รายงานวิจัยฉบับสมบูรณ์

โครงการ

ทฤษฎีบทจุดตรึงสำหรับการส่งแบบแอลฟา-อีตา-ไซ-เจราจตีคอนแทรกทีฟวางนัยทั่วไปในปริภูมิแอลฟา-อีตา-เมตริกบริบูรณ์

Fixed point theorems for generalized alpha-eta-psi-Geraghty contraction type mappings in alpha-eta-complete metric spaces

โดย ผู้ช่วยศาสตราจารย์ ดร.อัญชลีย์ แก้วเจริญ หัวหน้าโครงการ

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ผู้วิจัย

ผู้ช่วยศาสตราจารย์ ดร.อัญชลีย์ แก้วเจริญ (หัวหน้าโครงการวิจัย)
ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยนเรศวร

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ในโครงการนี้ผู้วิจัยได้แนะนำแนวคิดของการส่งหลายค่า (แอลฟา, อีตา, พไซ, ไซ)–คอนแทรกทีฟอย่างเข้มโดยใช้เงื่อนไขที่อ่อนว่าความต่อเนื่องสำหรับไซ ผู้วิจัยได้พิสูจน์ทฤษฎีบทจุดตรึงสำหรับการส่งดังกล่าวในปริภูมิเมตริกแอลฟา–อีตา–บริบูรณ์บางส่วน นอกจากนี้ยังได้พิสูจน์ทฤษฎีบทจุดตรึงในปริภูมิเมตริกบางส่วนบริบูรณ์กับความสัมพันธ์ทวินาม และกับกราฟโดยใช้ผลลัพธ์หลัก

คำสำคัญ: ปริภูมิเมตริกแอลฟา–อีตา–บริบูรณ์บางส่วน; แอลฟา–อีตา–ความต่อเนื่อง; การส่งหลายค่า (แอลฟา, อีตา, พไซ, ไซ)–คอนแทรกทีฟ; การส่งหลายค่าแอลฟา–แอดมิสสิเบิลซึ่งสัมพันธ์กับอีตา

CHAPTER I

EXECUTIVE SUMMARY

Let X be a set and $T : X \to X$ a mapping. The solutions we seek are represented by points invariant under T. These are the points satisfying

$$x = Tx. \tag{1}$$

Such points are said to be fixed under T or fixed points of T. The set of all solutions of (1) is called the fixed point set of T and denoted by Fix T. If the mapping T does not have a fixed point we often say that T is fixed point free.

Fundamental to the study of Fixed Point Theory is the attempt to identify conditions which may be imposed on the set X and/or the mapping T that will assure Fix $T \neq \varnothing$. Usually it is more efficient to study a family $\mathcal{T}$ of mapping satisfying some common conditions rather than an individual mapping. If all the mapping $T \in \mathcal{T}$ have fixed points, then we say that X has the fixed point property with respect to $\mathcal{T}$. The term " fixed point property" is often abbreviated as fpp, and if we are dealing with the fixed specific family $\mathcal{T}$ the words " with respect to $\mathcal{T}$ " are omitted.

Typically, a fixed point theorem has the following form.

Generic Theorem. *Let X be a set having structure* A *and let $\mathcal{T}$ be the family of mappings $T : X \to X$ satisfying condition* B. *Then each mapping $T \in \mathcal{T}$ has a fixed point.*

Fixed point theory has fascinated many researchers since 1922 with the celebrated Banach fixed point theorem. There exists a vast literature on the topic field and this is very active field of research at present. Fixed point theorems are very important tools for proving the existence and uniqueness of the solutions to various mathematical models (integral and partial equations, variational inequalities etc). It can be applied to, for examples, variational inequalities, optimization, and approximation theory. The fixed point theory has been continually studied by many researchers. It is well-known that the contractive-type conditions are very indispensable in the study of fixed point theory. The first important result on fixed points for contractive-type mappings was the well-known Banach-Caccioppoli theorem which was published in 1922. Later in 1968, Kannan studied a new type of contractive mappings. Since then, there have been many results related to mappings satisfying various types of contractive inequalities.

Recently, Samet et al. introduced a new category of contractive type mappings known as α-ψ contractive type mappings. The results obtained by Samet et al. extended and generalized the existing fixed point results in the literature, in particular the Banach contraction principle. Salimi et al. and Karapinar and Samet generalized the α-ψ contractive type mappings and obtained various fixed point theorems for this generalized class of contractive mappings. In most of papers have considered the α-ψ contractive type mapping for a nondecreasing mapping $\psi : [0, +\infty) \to [0, +\infty)$ with $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t \in (0, +\infty)$. The convergence of $\sum_{n=1}^{\infty} \psi^n(t)$ and nondecreasing condition for ψ are restrictive and it is a fact that such a mapping is differentiable almost everywhere and hence continuous why was one of our aims to write this article in order to consider a family of mappings $\psi : [0, +\infty) \to [0, +\infty)$ by relaxing nondecreasing condition and the convergence of the series $\sum_{n=1}^{\infty} \psi^n(t)$. This article inspired and motivated by above research works, we will introduce a new family of mappings on $[0, +\infty)$ and prove the fixed point theorems for mappings using properties of this new family in complete metric spaces. By applying our obtained results, we also assure the fixed point theorems in partially ordered complete metric spaces and give the applications to ordinary differential equations.

