\ &\Leftrightarrow \quad g(\hat{x}) = 0. \end{aligned}$$

And we also consider

$$\begin{aligned} \hat{x} \in \Gamma(\hat{x}) \quad &\Leftrightarrow \quad \langle F(\hat{x}), \eta(\hat{x}, \hat{x})\rangle = g(\hat{x}) \\ &\Leftrightarrow \quad 0 = g(\hat{x}). \end{aligned}$$

Similarly, we can obtain (ii). □

Proposition 3.2. *If F is η-pseudomonotone on C, $C^\eta \subseteq C_\eta$.*

Proof. Immediate from the definitions. □

The following proposition we present a sufficient condition for F to be constant on C^η.

Proposition 3.3. *Let F be η-pseudomonotone$^+$ on C^η. Then F is constant on C^η*

Proof. Let $x_1, x_2 \in C^\eta$. Since F is η-pseudomonotone$^+$ on C^η, we have

$$\langle F(x_1), \eta(x_2, x_1)\rangle \geq 0 \text{ and } \langle F(x_2), \eta(x_1, x_2)\rangle \geq 0.$$

By pseudomonotonicity of F on C^η, we have

$$\langle F(x_1), \eta(x_1, x_2)\rangle \geq 0 \text{ it follows that } \langle F(x_1), \eta(x_1, x_2)\rangle = 0.$$

Since F is η-pseudomonotone$^+$ on C^η and $\langle F(x_2), \eta(x_1, x_2)\rangle \geq 0$, implies that $F(x_1) = F(x_2)$. Hence, F is constant on C^η. □

Proposition 3.4. *Let F be η-pseudomonotone$^+$ on C and $x^* \in C^\eta$. Then $C^\eta = \Lambda(x^*)$ and F is constant on $\Lambda(x^*)$.*

Proof. First, we prove that F is constant on $\Lambda(x^*)$. For $x^* \in C^\eta$ and $c \in C$, we have

$\langle F(x^*), \eta(c, x^*)\rangle \geq 0$. Since F is pseudomonotone, we get that

$$\langle F(c), \eta(c, x^*)\rangle \geq 0, \forall c \in C. \text{ It follows that } G(x^*) = 0.$$

For $c \in \Lambda(x^*)$, we have

$$\langle F(c), \eta(c, x^*)\rangle = -G(x^*) = 0 \text{ and hence } F(c) = F(x^*).$$

It sufficient to show that $C^\eta = \Lambda(x^*)$.
($\subseteq$): Let $y^* \in C^\eta$. Then $\langle F(y^*), \eta(x^*, y^*)\rangle \geq 0$. Since $x^* \in C^\eta \subseteq C_\eta$, we have

$$\langle F(z), \eta(z, x^*)\rangle \geq 0, \forall z \in C.$$

It follows that $G(x^*) = 0$, and $\langle F(y^*), \eta(y^*, x^*)\rangle \geq 0$. Therefore,

$$\langle F(y^*), \eta(x^*, y^*)\rangle = 0 = G(x^*), \quad \text{that is,} \quad y^* \in \Lambda(x^*). \quad \text{Thus} \quad C^\eta \subseteq \Lambda(x^*).$$

($\supseteq$): Let $y^* \in \Lambda(x^*)$. Then $\langle F(y^*), \eta(x^*, y^*)\rangle = G(x^*) \geq 0$. Since $x^* \in C^\eta$, we have

$$\langle F(x^*), \eta(y, x^*)\rangle \geq 0, \forall y \in C.$$

Note that $x^* \in \Lambda(x^*)$, we have $F(x^*) = F(y^*)$. Consider, for all $y \in C$,

$$0 \leq \langle F(y^*), \eta(y, x^*)\rangle = \langle F(y^*), \eta(y, y^*)\rangle + \langle F(y^*), \eta(y^*, x^*)\rangle$$

implies $0 \leq \langle F(y^*), \eta(x^*, y^*)\rangle \leq \langle F(y^*), \eta(y, y^*)\rangle, \forall y \in C$. Therefore, $C^\eta = \Lambda(x^*)$. □

Proposition 3.5. *Suppose that F be η-pseudomonotone on C and $x^* \in C^\eta$. If F is constant on $\Gamma(x^*)$ then F is constant on C^η. And hence*

$$C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*).$$

Proof. Since F is η-pseudomonotone on C, we have $C^\eta \subseteq C_\eta$. Let $y^* \in C_\eta$. Then

$$\langle F(x^*), \eta(x^*, y^*)\rangle \geq 0.$$

By assumption, we obtain that $g(x^*) = 0$ and hence $\langle F(x^*), \eta(y^*, x^*)\rangle \geq 0$. It follows that

$$\langle F(x^*), \eta(x^*, y^*)\rangle = 0 = g(x^*). \quad \text{Thus} \quad y^* \in \Gamma(x^*).$$

Therefore $C^\eta \subseteq C_\eta \subseteq \Gamma(x^*)$. Let $z^* \in \Gamma(x^*)$. Then $\langle F(x^*), \eta(x^*, z^*)\rangle = g(x^*) = 0$.
From above $x^* \in C^\eta \subseteq \Gamma(x^*)$ and F is constant on $\Gamma(x^*)$, we obtain that $F(x^*) = F(z^*)$. Since $x^* \in C^\eta$, we have $\langle F(x^*), \eta(z, x^*)\rangle \geq 0, \forall z \in C$.

It follows that, for all $z \in C$,

$$\begin{aligned} 0 &\leq \langle F(z^*), \eta(z, x^*)\rangle \\ &= \langle F(z^*), \eta(z, z^*)\rangle + \langle F(z^*), \eta(z^*, x^*)\rangle \\ &= \langle F(z^*), \eta(z, z^*)\rangle + \langle F(x^*), \eta(z^*, x^*)\rangle \\ &= \langle F(z^*), \eta(z, z^*)\rangle. \end{aligned}$$

This implies that $z^* \in C^\eta$. Thus $\Gamma(x^*) \subseteq C^\eta$ and hence

$$C^\eta = C_\eta = \Gamma(x^*).$$

It sufficient to prove that $\Gamma(x^*) = \Lambda(x^*)$. For $c \in \Gamma(x^*)$, we have

$$\langle F(x^*), \eta(x^*, c)\rangle = g(x^*) = 0,$$

so $\langle F(c), \eta(x^*, c)\rangle = 0 = G(x^*)$. Therefore,

$$c \in \Lambda(x^*), \quad \text{which implies that} \quad \Gamma(x^*) \subseteq \Lambda(x^*).$$

Now let $c \in \Lambda(x^*)$. Then

$$\langle F(c), \eta(x^*, c)\rangle = G(x^*) = 0.$$

The pseudomonotonicity of F on C implies that $\langle F(x^*), \eta(x^*, c)\rangle \geq 0$. In this case,

$$\langle F(x^*), \eta(x^*, c)\rangle = 0 = g(x^*) \text{ since } x^* \in C^\eta.$$

Thus $c \in \Gamma(x^*)$ and hence $\Lambda(x^*) \subseteq \Gamma(x^*)$. Therefore

$$C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*).$$

□

Proposition 3.6. *Let F be η-pseudomonotone$^+$ on C. Then, for $x^* \in C^\eta$, F is constant on $\Gamma(x^*)$ if and only if*

$$C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*).$$

Proof. ($\Rightarrow$)Suppose that F is constant on $\Gamma(x^*)$. By Proposition 3.5, we obtain that

$$C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*).$$

($\Leftarrow$)Assume that $C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*)$. Let $x_1, x_2 \in \Gamma(x^*)$. Then

$$\langle F(x_1), \eta(x_2, x_1)\rangle \geq 0 \text{ and } \langle F(x_2), \eta(x_1, x_2)\rangle \geq 0 \text{ , because } x_1, x_2 \in C^\eta.$$

Since F is η-pseudomonotone and $\langle F(x_2), \eta(x_1, x_2)\rangle \geq 0$, we obtain that

$$\langle F(x_1), \eta(x_1, x_2)\rangle \geq 0 \text{ , that is , } \langle F(x_1), \eta(x_2, x_1)\rangle \leq 0.$$

Thus $\langle F(x_1), \eta(x_2, x_1)\rangle = 0$. Since F is η-pseudomonotone$^+$ on C, we have $F(x_1) = F(x_2)$. □

Proposition 3.7. *Let F be η-pseudomonotone$^+$ on C. Then the following are equivalent:*

(i) *F is constant on $\Gamma(x^*)$ for each $x^* \in C^\eta$.*
(ii) *$C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*)$ for each $x^* \in C^\eta$.*
(iii) *$C^\eta = \Gamma(x^*) = \Lambda(x^*)$ for each $x^* \in C^\eta$.*
(iv) *$C^\eta = \Gamma(x^*)$ for each $x^* \in C^\eta$.*

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ are immediate. It suffices to show $(iv) \Rightarrow (i)$. Suppose that $C^\eta = \Gamma(x^*)$ for each $x^* \in C^\eta$. Let $x^* \in C^\eta$ and $x_1, x_2 \in \Gamma(x^*)$. Then $x_1, x_2 \in C^\eta$ and $\langle F(x_1), \eta(x_2, x_1)\rangle \geq 0$ and $\langle F(x_2), \eta(x_1, x_2)\rangle \geq 0$.

