สัญญาเลขที่ R2561B081

รายงานวิจัยฉบับสมบูรณ์

โครงการ : ทฤษฎีบทจุดตรึงสำหรับการส่ง แอลฟา-อีตา-ไซ-เจรจาจตีชนิดคอนแทรกทีฟวางนัยทั่วไปในปริภูมิบีเมตริกบางส่วน

Some fixed point theorems for generalized alpha-eta-psi-Garaghty contractive type mappings in partial b-metric spaces

ผู้วิจัย

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สนับสนุนโดยมหาวิทยาลัยนเรศวร จากงบประมาณแผ่นดินมหาวิทยาลัยนเรศวร ประจำปี 2561

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หน้า

ABSTRACT

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In this project, we introduce a notion of generalized contractions in the setting of partial rectangular metric spaces. The existence of fixed point theorems for generalized contractions with triangular alpha-orbital admissible mappings with respect to eta in the complete partial rectangular metric spaces is proven. Moreover, we also give the example for supporting our main result.

Keywords : Partial rectangular metric spaces, triangular alpha-orbital admissible mappings with respect to eta, alpha-orbital attractive mappings with respect to eta.

บทคัดย่อ

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ชื่อโครงการ : ทฤษฎีบทจุดตรึงสำหรับการส่งแอลฟา–อีตา–ไซ–เจราจตีชนิดคอนแทรกทีฟวางนัยทั่วไปในปริภูมิบีเมตริกบางส่วน

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ในโครงการนี้ ผู้วิจัยได้แนะนำคอนแทรกทีฟวางนัยทั่วไปในปริภูมิเมตริกเรแทนกูลาร์บางส่วน และผู้วิจัยได้พิสูจน์การมีจริงของทฤษฎีบทจุดตรึงสำหรับการส่งคอนแทรกทีฟวางนัยทั่วไปดังกล่าวกับการส่งเรกแทนกูลาร์ แอลฟา–ออบิทอล แอดมิสสิเบิลซึ่งสอดคล้องกับอีตาในปริภูมิเมตริกเรกแทนกูลาร์บางส่วนบริบูรณ์ นอกจากนี้ผู้วิจัยยังได้ยกตัวอย่างเพื่อสนับสนุนผลลัพธ์หลัก

คำสำคัญ : ปริภูมิเมตริกเรแทนกูลาร์บางส่วน, การส่งเรกแทนกูลาร์ แอลฟา–ออบิทอล แอดมิสสิเบิลซึ่งสอดคล้องกับอีตา, การส่งแอทแทรกทีฟ แอลฟา–ออบิทอล ซึ่งสอดคล้องกับอีตา

CHAPTER I

EXECUTIVE SUMMARY

In 2000, Branciari presented a class of generalized (rectangular) metric spaces and proved the interesting topological properties in such spaces. The author also assured the Banach contraction principle in the setting of complete rectangular metric spaces. After that, many authors extended and improved the existence of fixed point theorems in complete rectangular metric spaces.

Recently, Arshad et al. extended the results proved by Jleli et al. in the setting of complete rectangular metric spaces. On the other hand, Matthew presented the concept of partial metric spaces as a part of the study of denotational semantics of data flow network. In this space, the usual metric is replaced by a partial metric with an interesting property that the self-distance of any point of a space may not be zero. Later on, Shukla introduced the partial rectangular metric spaces as a generalization of the concept of rectangular metric spaces and extended the concept of partial metric spaces.

Let X be a nonempty set. We say that a mapping $d : X \times X \to \mathbb{R}$ is a Branciari metric on X if d satisfies the following:

(d1) $0 \leq d(x, y)$, for all $x, y \in X$;

(d2) $d(x, y) = 0$ if and only if $x = y$;

(d3) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(d4) $d(x, y) \leq d(x, w) + d(w, z) + d(z, y)$, for all $x, y \in X$ and for all distinct points $w, z \in X \backslash \{x, y\}$.

If d is a Branciari metric on X, then a pair (X, d) is called a Branciari metric space (or for short BMS). As mentioned before, Branciari metric spaces are also called rectangular metric spaces in the literature. A sequence $\{x_n\}$ in X converges to $x \in X$ if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq n_0$. A sequence $\{x_n\}$ is called a Cauchy sequence if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq n_0$. A rectangular metric space (X, d) is called complete if every Cauchy sequence in X converges in X.

Shukla introduced a concept of the partial rectangular metric spaces as the following:

Let X be a nonempty set. We say that a mapping $p : X \times X \to \mathbb{R}$ is a partial rectangular metric on X if p satisfies the following:

(p1) $p(x, y) \geq 0$, for all $x, y \in X$;

(p2) $x = y$ if and only if $p(x, y) = p(x, x) = p(y, y)$, for all $x, y \in X$;

(p3) $p(x, x) \leq p(x, y)$, for all $x, y \in X$;

(p4) $p(x, y) = p(y, x)$, for all $x, y \in X$;

(p5) $p(x, y) \leq p(x, w) + p(w, z) + p(z, y) - p(w, w) - p(z, z)$, for all $x, y \in X$ and for all distinct points $w, z \in X \backslash \{x, y\}$.

If p is a partial rectangular metric on X, then a pair (X, p) is called a partial rectangular metric space.

In 2016, Chuadchawna introduced the notion of triangular α-orbital admissible mappings with respect to η and proved the key lemma which will be used for proving our main results.

Let $T : X \to X$ be a mapping and $\alpha, \eta : X \times X \to [0, \infty)$ be functions. Then T is said to be α-orbital admissible with respect to η if for all $x \in X$,

$$\alpha(x, Tx) \geq \eta(x, Tx) \quad \text{implies} \quad \alpha(Tx, T^2x) \geq \eta(Tx, T^2x).$$

Let $T : X \to X$ be a mapping and $\alpha, \eta : X \times X \to [0, \infty)$ be functions. Then T is said to be triangular α-orbital admissible with respect to η if

(T1) T is α-orbital admissible with respect to η;

(T2) for all $x, y \in X$, $\alpha(x, y) \geq \eta(x, y)$ and $\alpha(y, Ty) \geq \eta(y, Ty)$ imply

$$\alpha(x, Ty) \geq \eta(x, Ty).$$

In this project, we introduce a notion of generalized contractions in the setting of partial rectangular metric spaces. The existence of fixed point theorems for generalized contractions with triangular α-orbital admissible mappings with respect to η in the complete partial rectangular metric spaces is proven. Moreover, we also give the example for supporting our main result.

CHAPTER II

CONTENT OF RESEARCH

In this project, we obtain one publication that published in Journal of Computational Analysis and Applications as the following:

Suparat Baiya and **Anchalee Kaewcharoen**, Generalized contractions with triangular -orbital admissible mappings with respect to on partial rectangular metric spaces, Journal of Computational Analysis and Applications, vol.26 no.1 (2019), 91-109. SJR: Q3

1. **Theorem** : Let (X,p) be a partial rectangular metric space and $\{x_n\}$ be a sequence in (X,p) such that $p(x_n,x) \to p(x,x)$ as $n \to \infty$ for some $x \in X$, $p(x,x) = 0$ and $\lim_{n\to\infty} p(x_n,x_{n+1}) = 0$. Then $p(x_n,y) \to p(x,y)$ as $n \to \infty$ for all $y \in X$.

2. **Theorem** : Let (X,p) be a complete partial rectangular metric space, $T : X \to X$ be a mapping and let $\alpha, \eta : X \times X \to [0,\infty)$ be functions. Suppose that the following conditions hold :
(1) there exist $\theta \in \Theta$ and $\lambda \in (0,1)$ such that for all $x,y \in X$,
$$p(Tx,Ty) > 0 \text{ and } \alpha(x,y) \geq \eta(x,y) \text{ imply } \theta(p(Tx,Ty)) \leq [\theta(R(x,y))]^{\lambda}, \quad (1)$$
where
$$R(x,y) = \max\left\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Tx)p(y,Ty)}{1+p(x,y)}\right\};$$
(2) there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq \eta(x_1,Tx_1)$;
(3) T is a triangular α-orbital admissible mapping with respect to η;
(4) if $\{T^n x_1\}$ is a sequence in X such that $\alpha(T^n x_1, T^{n+1}x_1) \geq \eta(T^n x_1, T^{n+1}x_1)$ for all $n \in \mathbb{N}$ and $T^n x_1 \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{T^{n(k)}x_1\}$ of $\{T^n x_1\}$ such that $\alpha(T^{n(k)}x_1, x) \geq \eta(T^{n(k)}x_1, x)$ for all $k \in \mathbb{N}$;
(5) θ is continuous;
(6) if z is a periodic point T, then $\alpha(z,Tz) \geq \eta(z,Tz)$.
Then T has a fixed point.

3. **Example** :*Let $X = \{0,1,2,3,4,5\}$ and define $p : X \times X \to [0,+\infty)$ such that*
$$p(x,y) = \begin{cases} x & \text{if } x = y; \\ \frac{2x+y}{2} & \text{if } x,y \in \{0,1,2\},\ x \neq y; \\ \frac{2+x+2y}{2} & \text{otherwise.} \end{cases}$$
Then (X,p) is a complete partial rectangular metric space. Since, for all $x \in X$ and $x > 0$, then we have $p(x,x) = x > 0$. Therefore (X,p) is not

a rectangular metric space.
Define a mapping $T : X \to X$ *by*

$$T0 = T1 = T4 = 0, T2 = T3 = 2, \text{ and } T5 = 4.$$

We can see that 0 *and* 2 *are periodic points of* T. *Let* $\alpha, \eta : X \times X \to [0, +\infty)$ *be functions defined by*

$$\alpha(x,y) = \begin{cases} 1 \text{ if } x, y \in \{0,1,2,3\}; \\ 0 \text{ otherwise.} \end{cases}$$

$$\eta(x,y) = \begin{cases} \frac{1}{2} \text{ if } x, y \in \{0,1,2,3\}; \\ 1 \text{ otherwise.} \end{cases}$$

Also define $\theta : (0,\infty) \to (1,\infty)$ *by* $\theta(t) = e^{\sqrt{t}}$. *We next illustrate that all conditions in Theorem 1 hold. Taking* $x_1 = 1$, *we have* $\alpha(1, T1) = \alpha(1,0) = 1 \geq \frac{1}{2} = \eta(1,0) = \eta(1,T1)$. *Next, we prove that* T *is* α*-orbital admissible with respect to* η. *Let* $\alpha(x, Tx) \geq \eta(x, Tx)$. *Thus* $x, Tx \in \{0,1,2,3\}$. *By the definitions of* a, η, *we obtain that*

$$\alpha(T0, T^2 0) = \alpha(0,0) = 1 \geq \frac{1}{2} = \eta(0,0) = \eta(T0, T^2 0),$$

$$\alpha(T1, T^2 1) = \alpha(0,0) = 1 \geq \frac{1}{2} = \eta(0,0) = \eta(T1, T^2 1),$$

$$\alpha(T2, T^2 2) = \alpha(2,2) = 1 \geq \frac{1}{2} = \eta(2,2) = \eta(T2, T^2 2),$$

$$\alpha(T3, T^2 3) = \alpha(2,2) = 1 \geq \frac{1}{2} = \eta(2,2) = \eta(T3, T^2 3).$$