One of the most important results in fixed point theory is the Banach contraction principle introduced by Banach. There were many authors have studied and proved the results for fixed point theory by generalizing the Banach contraction principle in several directions. One of the remarkable result is Geraghty's theorem given by Geraghty. In 2013, Cho et al. introduced the notion of α-Geraghty contraction type mappings and assured the unique fixed point theorems for such mappings in complete metric spaces. Recently, Popescu defined the concept of triangular α-orbital admissible mappings and proved the unique fixed point theorems for the mentioned mappings which are generalized α-Geraghty contraction type mappings. On the other hand, Karapinar proved the existence of a unique fixed point for a triangular α-admissible mapping which is a generalized α-ψ-Geraghty contraction type mapping.

Fixed point theory in metric spaces is one of the most important tools for proving the existence and uniqueness of the solutions to various mathematical models. Later in 1993 Czerwik, generalized the notion of metric spaces by introducing the notion of b-metric spaces. On the other hand, Samet et al. proved the fixed point theorems for α-admissible mappings which are α-φ-contractive mappings in complete metric spaces. Salimi et al. and Hussain et al. modified these notions and assured the fixed point theorems. Recently, Hussain et al. established fixed point theorems for modified α-φ-rational contractive mappings in α-complete metric spaces and proved the existence of solutions of integral equations.

In 1994, Matthews introduced the partial metric spaces and proved the Banach contraction principle in such spaces. Later on, the researchers have studied the fixed point theorems for mappings in complete partial metric spaces. On the other hand, Nadler proved the multi-valued version of Banach contraction principle. Since then the metric fixed point theory of single-valued mappings has been extended to multi-valued mappings. Recently, Kutbi and Sintunavarat proved the existence of fixed point theorems for strictly (α, ψ, ξ)-contractive multi-valued mappings satisfying some certain contractive condition in the setting of α-complete metric spaces.

In this project, we relax the continuity of ξ to be the upper semicontinuity from the right at 0 and introduce the notion of strictly $(\alpha, \eta, \psi, \xi)$-contractive mappings. We also prove the existence of fixed point theorems for such mappings in the setting of α-η-complete partial metric spaces. Our results extend the results proved by Kutbi and Sintunavarat. Furthermore, we assure the fixed point theorems in partial complete metric spaces endowed with an arbitrary binary relation and with a graph using our obtained results.

CHAPTER II

CONTENTS OF RESEARCH

In this project, we obtain two publications that published in the international journal as the followings:

Ali Farajzadeh, Preeyaluk Chuadchawna and Anchalee Kaewcharoen, - Fixed point theorems for (alpha, eta, psi, xi)-contractive multi-valued mappings in alpha-eta-complete partial metric spaces, Journal of Nonlinear Science and Applications, 9 (2016), 1977-1990. (Impact Factor := 1.176)

In this paper, we relax the continuity of $\xi \in \Xi$ to be the upper semicontinuity from the right at 0. Let Ξ' denote the family of functions $\xi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(ξ_1') ξ is upper semicontinuous from the right at 0;

(ξ_2') ξ is nondecreasing on $[0, \infty)$;

(ξ_3') $\xi(t) = 0$ if and only if $t = 0$;

(ξ_4') ξ is subadditive.

1. **Theorem** : Let (X, p) be a partial metric space, A and B be nonempty closed bounded subsets of X, $\xi \in \Xi'$ and $h > 1$. Then for all $a \in A$ such that $\xi(p(a, B)) > 0$, there exists $b \in B$ such that $\xi(p(a, b)) < h(\xi(p(a, B)))$.

2. **Theorem** : Let (X, p) be an α-η-complete partial metric space and $T : X \to CB^p(X)$ be a strictly $(\alpha, \eta, \psi, \xi)$-contractive mapping. Assume that the following conditions hold:

   (i) T is an α-admissible mapping with respect to η;

   (ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$;

   (iii) T is an α-η-continuous mapping on (X, p);

   (iv) if $\{x_n\}$ is a sequence in X converging to a point x in (X, p) such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$, then we have $\alpha(x, x) \geq \eta(x, x)$.

   Then T has a fixed point.

3. **Corollary** : Let (X, p) be an α-complete partial metric space and $T : X \to CB^p(X)$ be a strictly (α, ψ, ξ)-contractive mapping. Assume that the following conditions hold:

   (i) T is an α-admissible mapping;

   (ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;

   (iii) T is an α-continuous mapping on (X, p);

(iv) if $\{x_n\}$ is a sequence in X be converges to a point x in (X, p) such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then we have $\alpha(x, x) \geq 1$.

Then T has a fixed point.

4. **Theorem** : Let (X, p) be an α-η-complete partial metric space and $T : X \to CB^p(X)$ be a strictly $(\alpha, \eta, \psi, \xi)$-contractive mapping. Assume that the following conditions hold:

 (i) T is an α-admissible mapping with respect to η;

 (ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$;

 (iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N} \cup \{0\}$.

Then T has a fixed point.

5. **Theorem** : Let (X, p) be an α-complete partial metric space and $T : X \to CB^p(X)$ be a strictly (α, ψ, ξ)-contractive mapping. Assume that the following conditions hold:

 (i) T is an α-admissible mapping;

 (ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;

 (iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then T has a fixed point.

6. **Theorem** : Let (X, d) be an α-complete metric space and $T : X \to CB(X)$ be a strictly (α, ψ, ξ)-contractive mapping. Assume that the following conditions hold:

 (i) T is an α-admissible mapping;

 (ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;

 (iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to^d x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then T has a fixed point.

7. **Theorem** : Let (X, p) be a partial metric space, $\mathcal{R}$ be a binary relation over X and $T : X \to CB^p(X)$ be a strictly $(\mathcal{S}, \psi, \xi)$-contractive mapping. Assume that the following conditions hold:

 (i) (X, p) is an $\mathcal{S}$-*complete* partial metric space;

 (ii) T is a *weakly comparative* mapping;

 (iii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0 \mathcal{S} x_1$;

 (iv) T is an $\mathcal{S}$-*continuous* multi-valued mapping;

(v) if $\{x_n\}$ is a sequence in X converging to a point x in (X,p), where $x_n\mathcal{S}x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, then we have $x\mathcal{S}x$.