Since F is η-pseudomonotone and $\langle F(x_2), \eta(x_1, x_2)\rangle \geq 0$, we obtain that

$$\langle F(x_1), \eta(x_1, x_2)\rangle \geq 0 \text{ , that is , } \langle F(x_1), \eta(x_2, x_1)\rangle \leq 0.$$

Thus $\langle F(x_1), \eta(x_2, x_1)\rangle = 0$. Since F is η-pseudomonotone$^+$ on C, we have $F(x_1) = F(x_2)$. □

Next we prove that if F is η-locally Lipschitz on C^η, then so is g.

Lemma 3.8. *Let C be compact. If F is η-locally Lipschitz on C^η, then g is also η-locally Lipschitz on C^η.*

Proof. Suppose that F is η-locally Lipschitz on C^η. Let x^* be any element in C^η. Then there exist $\delta > 0$ and $L_0 \geq 0$ such that

$$\|F(x) - F(y)\| \leq L_0\|\eta(x, y)\| \text{ and } \|F(x)\| \leq L_0 \text{ for all } x, y \in B(x^*, \delta).$$

Let $c \in \Gamma(x)$ with $x \in B(x^*, \delta)$. Then

$$\begin{aligned} g(x) - g(y) &\leq \langle F(x), \eta(x,c)\rangle - \langle F(y), \eta(y,c)\rangle \\ &= \langle F(x), \eta(x,y)\rangle + \langle F(x), \eta(y,c)\rangle - \langle F(y), \eta(y,c)\rangle \\ &= \langle \eta(x,y)\rangle + \langle F(x) - F(y), \eta(y,c)\rangle \\ &\leq \|F(x)\|\|\eta(x,y)\| + \|F(x) - F(y)\|\|\eta(y,c)\| \\ &\leq L_0\|\eta(x,y)\| + L_0\|\eta(x,y)\|\|\eta(y,c)\|. \end{aligned}$$

By the compactness of C and definition of η implies that there exists a constant $M \geq 0$ such that

$$\|\eta(y,c)\| \leq M \quad \text{for all} \quad y \in B(x^*, \delta) \quad \text{and} \quad c \in C.$$

We set $L = L_0 + L_0 M$, we obtain that

$$g(x) - g(y) \leq L\|\eta(x,y)\|.$$

We can conclude that g is η-locally Lipschitz on C^η. □

The following Proposition 3.2 we present the η-subdifferential of g at $x^* \in C^\eta$ is a singleton under sufficient condition.

Proposition 3.9. *Let F be η-monotone on X and $x^* \in C^\eta$. Suppose that g is finite on X and η-Gateaux differentiable at x^*. Then $\partial_\eta g(x^*) = \{F(x^*)\}$.*

Proof. Since $x^* \in C^\eta$, we have $g(x^*) = 0$. For each $y \in X$ and F is η-monotone, we obtain that

$$g(y) - g(x^*) \geq \langle F(y), \eta(y,x^*)\rangle \geq \langle F(x^*), \eta(y,x^*)\rangle.$$

Hence $F(x^*) \in \partial_\eta g(x^*)$.

Let $z \in \partial_\eta g(x^*)$. Then for each $v \in X$ and $t > 0$, we get that

$$g(x^* + tv) - g(x^*) \geq \langle z, \eta(x^* + tv, x^*)\rangle = t\langle z, \eta(v,0)\rangle,$$

that is,

$$\frac{g(x^* + tv) - g(x^*)}{t} \geq \langle z, \eta(v,0)\rangle.$$

By the η-Gateaux differentiability of g at x^* implies that

$$\langle \nabla_\eta g(x^*), \eta(v,0)\rangle = \lim_{t\to 0} \frac{g(x^* + tv) - g(x^*)}{t} \geq \langle z, \eta(v,0)\rangle.$$

Therefore, $\langle z - \nabla_\eta g(x^*), \eta(v,0)\rangle \leq 0$, for all $v \in X$. By definition of η we can set $\eta(v,0) = z - \nabla_\eta g(x^*)$, we have $\|z - \nabla_\eta g(x^*)\|^2 \leq 0$. This implies that $z = \nabla_\eta g(x^*)$, and hence $\partial_\eta g(x^*) = \{\nabla_\eta g(x^*)\} = \{F(x^*)\}$. □

4. Weak sharpness of C^η

Throughout this paper, we assume that C^η and C_η are nonempty and that E is a reflexive, strictly convex, and smooth Banach space. We introduce the notion of weak sharpness solution for generalized variational inequality(η-VIP).

Definition 4.1. The solution set C^η is said to be weakly sharp, if F satisfies

$$-F(x^*) \in int \bigcap_{x \in C^\eta} [T_C(x) \cap J^* N_{C^\eta}(x)]^\circ \quad \text{for all} \quad x^* \in C^\eta.$$

Theorem 4.2. *Let F be η-monotone on E and constant on $\Gamma(x^*)$ for some $x^* \in C^\eta$. Suppose that g is η-Gateaux differentiable, η-locally Lipschitz on C^η, and $g(x) < +\infty$ for all $x \in E$. Then C^η is weakly sharp if and only if there exists a positive number α such that*

$$\alpha d^\eta_{C^\eta}(x) \le g(x) \quad \text{for all} \quad x \in C, \tag{4.1}$$

where $d^\eta_{C^\eta}(x) := \inf_{y \in C^\eta} \|\eta(x,y)\|$.

Proof. On the given assumption and by Proposition 3.5, we obtain that

$$C^\eta = C_\eta = \Gamma(x^*) = \Lambda(x^*).$$

If C^η is weakly sharp, then for any $x^* \in C^\eta$ there exists $\alpha > 0$ such that

$$\alpha B_{E^*} \subseteq F(x^*) + \bigcap_{x \in C^\eta} [T_C(x) \cap J^* N_{C^\eta}(x)]^\circ, \tag{4.2}$$

where B_{E^*} is the open unit ball in E^*.

Since F is constant on $\Gamma(x^*)$, α satisfies (4.2) for all $x^* \in C^\eta$. Therefore, for every $y \in B_{E^*}$, we have

$$\alpha y - F(x^*) \in \bigcap_{x \in C^\eta} [T_C(x) \cap J^* N_{C^\eta}(x)]^\circ \subseteq [T_C(x^*) \cap J^* N_{C^\eta}(x^*)]^\circ.$$

Thus, for every $z \in [T_C(x^*) \cap J^* N_{C^\eta}(x^*)]$. It follows that

$$\langle \alpha y - F(x^*), z \rangle \le 0. \tag{4.3}$$

Taking $y = \frac{Jz}{\|Jz\|_*}$ in (4.3), we get that, for each $x^* \in C^\eta$,

$$\alpha \|Jz\|_* = \frac{\alpha}{\|Jz\|_*} \langle Jz, z \rangle \le \langle F(x^*), z \rangle.$$

This implies that for every $z \in [T_C(x^*) \cap J^* N_{C^\eta}(x^*)]$, we have

$$\alpha \|z\| \le \langle F(x^*), z \rangle.$$

For any $x \in C$, set $\overline{x} = P_{C^\eta}(x)$, we have $\eta(x, \overline{x}) \in T_C(\overline{x}) \cap J^* N_{C^\eta}(\overline{x})$ by Proposition 2.5 and lemma 2.3. Therefore,

$$\langle F(x^*), \eta(x, \overline{x}) \rangle \ge \alpha \|\eta(x, \overline{x})\| = \alpha d^\eta_{C^\eta}(x).$$

Conversely, suppose that there exists $\alpha > 0$ such that

$$\alpha d^\eta_{C^\eta}(x) \le g(x) \quad \text{for each} \quad x \in C.$$

We claim that

$$\alpha B_{E^*} \subseteq F(x^*) + [T_C(x^*) \cap J^* N_{C^\eta}(x^*)]^\circ \quad \text{for each} \quad x^* \in C^\eta. \tag{4.4}$$