It follows that T *is* α*-orbital admissible with respect to* η. *Let* $\alpha(x,y) \geq \eta(x,y)$ *and* $\alpha(y, Ty) \geq \eta(y, Ty)$. *By definitions of* α, η, *we have* $x, y, Ty \in \{0,1,2,3\}$. *This yields*

$$\alpha(0,0) \geq \eta(0,0) \text{ and } \alpha(0,T0) \geq \eta(0,T0) \text{ imply } \alpha(0,T0) \geq \eta(0,T0),$$

$$\alpha(0,1) \geq \eta(0,1) \text{ and } \alpha(1,T1) \geq \eta(1,T1) \text{ imply } \alpha(0,T1) \geq \eta(0,T1),$$

$$\alpha(0,2) \geq \eta(0,2) \text{ and } \alpha(2,T2) \geq \eta(2,T2) \text{ imply } \alpha(0,T2) \geq \eta(0,T2),$$

$$\alpha(0,3) \geq \eta(0,3) \text{ and } \alpha(3,T3) \geq \eta(3,T3) \text{ imply } \alpha(0,T3) \geq \eta(0,T3),$$

$$\alpha(1,0) \geq \eta(1,0) \text{ and } \alpha(0,T0) \geq \eta(0,T0) \text{ imply } \alpha(1,T0) \geq \eta(1,T0),$$

$$\alpha(1,1) \geq \eta(1,1) \text{ and } \alpha(1,T1) \geq \eta(1,T1) \text{ imply } \alpha(1,T1) \geq \eta(1,T1),$$

$$\alpha(1,2) \geq \eta(1,2) \text{ and } \alpha(2,T2) \geq \eta(2,T2) \text{ imply } \alpha(1,T2) \geq \eta(1,T2),$$

$$\alpha(1,3) \geq \eta(1,3) \text{ and } \alpha(3,T3) \geq \eta(3,T3) \text{ imply } \alpha(1,T3) \geq \eta(1,T3),$$

$$\alpha(2,0) \geq \eta(2,0) \text{ and } \alpha(0,T0) \geq \eta(0,T0) \text{ imply } \alpha(2,T0) \geq \eta(2,T0),$$

$$\alpha(2,1) \geq \eta(2,1) \text{ and } \alpha(1,T1) \geq \eta(1,T1) \text{ imply } \alpha(2,T1) \geq \eta(2,T1),$$

$$\alpha(2,2) \geq \eta(2,2) \text{ and } \alpha(2,T2) \geq \eta(2,T2) \text{ imply } \alpha(2,T2) \geq \eta(2,T2),$$
$$\alpha(2,3) \geq \eta(2,3) \text{ and } \alpha(3,T3) \geq \eta(3,T3) \text{ imply } \alpha(2,T3) \geq \eta(2,T3).$$
$$\alpha(3,0) \geq \eta(3,0) \text{ and } \alpha(0,T0) \geq \eta(0,T0) \text{ imply } \alpha(3,T0) \geq \eta(3,T0),$$
$$\alpha(3,1) \geq \eta(3,1) \text{ and } \alpha(1,T1) \geq \eta(1,T1) \text{ imply } \alpha(3,T1) \geq \eta(3,T1),$$
$$\alpha(3,2) \geq \eta(3,2) \text{ and } \alpha(2,T2) \geq \eta(2,T2) \text{ imply } \alpha(3,T2) \geq \eta(3,T2),$$
$$\alpha(3,3) \geq \eta(3,3) \text{ and } \alpha(3,T3) \geq \eta(3,T3) \text{ imply } \alpha(3,T3) \geq \eta(3,T3).$$

This implies that T is triangular α-orbital admissible with respect to η. Afterward, let $\{T^n x_1\}$ be a sequence such that $\alpha(T^n x_1, T^{n+1}x_1) \geq \eta(T^n x_1, T^{n+1}x_1)$ for all $n \in \mathbb{N}$ and $T^n x_1 \to x$ as $n \to \infty$. By the definitions of α, η for each $n \in \mathbb{N}$, we get $T^n x_1 \in \{0,1,2,3\}$. We obtain that $x \in \{0,1,2,3\}$. Thus we have $\alpha(T^n x_1, x) \geq \eta(T^n x_1, x)$ for each $n \in \mathbb{N}$. Let $x, y \in X$ be such that $p(Tx,Ty) > 0$. We could observe that if $x, y \in \{0,1,4\}$, then $Tx = Ty = 0$. This implies that $p(Tx,Ty) = 0$. So we consider the following cases:

- $x \in \{0,1,4\}$ *and* $y \in \{2,3\}$ *or*
- $x \in \{0,1,4\}$ *and* $y = 5$ *or*
- $x = \{2,3\}$ *and* $y = 5$.

If $x = 4$ and $y \in \{2,3\}$ or $x \in \{0,1,4\}$ and $y = 5$ or $x = \{2,3\}$ and $y = 5$, then we have $\alpha(x,y) \not\geq \eta(x,y)$. We divide the proof into four cases as follows:

(1) If $(x,y) \in \{(0,2),(2,0)\}$, then

$$\begin{aligned} R(0,2) &= \max\left\{p(0,2), p(0,0), p(2,2), \frac{p(0,0)p(2,2)}{1+p(0,2)}\right\} \\ &= \max\left\{1,0,2,0\right\} \\ &= 2. \end{aligned}$$

This implies that

$$\psi(p(T0,T2)) = \psi(p(0,2)) = \psi(1) = e^{\sqrt{1}} \leq [e^{\sqrt{2}}]^{0.71} = [\psi(2)]^{0.71} \leq [\psi(R(0,2))]^{0.71}.$$

Therefore

$$\psi(p(T0,T2)) \leq [\psi(R(0,2))]^{0.71}.$$

Since $p(x,y) = p(y,x)$ for all $x, y \in X$, we also obtain that

$$\psi(p(T2,T0)) \leq [\psi(R(2,0))]^{0.71}.$$

(2) If $(x,y) \in \{(1,2),(2,1)\}$, then

$$\begin{aligned} R(1,2) &= \max\left\{p(1,2), p(1,0), p(2,2), \frac{p(1,0)p(2,2)}{1+p(1,2)}\right\} \\ &= \max\left\{2,1,2,\frac{2}{3}\right\} \\ &= 2. \end{aligned}$$

This implies that

$$\psi(p(T1,T2)) = \psi(p(0,2)) = \psi(1) = e^{\sqrt{1}} \le [e^{\sqrt{2}}]^{0.71} = [\psi(2)]^{0.71} \le [\psi(R(1,2))]^{0.71}.$$

Therefore

$$\psi(p(T1,T2)) \le [\psi(R(1,2))]^{0.71}.$$

Since $p(x,y) = p(y,x)$ *for all* $x,y \in X$, *we also obtain that*

$$\psi(p(T2,T1)) \le [\psi(R(2,1))]^{0.71}.$$

(3) If $(x,y) \in \{(0,3),(3,0)\}$, *then*

$$\begin{aligned} R(0,3) &= \max\left\{p(0,3), p(0,0), p(3,2), \frac{p(0,0)p(3,2)}{1+p(0,3)}\right\} \\ &= \max\left\{4, 0, \frac{9}{2}, 0\right\} \\ &= \frac{9}{2}. \end{aligned}$$

This implies that

$$\psi(p(T0,T3)) = \psi(p(0,2)) = \psi(1) = e^{\sqrt{1}} \le [e^{\sqrt{\frac{9}{2}}}]^{0.5} = [\psi(\frac{9}{2})]^{0.5} \le [\psi(R(0,3))]^{0.5}.$$

Therefore

$$\psi(p(T0,T3)) \le [\psi(R(0,3))]^{0.5}.$$

Since $p(x,y) = p(y,x)$ *for all* $x,y \in X$, *we also obtain that*

$$\psi(p(T3,T0)) \le [\psi(R(3,0))]^{0.5}.$$

(4) If $(x,y) \in \{(1,3),(3,1)\}$, *then*

$$\begin{aligned} R(1,3) &= \max\left\{p(1,3), p(1,0), p(3,2), \frac{p(1,0)p(3,2)}{1+p(1,3)}\right\} \\ &= \max\left\{\frac{9}{2}, 1, \frac{9}{2}, \frac{9}{11}\right\} \\ &= \frac{9}{2}. \end{aligned}$$

This implies that

$$\psi(p(T1,T3)) = \psi(p(0,2)) = \psi(1) = e^{\sqrt{1}} \le [e^{\sqrt{\frac{9}{2}}}]^{0.5} = [\psi(\frac{9}{2})]^{0.5} \le [\psi(R(1,3))]^{0.5}.$$

Therefore

$$\psi(p(T1,T3)) \le [\psi(R(1,3))]^{0.5}.$$

Since $p(x,y) = p(y,x)$ *for all* $x,y \in X$, *we also obtain that*

$$\psi(p(T3,T1)) \le [\psi(R(3,1))]^{0.5}.$$

It follows that if $x,y \in X$, $p(Tx,Ty) > 0$ *and* $\alpha(x,y) \ge \eta(x,y)$, *Then* $\psi(p(Tx,Ty)) \le [\psi(R(x,y))]^{\lambda}$. *Hence all assumptions in the main result are satisfied and thus* T *has a fixed point which are* $x = 0$ *and* $x = 2$.

4. **Theorem** : Let (X,p) be a complete partial rectangular metric space, $T: X \to X$ be a mapping and let $\alpha, \eta : X \times X \to [0,\infty)$ be functions. Suppose that the following conditions hold :
(1) there exist $\theta \in \Theta$ and $\lambda \in (0,1)$ such that for all $x, y \in X$,

$$p(Tx,Ty) > 0 \text{ and } \alpha(x,y) \geq \eta(x,y) \text{ imply } \theta(p(Tx,Ty)) \leq [\theta(R(x,y))]^{\lambda}, \tag{2}$$

where

$$R(x,y) = \max\left\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Tx)p(y,Ty)}{1+p(x,y)}\right\};$$

(2) there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq \eta(x_1,Tx_1)$ and $\alpha(x_1,T^2x_1) \geq \eta(x_1,T^2x_1)$;
(3) T is an α-orbital admissible mapping with respect to η;
(4) T is an α-orbital attractive mapping with respect to η;
(5) θ is continuous;
(6) if z is a periodic point of T, then $\alpha(z,Tz) \geq \eta(z,Tz)$.
Then T has a fixed point.