Then T has a fixed point.

8. **Theorem** : Let (X,p) be a partial metric space, $\mathcal{R}$ be a binary relation over X and $T : X \to CB^p(X)$ be a strictly $(\mathcal{S},\psi,\xi)$-contractive mapping. Assume that the following conditions hold:

   (i) (X,p) is an $\mathcal{S}$-*complete* partial metric space;

   (ii) T is a weakly comparative mapping;

   (iii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0\mathcal{S}x_1$;

   (iv) if $\{x_n\}$ is a sequence in X with $x_n \to x \in X$ as $n \to \infty$ and $x_n\mathcal{S}x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, then we have $x_n\mathcal{S}x$.

   Then T has a fixed point.

9. **Theorem** : Let (X,p) be a partial metric space endowed with a graph G and $T : X \to CB^p(X)$ be a strictly $(E(G),\psi,\xi)$-contractive mapping. Assume that the following conditions hold:

   (i) (X,p) is an $E(G)$-*complete* partial metric space;

   (ii) T weakly preserves edges;

   (iii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0,x_1) \in E(G)$;

   (iv) T is an $E(G)$-continuous mapping on (X,p);

   (v) if $\{x_n\}$ is a sequence in X converging to a point x in (X,p), where $(x_n,x_{n+1}) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$, then we have $(x,x) \in E(G)$.

   Then T has a fixed point.

10. **Theorem** : Let (X,p) be a partial metric space endowed with a graph G and $T : X \to CB^p(X)$ be a strictly $(E(G),\psi,\xi)$-contractive mapping. Assume that the following conditions hold:

   (i) (X,p) is an $E(G)$-*complete* partial metric space;

   (ii) T weakly preserves edges;

   (iii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0,x_1) \in E(G)$;

   (iv) if $\{x_n\}$ is a sequence in X with $x_n \to x \in X$ as $n \to \infty$ and $(x_n,x_{n+1}) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$, then we have $(x_n,x) \in E(G)$.

   Then T has a fixed point.

CHAPTER III

OUTPUT

ผลลัพธ์จากโครงการวิจัยที่ได้รับทุนจากงบประมาณแผ่นดินมหาวิทยาลัยนเรศวร ประจำปี 2559

1. ผลงานวิจัยตีพิมพ์ในวารสารวิชาการนานาชาติ

Ali Farajzadeh, Preeyaluk Chaudchawna and Anchalee Kaewcharoen, Fixed point theorems for (alpha, eta, psi, xi)-contractive multi-valued mappings in alpha-eta-complete partial metric spaces, Journal of Nonlinear Science and Applications, 9 (2016), 1977-1990. (Impact Factor := 1.176)

2. การนำผลงานวิจัยไปใช้ประโยชน์

มีการนำไปใช้ประโยชน์ทั้งเชิงวิชาการและเชิงสาธารณะโดยทำให้มีการพัฒนาการเรียนการสอน รวมทั้งมีการสร้างเครือข่ายความร่วมมือในการทำวิจัย

ภาคผนวก 1

Fixed point theorems for (alpha, eta; psi, xi)-contractive multi-valued mappings in alpha-eta-complete partial metric spaces

Anchalee Kaewcharoen

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Fixed point theorems for $(\alpha, \eta, \psi, \xi)$-contractive multi-valued mappings on α-η-complete partial metric spaces

Ali Farajzadeh[a], Preeyaluk Chuadchawna[b], Anchalee Kaewcharoen[b,*]

[a]Department of Mathematics, Faculty of Science, Razi University, Kermanshah, 67149, Iran.
[b]Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand.

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Abstract

In this paper, the notion of strictly $(\alpha, \eta, \psi, \xi)$-contractive multi-valued mappings is introduced where the continuity of ξ is relaxed. The existence of fixed point theorems for such mappings in the setting of α-η-complete partial metric spaces are provided. The results of the paper can be viewed as the extension of the recent results obtained in the literature. Furthermore, we assure the fixed point theorems in partial complete metric spaces endowed with an arbitrary binary relation and with a graph using our obtained results.

Keywords: α-η-complete partial metric spaces, α-η-continuity, $(\alpha, \eta, \psi, \xi)$-contractive multi-valued mappings, α-admissible multi-valued mappings with respect to η.
2010 MSC: 47H10, 54H25.

1. Introduction and Preliminaries

The metric fixed point theory is one of the most important tools for proving the existence and uniqueness of the solution to various mathematical models. There are many authors who have generalized the metric spaces in many directions. In 1994, Matthews [12] introduced the partial metric spaces and proved the Banach contraction principle in such spaces. Later on, the researchers have studied the fixed point theorems

*Corresponding author
Email addresses: farajzadehali@gmail.com (Ali Farajzadeh), Chuadchawna@hotmail.com (Preeyaluk Chuadchawna), anchaleeka@nu.ac.th (Anchalee Kaewcharoen)

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for mappings in complete partial metric spaces, see for examples [3, 4, 6, 7, 8] and references contained therein. On the other hand, Nadler [14] proved the multi-valued version of Banach contraction principle. Since then the metric fixed point theory of single-valued mappings has been extended to multi-valued mappings, see for examples [11, 17]. Recently, Kutbi and Sintunavarat [11] proved the existence of fixed point theorems for strictly (α, ψ, ξ)-contractive multi-valued mappings satisfying some certain contractive conditions in the setting of α-complete metric spaces.