If $T_C(x^*) \cap J^* N_{C^\eta}(x^*) = \{0\}$ for $x^* \in C^\eta$, then $[T_C(x^*) \cap J^* N_{C^\eta}(x^*)]^\eta = E$ and $\alpha B_{E^*} \subseteq F(x^*) + [T_C(x^*) \cap J^* N_{C^\eta}(x^*)]^\eta$, trivially. So it suffices to prove (4.4) to hold if $T_C(x^*) \cap J^* N_{C^\eta}(x^*) \ne \{0\}$ for $x^* \in C^\eta$. Now let $0 \ne z \in T_C(x^*) \cap J^* N_{C^\eta}(x^*)$. By definition of η there exists a unique $v \in E$ such that $z = \eta(v, 0)$. Then

$$\langle J(\eta(v,0)), \eta(v,0) \rangle > 0 \quad \text{and} \quad \langle J(\eta(v,0)), \eta(y^*, x^*) \rangle \le 0 \quad \text{for each} \quad y^* \in C^\eta,$$

which implies that C^η is separated from $x^* + v$ by the hyperplane

$$H_v = \{x \in E : \langle J(\eta(v,0)), \eta(x, x^*) \rangle = 0\} = \{x \in E : \langle J(\eta(v,0)), \eta(x,0) \rangle = \langle J(\eta(v,0)), \eta(x^*,0) \rangle\}.$$

Thus we can write

$$H_v = \{x \in E : \langle J(\eta(v,0)), \eta(x,0)\rangle = \beta\}, \quad \text{where} \quad \beta = \langle J(\eta(v,0)), \eta(x^*,0)\rangle.$$

Since $\eta(v,0) \in T_C(x^*)$, for each positive sequence $\{t_i\}$ decreasing to 0, there exists a sequence $\{v_i\}$ such that $\{\eta(v_i,0)\}$ converging to $\eta(v,0)$ and $x^* + t_i v_i \in C$ for sufficiently large i . By definition of η, we obtain that v_i converging to v. Thus $\langle J(\eta(v,0)), \eta(v_i,0)\rangle > 0$ holds for sufficiently large i, and hence we suppose that $x^* + t_i v_i$ lies in the open set $\{x \in E : \langle J(\eta(v,0)), \eta(x,x^*)\rangle > 0\}$. Therefore,

$$d^{\eta}_{C^{\eta}}(x^* + t_i v_i) \geq d^{\eta}_{H_v}(x^* + t_i v_i). \tag{4.5}$$

For each $x \in E$. We set

$$y := x - \left[\frac{\langle J(\eta(v,0)), \eta(x,0)\rangle - \beta}{\|J(\eta(v,0))\|_*^2}\right] v.$$

A straightforward computation show that $\langle J(\eta(v,0)), \eta(y,0)\rangle = \beta$, i.e., $y \in H_v$.
Furthermore, for any $z \in H_v$, we have

$$\begin{aligned}\langle J(\eta(x,y)), \eta(z,y)\rangle &= \left[\frac{\langle J(\eta(v,0)), \eta(x,0)\rangle - \beta}{\|J(\eta(v,0))\|_*^2}\right] \left(\langle J(\eta(v,0)), \eta(z,0)\rangle - \langle J(\eta(v,0)), \eta(y,0)\rangle\right) \\ &= 0.\end{aligned}$$

By Lemma 2.3, we have

$$\begin{aligned} d^{\eta}_{H_v}(x^* + t_i v_i) &= \|\eta(x^* + t_i v_i, y)\| \\ &= \frac{t_i\langle J(\eta(v,0)), \eta(v_i,0)\rangle}{\|J(\eta(v,0))\|_*^2}\|\eta(v,0)\| \\ &= \frac{t_i\langle J(\eta(v,0)), \eta(v_i,0)\rangle}{\|\eta(v,0)\|}, \end{aligned}$$

and hence, by (4.1),

$$g(x^* + t_i v_i) \geq \alpha d_{C^{\eta}}(x^* + t_i v_i) \geq \alpha t_i \frac{\langle J\eta(v,0), \eta(v_i,0)\rangle}{\|\eta(v,0)\|}.$$

By Proposition 3.1, $g(x^*) = 0$ for any $x^* \in C^{\eta}$, so

$$g(x^* + t_i v_i) - g(x^*) = g(x^* + t_i v_i) \geq \alpha t_i \frac{\langle J\eta(v,0), \eta(v_i,0)\rangle}{\|\eta(v,0)\|}.$$

Since g is η-locally Lipschitz and η-Gateaux differentiable on C^{η}, there hold

$$\|g(x^* + t_i v_i) - g(x^* + t_i v)\| \leq L t_i \|\eta(v_i, v)\|$$

for some $L > 0$ and all sufficiently large i and

$$\begin{aligned}\langle \nabla_{\eta} g(x^*), \eta(v,0)\rangle &= \lim_{i\to\infty} \frac{g(x^* + t_i v) - g(x^*)}{t_i} \\ &= \lim_{i\to\infty} \frac{g(x^* + t_i v_i) - g(x^*)}{t_i} \geq \alpha \|\eta(v,0)\|.\end{aligned}$$

By Proposition 3.9, $\nabla_{\eta} g(x^*) = F(x^*)$. Thus

$$\langle F(x^*), \eta(v,0)\rangle \geq \alpha \|\eta(v,0)\|.$$

This implies that for each $w \in B_{E^*}$,

$$\langle \alpha w - F(x^*), \eta(v,0)\rangle = \langle \alpha w, \eta(v,0)\rangle - \langle F(x^*), \eta(v,0)\rangle \leq \alpha\|\eta(v,0)\| - \alpha\|\eta(v,0)\| = 0.$$

Hence $\alpha B_{E^*} - F(x^*) \subseteq [T_C(x^*) \cap J^* N_{C^\eta}(x^*)]^\circ$, that is,

$$\alpha B_{E^*} \subseteq F(x^*) + [T_C(x^*) \cap J^* N_{C^\eta}(x^*)]^\circ.$$

This shows that C^η is weakly sharp since F is constant on C^η. □

Corollary 4.3 ([5]). *Let F be monotone on R^n and constant on $\Gamma(x^*)$ for some $x^* \in C^*$. Suppose that g is Gateaux differentiable, locally Lipschitz on C^*, and $g(x) < +\infty$ for all $x \in R^n$. Then C^* is weakly sharp if and only if there exists a positive number α such that*

$$\alpha d_{C^*}(x) \leq g(x) \quad \text{for all} \quad x \in C.$$

Proof. By applying above Theorem 4.2, if we define $\eta(x,y) = x - y$, for all $x, y \in E$ and space $E = \mathbb{R}^n$, then C^η can be reduce to C^*, where C^* is the solution set of variational inequalities. Moreover, the mapping g is Gateaux differentiable and locally Lipschitz on C^*. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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ภาคผนวก

Computational and applicative approach through (F,\psi)-rational type contraction for existence of non-linear problems with two partial ordering

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(Submitted)

Computational and applicative approach through (F, ψ)-rational type contraction for existence of non-linear problems with two partial ordering

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Abstract: In this paper, we introduce the concept of (F, ψ)-rational type contraction and solve a fixed point problem for such mappings in a complete metric space endowed with two partial orders. Some examples are given to illustrate the usability of the established concept. Three applications to dynamic programming, fractional differential equation and integral equation are included here to highlight the usability of the obtained results. Using the derived results application to the system of dynamic programming along with an example is discussed. We also explain an illustrative example with graphical representation to validate the application of our result to integral equation, which includes some surfaces demonstrating the justification of approximate solution of the integral equation along with error function. Along this implementation, we give an entrance to the theory of fixed point with some relevant and innovative applications.

Keywords : Constraint inequalities; partial order; F-contraction; Fixed point.
AMS Mathematics subject classification 2010 : $47H10, 54H25$.

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1 Introduction

In recent times, many results developed related to metric fixed point theory endowed with a partial order. An early result in this direction was established by Ran and Reurings [1], where they presented a fixed point result, which can be considered as a junction of two fixed point theorems: Banach contraction principle and Knaster-Tarski fixed point theorem. Moreover, the result achieved in [1] was extended and generalized by many researchers, some of which are in ([2]-[4], [7], [9]).

On the other hand, Wardowski [5] introduced the notion of F-contraction. This kind of contractions generalizes the Banach contraction. Newly, Piri and Kumam [8] enhanced the results of Wardowski [5] by launching the concept of an F-Suzuki contraction and obtained some curious fixed point results. Several extensions of this result have appeared in the reference therein; see in ([9], [10], [12])

Very recently, Jleli and Samet [11] presented a fixed point problem under two constraints inequalities. Following this direction of research in this manuscript, we introduce the concept of (F,ψ)-rational type contraction in the setup of metric space and examine the existence of fixed points for such type of contraction. Some examples and applications are given to illustrate the realized improvement.

2 Preliminaries

Let us introduce some definitions and recall some basic preliminary results which will be needed in the following sections. Throughout the article, we denote by $\mathbb{R}$ the set of all real numbers, by $\mathbb{R}^+$ the set of all positive real numbers and by $\mathbb{N}$ the set of all positive integers.