CHAPTER III

OUTPUT

ผลลัพธ์จากโครงการวิจัยที่ได้รับทุนจากงบประมาณแผ่นดินมหาวิทยาลัยนเรศวร ประจำปี 2561

1. ผลงานวิจัยตีพิมพ์ในวารสารวิชาการนานาชาติ

Suparat Baiya and **Anchalee Kaewcharoen,** Generalized contractions with triangular α-orbital admissible mappings with respect to η on partial rectangular metric spaces, Journal of Computational Analysis and Applications, vol.26 no.1 (2019), 91–109. SJR: Q3

2. การนำผลงานวิจัยไปใช้ประโยชน์

มีการนำไปใช้ประโยชน์ทั้งเชิงวิชาการและเชิงสาธารณะโดยทำให้มีการพัฒนาการเรียนการสอน รวมทั้งมีการสร้างเครือข่ายความร่วมมือในการทำวิจัย

ภาคผนวก

Generalized contractions with triangular α-orbital admissible mappings with respect to η on partial rectangular metric spaces

Suparat Baiya

Anchalee Kaewcharoen

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Subject Area and Category	Mathematics Computational Mathematics	
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Scope

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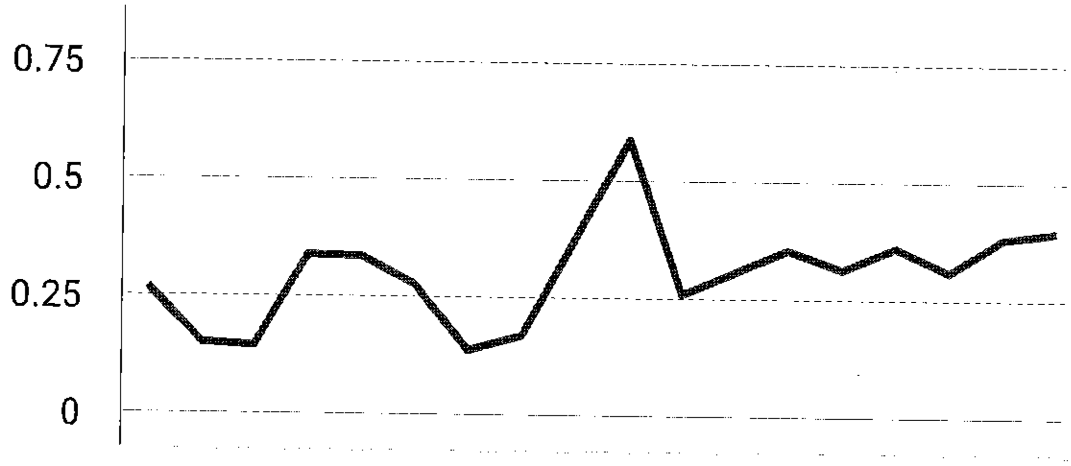

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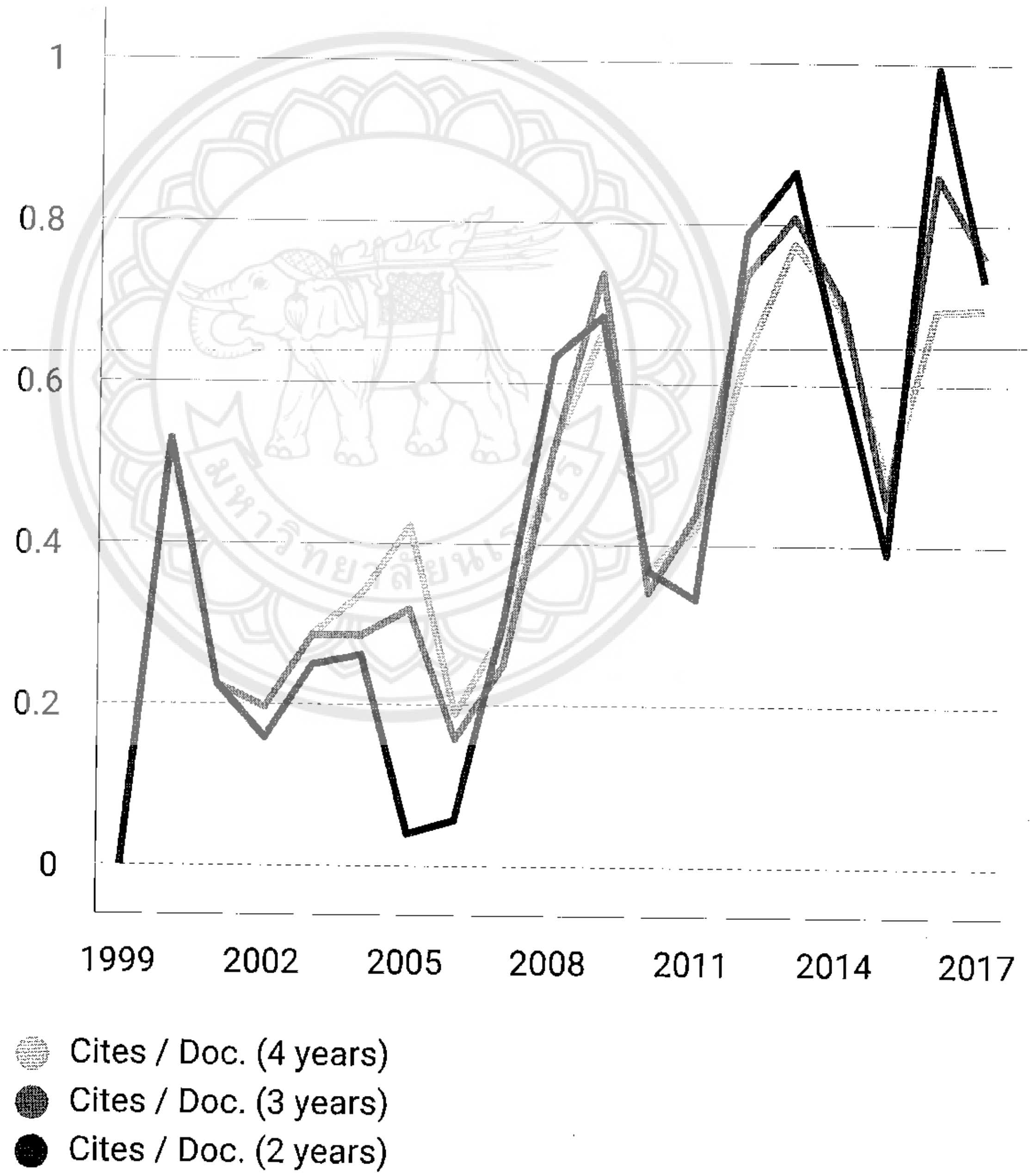

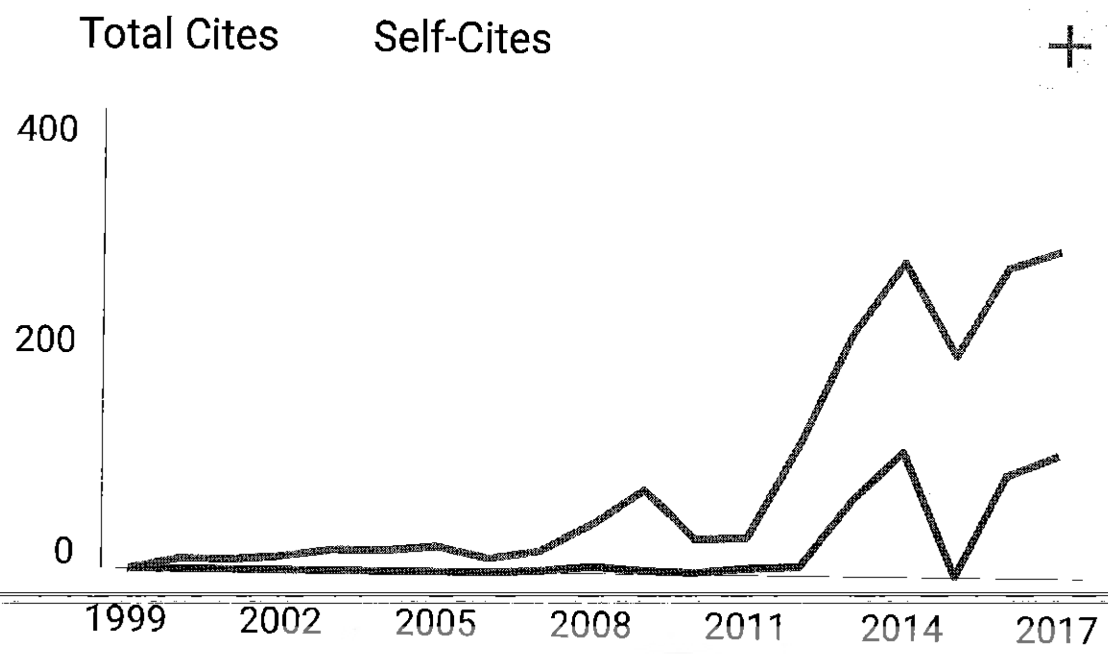

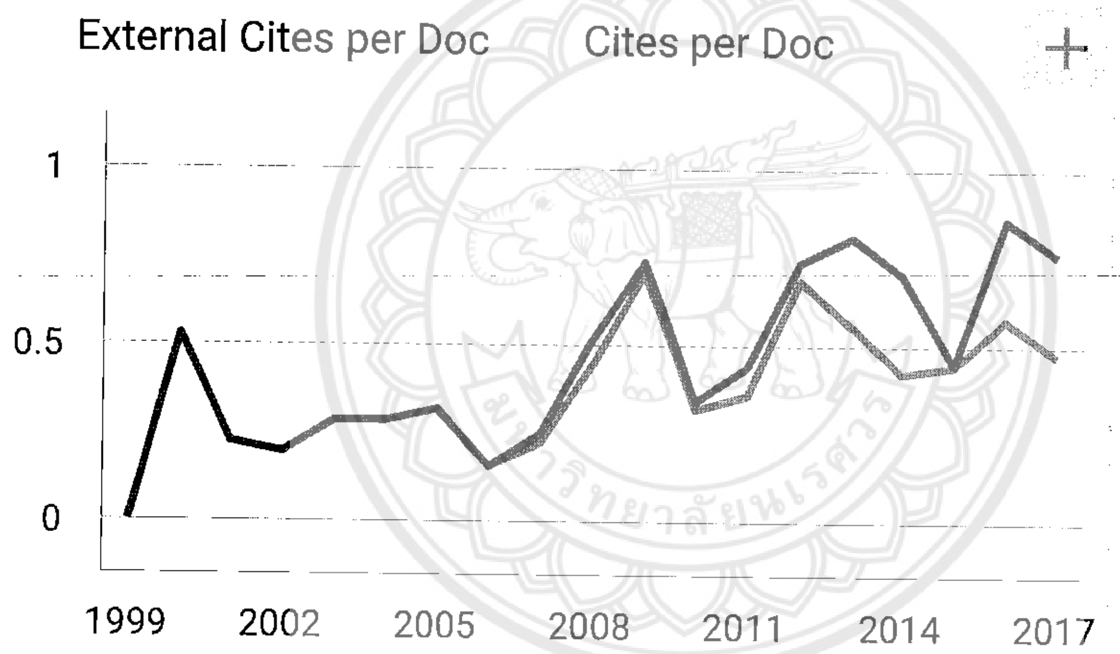

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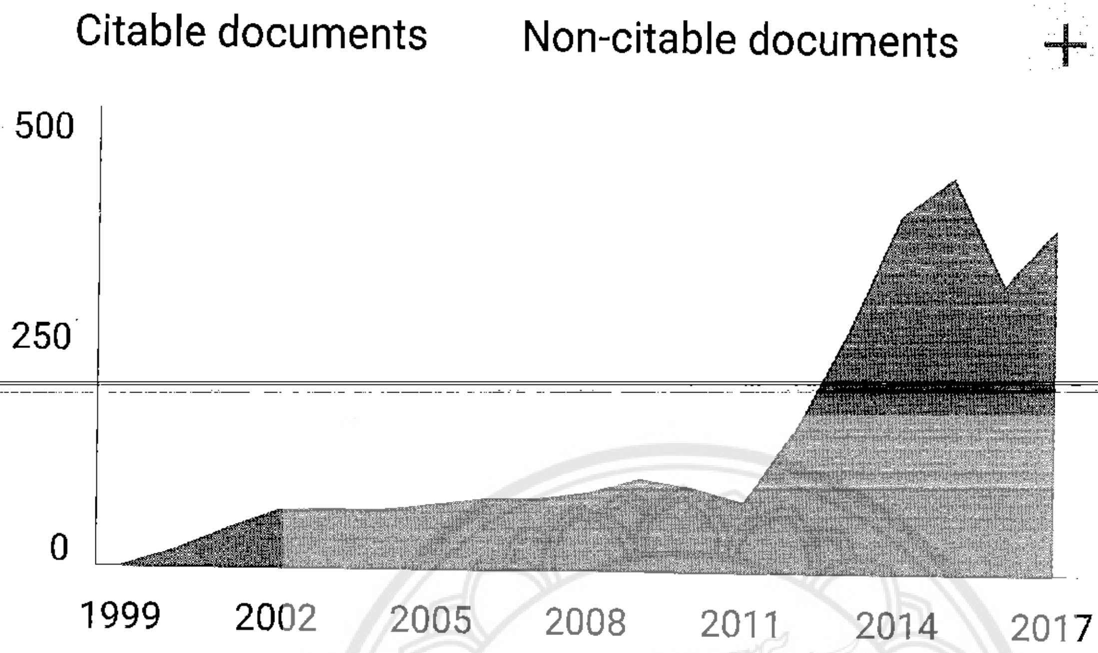