In this paper, we relax the continuity of ξ to be the upper semicontinuity from the right at 0 and introduce the notion of strictly $(\alpha, \eta, \psi, \xi)$-contractive mappings. We also prove the existence of fixed point theorems for such mappings in the setting of α-η-complete partial metric spaces. Our results extend the results proved by Kutbi and Sintunavarat [11]. Furthermore, we assure the fixed point theorems in partial complete metric spaces endowed with an arbitrary binary relation and with a graph using our obtained results.

We now recall some definitions and lemmas that will be used in the sequel.

Definition 1.1 ([12]). A partial metric on a nonempty set X is a mapping $p : X \times X \to [0, +\infty)$ such that for all $x, y, z \in X$, the following conditions are satisfied:

(P1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;
(P2) $p(x, x) \leq p(x, y)$;
(P3) $p(x, y) = p(y, x)$;
(P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A set X equipped with a partial metric p is called a partial metric space and denoted by a pair (X, p).

Lemma 1.2 ([1]). *Let (X, p) be a partial metric space. If $p(x, y) = 0$, then $x = y$.*

For each partial metric p on X, the function $p^s : X \times X \to [0, +\infty)$ defined by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a metric on X.

Definition 1.3 ([12]). Let (X, p) be a partial metric space.

(i) A sequence $\{x_n\}$ in a partial metric space (X, p) is convergent to a point $x \in X$ if $\lim_{n\to\infty} p(x, x_n) = p(x, x)$.
(ii) A sequence $\{x_n\}$ in a partial metric space (X, p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and is finite).
(iii) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ that is,
$$\lim_{n,m\to\infty} p(x_n, x_m) = p(x, x).$$

Lemma 1.4 ([12]). *Let (X, p) be a partial metric space. Then*

(i) a sequence $\{x_n\}$ in a partial metric space (X, p) is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space (X, p^s);
(ii) a partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Moreover, $\lim_{n\to\infty} p^s(x, x_n) = 0$ if and only if
$$\lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m) = p(x, x);$$
(iii) a subset E of a partial metric space (X, p) is closed if whenever $\{x_n\}$ is a sequence in E such that $\{x_n\}$ converges to some $x \in X$, then $x \in E$.

Aydi et al. [8] defined a partial Hausdorff metric as follows. Let (X, p) be a partial metric space. Let$CB^p(X)$ be the family of all nonempty closed bounded subsets of a partial metric space (X, p). For any

$A, B \in CB^p(X)$ and $x \in X$, define

$$\delta_p(A,B) = \sup\{p(a,B) : a \in A\} \text{ and } \delta_p(B,A) = \sup\{p(b,A) : b \in B\},$$

where

$$p(x,A) = \inf\{p(x,a), a \in A\}.$$

The mapping $H_p : CB^p(X) \times CB^p(X) \to [0,+\infty)$ defined by

$$H_p(A,B) = \max\{\delta_p(A,B), \delta_p(B,A)\}$$

is called a partial Hausdorff metric induced by p.

Remark 1.5 ([3]). Let (X,p) be a partial metric space. If A is a nonempty set in (X,p), then

$$a \in \bar{A} \text{ if and only if } p(a,A) = p(a,a),$$

where $\bar{A}$ is the closure of A with respect to the partial metric p.

Lemma 1.6 ([8]). *Let (X,p) be a partial metric space and $T : X \to CB^p(X)$ be a multi-valued mapping. If $\{x_n\}$ is a sequence in X such that $x_n \to z$ and $p(z,z) = 0$, then*

$$\lim_{n\to\infty} p(x_n, Tz) = p(z, Tz).$$

In this paper, we denote by Ψ the class of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(ψ_1) ψ is a nondecreasing function;
(ψ_2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the nth iteration of ψ.

A function $\psi \in \Psi$ is known in the literature as Bianchini-Grandolfi gauge functions (see e.g. [9] and [15]).

Remark 1.7 ([11]). For each $\psi \in \Psi$, the following statements are satisfied,

(i) $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$;
(ii) $\psi(t) < t$ for each $t > 0$;
(iii) $\psi(0) = 0$.

Recently, Ali et al. [2] introduced the family Ξ of functions $\xi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(ξ_1) ξ is continuous;
(ξ_2) ξ is nondecreasing on $[0,\infty)$;
(ξ_3) $\xi(t) = 0$ if and only if $t = 0$;
(ξ_4) ξ is subadditive.

They [2] also introduced the concept of (α,ψ,ξ)-contractive multi-valued mappings as follows.

Definition 1.8 ([2]). Let (X,d) be a metric space. A multi-valued mapping $T : X \to CB(X)$ is called an (α,ψ,ξ)-contractive mapping if there exist three functions $\psi \in \Psi$, $\xi \in \Xi$ and $\alpha : X \times X \to [0,\infty)$ such that for all $x, y \in X$,

$$\alpha(x,y) \geq 1 \text{ implies } \xi(H(Tx,Ty)) \leq \psi(\xi(M(x,y))),$$

where

$$M(x,y) = \max\{d(x,y), d(x;Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}.$$

In the case when $\psi \in \Psi$ is strictly increasing, the (α,ψ,ξ)-contractive mapping is called a strictly (α,ψ,ξ)-contractive mapping.

On the other hand, Mohamadi et al. [13] introduced the concept of α-admissible multi-valued mappings as follows.

Definition 1.9 ([13]). Let X be a nonempty set, $T : X \to N(X)$ where $N(X)$ is a set of nonempty subsets of X and $\alpha : X \times X \to [0,\infty)$. T is α-admissible whenever for each $x \in X$ and $y \in Tx$ with $\alpha(x,y) \geq 1$, we have $\alpha(y,z) \geq 1$ for all $z \in Ty$.