Definition 2.1. *Let X be a nonempty set and let $\preceq$ be a binary relation on X. We say that $\preceq$ is a partial order on X if the following conditions are satisfied:*

(i) For every $x \in X$, we have $x \preceq x$.

(ii) For every $x, y, z \in X$, we have $x \preceq y, \quad y \preceq x \Longrightarrow \quad x = y.$

(iii) For every $x, y, z \in X$, we have $x \preceq y, \quad y \preceq z \Longrightarrow \quad x \preceq z.$

Definition 2.2. *[11] Let (X,d) be a metric space and $\preceq$ be a partial order on X. We say that the partial order $\preceq$ is d-regular if the following condition is satisfied:*

For every sequences $\{a_n\}, \{b_n\} \subset X$, *we have*

$$\lim_{n\to\infty} d(a_n, a) = \lim_{n\to\infty} d(b_n, b) = 0, a_n \preceq b_n, \text{ for all } n \Longrightarrow a \preceq b,$$

where $(a, b) \in X \times X$.

Definition 2.3. *[11] Let* X *be a nonempty set endowed with two partial orders* $\preceq_1$ *and* $\preceq_2$. *Let* $T, A, B, C, D : X \to X$ *be given operators. We say that the operator* T *is* $((A, B, \preceq_1), (C, D, \preceq_2))$*-stable, if the following condition is satisfied:*

$$\forall x \in X, \quad Ax \preceq_1 Bx \Longrightarrow CTx \preceq_2 DTx.$$

In this paper, we will denote Δ_F for the set of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:

(F1) F is strictly increasing, that is, for $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta$ implies $F(\alpha) < F(\beta)$;

(F2) for each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive real numbers,

$$\lim_{n\to\infty} F(\alpha_n) = -\infty \text{ if and only if } \lim_{n\to\infty} \alpha_n = 0;$$

(F3) F is continuous on $(0, \infty)$.

Remark 2.1. *From [6], condition (F2) from above may replaced by*

(F2′) $\inf F = -\infty$

or, also, by

(F2″) there is $\{\alpha_n\} \subseteq \mathbb{R}^+$ *such that* $\lim_{n\to\infty} F(\alpha_n) = -\infty$

For more information on the class of Δ_F mappings, the reader may see [5, 6, 8]. Furthermore, we will also use the following notation

$\Psi = \{\psi : [0, \infty) \to [0, \infty) : \psi$ is upper semi continuous and non-decreasing with $\psi(t) < t$ for each $t > 0\}$.

3 Fixed point problem under two constraint inequalities for (F, ψ)-rational type contraction

In this section, the following problem will be discussed: find $x \in X$ such that

$$\begin{cases} x = Tx, \\ Ax \preceq_1 Bx, \\ Cx \preceq_2 Dx. \end{cases} \tag{3.1}$$

Now, we introduce the following definition:

Definition 3.1. *Let (X, d) be a metric space endowed with two partial orders $\preceq_1$ and $\preceq_2$. Let $T, A, B, C, D : X \to X$ are given operators. We say that T is (F, ψ)-rational type contraction on a metric space X, if there exist $F \in \Delta_F$, $\tau > 0$ and $\psi \in \Psi$ such that for all $x, y \in X$ with $Tx \neq Ty$, we have*

$$Ax \preceq_1 Bx,\ Cy \preceq_2 Dy \implies F\Big(d(Tx, Ty)\Big) \leq F\Big(\psi\Big(M(x,y)\Big)\Big) - \tau \tag{3.2}$$

where

$$M(x,y) = \max\Big\{d(x,y), \frac{d(y,Ty)[1 + d(x,Tx)]}{1 + d(x,y)}, \frac{d(y,Tx)[1 + d(x,Ty)]}{1 + d(x,y)}\Big\}. \tag{3.3}$$

The following examples are presented, in the favour of aforementioned notion.

Example 3.1. *Let $X = [0, \infty)$ be equipped with two partial orders " $\preceq_1 = \geq$ " and " $\preceq_2 = \leq$ ". Define a metric d on X by*

$$d(x,y) = \begin{cases} \max\{x, y\}, & if\ \ x \neq y; \\ 0, & if\ \ x = y. \end{cases}$$

Thus (X, d) is a complete metric space. Let $A, B, C, D, T : X \to X$ be given by

$$Ax = \log((\sqrt{x^2 + 3})^{\sqrt{x}}) + 5x; \qquad Bx = \log((x^4 + 2)^{\sqrt{x}}) + x;$$
$$Cx = \log((x^3 + 7e^{-x})^2); \qquad Dx = 10\log(x + 3e^x);$$
$$Tx = \frac{x}{\sqrt[n]{150 + \sqrt[n]{x}}}, \qquad where\ n = \{1, 2, 3, 4, \ldots, 70\}.$$

Simple calculations show that $Ax \geq Bx$ and $Cy \leq Dy$, for all $x, y \in X$. That is,

$$Ax \preceq_1 Bx \text{ and } Cy \preceq_2 Dy, \text{ for all } x, y \in X$$

Now, we claim that T is (F, ψ)-rational type contraction on a metric space X with

$$F(t) = \log(t^2 + t), t > 0 \text{ and } \psi(t) = \frac{100t + 5}{106}.$$

With out loss of generality we assume that $x > y$. Then it follows from inequality (3.2) that

$$\begin{aligned} F(d(Tx, Ty)) &= F\Big(d\Big(\frac{x}{\sqrt[n]{150 + \sqrt[n]{x}}}, \frac{y}{\sqrt[n]{150 + \sqrt[n]{y}}}, \Big)\Big) \\ &= F\Big(\frac{x}{\sqrt[n]{150 + \sqrt[n]{x}}}\Big) \\ &= \log\Big(\Big(\frac{x}{\sqrt[n]{150 + \sqrt[n]{x}}}\Big)^2 + \frac{x}{\sqrt[n]{150 + \sqrt[n]{x}}}\Big), \end{aligned}$$

where $n = \{1, 2, 3, 4, \ldots, 70\}$. *In view of the inequality (3.3), it is easy to verify that* $M(x, y) = x$. *Consider,*

$$\begin{aligned} F(\psi(M(x,y))) - \tau &= F(\psi(x)) - \tau \\ &= F\Big(\frac{100x+5}{106}\Big) - \tau \\ &= \log\Big(\Big(\frac{100x+5}{106}\Big)^2 + \frac{100x+5}{106}\Big) - \tau. \end{aligned}$$

It is easy to calculate that for $n = 1, 2, 3, 4, \ldots, 70$, *there exist some* τ *such that inequality (3.2) hold, for all* $x \in X$. *Here, few choices of* $n \in \{1, 2, 3, 4, \ldots, 70\}$ *and the corresponding values of* τ *are given in the Table 1.*

Table 1: Few choices of n and τ for the verification of inequality (3.2)

n	τ	n	τ	n	τ
1	(0,2.3]	10	(0,0.25]	19	(0,0.15]
2	(0,1.2]	11	(0,0.25]	20	(0,0.15]
3	(0,0.85]	12	(0,0.2]	21	(0,0.1]
4	(0,0.65]	13	(0,0.2]	30	(0,0.08]
5	(0,0.5]	14	(0,0.2]	40	(0,0.05]
6	(0,0.4]	15	(0,0.15]	50	(0,0.03]
7	(0,0.4]	16	(0,0.15]	60	(0,0.02]
8	(0,0.3]	17	(0,0.15]	70	(0,0.02]
9	(0,0.3]	18	(0,0.15]		

Our main result runs as follows.

Theorem 3.1. *Let (X, d) be a complete metric space endowed with two partial orders $\preceq_1$ and $\preceq_2$. Let $T, A, B, C, D : X \to X$ are given operators. Assume that the following assumptions are true:*

1. *$\preceq_i$ $(i = 1, 2)$ is d-regular ;*
2. *T, A, B, C, D are continuous;*
3. *there exists $x_0 \in X$ such that $Ax_0 \preceq_1 Bx_0$;*
4. *T is $((A, B, \preceq_1), (C, D, \preceq_2))$-stable;*
5. *T is $((C, D, \preceq_2), (A, B, \preceq_1))$-stable;*
6. *T is (F, ψ)-rational type contraction on X.*

Then, the sequence $\{T^n x_0\}$ converges to some $u \in X$ and such $u \in X$ is a solution of the problem (3.1).