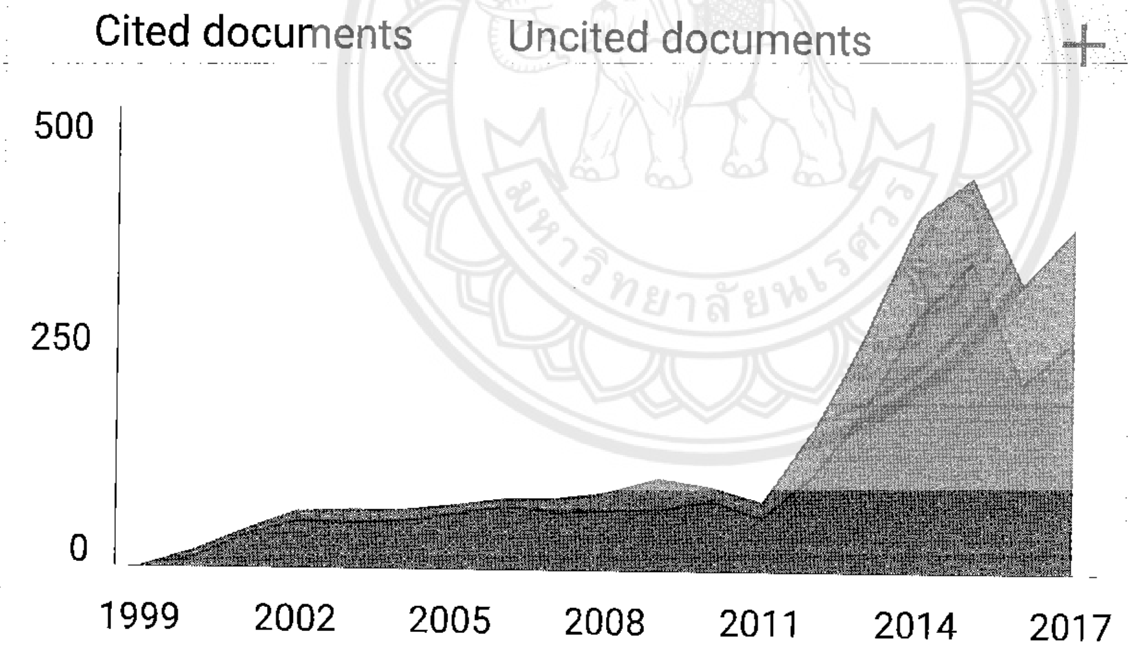

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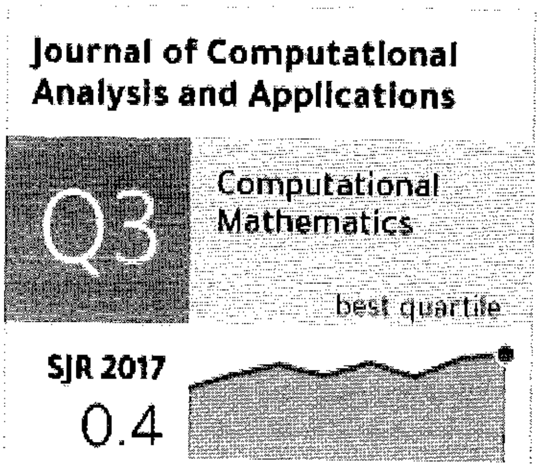

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Generalized contractions with triangular α-orbital admissible mappings with respect to η on partial rectangular metric spaces

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Abstract

In this paper, we introduce a notion of generalized contractions in the setting of partial rectangular metric spaces. The existence of fixed point theorems for generalized contractions with triangular α-orbital admissible mappings with respect to η in the complete partial rectangular metric spaces is proven. Moreover, we also give the example for supporting our main result.

Keywords: Partial rectangular metric spaces, triangular α-orbital admissible mappings with respect to η, α-orbital attractive mappings with respect to η.

1 Introduction and preliminaries

In 2000, Branciari [2] presented a class of generalized (rectangular) metric spaces and proved the interesting topological properties in such spaces. The author [2] also assured the Banach contraction principle in the setting of complete rectangular metric spaces. After that, many authors extended and improved the existence of fixed point theorems in complete rectangular metric spaces, see [4, 5, 6, 7, 8, 9, 10, 11, 15] and the references contained therein.

Recently, Arshad et al. [1] extended the results proved by Jleli et al. [6, 7] in the setting of complete rectangular metric spaces. On the other hand, Matthew [12] presented the concept of partial metric spaces as a part of the study of denotational semantics of data flow network. In this space, the usual metric is replaced by a partial metric with an interesting property that the self-distance of

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any point of a space may not be zero. Later on, Shukla [16] introduced the partial rectangular metric spaces as a generalization of the concept of rectangular metric spaces and extended the concept of partial metric spaces.

In this paper, we introduce a notion of generalized contractions appeared in [1] in the setting of partial rectangular metric spaces. The existence of fixed point theorems for generalized contractions with triangular α-orbital admissible mappings with respect to η in the complete partial rectangular metric spaces is proven. Moreover, we also give the example for supporting our main result.

We now recall some definitions, lemmas and propositions that will be used in the sequel.

Definition 1.1 [2] Let X be a nonempty set. We say that a mapping $d : X \times X \to \mathbb{R}$ is a Branciari metric on X if d satisfies the following:

(d1) $0 \leq d(x,y)$, for all $x, y \in X$;

(d2) $d(x,y) = 0$ if and only if $x = y$;

(d3) $d(x,y) = d(y,x)$, for all $x, y \in X$;

(d4) $d(x,y) \leq d(x,w) + d(w,z) + d(z,y)$, for all $x, y \in X$ and for all distinct points $w, z \in X\backslash\{x,y\}$.

If d is a Branciari metric on X, then a pair (X,d) is called a Branciari metric space (or for short BMS). As mentioned before, Branciari metric spaces are also called rectangular metric spaces in the literature. A sequence $\{x_n\}$ in X converges to $x \in X$ if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq n_0$. A sequence $\{x_n\}$ is called a Cauchy sequence if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq n_0$. A rectangular metric space (X,d) is called complete if every Cauchy sequence in X converges in X.

Shukla [16] introduced a concept of the partial rectangular metric spaces as the following:

Definition 1.2 [16] Let X be a nonempty set. We say that a mapping $p : X \times X \to \mathbb{R}$ is a partial rectangular metric on X if p satisfies the following:

(p1) $p(x,y) \geq 0$, for all $x, y \in X$;

(p2) $x = y$ if and only if $p(x,y) = p(x,x) = p(y,y)$, for all $x, y \in X$;

(p3) $p(x,x) \leq p(x,y)$, for all $x, y \in X$;

(p4) $p(x,y) = p(y,x)$, for all $x, y \in X$;

(p5) $p(x,y) \leq p(x,w) + p(w,z) + p(z,y) - p(w,w) - p(z,z)$, for all $x, y \in X$ and for all distinct points $w, z \in X\backslash\{x,y\}$.

If p is a partial rectangular metric on X, then a pair (X,p) is called a partial rectangular metric space.

Remark 1.3 [16] In a partial rectangular metric space (X,p), if $x, y \in X$ and $p(x,y) = 0$, then $x = y$ but the converse may not be true.

Remark 1.4 [16] It is clear that every rectangular metric space is a partial rectangular metric space with zero self-distance. However, the converse of this fact need not hold.

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Example 1.5 [16] Let $X = [0, d], \alpha \geq d \geq 3$ and define a mapping $p : X \times X \to [0, \infty)$ by

$$p(x, y) = \begin{cases} x & \text{if } x = y; \\ \frac{3\alpha + x + y}{2} & \text{if } x, y \in \{1, 2\},\ x \neq y; \\ \frac{\alpha + x + y}{2} & \text{otherwise.} \end{cases}$$

Then (X, p) is a partial rectangular metric space but it is not a rectangular metric space. Moreover, (X, p) is not a partial metric space.

Proposition 1.6 [16] *For each partial rectangular metric space (X, p), the pair (X, d_p) is a rectangular metric space where*

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

for all $x, y \in X$.

Definition 1.7 [16] Let (X, p) be a partial rectangular metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then,

(i) the sequence $\{x_n\}$ is said to converges to $x \in X$ if $\lim_{n\to\infty} p(x_n, x) = p(x, x)$;

(ii) the sequence $\{x_n\}$ is said to be a Cauchy sequence in (X, p) if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite;

(iii) (X, p) is said to be a complete partial rectangular metric space if for every Cauchy sequence $\{x_n\}$ in X, there exists $x \in X$ such that

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x, x).$$

Lemma 1.8 [16] *Let (X, p) be a partial rectangular metric space and let $\{x_n\}$ be a sequence in X. Then $\lim_{n\to\infty} d_p(x_n, x) = 0$ if and only if $\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, x_n) = p(x, x)$.*

Lemma 1.9 [16] *Let (X, p) be a partial rectangular metric space and let $\{x_n\}$ be a sequence in X. Then the sequence $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in (X, d_p).*

Lemma 1.10 [16] *A partial rectangular metric space (X, p) is complete if and only if a rectangular metric space (X, d_p) is complete.*

In 2014, Popescu [13] introduced the definitions of α-orbital admissible mappings and triangular α-orbital admissible mappings including α-orbital attractive mappings.

Definition 1.11 [13] Let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Then T is said to be α-orbital admissible if for all $x \in X$,

$$\alpha(x, Tx) \geq 1 \text{ implies } \alpha(Tx, T^2x) \geq 1.$$

Definition 1.12 [13] Let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Then T is said to be triangular α-orbital admissible if:

(T3) T is α-orbital admissible;

(T4) for all $x, y \in X, \alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1$ imply $\alpha(x, Ty) \geq 1$.

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Definition 1.13 [13] Let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Then T is said to be α-orbital attractive if for all $x \in X$,

$\alpha(x, Tx) \geq 1$ implies $\alpha(x, y) \geq 1$ or $\alpha(y, Tx) \geq 1$ for all $y \in X$.

We denote by Θ the set of all functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

(Θ1) θ is non-decreasing;

(Θ2) for each sequence $\{t_n\} \subset (0, \infty)$,

$$\lim_{n\to\infty} \theta(t_n) = 1 \text{ if and only if } \lim_{n\to\infty} t_n = 0^+;$$

(Θ3) there exist $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = \ell$.