Hussain et al. [10] introduced the concept of an α-completeness of a metric space which is weaker than the concept of a completeness.

Definition 1.10 ([10]). Let (X,d) be a metric space and $\alpha : X \times X \to [0,\infty)$ be a mapping. The metric space X is said to be α-complete if and only if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ converges in X.

Recently, Kutbi and Sintunavarat [11] introduced the concept of an α-continuities for multi-valued mappings in metric spaces and proved the fixed point theorems for strictly (α, ψ, ξ)-contractive mappings in α-complete metric spaces.

Definition 1.11 ([11]). Let (X,d) be a metric space, $\alpha : X \times X \to [0,\infty)$ and $T : X \to CB(X)$ be two given mappings. T is an α-continuous multi-valued mapping if for all sequence $\{x_n\}$ in X with $x_n \xrightarrow{d} x \in X$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $Tx_n \xrightarrow{H} Tx \in X$ as $n \to \infty$, that is

$$\lim_{n\to\infty} d(x_n, x) = 0 \text{ and } \alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \text{ imply } \lim_{n\to\infty} H(Tx_n, Tx) = 0.$$

Theorem 1.12 ([11]). *Let (X, d) be an α-complete metric space and $T : X \to CB(X)$ be a strictly (α, ψ, ξ)-contractive mapping. Assume that the following conditions hold:*

(i) T is an α-admissible mapping;
(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then T has a fixed point.

2. Main results

In this paper, we relax the continuity of $\xi \in \Xi$ to be the upper semicontinuity from the right at 0. Let Ξ' denote the family of functions $\xi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(ξ_1') ξ is upper semicontinuous from the right at 0;
(ξ_2') ξ is nondecreasing on $[0,\infty)$;
(ξ_3') $\xi(t) = 0$ if and only if $t = 0$;
(ξ_4') ξ is subadditive.

Example 2.1. The floor function $\xi(x) = \lfloor x \rfloor$ is upper semicontinuous function from the right at 0 and nondecreasing but is not continuous.

The following example illustrates that (ξ_1') is independent from the conditions (ξ_2') – (ξ_4'). Roughly, we cannot obtain (ξ_1') by using (ξ_2') – (ξ_4').

Example 2.2. Let $\xi : [0,\infty) \to [0,\infty)$ be defined by

$$\xi(t) = \begin{cases} 0, & \text{if } x = 0\ ; \\ 2t + 3, & \text{if otherwise.} \end{cases}$$

We see that ξ is nondecreasing, subadditive, $\xi(t) = 0$ if and only if $t = 0$. Moreover, ξ is not upper semicontinuous from the right at 0 since

$$\limsup_{n\to\infty} f(\frac{1}{n}) = \limsup_{n\to\infty}(\frac{2}{n} + 3) = 3 > \xi(0).$$

The following example shows that (ξ_2') is independent from the conditions $(\xi_1'), (\xi_3')$ and (ξ_4').

Example 2.3. Let $\xi : [0,\infty) \to [0,\infty)$ be defined by

$$\xi(t) = \begin{cases} \frac{1}{n}, & \text{if } t = \frac{1}{n}\ ; \\ 0, & \text{if otherwise.} \end{cases}$$

Therefore ξ is upper semicontinuous from the right at 0, subadditive, $\xi(t) = 0$ if and only if $t = 0$, but not nondecreasing.

We now introduce the concepts of α-η-complete partial metric spaces and α-η-continuous multi-valued mappings in partial metric spaces.

Definition 2.4. Let (X,p) be a partial metric space and $\alpha, \eta : X \times X \to [0,\infty)$. The partial metric space X is said to be α-η-complete if and only if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, converges in (X,p).

Definition 2.5. Let (X,p) be a partial metric space, $\alpha, \eta : X \times X \to [0,\infty)$ and $T : X \to CB^p(X)$. T is an α-η-continuous multi-valued mapping if, for all sequence $\{x_n\}$ with $\lim_{n\to\infty} p(x_n, x) = p(x,x)$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} H_p(Tx_n, Tx) = H_p(Tx, Tx).$$

We now prove the key lemma that will be used in proving our main results.

Lemma 2.6. *Let (X,p) be a partial metric space, A and B be nonempty closed bounded subsets of X, $\xi \in \Xi'$ and $h > 1$. Then for all $a \in A$ such that $\xi(p(a,B)) > 0$, there exists $b \in B$ such that $\xi(p(a,b)) < h(\xi(p(a,B)))$.*

Proof. Let $a \in A$ be such that $\xi(p(a,B)) > 0$. By (ξ_3'), we have $p(a,B) > 0$. We can construct a sequence $\{b_n\}$ in B such that $\lim_{n\to\infty} p(a,b_n) = p(a,B)$. Using (ξ_4'), we have

$$\xi(p(a,b_n)) \leq \xi(p(a,b_n) - p(a,B)) + \xi(p(a,B)).$$

This implies that

$$\xi(p(a,b_n)) - \xi(p(a,B)) \leq \xi(p(a,b_n) - p(a,B)).$$

Since ξ is upper semicontinuous from the right at 0 and $\lim_{n\to\infty}(p(a,b_n) - p(a,B)) = 0$, we obtain that

$$\limsup_{n\to\infty}(\xi(p(a,b_n)) - \xi(p(a,B)) \leq \limsup_{n\to\infty} \xi(p(a,b_n) - p(a,B)) \leq \xi(0) = 0.$$

This yields

$$\limsup_{n\to\infty} \xi(p(a,b_n)) \leq \xi(p(a,B)) < h\xi(p(a,B)).$$

It follows that there exists $N \in \mathbb{N}$ such that $\xi(p(a,b_N)) < h\xi(p(a,B))$. This completes the proof. □

Next, we introduce the concepts of α-admissibility with respect to η and (α,η,ψ,ξ)-contractive multi-valued mappings on α-η-partial metric spaces.