Proof. By assumption (3), there exists a point $x_0 \in X$ such that $Ax_0 \preceq_1 Bx_0$. We construct a sequence $\{x_n\}$ by

$$x_{n+1} = Tx_n \qquad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

In veiw of assumptions (4), we will have $Cx_1 \preceq_2 Dx_1$. Subsequently, by assumption (5), we get $Ax_2 \preceq_1 Bx_2$. By repeating the process above, we derive

$$Ax_{2n} \preceq_1 Bx_{2n} \text{ and } Cx_{2n+1} \preceq_2 Dx_{2n+1}, \quad n \in \mathbb{N} \cup \{0\}. \tag{3.4}$$

If there exists $n \in \mathbb{N} \cup \{0\}$ such that $x_{n_0+1} = x_{n_0}$, then x_{n_0} is a solution of the problem (3.1), which complete the proof. Consequently, we will assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. by assumption (6), we obtain that

$$F\Big(d(x_n, x_{n+1})\Big) = F\Big(d(Tx_{n-1}, Tx_n)\Big) \leq F\Big(\psi\Big(M(x_{n-1}, x_n)\Big)\Big) - \tau \tag{3.5}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left\{d(x_{n-1}, x_n), \frac{d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)}, \frac{d(x_n, x_n)[1 + d(x_{n-1}, x_{n+1})]}{1 + d(x_{n-1}, x_n)}\right\} \\ &= \max\Big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\Big\}. \end{aligned}$$

We will claim that $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$, for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$. Then by using the definition of function ψ together with the inequality (3.5), we would have

$$F\Big(d(x_n, x_{n+1})\Big) \leq F\Big(\psi\Big(d(x_n, x_{n+1})\Big)\Big) - \tau.$$

This implies

$$F\Big(d(x_n, x_{n+1})\Big) < F\Big(d(x_n, x_{n+1})\Big),$$

which leads to a contradiction.

Therefore for each $n \in \mathbb{N}$, by repeating the same technique as mentioned above, we speculate that

$$\begin{aligned} F\Big(d(x_n, x_{n+1})\Big) &\leq F\Big(\psi\Big(d(x_{n-1}, x_n)\Big)\Big) - \tau \\ &< F\Big(d(x_{n-1}, x_n)\Big). \end{aligned} \tag{3.6}$$

Thus, since F is strictly increasing, we obtain

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n).$$

That is $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of positive real numbers. Moreover, from (3.6) since $\psi(t) < t$, for all $t > 0$, we have

$$F\Big(d(x_n, x_{n+1})\Big) < F\Big(d(x_{n-1}, x_n)\Big) - \tau. \quad \text{for } n \in \mathbb{N}, \tag{3.7}$$

Noted that, by the repeated use of (3.7), it establishes that

$$\begin{aligned} F\Big(d(x_n, x_{n+1})\Big) &< F\Big(d(x_{n-1}, x_n)\Big) - \tau \\ &< F\Big(d(x_{n-2}, x_{n-1})\Big) - 2\tau \\ &\cdots \\ &< F\Big(d(x_0, x_1)\Big) - n\tau. \end{aligned}$$

Which implies that

$$F\Big(d(x_n, x_{n+1})\Big) < F\Big(d(x_0, x_1)\Big) - n\tau.$$

Letting the limit as $n \to \infty$, the above inequality turns into $\lim_{n\to\infty} F\Big(d(x_n, x_{n+1})\Big) = -\infty$. Subsequently, by $F \in \Delta_F$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.8}$$

Next, we are going to prove that $\{x_n\}$ is a Cauchy sequence in (X,d). We argue it by contradiction. Assume that $\{x_n\}$ is not a Cauchy sequence. In this case, there exists $\epsilon > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that for all positive integer k with $n(k) > m(k) > k$, we have

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon, \tag{3.9}$$

and

$$d(x_{m(k)}, x_{n(k)-1}) < \epsilon.$$

Now, inequality (3.9), turns into

$$\begin{aligned}\epsilon &\leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})\\ &\leq \epsilon + d(x_{n(k)-1}, x_{n(k)}).\end{aligned}$$

By taking the limit as $k \to \infty$ in above inequality and using (3.8), we obtain

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon. \tag{3.10}$$

Further, from inequality (3.10), it is easy to see that

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \epsilon, \tag{3.11}$$

and

$$\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}) = \epsilon. \tag{3.12}$$

On the other hand, from (3.8), there exists a natural number $K \in \mathbb{N}$ such that for all $k \geq K$, we have

$$d(x_{m(k)}, x_{m(k)+1}) < \frac{\epsilon}{4} \ \ and \ \ d(x_{n(k)}, x_{n(k)+1}) < \frac{\epsilon}{4}. \tag{3.13}$$

Next, we will show that

$$d(Tx_{m(k)}, Tx_{n(k)}) = d(x_{m(k)+1}, x_{n(k)+1}), \tag{3.14}$$

for all $k \geq K$, reasoning by contradiction. Assume that, there exists $r \geq K$, such that

$$d(x_{m(r)+1}, x_{n(r)+1}) = 0. \tag{3.15}$$

In account of (3.9), (3.13) and (3.15), we arrive at

$$\begin{aligned}\epsilon \leq d(x_{m(r)}, x_{n(r)}) &\leq d(x_{m(r)}, x_{n(r)+1}) + d(x_{n(r)+1}, x_{n(r)})\\ &\leq d(x_{m(r)}, x_{m(r)+1}) + d(x_{m(r)+1}, x_{n(r)+1}) + d(x_{n(r)+1}, x_{n(r)})\\ &< \frac{\epsilon}{4} + 0 + \frac{\epsilon}{4} = \frac{\epsilon}{2},\end{aligned}$$

which is impossible. This means that (3.14) is proved. Thus, from (3.2), we have

$$\begin{aligned} F\Big(d(x_{m(k)+1}, x_{n(k)+1})\Big) &= F\Big(d(Tx_{m(k)}, Tx_{n(k)})\Big) \\ &\leq F\Big(\psi\Big(M(x_{m(k)}, x_{n(k)})\Big)\Big) - \tau \end{aligned} \tag{3.16}$$

in which,

$$M(x_{m(k)}, x_{n(k)}) = \max\Big\{d(x_{m(k)}, x_{n(k)}), \frac{d(x_{n(k)}, x_{n(k)+1})[1 + d(x_{m(k)}, x_{m(k)+1})]}{1 + d(x_{m(k)}, x_{n(k)})}, \\ \frac{d(x_{n(k)}, x_{m(k)+1})[1 + d(x_{m(k)}, x_{n(k)+1})]}{1 + d(x_{m(k)}, x_{n(k)})}\Big\},$$

for all $k \geq K$. Letting $k \to \infty$ and using (3.8), (3.10), (3.11) and (3.12), then above inequality deduce to

$$\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}) = \epsilon. \tag{3.17}$$

Making the limit as $k \to \infty$ in (3.16) and using (3.10), (3.17) and upper semi-continuity of ψ, we get

$$\begin{aligned} F(\epsilon) &\leq F(\psi(\epsilon)) - \tau \\ &< F(\epsilon) - \tau, \end{aligned}$$

which is impossible, since $\tau > 0$. This contradiction must verify that $\{x_n\}$ is a Cauchy sequence in a complete metric space X.

Next, the completeness of X assures that there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u. \tag{3.18}$$

Due to continuity of A and B, from (3.18), we obtain that

$$\lim_{n\to\infty} d(Ax_{2n}, Au) = 0 = \lim_{n\to\infty} d(Bx_{2n}, Bu).$$

As $\preceq_1$ is d-regular, in view of (3.4), we get

$$Au \preceq_1 Bu.$$

By repeating the same technique as mentioned above, one can get

$$Cu \preceq_2 Du.$$

Moreover, by (3.18), the continuity of T asserts that

$$d(Tu, u) = \lim_{n\to\infty} d(Tx_n, x_n) = \lim_{n\to\infty} (x_{n+1}, x_n) = 0.$$

This means

$$Tu = u.$$

Hence, we conclude that $u \in X$ is a solution of the problem (3.1). This completes the proof.

From Theorem 3.1, if $A = D = I_X$ and $B = C = T$ then we deduce the following corollary

Corollary 3.1. *Let (X,d) be a complete metric space endowed with two partial orders $\preceq_1$ and $\preceq_2$. Let $T : X \to X$ be a given operators. Assume that the following assumptions are true:*

1. *$\preceq_i$ $(i = 1, 2)$ is d-regular;*

2. *T is continuous;*

3. *there exists $x_0 \in X$ such that $x_0 \preceq_1 Tx_0$;*

4. *for all $x \in X$, we have $x \preceq_1 Tx \Longrightarrow T^2x \preceq_2 Tx$;*

5. *for all $x \in X$, we have $Tx \preceq_2 x \Longrightarrow Tx \preceq_1 T^2x$;*

6. *if there exist $F \in \Delta_F$, $\tau > 0$ and $\psi \in \Psi$ such that for all $x, y \in X$ with $Tx \neq Ty$, we have*

$$x \preceq_1 Tx,\ Ty \preceq_2 y \Longrightarrow F\Big(d(Tx,Ty)\Big) \leq F\Big(\psi\Big(M(x,y)\Big)\Big) - \tau \tag{3.19}$$

where $M(x,y) = \max\left\{d(x,y), \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Ty)]}{1+d(x,y)}\right\}$.