Example 1.14 [6] The following functions $\theta : (0, \infty) \to (1, \infty)$ are in Θ:

(1) $\theta(t) = e^{\sqrt{t}}$;

(2) $\theta(t) = e^{\sqrt{te^t}}$;

(3) $\theta(t) = 2 - \frac{2}{\pi} \arctan(\frac{1}{t^\gamma})$ where $0 < \gamma < 1$.

Very recently Jleli et al. [6, 7] established the following generalization of the Banach fixed point theorem in the setting of complete rectangular metric spaces.

Theorem 1.15 [6] *Let (X, d) be a complete rectangular metric space and $T : X \to X$ be a mapping. Suppose that there exist $\theta \in \Theta$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \neq 0 \text{ implies } \theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^\lambda.$$

Then T has a unique fixed point.

Theorem 1.16 [7] *Let (X, d) be a complete rectangular metric space and $T : X \to X$ be a mapping. Suppose that there exist $\theta \in \Theta$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \neq 0 \text{ implies } \theta(d(Tx, Ty)) \leq [\theta(M(x, y))]^\lambda,$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

Then T has a unique fixed point.

Later, Arshad et al. [1] extended the results proved by Jleli et al. [6, 7] by using the concept of triangular α-orbital admissible mappings.

Theorem 1.17 [1] *Let (X, d) be a complete rectangular metric space, $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Suppose that the*

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following conditions hold :
(1) there exist $\theta \in \Theta$ and $\lambda \in (0,1)$ such that for all $x, y \in X$,

$$d(Tx,Ty) \neq 0 \text{ implies } \alpha(x,y) \cdot \theta(d(Tx,Ty)) \leq [\theta(R(x,y))]^{\lambda},$$

where

$$R(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\};$$

(2) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$ and $\alpha(x_1, T^2x_1) \geq 1$;
(3) T is a triangular α-orbital admissible mapping;
(4) if $\{T^n x_1\}$ is a sequence in X such that $\alpha(T^n x_1, T^{n+1}x_1) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{T^{n(k)}x_1\}$ of $\{T^n x_1\}$ such that $\alpha(T^{n(k)}x_1, x) \geq 1$ for all $k \in \mathbb{N}$;
(5) θ is continuous;
(6) if z is a periodic point T, then $\alpha(z, Tz) \geq 1$.
Then T has a fixed point.

Theorem 1.18 [1] *Let (X,d) be a complete rectangular metric space, $T : X \to X$ be a mapping and $\alpha : X \times X \to [0,\infty)$ be a function. Suppose that the following conditions hold :*
(1) there exist $\theta \in \Theta$ and $\lambda \in (0,1)$ such that for all $x, y \in X$,

$$d(Tx,Ty) \neq 0 \text{ implies } \alpha(x,y) \cdot \theta(d(Tx,Ty)) \leq [\theta(R(x,y))]^{\lambda},$$

where

$$R(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\};$$

(2) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$ and $\alpha(x_1, T^2x_1) \geq 1$;
(3) T is an α-orbital admissible mapping;
(4) T is an α-orbital attractive mapping;
(5) θ is continuous;
(6) if z is a periodic point T, then $\alpha(z, Tz) \geq 1$.
Then T has a fixed point.

In 2016, Chuadchawna [3] introduced the notion of triangular α-orbital admissible mappings with respect to η and proved the key lemma which will be used for proving our main results.

Definition 1.19 [3] Let $T : X \to X$ be a mapping and $\alpha, \eta : X \times X \to [0,\infty)$ be functions. Then T is said to be α-orbital admissible with respect to η if for all $x \in X$,

$$\alpha(x,Tx) \geq \eta(x,Tx) \text{ implies } \alpha(Tx,T^2x) \geq \eta(Tx,T^2x).$$

Definition 1.20 [3] Let $T : X \to X$ be a mapping and $\alpha, \eta : X \times X \to [0,\infty)$ be functions. Then T is said to be triangular α-orbital admissible with respect

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to η if

(T1) T is α-orbital admissible with respect to η;

(T2) for all $x, y \in X, \alpha(x,y) \geq \eta(x,y)$ and $\alpha(y,Ty) \geq \eta(y,Ty)$ imply

$$\alpha(x,Ty) \geq \eta(x,Ty).$$

Remark 1.21 If we suppose that $\eta(x,y) = 1$ for all $x, y \in X$, then Definition 1.19 and Definition 1.20 reduces to Definition 1.11 and Definition 1.12, respectively.

Lemma 1.22 [3] *Let $T : X \to X$ be a triangular α-orbital admissible mapping with respect to η. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n, x_m) \geq \eta(x_n, x_m)$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Definition 1.23 Let $T : X \to X$ be a mapping and $\alpha, \eta : X \times X \to [0, \infty)$ be functions. Then T is said to be α-orbital attractive with respect to η if for all $x \in X$,

$\alpha(x,Tx) \geq \eta(x,Tx)$ implies $\alpha(x,y) \geq \eta(x,y)$ or $\alpha(y,Tx) \geq \eta(y,Tx)$ for all $y \in X$.

2 Main results

We now prove the following lemma needed in proving our result. The idea comes from [10] but the proof is slightly different.

Lemma 2.1 *Let (X,p) be a partial rectangular metric space and $\{x_n\}$ be a sequence in (X,p) such that $p(x_n,x) \to p(x,x)$ as $n \to \infty$ for some $x \in X$, $p(x,x) = 0$ and $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$. Then $p(x_n,y) \to p(x,y)$ as $n \to \infty$ for all $y \in X$.*

Proof. Suppose that $x \neq y$. If $x_n = y$ for arbitrarily large n, then $p(y,x) = p(x,x) = p(y,y)$. Therefore $x = y$. Assume that $y \neq x_n$ for all $n \in \mathbb{N}$. We also suppose that $x_n \neq x$ for infinitely many n. Otherwise, the result is complete. It follows that we may assume that $x_n \neq x_m \neq x$ and $x_n \neq x_m \neq y$ for all $m, n \in \mathbb{N}$ with $m \neq n$. By the partial rectangular inequality, we have

$$\begin{aligned} p(y,x) &\leq p(y,x_n) + p(x_n,x_{n+1}) + p(x_{n+1},x) - p(x_n,x_n) - p(x_{n+1},x_{n+1}) \\ &\leq p(y,x_n) + p(x_n,x_{n+1}) + p(x_{n+1},x) \end{aligned}$$

and

$$\begin{aligned} p(y,x_n) &\leq p(y,x) + p(x,x_{n+1}) + p(x_{n+1},x_n) - p(x,x) - p(x_{n+1},x_{n+1}) \\ &\leq p(y,x) + p(x,x_{n+1}) + p(x_{n+1},x_n). \end{aligned}$$

Since $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$ and taking the limit as $n \to \infty$ in the above inequalities, we have

$$\limsup_n p(y,x_n) \leq p(y,x) \leq \liminf_n p(y,x_n).$$

Hence the proof is complete. ■

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Theorem 2.2 *Let (X,p) be a complete partial rectangular metric space, $T : X \to X$ be a mapping and let $\alpha, \eta : X \times X \to [0,\infty)$ be functions. Suppose that the following conditions hold :*
(1) there exist $\theta \in \Theta$ and $\lambda \in (0,1)$ such that for all $x, y \in X$,

$$p(Tx,Ty) > 0 \text{ and } \alpha(x,y) \geq \eta(x,y) \text{ imply } \theta(p(Tx,Ty)) \leq [\theta(R(x,y))]^{\lambda}, \tag{2.1}$$

where

$$R(x,y) = \max\left\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Tx)p(y,Ty)}{1+p(x,y)}\right\};$$

(2) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$;
(3) T is a triangular α-orbital admissible mapping with respect to η;
(4) if $\{T^n x_1\}$ is a sequence in X such that $\alpha(T^n x_1, T^{n+1}x_1) \geq \eta(T^n x_1, T^{n+1}x_1)$ for all $n \in \mathbb{N}$ and $T^n x_1 \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{T^{n(k)}x_1\}$ of $\{T^n x_1\}$ such that $\alpha(T^{n(k)}x_1, x) \geq \eta(T^{n(k)}x_1, x)$ for all $k \in \mathbb{N}$;
(5) θ is continuous;
(6) if z is a periodic point T, then $\alpha(z,Tz) \geq \eta(z,Tz)$.
Then T has a fixed point.

Proof. By (2), there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. Define the sequence $\{x_n\}$ in X by $x_n = Tx_{n-1} = T^n x_1$ for all $n \in \mathbb{N}$. By Lemma 1.22, we obtain that

$$\alpha(T^n x_1, T^{n+1}x_1) \geq \eta(T^n x_1, T^{n+1}x_1) \quad \text{for all } n \in \mathbb{N}. \tag{2.2}$$

If $T^n x_1 = T^{n+1}x_1$ for some $n \in \mathbb{N}$, then $T^n x_1$ is a fixed point of T. Thus we suppose that $T^n x_1 \neq T^{n+1}x_1$ for all $n \in \mathbb{N}$. That is $p(T^n x_1, T^{n+1}x_1) > 0$. Applying (2.1), for each $n \in \mathbb{N}$, we have

$$\begin{aligned}\theta(p(T^n x_1, T^{n+1}x_1)) &= \theta(p(T(T^{n-1}x_1), T(T^n x_1))) \\ &\leq [\theta(R(T^{n-1}x_1, T^n x_1))]^{\lambda},\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}R(T^{n-1}x_1, T^n x_1) &= \max\Big\{p(T^{n-1}x_1, T^n x_1), p(T^{n-1}x_1, T(T^{n-1}x_1)), p(T^n x_1, T(T^n x_1)), \\ &\quad \frac{p(T^{n-1}x_1, T(T^{n-1}x_1))p(T^n x_1, T(T^n x_1))}{1+p(T^{n-1}x_1, T^n x_1)}\Big\} \\ &= \max\Big\{p(T^{n-1}x_1, T^n x_1), p(T^{n-1}x_1, T^n x_1), p(T^n x_1, T^{n+1}x_1), \\ &\quad \frac{p(T^{n-1}x_1, T^n x_1)p(T^n x_1, T^{n+1}x_1)}{1+p(T^{n-1}x_1, T^n x_1)}\Big\} \\ &= \max\{p(T^{n-1}x_1, T^n x_1), p(T^n x_1, T^{n+1}x_1)\}.\end{aligned}$$

If $R(T^{n-1}x_1, T^n x_1) = p(T^n x_1, T^{n+1}x_1)$. By (2.3), we have

$$\theta(p(T^n x_1, T^{n+1}x_1)) \leq [\theta(p(T^n x_1, T^{n+1}x_1))]^{\lambda}.$$

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This implies that

$$\ln[\theta(p(T^n x_1, T^{n+1}x_1))] \leq \lambda \ln[\theta(p(T^n x_1, T^{n+1}x_1))],$$

which is a contradiction with $\lambda \in (0,1)$. This implies that $R(T^{n-1}x_1, T^n x_1) = p(T^{n-1}x_1, T^n x_1)$ for all $n \in \mathbb{N}$. From (2.3), we obtain that

$$\theta(p(T^n x_1, T^{n+1}x_1)) \leq [\theta(p(T^{n-1}x_1, T^n x_1))]^{\lambda} \quad \text{for all } n \in \mathbb{N}.$$

It follows that

$$1 \leq \theta(p(T^n x_1, T^{n+1}x_1)) \leq \cdots \leq [\theta(p(x_1, Tx_1))]^{\lambda^n} \quad \text{for all } n \in \mathbb{N}. \tag{2.4}$$