Definition 2.7. Let X be a nonempty set, $T : X \to N(X)$ where $N(X)$ is a set of nonempty subsets of X and $\alpha, \eta : X \times X \to [0,\infty)$. T is α-admissible with respect to η whenever for each $x \in X$ and $y \in Tx$ with $\alpha(x,y) \geq \eta(x,y)$, we have $\alpha(y,z) \geq \eta(y,z)$ for all $z \in Ty$.

Definition 2.8. Let (X,p) be a partial metric space. A multi-valued mapping $T : X \to CB^p(X)$ is called an (α,η,ψ,ξ)-contractive mapping if there exist $\psi \in \Psi$, $\xi \in \Xi'$ and $\alpha,\eta : X \times X \to [0,\infty)$ such that for all $x,y \in X$,

$$\alpha(x,y) \geq \eta(x,y) \Rightarrow \xi(H_p(Tx,Ty)) \leq \psi(\xi(M(x,y))),$$

where

$$M(x,y) = \max\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Ty)+p(y,Tx)}{2}\}.$$

In the case when $\psi \in \Psi$ is strictly increasing, the (α,η,ψ,ξ)-contractive mapping is called a strictly (α,η,ψ,ξ)-contractive mapping.

We now prove the existence of fixed point theorems for strictly (α,η,ψ,ξ)-contractive mappings in α-η-complete partial metric spaces.

Theorem 2.9. *Let (X,p) be an α-η-complete partial metric space and $T : X \to CB^p(X)$ be a strictly (α,η,ψ,ξ)-contractive mapping. Assume that the following conditions hold:*

(i) T is an α-admissible mapping with respect to η;
(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq \eta(x_0,x_1)$;
(iii) T is an α-η-continuous mapping on (X,p);
(iv) if $\{x_n\}$ is a sequence in X converging to a point x in (X,p) such that $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}\cup\{0\}$, then we have $\alpha(x,x) \geq \eta(x,x)$.

Then T has a fixed point.

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$ be such that $\alpha(x_0,x_1) \geq \eta(x_0,x_1)$. If $x_0 = x_1$, then x_0 is a fixed point of T. Assume that $x_0 \neq x_1$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T. Assume that $x_1 \notin Tx_1$. Since T is a strictly (α,η,ψ,ξ)-contractive mapping, we obtain that

$$\begin{aligned}\xi(H_p(Tx_0,Tx_1)) &\leq \psi(\xi(M(x_0,x_1)))\\ &= \psi(\xi(\max\{p(x_0,x_1), p(x_0,Tx_0), p(x_1,Tx_1), \frac{p(x_0,Tx_1)+p(x_1,Tx_0)}{2}\}))\\ &\leq \psi(\xi(\max\{p(x_0,x_1), p(x_0,x_1), p(x_1,Tx_1), \frac{p(x_0,Tx_1)+p(x_1,x_1)}{2}\}))\\ &\leq \psi(\xi(\max\{p(x_0,x_1), p(x_1,Tx_1),\\ &\qquad \frac{1}{2}[p(x_0,x_1)+p(x_1,Tx_1)-p(x_1,x_1)+p(x_1,x_1)]\}))\\ &\leq \psi(\xi(\max\{p(x_0,x_1), p(x_1,Tx_1), \frac{p(x_0,x_1)+p(x_1,Tx_1)}{2}\}))\\ &= \psi(\xi(\max\{p(x_0,x_1), p(x_1,Tx_1)\})).\end{aligned} \tag{2.1}$$

If $\max\{p(x_0,x_1), p(x_1,Tx_1)\} = p(x_1,Tx_1)$, then we have

$$\begin{aligned}0 < \xi(p(x_1,Tx_1)) \leq \xi(H_p(Tx_0,Tx_1)) &\leq \psi(\xi(\max\{p(x_0,x_1), p(x_1,Tx_1)\}))\\ &\leq \psi(\xi(p(x_1,Tx_1)))\\ &< \xi(p(x_1,Tx_1)),\end{aligned}$$

which is a contradiction. Therefore, $\max\{p(x_0,x_1), p(x_1,Tx_1)\} = p(x_0,x_1)$. By (2.1), we have

$$0 < \xi(p(x_1,Tx_1)) \leq \xi(H_p(Tx_0,Tx_1)) \leq \psi(\xi(p(x_0,x_1))). \tag{2.2}$$

Fix $h > 1$ and by using Lemma 2.6, there exists $x_2 \in Tx_1$ such that

$$0 < \xi(p(x_1,x_2)) < h(\xi(p(x_1,Tx_1))). \tag{2.3}$$

By (2.2) and (2.3), we have

$$0 < \xi(p(x_1, x_2)) < h\psi(\xi(p(x_0, x_1))). \tag{2.4}$$

Since ψ is a strictly increasing mapping, we have

$$0 < \psi(\xi(p(x_1, x_2))) < \psi(h\psi(\xi(p(x_0, x_1)))). \tag{2.5}$$

By setting

$$h_1 = \frac{\psi(h\psi(\xi(p(x_0, x_1))))}{\psi(\xi(p(x_1, x_2)))}, \text{ we obtain that } h_1 > 1.$$