Then, T has a fixed point.

Remark 3.1. *Theorem 3.1 generalizes, improves and extends the Theorem 2.1 of H. Piri and P. Kumam [8] for two partial orders $\preceq_1$ and $\preceq_2$ along with rational type F-contraction.*

Remark 3.2. *By introducing Theorem 3.1, we generalized the results of Jleli et al. [11] and obtained the F-contraction version of [11].*

4 Applications

In this section, we will apply the Theorem 3.1 to obtain the existence theorems of some well known problems

4.1 Application to dynamic programming

The dynamic processing gives fruitful tools for mathematical optimization and computer programming. We suppose that $W \subseteq U$ is a state space, $D \subseteq V$ is a decision space, where U and V are Banach spaces. In aforesaid system, the problem of dynamic programming associated to multistage process reduces to the problem of solving the functional equations:

$$\begin{aligned} h(x) &= \sup_{y\in D}\{f(x,y) + G(x,y,h(\rho(x,y)))\}, \quad x \in W; \\ g_i(x) &= \sup_{y\in D}\{f(x,y) + G_i(x,y,g_i(\rho(x,y)))\}, \quad x \in W,\ i = 1,2,3,4, \end{aligned} \tag{4.1}$$

where $\rho : W \times D \to W$, $f : W \times D \to \mathbb{R}$ and $G, G_1, G_2, G_3, G_4 : W \times D \times \mathbb{R} \to \mathbb{R}$.

Let $B(W)$ denote the set of all bounded real valued functions on W and for an arbitrary $h \in B(W)$, define

$$||h|| = \sup_{x\in W} |h(x)|.$$

Clearly, the pair $(B(W), ||.||)$ with the metric d defined by

$$d(h,k) = \sup_{x\in W} |h(x) - k(x)|,$$

for all $h, k \in B(W)$, is a Banach space. Precisely, the convergence in the space $B(W)$ with respect to $||.||$ is uniform and thus, if we consider a Cauchy sequence $\{h_n\}$ in $B(W)$, then $\{h_n\}$ converges uniformly to a function, say h^*, that is bounded and so $h^* \in B(W)$.

The mappings $T, H_1, H_2, H_3, H_4 : B(W) \to B(W)$ which are defined as follows:

$$\begin{aligned} T(h)(x) &= \sup_{y\in D}\{f(x,y) + G(x,y,h(\rho(x,y)))\}; \\ H_i(k)(x) &= \sup_{y\in D}\{f(x,y) + G_i(x,y,k(\rho(x,y)))\}, \end{aligned} \tag{4.2}$$

for all $h, k \in B(W)$, $x \in W$ and $i = 1,2,3,4$. Also consider $B(W)$ is equipped with two partial orders $\preceq_1$ and $\preceq_2$ in the following sense:

$$h(x) \preceq_1 k(x) \quad \text{implies} \quad h(x) \leq k(x);$$

$$h(x) \preceq_2 k(x) \quad \text{implies} \quad h(x) \geq k(x),$$

for all $h, k \in B(W)$.

Theorem 4.1. *Suppose that the following conditions are satisfied:*

(1) $G(.,.,0), G_1(.,.,0), G_2(.,.,0), G_3(.,.,0), G_4(.,.,0) : W \times D \to \mathbb{R}$ *and* $f : W \times D \to \mathbb{R}$ *are*

continuous and bounded functions;

(2) $H_1(h)(x) \leq H_2(h)(x), H_3(k)(x) \geq H_4(k)(x) \implies$

$$|G(x,y,h(x)) - G(x,y,k(x))| \leq e^{-\tau}\psi(|h(x) - k(x)|)$$

for all $h(x), k(x) \in B(W)$, $x \in W$ *and* $y \in D$. *Where* $\psi \in \Psi$ *is defined as in Theorem 3.1 and*

$$\tau = \{|h(x) - k(x)|, |h(x) - Th(x)|, |k(x) - Tk(x)|, |k(x) - Th(x)|\} > 0;$$

(3) for every sequences $\{h_n\}, \{k_n\} \subset B(W)$ *and* $h, k \in B(W)$, *if* $\lim_{n\to\infty} \sup_{x\in W} |h_n(x) - h(x)| = \lim_{n\to\infty} \sup_{x\in W} |k_n(x) - k(x)| = 0$ *and* $h_n(x) \leq k_n(x)$ *for all* $n \in \mathbb{N}$, *we have* $h \leq l$.

(4) there exists $x_0 \in W$ *such that*

$$\sup_{y\in D}\{f(x,y) + G_1(x,y,h(\rho(x_0,y)))\} \leq \sup_{y\in D}\{f(x,y) + G_2(x,y,h(\rho(x_0,y)))\}$$

(5) there exists $h, k \in B(W)$ *such that*

$$H_1(h)(x) \leq H_2(h)(x) \implies H_3T(h)(x) \geq H_4T(h)(x);$$

$$H_3(k)(x) \geq H_4(k)(x) \implies H_1T(k)(x) \leq H_2T(k)(x).$$

Then the functional equation (4.1) has a bounded solution.

Proof. It follows from condition (1) of Theorem 4.1 that $T, H_1, H_2, H_3, H_4 : B(W) \to B(W)$ are continuous mappings. Moreover, condition (5), we have T is $((H_1, H_2, \leq), (H_3, H_4, \geq))$-stable and also it is $((H_3, H_4, \geq)(H_1, H_2, \leq))$-stable. Moreover, by virtue of condition (3) the relation $\preceq_1 = \leq$ and $\preceq_2 = \geq$ are d-regular.

Let λ be an arbitrary positive number, $x \in W$ and $h_1, h_2 \in B(W)$. Then there exist $y_1, y_2 \in D$ such that $H_1(h_1)(x) \leq H_2(h_1)(x)$, $H_3(h_2)(x) \geq H_4(h_2)(x)$ imply

$$T(h_1)(x) < f(x,y_1) + G(x,y_1,h_1(\rho(x,y_1))) + \lambda,$$

$$T(h_2)(x) < f(x,y_2) + G(x,y_2,h_2(\rho(x,y_2))) + \lambda,$$

$$T(h_1)(x) \geq f(x,y_2) + G(x,y_2,h_1(\rho(x,y_2))),$$

$$T(h_2)(x) \geq f(x,y_1) + G(x,y_1,h_2(\rho(x,y_1))).$$

Which yields

$$\begin{aligned} T(h_1)(x) - T(h_2)(x) &< G(x,y_1,h_1(\rho(x,y_1))) - G(x,y_1,h_2(\rho(x,y_1))) + \lambda \\ &\leq |G(x,y_1,h_1(\rho(x,y_1))) - G(x,y_1,h_2(\rho(x,y_1)))| + \lambda \\ &\leq e^{-\tau}\psi(|h_1(x) - h_2(x)|) + \lambda, \end{aligned}$$

where τ is defined in assumption(2).

In a same manner, we arrive at

$$T(h_2)(x) - T(h_1)(x) \leq e^{-\tau}\psi(|h_2(x) - h_1(x)|) + \lambda.$$

Therefore

$$|T(h_1)(x) - T(h_2)(x)| \leq e^{-\tau}\psi(|h_1(x) - h_2(x)|) + \lambda.$$

As a consequence,

$$d(T(h_1), T(h_2)) = \sup_{y \in D} |T(h_1)(x) - T(h_2)(x)| \leq e^{-\tau}\psi(d(h_1, h_2)) + \lambda.$$

Which yields

$$d(T(h_1), T(h_2)) \leq e^{-\tau}\psi(M(h_1, h_2)) + \lambda,$$

where

$$M(h_1, h_2) = \max\left\{d(h_1, h_2), \frac{d(h_2, Th_2)[1 + d(h_1, Th_1)]}{1 + d(h_1, h_2)}, \frac{d(h_2, Th_1)[1 + d(h_1, Th_2)]}{1 + d(h_1, h_2)}\right\}.$$

Since the above inequality does not depend on $x \in W$ and $\lambda > 0$ is taken arbitrary, then we conclude that

$$d(T(h_1), T(h_2)) \leq e^{-\tau}\psi(M(h_1, h_2)).$$

By using the property of logarithm function, the above inequality turns into the following

$$\log d(T(h_1), T(h_2)) \leq \log \psi(M(h_1, h_2)) - \tau.$$

Hence, we conclude that T is an (F, ψ)-rational type contraction. Notice that from condition (4), we have $H_1(h)(x_0) \leq H_2(h)(x_0)$. Thus all the conditions of Theorem 3.1 are satisfied. Due to Theorem 3.1, T has a fixed point $h^* \in B(W)$, that is h^* is a bounded solution of the functional equation (4.1). □

4.2 Application to fractional differential equation

Firstly, we present some definitions from the theory of fractional calculus.