Taking the limit as $n \to \infty$ in the above inequality, we obtain that

$$\lim_{n\to\infty} \theta(p(T^n x_1, T^{n+1}x_1)) = 1. \tag{2.5}$$

By using condition (Θ2), we have

$$\lim_{n\to\infty} p(T^n x_1, T^{n+1}x_1) = 0. \tag{2.6}$$

From condition (Θ3), there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(p(T^n x_1, T^{n+1}x_1)) - 1}{[p(T^n x_1, T^{n+1}x_1)]^r} = \ell.$$

Assume that $\ell < \infty$. Let $B = \frac{\ell}{2} > 0$. It follows that there exists $n_0 \in \mathbb{N}$ such that

$$\left|\frac{\theta(p(T^n x_1, T^{n+1}x_1)) - 1}{[p(T^n x_1, T^{n+1}x_1)]^r} - \ell\right| \leq B \quad \text{for all } n \geq n_0.$$

This implies that

$$\frac{\theta(p(T^n x_1, T^{n+1}x_1)) - 1}{[p(T^n x_1, T^{n+1}x_1)]^r} \geq \ell - B = B \quad \text{for all } n \geq n_0.$$

Thus we have

$$n[p(T^n x_1, T^{n+1}x_1)]^r \leq An[\theta(p(T^n x_1, T^{n+1}x_1)) - 1] \quad \text{for all } n \geq n_0,$$

where $A = \frac{1}{B}$. Assume that $\ell = \infty$. Let $B > 0$ be an arbitrary positive number. It follows that there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\theta(p(T^n x_1, T^{n+1}x_1)) - 1}{[p(T^n x_1, T^{n+1}x_1)]^r} \geq B \quad \text{for all } n \geq n_0.$$

This implies that

$$n[p(T^n x_1, T^{n+1}x_1)]^r \leq An[\theta(p(T^n x_1, T^{n+1}x_1)) - 1] \quad \text{for all } n \geq n_0,$$

where $A = \frac{1}{B}$. From the above two cases, there exist $A > 0$ and $n_0 \in N$ such that

$$n[p(T^n x_1, T^{n+1}x_1)]^r \leq An[\theta(p(T^n x_1, T^{n+1}x_1)) - 1] \quad \text{for all } n \geq n_0.$$

Using (2.4), we have

$$n[p(T^n x_1, T^{n+1}x_1)]^r \leq An([\theta(p(x_1, Tx_1))]^{\lambda^n} - 1) \quad \text{for all } n \geq n_0. \tag{2.7}$$

Taking the limit as $n \to \infty$ in the above inequality, we get that

$$\lim_{n\to\infty} n[p(T^n x_1, T^{n+1}x_1)]^r = 0.$$

This implies that there exists $n_1 \in \mathbb{N}$ such that

$$p(T^n x_1, T^{n+1}x_1) \leq \frac{1}{n^{1/r}} \quad \text{for all } n \geq n_1. \tag{2.8}$$

We now prove that T has a periodic point. Suppose that T does not have periodic points. Thus $T^n x_1 \neq T^m x_1$ for all $n, m \in \mathbb{N}$ such that $n \neq m$. Using Lemma 1.22 and (2.1), we get that

$$\begin{aligned}\theta(p(T^n x_1, T^{n+2}x_1)) &= \theta(p(T(T^{n-1}x_1), T(T^{n+1}x_1))) \\ &\leq [\theta(R(T^{n-1}x_1, T^{n+1}x_1))]^\lambda,\end{aligned}$$

where

$$\begin{aligned}R(T^{n-1}x_1, T^{n+1}x_1) &= \max\Big\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T(T^{n-1}x_1)), p(T^{n+1}x_1, T(T^{n+1}x_1)), \\ &\quad \frac{p(T^{n-1}x_1, T(T^{n-1}x_1))p(T^{n+1}x_1, T(T^{n+1}x_1))}{1 + p(T^{n-1}x_1, T^{n+1}x_1)}\Big\} \\ &= \max\Big\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T^n x_1), p(T^{n+1}x_1, T^{n+2}x_1), \\ &\quad \frac{p(T^{n-1}x_1, T^n x_1)p(T^{n+1}x_1, T^{n+2}x_1)}{1 + p(T^{n-1}x_1, T^{n+1}x_1)}\Big\} \\ &= \max\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T^n x_1), p(T^{n+1}x_1, T^{n+2}x_1)\}.\end{aligned}$$

Thus we have

$$\theta(p(T^n x_1, T^{n+2}x_1)) \leq [\theta(\max\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T^n x_1), p(T^{n+1}x_1, T^{n+2}x_1)\})]^\lambda.$$

It follows that

$$\theta(p(T^n x_1, T^{n+2}x_1)) \leq [\max\{\theta(p(T^{n-1}x_1, T^{n+1}x_1)), \theta(p(T^{n-1}x_1, T^n x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}]^\lambda. \tag{2.9}$$

Let I be the set of $n \in \mathbb{N}$ such that
$u_n := \max\{\theta(p(T^{n-1}x_1, T^{n+1}x_1)), \theta(p(T^{n-1}x_1, T^n x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}$

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$= \theta(p(T^{n-1}x_1, T^{n+1}x_1))$.

If $|I| < \infty$, then there exists $N \in \mathbb{N}$ such that, for every $n \geq N$,

$$\max\{\theta(p(T^{n-1}x_1, T^{n+1}x_1)), \theta(p(T^{n-1}x_1, T^{n}x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}$$
$$= \max\{\theta(p(T^{n-1}x_1, T^{n}x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}.$$

For all $n \geq N$, from (2.9), we obtain that

$$1 \leq \theta(p(T^{n}x_1, T^{n+2}x_1)) \leq [\max\{\theta(p(T^{n-1}x_1, T^{n}x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}]^{\lambda}.$$

Taking the limit as $n \to \infty$ in the above inequality and using (2.5), we get that

$$\lim_{n\to\infty} \theta(p(T^{n}x_1, T^{n+2}x_1)) = 1.$$

If $|I| = \infty$, then we can find a subsequence of $\{u_n\}$, denoted by $\{u_n\}$, such that $u_n = \theta(p(T^{n-1}x_1, T^{n+1}x_1))$ for large n. From (2.9), we have

$$1 \leq \theta(p(T^{n}x_1, T^{n+2}x_1)) \leq [\theta(p(T^{n-1}x_1, T^{n+1}x_1))]^{\lambda} \leq [\theta(p(T^{n-2}x_1, T^{n}x_1))]^{\lambda^2}$$
$$\leq \cdots \leq [\theta(p(x_1, T^{2}x_1))]^{\lambda^n},$$

for large n. Taking the limit as $n \to \infty$ in the above inequality, we obtain that

$$\lim_{n\to\infty} \theta(p(T^{n}x_1, T^{n+2}x_1)) = 1. \tag{2.10}$$

Then in all cases, we obtain that (2.10) holds. By using (2.10) and (Θ2), we get that

$$\lim_{n\to\infty} p(T^{n}x_1, T^{n+2}x_1) = 0.$$

As an analogous proof as above, from (Θ3), there exists $n_2 \in \mathbb{N}$ such that

$$p(T^{n}x_1, T^{n+2}x_1) \leq \frac{1}{n^{1/r}} \quad \text{for all } n \geq n_2. \tag{2.11}$$

Let $M = \max\{n_1, n_2\}$. We consider the following two cases.

Case 1: If $m > 2$ is odd, then $m = 2L + 1$ for some $L \geq 1$. Using (2.8), for all $n \geq M$, we get that

$$\begin{aligned}
p(T^{n}x_1, T^{n+m}x_1) &\leq p(T^{n}x_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+2L+1}x_1) - \\
&\quad p(T^{n+1}x_1, T^{n+1}x_1) - p(T^{n+2}x_1, T^{n+2}x_1) \\
&\leq p(T^{n}x_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+2L+1}x_1) \\
&\leq p(T^{n}x_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+3}x_1) + \\
&\quad p(T^{n+3}x_1, T^{n+4}x_1) + p(T^{n+4}x_1, T^{n+2L+1}x_1) \\
&\;\vdots \\
&\leq p(T^{n}x_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + \cdots + p(T^{n+2L}x_1, T^{n+2L+1}x_1) \\
&\leq \frac{1}{n^{1/r}} + \frac{1}{(n+1)^{1/r}} + \cdots + \frac{1}{(n+2L)^{1/r}} \qquad (2.12) \\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}.
\end{aligned}$$

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Case 2: If $m > 2$ is even, then $m = 2L$ for some $L \geq 2$. Using (2.8) and (2.11), for all $n \geq M$, we get that

$$\begin{aligned}
p(T^n x_1, T^{n+m} x_1) &\leq p(T^n x_1, T^{n+2} x_1) + p(T^{n+2} x_1, T^{n+3} x_1) + p(T^{n+3} x_1, T^{n+2L} x_1) - \\
&\quad p(T^{n+2} x_1, T^{n+2} x_1) - p(T^{n+3} x_1, T^{n+3} x_1) \\
&\leq p(T^n x_1, T^{n+2} x_1) + p(T^{n+2} x_1, T^{n+3} x_1) + p(T^{n+3} x_1, T^{n+2L} x_1) \\
&\leq p(T^n x_1, T^{n+2} x_1) + p(T^{n+2} x_1, T^{n+3} x_1) + p(T^{n+3} x_1, T^{n+4} x_1) + \\
&\quad p(T^{n+4} x_1, T^{n+5} x_1) + p(T^{n+5} x_1, T^{2L} x_1) \\
&\vdots \\
&\leq p(T^n x_1, T^{n+2} x_1) + p(T^{n+2} x_1, T^{n+3} x_1) + \cdots + p(T^{n+2L-1} x_1, T^{n+2L} x_1) \\
&\leq \frac{1}{n^{1/r}} + \frac{1}{(n+2)^{1/r}} + \cdots + \frac{1}{(n+2L-1)^{1/r}} \qquad (2.13) \\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}.
\end{aligned}$$

From Case 1 and Case 2, we have

$$p(T^n x_1, T^{n+m} x_1) \leq \frac{1}{n^{1/r}} + \frac{1}{(n+1)^{1/r}} + \cdots + \frac{1}{(n+2L)^{1/r}} \quad \text{for all } n \geq M. \tag{2.14}$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{1/r}}$ is convergent (since $\frac{1}{r} > 1$) and (2.14), we have

$$\lim_{n,m \to \infty} p(T^n x_1, T^{n+m} x_1) = 0.$$

This implies that $\{T^n x_1\}$ is a Cauchy sequence in (X, p). By Lemma 1.9, we have $\{T^n x_1\}$ is a Cauchy sequence in (X, d_p). Since (X, p) is complete, then (X, d_p) is complete. This implies that there exists $z \in X$ such that $\lim_{n \to \infty} d_p(T^n x_1, z) = 0$. Using Lemma 1.8, we have

$$\lim_{n \to \infty} p(T^n x_1, z) = \lim_{n \to \infty} p(T^n x_1, T^n x_1) = p(z, z).$$

By applying Proposition 1.6, we obtain that

$$\begin{aligned}
2p(T^n x_1, z) &= d_p(T^n x_1, z) + p(T^n x_1, T^n x_1) + p(z, z) \\
&\leq d_p(T^n x_1, z) + p(T^n x_1, T^{n+1} x_1) + p(T^n x_1, z).
\end{aligned}$$