If $x_1 = x_2$ or $x_2 \in Tx_2$, then T has a fixed point. Assume that $x_1 \neq x_2$ and $x_2 \notin Tx_2$. Since $x_1 \in Tx_0$, $x_2 \in Tx_1$, $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$ and T is an α-admissible mapping with respect to η, we have $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$. Since T is a strictly $(\alpha, \eta, \psi, \xi)$-contractive mapping, we obtain that

$$\begin{aligned}
\xi(H_p(Tx_1, Tx_2)) &\leq \psi(\xi(M(x_1, x_2))) \\
&= \psi(\xi(\max\{p(x_1, x_2), p(x_1, Tx_1), p(x_2, Tx_2), \frac{p(x_1, Tx_2) + p(x_2, Tx_1)}{2}\})) \\
&\leq \psi(\xi(\max\{p(x_1, x_2), p(x_1, x_2), p(x_2, Tx_2), \frac{p(x_1, Tx_2) + p(x_2, x_2)}{2}\})) \\
&\leq \psi(\xi(\max\{p(x_1, x_2), p(x_2, Tx_2), \\
&\qquad \frac{1}{2}[p(x_1, x_2) + p(x_2, Tx_2) - p(x_2, x_2) + p(x_2, x_2)]\})) \\
&\leq \psi(\xi(\max\{p(x_1, x_2), p(x_2, Tx_2), \frac{p(x_1, x_2) + p(x_2, Tx_2)}{2}\})) \\
&= \psi(\xi(\max\{p(x_1, x_2), p(x_2, Tx_2)\})).
\end{aligned} \tag{2.6}$$

Assume that $\max\{p(x_1, x_2), p(x_2, Tx_2)\} = p(x_2, Tx_2)$. By (2.6), we have

$$\begin{aligned}
0 < \xi(p(x_2, Tx_2)) \leq \xi(H_p(Tx_1, Tx_2)) &\leq \psi(\xi(\max\{p(x_1, x_2), p(x_2, Tx_2)\})) \\
&\leq \psi(\xi(p(x_2, Tx_2))) \\
&< \xi(p(x_2, Tx_2)),
\end{aligned}$$

which is a contradiction. Then $\max\{p(x_1, x_2), p(x_2, Tx_2)\} = p(x_1, x_2)$. Using (2.6), we obtain that

$$0 < \xi(p(x_2, Tx_2)) \leq \xi(H_p(Tx_1, Tx_2)) \leq \psi(\xi(p(x_1, x_2))). \tag{2.7}$$

By using Lemma 2.6 with $h_1 > 1$, there exists $x_3 \in Tx_2$ such that

$$0 < \xi(p(x_2, x_3)) < h_1(\xi(p(x_2, Tx_2))). \tag{2.8}$$

By (2.7) and (2.8), we have

$$\begin{aligned}
0 < \xi(p(x_2, x_3)) < h_1\psi(\xi(p(x_1, x_2))) &= \frac{\psi(h\psi(\xi(p(x_0, x_1))))}{\psi(\xi(p(x_1, x_2)))}\psi(\xi(p(x_1, x_2))) \\
&= \psi(h\psi(\xi(p(x_0, x_1)))).
\end{aligned}$$

Since ψ is a strictly increasing mapping, we have

$$0 < \psi(\xi(p(x_2, x_3))) < \psi^2(h\psi(\xi(p(x_0, x_1)))). \tag{2.9}$$

Continuing this process, we can construct a sequence $\{x_n\}$ in X such that $x_n \neq x_{n+1} \in Tx_n$,

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \tag{2.10}$$

and

$$0 < \xi(p(x_{n+1}, x_{n+2})) < \psi^n(h\psi(\xi(p(x_0, x_1)))) \tag{2.11}$$

for all $n \in \mathbb{N} \cup \{0\}$. Let $m > n$. Then by the triangular inequality, we have

$$\begin{aligned}
\xi(p(x_n, x_m)) &\leq \xi(p(x_n, x_{n+1}) + p(x_{n+1}, x_m) - p(x_{n+1}, x_{n+1})) \\
&\leq \xi(p(x_n, x_{n+1}) + p(x_{n+1}, x_m)) \\
&\leq \xi(p(x_n, x_{n+1}) + \xi(p(x_{n+1}, x_m)) \\
&\leq \xi(p(x_n, x_{n+1})) + \xi(p(x_{n+1}, x_{n+2})) + \xi(p(x_{n+2}, x_m)) \\
&\leq \xi(p(x_n, x_{n+1})) + \xi(p(x_{n+1}, x_{n+2})) + \xi(p(x_{n+2}, x_{n+3})) + \cdots + \xi(p(x_{m-1}, x_m)) \\
&= \Sigma_{i=n}^{m-1} \xi(p(x_i, x_{i+1})) \\
&< \Sigma_{i=n}^{m-1} \psi^{i-1}(h\psi(\xi(p(x_0, x_1)))) \\
&\leq \Sigma_{i=n}^{\infty} \psi^{i-1}(h\psi(\xi(p(x_0, x_1)))).
\end{aligned}$$

Since $\psi \in \Psi$, we have $\lim_{m,n\to\infty} \xi(p(x_n, x_m)) = 0$. If $\lim_{m,n\to\infty} p(x_n, x_m) \neq 0$, then there exist $\varepsilon > 0$ and two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $m(k) > n(k) \geq k$ such that

$$p(x_{n(k)}, x_{m(k)}) \geq \varepsilon.$$

Since ξ is nondecreasing, we have $\lim_{k\to\infty} \xi(p(x_{n(k)}, x_{m(k)})) \geq \xi(\varepsilon) > 0$ which is a contradiction. Therefore $\lim_{m,n\to\infty} p(x_n, x_m) = 0$. Then $\{x_n\}$ is a Cauchy sequence in (X, p). By Lemma 1.4, we have $\{x_n\}$ is a Cauchy sequence in metric space (X, p^s). Since (X, p) is α-η-complete, we obtain that (X, p^s) is α-η-complete. Then there exists $z \in X$ such that

$$\lim_{n\to\infty} p^s(x_n, z) = 0. \tag{2.12}$$

Since $\lim_{m,n\to\infty} p(x_n, x_m) = 0$, from Lemma 1.4, we have

$$\lim_{n\to\infty} p(x_n, z) = \lim_{m,n\to\infty} p(x_n, x_m) = p(z, z) = 0. \tag{2.13}$$