The Reiman-Liouville fractional derivative of order $\beta > 0$ for a function $u \in C[0, 1]$ is defined by

$$D^{\beta}u(t) = \frac{1}{\Gamma(n - \beta)} \frac{d^n}{dt^n} \int_0^t \frac{u(s)\, ds}{(t - s)^{\beta - n + 1}}$$

provided that the right hand side is pointwise defined on [0,1]. Where $n = [\beta] + 1$ and $[\beta]$ means the integral part of the number β and Γ is the Euler gamma function.

Consider the following fractional boundary value problem

$$^{c}D^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 \geq t \geq 1, 1 < \alpha \geq 2;$$
$$u(0) = u(1) = 0, \tag{4.3}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous function and $^{c}D^{\alpha}$ represents the Caputo fractional derivative of order α and it is defined by

$$^{c}D^{\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds.$$

We consider the space $X = C([0,1], \mathbb{R})$ of all continuous functions defined on [0,1] and

$$||u|| = \sup_{t\in[0,1]} |u(t)|, \text{ for all } u \in X.$$

Obviously, this space with the metric given by

$$d(u,v) = \sup_{t\in[0,1]} |u(t) - v(t)|, \quad u, v \in X$$

and it is a complete metric space.

Theorem 4.2. *Consider, the nonlinear fractional differential equation (4.3). Assume that the following assertions hold:*

(i) there exist $\psi \in \Psi$ and $\tau > 0$ such that for all $u, v \in \mathbb{R}$, $u \leq v$

$$f(t,u) - f(t,v) \geq 0 \;\; \text{and} \;\; |f(t,u) - f(t,v)| \leq e^{-\tau}\psi(|v-u|), \quad \text{for all } t \in [0,1];$$

(ii) there exists $u_0 \in X$ with $X = C([0,1], \mathbb{R})$ such that

$$u_0(t) \leq \int_0^1 G(t,s) f(s, u_0(s))\, ds;$$

(iii) $\sup_{t\in[0,1]} \int_0^1 G(t,s)ds \leq 1$.

Then the problem (4.3) has at least one solution in X.

Proof. The problem (4.3) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s) f(s, u(s))\, ds,$$

for all $u \in X$ and $t \in [0,1]$, where

$$G(t,s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le T, \\ \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le T. \end{cases}$$

Consider the mapping $T : X \to X$ defined by

$$Tu(t) = \int_0^1 G(t,s) f(s,u(s))\, ds.$$

It is easy to note that if $u^* \in X$ is a fixed point of T then u^* is a solution of the problem (4.3). By the routine calculation from condition (i), for $u \in X$ with $u(t) \le Tu(t)$, for all $t \in [0,1]$, we have

$$\begin{aligned} Tu(t) - T^2u(t) &= \int_0^1 G(t,s)f(s,u(s))ds - \int_0^1 G(t,s)f(s,Tu(s))ds \\ &= \int_0^1 G(t,s)(f(s,u(s)) - f(s,Tu(s)))ds \ge 0, \end{aligned}$$

which yields that

$$Tu(t) \ge T^2u(t), \qquad \text{for all } t \in [0,1].$$

Similarly, one can show that for all $u \in X$, with $Tu(t) \le u(t)$, $t \in [0,1]$ implies

$$Tu(t) \le T^2u(t) \text{ for all } t \in [0,1].$$

Now, for $u, v \in X$ with $u(t) \le v(t)$ for all $t \in [0,1]$, we obtain

$$\begin{aligned} |Tu(t) - Tv(t)| &= |\int_0^1 G(t,s)f(s,u(s))\,ds - \int_0^1 G(t,s)f(s,v(s))\,ds| \\ &\le \int_0^1 G(t,s)|f(s,u(s)) - f(s,v(s))|ds \\ &\le \int_0^1 G(t,s)\, e^{-\tau}\psi(|v(s) - u(s)|)ds, \end{aligned}$$

Since, ψ is non-decreasing function, therefore we obtain

$$\begin{aligned} \psi(|v(s) - u(s)|) &\le \psi\Big(\sup_{s\in[0,1]} |u(s) - v(s)|\Big) \\ &= \psi\Big(d(u,v)\Big). \end{aligned}$$

Hence, from the above inequality, we arrive at

$$\begin{aligned} d(Tu,Tv) = \sup_{t\in[0,1]} |Tu(t) - Tv(t)| &\le e^{-\tau}\psi\Big(d(u,v)\Big) \sup_{t\in[0,1]} \int_0^1 G(t,s)ds \\ &\le \psi\Big(d(u,v)\Big)e^{-\tau} \\ &\le \psi\Big(M(u,v)\Big)e^{-\tau}, \end{aligned}$$

where $M(u,v) = \max\left\{d(u,v), \frac{d(v,Tv)[1+d(u,Tu)]}{1+d(u,v)}, \frac{d(v,Tu)[1+d(u,Tv)]}{1+d(u,v)}\right\}$.
By passing through a logarithms, we have

$$\log d\Big(Tu,Tv\Big) \leq \log \psi\Big(d(u,v)\Big) - \tau.$$

This implies

$$F\Big(d\Big(Tu,Tv\Big)\Big) \leq F\Big(\psi\Big(d(u,v)\Big)\Big) - \tau,$$

for $F(t) = \log t$. This concludes that the contractive condition of Corollary 3.1 is satisfied. Also, by the condition (ii) of Theorem 4.2, we deduce that $u_0 \leq Tu_0$.
Thus, as a result of Corollary 3.1,we can assert that T has a fixed point in X. That is, the fractional differential equation (4.3) has a solution. □

4.3 Application to integral equation

Consider the following integral equation:

$$u(t) = p(t) + \int_0^{\Omega} \lambda(t,s) f(s,u(s))ds. \tag{4.4}$$

We consider the space $X = C([0,\Omega],\mathbb{R})$ of all continuous functions defined on $[0,\Omega]$. Obviously, the space with the metric given by

$$d(u,v) = \sup_{t\in[0,\Omega]} |u(t)-v(t)|, \quad u,v \in X$$

is a complete metric space. Consider on $X = C([0,\Omega],\mathbb{R})$ equipped with the natural partial order relation, that is,

$$u,v \in X, \quad u \leq v \iff u(t) \leq v(t), \; t \in [0,\Omega].$$

Theorem 4.3. *Consider the problem (4.4) and assume that the following conditions are satisfied:*

(i) $f : [0,\Omega] \times \mathbb{R} \to \mathbb{R}$ *is continuous;*

(ii) $p : [0,\Omega] \to \mathbb{R}$ *is continuous;*

(iii) $\lambda : [0,\Omega] \times \mathbb{R} \to [0,\infty)$ *is continuous;*

(iv) there are $\psi \in \Psi$ *and* $\tau > 0$ *such that for all* $u,v \in \mathbb{R}$, $u \leq v$,

$$f(s,u) - f(s,v) \geq 0 \text{ and } |f(s,u) - f(s,v)| \leq e^{-\tau}\psi(|v-u|);$$

(v) assume that

$$\sup_{t\in[0,\Omega]}\int_0^{\Omega}\lambda(t,s)ds\leq 1;$$

(vi) there exists a $x_0\in X$ *with* $(X=C([0,\Omega],\mathbb{R}))$ *such that*

$$x_0(t)\leq p(t)+\int_0^{\Omega}\lambda(t,s)f(s,x_0(s))ds.$$

Then the integral equation (4.4) has a solution in X *with* $(X=C([0,\Omega],\mathbb{R}))$.