Therefore $p(T^n x_1, z) \leq d_p(T^n x_1, z) + p(T^n x_1, T^{n+1} x_1)$ for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, we obtain that $p(z, z) = \lim_{n \to \infty} p(T^n x_1, z) = 0$. We now suppose that $p(z, Tz) > 0$. By condition (4), there exists a subsequence $\{T^{n(k)} x_1\}$ of $\{T^n x_1\}$ such that $\alpha(T^{n(k)} x_1, z) \geq \eta(T^{n(k)} x_1, z)$ for all $k \in \mathbb{N}$. Since $T^n x_1 \neq T^m x_1$ for all $n, m \in \mathbb{N}$ with $m \neq n$, without loss of generality, we can assume that $T^{n(k)+1} x_1 \neq Tz$. And applying the condition (2.1), we obtain that

$$\begin{aligned}
\theta(p(T^{n(k)+1} x_1, Tz)) &= \theta(p(T(T^{n(k)} x_1), Tz)) \\
&\leq [\theta(R(T^{n(k)} x_1, z))]^{\lambda},
\end{aligned}$$

where

$$\begin{aligned} R(T^{n(k)}x_1, z) &= \max\Big\{p(T^{n(k)}x_1, z), p(T^{n(k)}x_1, T(T^{n(k)}x_1)), p(z, Tz), \\ &\qquad \frac{p(T^{n(k)}x_1, T(T^{n(k)}x_1))p(z, Tz)}{1 + p(T^{n(k)}x_1, z)}\Big\} \\ &= \max\Big\{p(T^{n(k)}x_1, z), p(T^{n(k)}x_1, T^{n(k)+1}x_1), p(z, Tz), \\ &\qquad \frac{p(T^{n(k)}x_1, T^{n(k)+1}x_1)p(z, Tz)}{1 + p(T^{n(k)}x_1, z)}\Big\}. \end{aligned}$$

Thus we have

$$\begin{aligned} \theta(p(T^{n(k)+1}x_1, Tz)) \le \Big[\theta\Big(\max\Big\{&p(T^{n(k)}x_1, z), p(T^{n(k)}x_1, T^{n(k)+1}x_1), p(z, Tz), \\ &\frac{p(T^{n(k)}x_1, T^{n(k)+1}x_1)p(z, Tz)}{1 + p(T^{n(k)}x_1, z)}\Big\}\Big)\Big]^{\lambda}. \end{aligned} \tag{2.15}$$

Taking the limit as $k \to \infty$ in (2.15), using the continuity of θ and Lemma **2.1**, we obtain that

$$\theta(p(z, Tz)) \le [\theta(p(z, Tz))]^{\lambda} < \theta(p(z, Tz)),$$

which is a contradiction. Thus we obtain that $p(z, Tz) = 0$. By Remark **1.3**, we have $Tz = z$, which contradicts to the assumption that T does not have a periodic point. Thus T has a periodic point, say z of period q. Suppose that the set of fixed points of T is empty. Then we have $q > 1$ and $p(z, Tz) > 0$. By using (2.1) and condition (6), we get that

$$\theta(p(z, Tz)) = \theta(p(T^q z, T^{q+1}z)) \le [\theta(p(z, Tz))]^{\lambda^q} < \theta(p(z, Tz)),$$

which is a contradiction. This implies that the set of fixed points of T is non-empty. Hence T has at least one fixed point. ∎

Example 2.3 Let $X = \{0, 1, 2, 3, 4, 5\}$ and define $p : X \times X \to [0, +\infty)$ such that

$$p(x, y) = \begin{cases} x & \text{if } x = y; \\ \frac{2x+y}{2} & \text{if } x, y \in \{0, 1, 2\},\ x \ne y; \\ \frac{2+x+2y}{2} & \text{otherwise.} \end{cases}$$

Then (X, p) is a complete partial rectangular metric space. Since, for all $x \in X$ and $x > 0$, then we have $p(x, x) = x > 0$. Therefore (X, p) is not a rectangular metric space. Define a mapping $T : X \to X$ by

$$T0 = T1 = T4 = 0, T2 = T3 = 2, \text{ and } T5 = 4.$$

We can see that 0 and 2 are periodic points of T. Let $\alpha, \eta : X \times X \to [0, +\infty)$ be functions defined by

$$\alpha(x, y) = \begin{cases} 1 \text{ if } x, y \in \{0, 1, 2, 3\}; \\ 0 \text{ otherwise.} \end{cases}$$

$$\eta(x,y) = \begin{cases} \frac{1}{2} \text{ if } x,y \in \{0,1,2,3\}; \\ 1 \text{ otherwise.} \end{cases}$$

Also define $\theta : (0,\infty) \to (1,\infty)$ by $\theta(t) = e^{\sqrt{t}}$. We next illustrate that all conditions in Theorem 2.1 hold. Taking $x_1 = 1$, we have $\alpha(1,T1) = \alpha(1,0) = 1 \geq \frac{1}{2} = \eta(1,0) = \eta(1,T1)$. Next, we prove that T is α-orbital admissible with respect to η. Let $\alpha(x,Tx) \geq \eta(x,Tx)$. Thus $x, Tx \in \{0,1,2,3\}$. By the definitions of a,η, we obtain that $\alpha(Tx,T^2x) \geq \eta(Tx,T^2x)$ for all $x \in \{0,1,2,3\}$. It follows that T is α-orbital admissible with respect to η. Let $\alpha(x,y) \geq \eta(x,y)$ and $\alpha(y,Ty) \geq \eta(y,Ty)$. By definitions of α,η, we have $x,y,Ty \in \{0,1,2,3\}$. This yields $\alpha(x,Ty) \geq \eta(x,Ty)$ for all $x,y \in \{0,1,2,3\}$. This implies that T is triangular α-orbital admissible with respect to η. Let $x,y \in X$ be such that $p(Tx,Ty) > 0$. We could observe that if $x,y \in \{0,1,4\}$, then $Tx = Ty = 0$. This implies that $p(Tx,Ty) = 0$. So we consider the following cases:

- $x \in \{0,1,4\}$ and $y \in \{2,3\}$ or
- $x \in \{0,1,4\}$ and $y = 5$ or
- $x = \{2,3\}$ and $y = 5$.

If $x = 4$ and $y \in \{2,3\}$ or $x \in \{0,1,4\}$ and $y = 5$ or $x = \{2,3\}$ and $y = 5$, then we have $\alpha(x,y) \ngeq \eta(x,y)$. We divide the proof into four cases as follows:
(1) If $(x,y) \in \{(0,2),(2,0)\}$, then

$$R(0,2) = \max\left\{p(0,2), p(0,0), p(2,2), \frac{p(0,0)p(2,2)}{1+p(0,2)}\right\} = \max\left\{1,0,2,0\right\} = 2.$$

This implies that

$$\psi(p(T0,T2)) = \psi(p(0,2)) = \psi(1) = e^{\sqrt{1}} \leq [e^{\sqrt{2}}]^{0.71} = [\psi(2)]^{0.71} \leq [\psi(R(0,2))]^{0.71}.$$

(2) If $(x,y) \in \{(1,2),(2,1)\}$, then

$$R(1,2) = \max\left\{p(1,2), p(1,0), p(2,2), \frac{p(1,0)p(2,2)}{1+p(1,2)}\right\} = \max\left\{2,1,2,\frac{2}{3}\right\} = 2.$$

This implies that

$$\psi(p(T1,T2)) = \psi(p(0,2)) = \psi(1) = e^{\sqrt{1}} \leq [e^{\sqrt{2}}]^{0.71} = [\psi(2)]^{0.71} \leq [\psi(R(1,2))]^{0.71}.$$

(3) If $(x,y) \in \{(0,3),(3,0)\}$, then

$$R(0,3) = \max\left\{p(0,3), p(0,0), p(3,2), \frac{p(0,0)p(3,2)}{1+p(0,3)}\right\} = \max\left\{4,0,\frac{9}{2},0\right\} = \frac{9}{2}.$$

This implies that

$$\psi(p(T0,T3)) = \psi(p(0,2)) = \psi(1) = e^{\sqrt{1}} \leq [e^{\sqrt{\frac{9}{2}}}]^{0.5} = [\psi(\frac{9}{2})]^{0.5} \leq [\psi(R(0,3))]^{0.5}.$$

(4) If $(x,y) \in \{(1,3),(3,1)\}$, then

$$R(1,3) = \max\left\{p(1,3), p(1,0), p(3,2), \frac{p(1,0)p(3,2)}{1+p(1,3)}\right\} = \max\left\{\frac{9}{2}, 1, \frac{9}{2}, \frac{9}{11}\right\} = \frac{9}{2}.$$

This implies that

$$\psi(p(T1,T3)) = \psi(p(0,2)) = \psi(1) = e^{\sqrt{1}} \leq [e^{\sqrt{\frac{9}{2}}}]^{0.5} = [\psi(\frac{9}{2})]^{0.5} \leq [\psi(R(1,3))]^{0.5}.$$

It follows that $\psi(p(Tx,Ty)) \leq [\psi(R(x,y))]^{\lambda}$. Hence all assumptions in Theorem 2.1 are satisfied and thus T has a fixed point which are $x=0$ and $x=2$.

We now prove the existence of the fixed point theorem by replacing triangular mappings and condition (4) in Theorem 2.2 by α-orbital attractive mappings.

Theorem 2.4 *Let (X,p) be a complete partial rectangular metric space, $T: X \to X$ be a mapping and let $\alpha, \eta : X \times X \to [0,\infty)$ be functions. Suppose that the following conditions hold :*
(1) there exist $\theta \in \Theta$ and $\lambda \in (0,1)$ such that for all $x, y \in X$,

$$p(Tx,Ty) > 0 \text{ and } \alpha(x,y) \geq \eta(x,y) \text{ imply } \theta(p(Tx,Ty)) \leq [\theta(R(x,y))]^{\lambda}, \tag{2.16}$$

where

$$R(x,y) = \max\left\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Tx)p(y,Ty)}{1+p(x,y)}\right\};$$

(2) there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq \eta(x_1,Tx_1)$ and $\alpha(x_1,T^2x_1) \geq \eta(x_1,T^2x_1)$;
(3) T is an α-orbital admissible mapping with respect to η;
(4) T is an α-orbital attractive mapping with respect to η;
(5) θ is continuous;
(6) if z is a periodic point of T, then $\alpha(z,Tz) \geq \eta(z,Tz)$.
Then T has a fixed point.