Proof. Consider the mapping $T:X\to X$ defined by

$$Tu(t)=p(t)+\int_0^{\Omega}\lambda(t,s)f(s,u(s))ds,$$

for all $u\in X$ and $t\in[0,\Omega]$. We will prove that all the conditions of Corollary 3.1 are satisfied. Clearly, $\preceq$ is d-regular and by the condition (iv) of the Theorem 4.3, for $x\in X$ with $x(t)\leq Tx(t)$, $t\in[0,\Omega]$, we have

$$\begin{aligned}Tx(t)-T^2x(t)&=\int_0^{\Omega}\lambda(t,s)f(s,x(s))ds-\int_0^{\Omega}\lambda(t,s)f(s,Tx(s))ds\\&=\int_0^{\Omega}\lambda(t,s)(f(s,x(s))-f(s,Tx(s)))ds\geq 0,\end{aligned}$$

which yields that

$$Tx(t)\geq T^2x(t),\qquad \text{for all } t\in[0,\Omega].$$

Similarly, one can show that for all $x\in X$, with $Tx(t)\leq x(t)$, $t\in[0,\Omega]$ implies

$$Tx(t)\leq T^2x(t) \text{ for all } t\in[0,\Omega].$$

Now, for $u,v\in X$ with $u\leq v$, we obtain

$$\begin{aligned}Tu(t)-Tv(t)&=\int_0^{\Omega}\lambda(t,s)f(s,u(s))ds-\int_0^{\Omega}\lambda(t,s)f(s,v(s))ds\\&=\int_0^{\Omega}\lambda(t,s)(f(s,u(s))-f(s,v(s)))ds\\&\leq e^{-\tau}\int_0^{\Omega}\lambda(t,s)\psi(|v(s)-u(s)|)ds,\end{aligned}$$

As ψ is non-decreasing function, we have

$$\begin{aligned}\psi(|v(s)-u(s)|)&\leq\psi\Big(\sup_{s\in[0,\Omega]}|u(s)-v(s)|\Big)\\&=\psi\Big(d(u,v)\Big).\end{aligned}$$

Hence, from the above inequality, we arrive at

$$\begin{aligned}(Tu,Tv) = \sup_{t\in[0,\Omega]} |Tu(t)-Tv(t)| &\le e^{-\tau}\psi\Big(d(u,v)\Big) \sup_{t\in[0,\Omega]} \int_0^{\Omega} \lambda(t,s)ds \\ &\le \psi\Big(d(u,v)\Big)e^{-\tau} \\ &\le \psi\Big(M(u,v)\Big)e^{-\tau},\end{aligned}$$

where $M(u,v) = \max\Big\{d(u,v), \frac{d(v,Tv)[1+d(u,Tu)]}{1+d(u,v)}, \frac{d(v,Tu)[1+d(u,Tv)]}{1+d(u,v)}\Big\}$.

Consequently, by passing to logarithms, one can obtain

$$\log d\Big(Tu,Tv\Big) \le \log\psi\Big(d(u,v)\Big) - \tau.$$

This turns up to

$$F\Big(d\Big(Tu,Tv\Big)\Big) \le F\Big(\psi\Big(d(u,v)\Big)\Big) - \tau.$$

This show that the contractive condition in Corollary 3.1 is satisfied.

Also, from condition *(vi)* of Theorem 4.3, we know that $x_0 \le Tx_0$.

As a result of Corollary 3.1, T has a fixed point in X, that is, the integral equation has a solution. □

The following example demonstrates the superiority of Theorem 4.3.

Example 4.1. *Consider the following integral equation in $X = C([0,1],\mathbb{R})$.*

$$u(t) = \frac{t^2+1}{t^3+0.1} + \frac{1}{3}\int_0^1 \frac{s^2}{(t+1)}\frac{1}{(1+u(s))}ds; \quad t\in[0,1]. \tag{4.5}$$

Observe that this equation is a special case of (4.4), in which

$$p(t) = \frac{t^2+1}{t^3+0.1}, \lambda(t,s) = \frac{s^2}{(t+1)} \text{ and } f(s,t) = \frac{1}{3(1+t)}.$$

Indeed, the function p, λ and f are continuous. Thus the assumptions (i)-(iii) *are satisfied. Further, for all $u,v\in\mathbb{R}$ with $u\le v$, we get*

$$\begin{aligned}0 \le |f(s,u)-f(s,v)| &\le |\frac{1}{3(1+u)} - \frac{1}{3(1+v)}| \\ &\le \frac{1}{3}|v-u| \\ &\le e^{-0.1}\frac{2}{3}(|v-u|) \\ &\le e^{-\tau}\psi(|v-u|)\end{aligned}$$

for $\tau = 0.1$ *and* $\psi(t) = \frac{2t}{3}$. *Hence, condition* (iv) *of Theorem 4.3 is fulfilled. For condition* (v), *we have*

$$\sup_{t\in[0,1]} \int_0^1 \lambda(t,s)ds = \sup_{t\in[0,1]} \int_0^1 \frac{s^2}{(t+1)} ds = \sup_{t\in[0,1]} \frac{1}{3(t+1)} \leq 1.$$

Thus, condition (v) *is proved. Consider* $x_0(t) = 1$, *then we arrive at*

$$\begin{aligned} p(t) + \int_0^1 \lambda(t,s) f(s, x_0(s))ds &= \frac{t^2+1}{t^3+0.1} + \int_0^1 \frac{s^2}{(t+1)} f(s,1)ds \\ &= \frac{t^2+1}{t^3+0.1} + \frac{1}{6}\int_0^1 \frac{s^2}{(t+1)}ds \\ &= \frac{t^2+1}{t^3+0.1} + \frac{1}{18(t+1)} \\ &> 1 \\ &= x_0(t), \end{aligned}$$

for all, $t \in [0,1]$. *This shows that, all the conditions of Theorem 4.3 are satisfied. Hence, the integral equation (4.5) has a solution in* $X = C([0,1],\mathbb{R})$. *Further,the approximate solution of the integral equation (4.5) is*

$$u(t) = \frac{1.02733\,t^3 + t^2 + t + 1.002733}{(t^3+0.1)(t+1)}. \tag{4.6}$$

The approximate solution of the integral equation (4.5) is represented by the following figure.

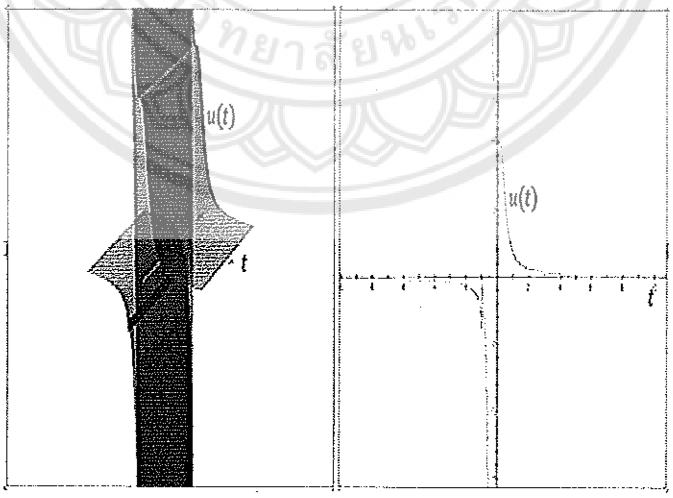

Figure 1: Approximate solution of (4.5)

For the justification of the approximate solution, from (4.5) with (4.6), we arrive at

$$u(t) = \frac{t^2+1}{t^3+0.1} + \frac{1}{3(t+1)} \int_0^1 \frac{s^2(s+1)(s^3+0.1)\,ds}{s^4+2.02733\,s^3+s^2+1.1s+1.102733}; \quad t \in [0,1]. \tag{4.7}$$

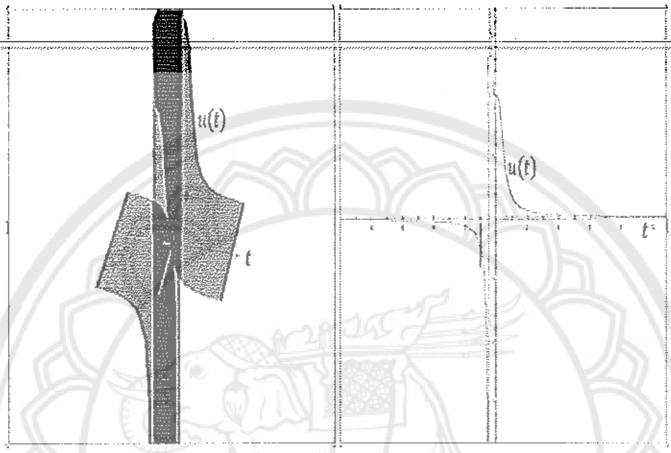

Figure 2: Plot of equality (4.7)

From ***Figure*** *1 and 2, one can easily deduce that the plot of approximate solution with purple surface is almost coincide with the value of u(t) with dark blue surface(see* ***Figure 2****). Hence,* ***Figure 1*** *and 2 confirm the validity of the approximate solution.*

The error between the approximate solution and the value of u(t) is given by the following figures.

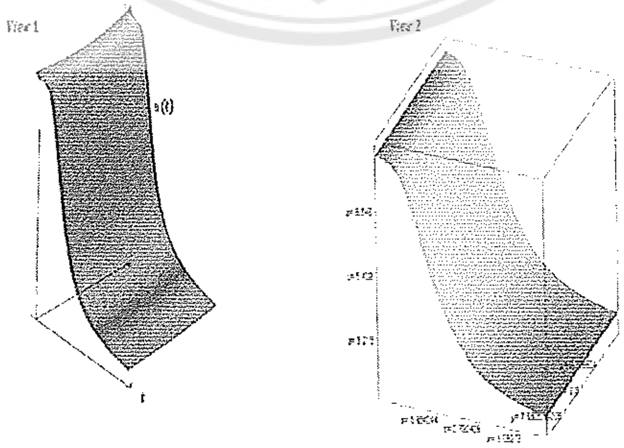

Figure 3: View 1 and view 2 for error function

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