Proof. By (2), there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq \eta(x_1,Tx_1)$ and $\alpha(x_1,T^2x_1) \geq \eta(x_1,T^2x_1)$. Define the iterative sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1} = T^nx_1$ for all $n \in \mathbb{N}$. Since T is an α-orbital admissible mapping with respect to η, we obtain that
$\alpha(x_1,Tx_1) \geq \eta(x_1,Tx_1)$ implies $\alpha(Tx_1,T^2x_1) \geq \eta(Tx_1,T^2x_1)$
and
$\alpha(x_1,T^2x_1) \geq \eta(x_1,T^2x_1)$ implies $\alpha(Tx_1,T^3x_1) \geq \eta(Tx_1,T^3x_1)$.
By continuing this process, we get that

$$\alpha(T^nx_1,T^{n+1}x_1) \geq \eta(T^nx_1,T^{n+1}x_1) \quad \text{for all } n \in \mathbb{N} \tag{2.17}$$

and

$$\alpha(T^nx_1,T^{n+2}x_1) \geq \eta(T^nx_1,T^{n+2}x_1) \quad \text{for all } n \in \mathbb{N}. \tag{2.18}$$

If $T^n x_1 = T^{n+1}x_1$ for some $n \in \mathbb{N}$, then $T^n x_1$ is a fixed point of T. Thus we suppose that $T^n x_1 \neq T^{n+1}x_1$ for all $n \in \mathbb{N}$. That is $p(T^n x_1, T^{n+1}x_1) > 0$. Applying (2.16) and (2.17), for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned}\theta(p(T^n x_1, T^{n+1}x_1)) &= \theta(p(T(T^{n-1}x_1), T(T^n x_1))) \\ &\leq [\theta(R(T^{n-1}x_1, T^n x_1))]^{\lambda}, \qquad (2.19)\end{aligned}$$

where

$$\begin{aligned}R(T^{n-1}x_1, T^n x_1) &= \max\Big\{p(T^{n-1}x_1, T^n x_1), p(T^{n-1}x_1, T^n x_1), p(T^n x_1, T^{n+1}x_1), \\ &\quad \frac{p(T^{n-1}x_1, T^n x_1)p(T^n x_1, T^{n+1}x_1)}{1 + p(T^{n-1}x_1, T^n x_1)}\Big\} \\ &= \max\{p(T^{n-1}x_1, T^n x_1), p(T^n x_1, T^{n+1}x_1)\}.\end{aligned}$$

If $R(T^{n-1}x_1, T^n x_1) = p(T^n x_1, T^{n+1}x_1)$. By using (2.19), we get that

$$\theta(p(T^n x_1, T^{n+1}x_1)) \leq [\theta(p(T^n x_1, T^{n+1}x_1))]^{\lambda}.$$

This implies that

$$\ln[\theta(p(T^n x_1, T^{n+1}x_1))] \leq \lambda \ln[\theta(p(T^n x_1, T^{n+1}x_1))],$$

which is a contradiction with $\lambda \in (0,1)$. It follows that $R(T^{n-1}x_1, T^n x_1) = p(T^{n-1}x_1, T^n x_1)$ for all $n \in \mathbb{N}$. From (2.19), we obtain that

$$\theta(p(T^n x_1, T^{n+1}x_1)) \leq [\theta(p(T^{n-1}x_1, T^n x_1))]^{\lambda} \quad \text{for all } n \in \mathbb{N}.$$

It follows that

$$1 \leq \theta(p(T^n x_1, T^{n+1}x_1)) \leq \cdots \leq [\theta(p(x_1, Tx_1))]^{\lambda^n} \quad \text{for all } n \in \mathbb{N}. \qquad (2.20)$$

Taking the limit as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} \theta(p(T^n x_1, T^{n+1}x_1)) = 1. \qquad (2.21)$$

By using condition (Θ2), we have

$$\lim_{n\to\infty} p(T^n x_1, T^{n+1}x_1) = 0.$$

As in the proof of Theorem 2.2, we can prove that there exists $n_1 \in \mathbb{N}$ such that

$$p(T^n x_1, T^{n+1}x_1) \leq \frac{1}{n^{1/r}} \quad \text{for all } n \geq n_1. \qquad (2.22)$$

We now prove that T has a periodic point. Suppose that T does not have periodic points. Thus $T^n x_1 \neq T^m x_1$ for all $n, m \in \mathbb{N}$ such that $n \neq m$. Using (2.16) and (2.18), we get that

$$\begin{aligned}\theta(p(T^n x_1, T^{n+2}x_1)) &= \theta(p(T(T^{n-1}x_1), T(T^{n+1}x_1))) \\ &\leq [\theta(R(T^{n-1}x_1, T^{n+1}x_1))]^{\lambda},\end{aligned}$$

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where

$$R(T^{n-1}x_1, T^{n+1}x_1) = \max\Big\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T^nx_1), p(T^{n+1}x_1, T^{n+2}x_1), \frac{p(T^{n-1}x_1, T^nx_1)p(T^{n+1}x_1, T^{n+2}x_1)}{1+p(T^{n-1}x_1, T^{n+1}x_1)}\Big\}$$
$$= \max\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T^nx_1), p(T^{n+1}x_1, T^{n+2}x_1)\}.$$

By the analogous proof in Theorem 2.2, we have

$$\lim_{n\to\infty} p(T^nx_1, T^{n+2}x_1) = 0$$

and there exists $n_2 \in \mathbb{N}$ such that

$$p(T^nx_1, T^{n+2}x_1) \leq \frac{1}{n^{1/r}} \quad \text{for all } n \geq n_2. \tag{2.23}$$

Let $h = \max\{n_1, n_2\}$. We consider the following two cases.

Case 1: If $m > 2$ is odd, then $m = 2L + 1$ for some $L \geq 1$. By using (2.22), for all $n \geq h$, we obtain that

$$\begin{aligned} p(T^nx_1, T^{n+m}x_1) &\leq p(T^nx_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+2L+1}x_1) - \\ &\quad p(T^{n+1}x_1, T^{n+1}x_1) - p(T^{n+2}x_1, T^{n+2}x_1) \\ &\;\vdots \\ &\leq p(T^nx_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + \cdots + p(T^{n+2L}x_1, T^{n+2L+1}x_1) \\ &\leq \frac{1}{n^{1/r}} + \frac{1}{(n+1)^{1/r}} + \cdots + \frac{1}{(n+2L)^{1/r}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}. \end{aligned}$$

Case 2: If $m > 2$ is even, then $m = 2L$ for some $L \geq 2$. By using (2.22) and (2.23), for all $n \geq h$, we get that

$$\begin{aligned} p(T^nx_1, T^{n+m}x_1) &\leq p(T^nx_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+3}x_1) + p(T^{n+3}x_1, T^{n+2L}x_1) - \\ &\quad p(T^{n+2}x_1, T^{n+2}x_1) - p(T^{n+3}x_1, T^{n+3}x_1) \\ &\;\vdots \\ &\leq p(T^nx_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+3}x_1) + \cdots + p(T^{n+2L-1}x_1, T^{n+2L}x_1) \\ &\leq \frac{1}{n^{1/r}} + \frac{1}{(n+2)^{1/r}} + \cdots + \frac{1}{(n+2L)^{1/r}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}. \end{aligned}$$

From Case 1 and Case 2, we obtain that

$$p(T^n x_1, T^{n+m} x_1) \leq \frac{1}{n^{1/r}} + \frac{1}{(n+1)^{1/r}} + \cdots + \frac{1}{(n+2L)^{1/r}} \quad \text{for all } n \geq h. \tag{2.24}$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{1/r}}$ is convergent (since $\frac{1}{r} > 1$) and (2.24), we have

$$\lim_{n,m\to\infty} p(T^n x_1, T^{n+m} x_1) = 0.$$

This implies that $\{T^n x_1\}$ is a Cauchy sequence in (X, p). By Lemma 1.9, we have $\{T^n x_1\}$ is a Cauchy sequence in (X, d_p). Since (X, p) is complete, then (X, d_p) is complete. This implies that there exists $z \in X$ such that $\lim_{n\to\infty} d_p(T^n x_1, z) = 0$. Using Lemma 1.8, we have

$$\lim_{n\to\infty} p(T^n x_1, z) = \lim_{n\to\infty} p(T^n x_1, T^n x_1) = p(z, z).$$

By applying Proposition 1.6, we obtain that

$$\begin{aligned} 2p(T^n x_1, z) &= d_p(T^n x_1, z) + p(T^n x_1, T^n x_1) + p(z, z) \\ &\leq d_p(T^n x_1, z) + p(T^n x_1, T^{n+1} x_1) + p(T^n x_1, z). \end{aligned}$$

Therefore $p(T^n x_1, z) \leq d_p(T^n x_1, z) + p(T^n x_1, T^{n+1} x_1)$ for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, we obtain that $p(z, z) = \lim_{n\to\infty} p(T^n x_1, z) = 0$. We now prove that $z = Tz$. Suppose that $z \neq Tz$. Since T is α-orbital attractive with respect to η, we obtain that for all $n \in \mathbb{N}$,

$\alpha(T^n x_1, z) \geq \eta(T^n x_1, z)$ or $\alpha(z, T^{n+1} x_1) \geq \eta(z, T^{n+1} x_1)$.

We divide the proof in two cases as follows.

(1) There exists an infinite subset J of $\mathbb{N}$ such that $\alpha(T^{n(k)} x_1, z) \geq \eta(T^{n(k)} x_1, z)$ for every $k \in J$.

(2) There exists an infinite subset L of $\mathbb{N}$ such that $\alpha(z, T^{n(k)+1} x_1) \geq \eta(z, T^{n(k)+1} x_1)$ for every $k \in L$.

For the case (1), since $T^n x_1 \neq T^m x_1$ for all $n, m \in \mathbb{N}$ with $n \neq m$, without loss of the generality, we can assume that $T^{n(k)+1} x_1 \neq z$ for all $k \in J$. Applying the condition (2.16), we get that

$$\begin{aligned} \theta(p(T^{n(k)+1} x_1, Tz)) &= \theta(p(T(T^{n(k)} x_1), Tz)) \\ &\leq [\theta(R(T^{n(k)} x_1, z))]^{\lambda}, \end{aligned}$$

where

$$\begin{aligned} R(T^{n(k)} x_1, z) &= \max\Big\{ p(T^{n(k)} x_1, z), p(T^{n(k)} x_1, T(T^{n(k)} x_1)), p(z, Tz), \\ &\qquad \frac{p(T^{n(k)} x_1, T(T^{n(k)} x_1)) p(z, Tz)}{1 + p(T^{n(k)} x_1, z)} \Big\} \\ &= \max\Big\{ p(T^{n(k)} x_1, z), p(T^{n(k)} x_1, T^{n(k)+1} x_1), p(z, Tz), \\ &\qquad \frac{p(T^{n(k)} x_1, T^{n(k)+1} x_1) p(z, Tz)}{1 + p(T^{n(k)} x_1, z)} \Big\}. \end{aligned}$$

Then we have

$$\theta(p(T^{n(k)+1}x_1, Tz)) \leq \Big[\theta\Big(\max\Big\{p(T^{n(k)}x_1, z), p(T^{n(k)}x_1, T^{n(k)+1}x_1), p(z, Tz), \frac{p(T^{n(k)}x_1, T^{n(k)+1}x_1)p(z, Tz)}{1 + p(T^{n(k)}x_1, z)}\Big\}\Big)\Big]^{\lambda}.$$

Taking the limit as $k \to \infty$ in the above equality, using the continuity of θ and Lemma 2.1, we obtain that

$$\theta(p(z, Tz)) \leq [\theta(p(z, Tz))]^{\lambda} < \theta(p(z, Tz)),$$

which is a contradiction. For the case (2), the proof is similar. Therefore $z = Tz$, which is a contradiction with the assumption that T does not have a periodic point. Thus T has a periodic point, say z of period q. Suppose that the set of fixed points of T is empty. Then we have $q > 1$ and $p(z, Tz) > 0$. Applying (2.16) and condition (6), we get that

$$\theta(p(z, Tz)) = \theta(p(T^q z, T^{q+1}z)) \leq [\theta(p(z, Tz))]^{\lambda} < \theta(p(z, Tz)),$$

which is a contradiction. Thus the set of fixed points of T is non-empty. Hence T has at least one fixed point. ■

Since a rectangular metric space is a partial rectangular metric space, we immediately obtain Theorem 17 and Theorem 19 in [1] by applying Theorem 2.2 and Theorem 2.4, respectively.

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