This implies that $\{x_n\}$ converges to z in (X, p). Since T is α-η-continuous on (X, p), we have

$$\lim_{n\to\infty} p(x_{n+1}, Tz) \leq \lim_{n\to\infty} H_p(Tx_n, Tz) = H_p(Tz, Tz). \tag{2.14}$$

Using the triangular inequality, we have

$$p(z, Tz) \leq p(z, x_{n+1}) + p(x_{n+1}, Tz).$$

Letting $n \to \infty$ and using (2.14), we get

$$p(z, Tz) \leq \lim_{n\to\infty} p(z, x_{n+1}) + \lim_{n\to\infty} p(x_{n+1}, Tz) \leq H_p(Tz, Tz).$$

So we have $p(z, Tz) \leq H_p(Tz, Tz)$. We will show that $z \in Tz$. Suppose that $z \notin Tz$. By Remark 1.5, we obtain that $p(z, Tz) \neq 0$. Since $\{x_n\}$ converges to z with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and by (iv), it follows that $\alpha(z, z) \geq \eta(z, z)$. This implies that

$$\begin{aligned}
\xi(H_p(Tz, Tz)) &\leq \psi(\xi(M(z, z))) \\
&\leq \psi(\xi(\max\{p(z, z), p(z, Tz), p(z, Tz), \frac{p(z, Tz) + p(Tz, z)}{2}\})) \\
&\leq \psi(\xi(\max\{p(z, z), p(z, Tz)\})) \\
&= \psi(\xi(p(z, Tz))) \\
&< \xi(p(z, Tz) \\
&\leq \xi(H_p(Tz, Tz)),
\end{aligned}$$

which is a contradiction. Therefore $z \in Tz$ and hence T has a fixed point. □

If we take $\eta(x,y) = 1$, we obtain the following results.

Corollary 2.10. *Let (X,p) be an α-complete partial metric space and $T : X \to CB^p(X)$ be a strictly (α,ψ,ξ)-contractive mapping. Assume that the following conditions hold:*

(i) T is an α-admissible mapping;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;

(iii) T is an α-continuous mapping on (X, p);

(iv) if $\{x_n\}$ be a sequence in X that converges to a point x in (X, p) such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then we have $\alpha(x, x) \geq 1$.

Then T has a fixed point.

We next substitute the α-η-continuity of T by some appropriate conditions.

Theorem 2.11. *Let (X,p) be an α-η-complete partial metric space and $T : X \to CB^p(X)$ be a strictly (α,η,ψ,ξ)-contractive mapping. Assume that the following conditions hold:*

(i) T is an α-admissible mapping with respect to η;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N} \cup \{0\}$.

Then T has a fixed point.

Proof. As in Theorem 2.9, we can construct a sequence $\{x_n\}$ such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$, $x_n \neq x_{n+1} \in Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$ and there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$ and $p(z,z) = 0$. From condition (iii), we have

$$\alpha(x_n, z) \geq \eta(x_n, z) \tag{2.15}$$

for all $n \in \mathbb{N} \cup \{0\}$. Suppose that $z \notin Tz$. By Remark 1.5, we have $p(z, Tz) > 0$. Since T is a strictly (α,η,ψ,ξ)-contractive mapping and (2.15), we obtain that

$$\begin{aligned}\xi(H_p(Tx_n, Tz)) &\leq \psi(\xi(M(x_n, z))) \\ &= \psi(\xi(\max\{p(x_n, z), p(x_n, Tx_n), p(z, Tz), \frac{p(x_n, Tz) + p(z, Tx_n)}{2}\}))\end{aligned} \tag{2.16}$$

for all $n \in \mathbb{N}$. Let $\varepsilon = \frac{p(z,Tz)}{2}$. Since $\{x_n\}$ converges to z in (X,p), There exists $N_1 \in \mathbb{N}$ such that

$$p(x_n, z) = |p(x_n, z) - p(z, z)| < \frac{p(z, Tz)}{2} \text{ for all } n \geq N_1. \tag{2.17}$$

Furthermore, we obtain that

$$p(Tx_n, z) \leq p(x_{n+1}, z) < \frac{p(z, Tz)}{2} \text{ for all } n \geq N_1. \tag{2.18}$$

Since $\{x_n\}$ is a Cauchy sequence in (X,p), there exists $N_2 \in \mathbb{N}$ such that

$$p(x_n, Tx_n) \leq p(x_n, x_{n+1}) < \frac{p(z, Tz)}{2} \text{ for all } n \geq N_2. \tag{2.19}$$

It follows from $x_n \to z$ as $n \to \infty$ and $p(z, z) = 0$ via Lemma 1.6, we have $p(x_n, Tz) \to p(z, Tz)$ as $n \to \infty$. This implies that there exists $N_3 \in \mathbb{N}$ such that

$$p(x_n, Tz) < \frac{3p(z, Tz)}{2} \tag{2.20}$$

for all $n \geq N_3$. Let $N = \max\{N_1, N_2, N_3\}$. Using (2.17)-(2.20), we have

$$\max\{p(x_n, z), p(x_n, Tx_n), p(z, Tz), \frac{p(x_n, Tz) + p(z, Tx_n)}{2}\} = p(z, Tz), \tag{2.21}$$

for all $n \geq N$. By (2.16) and the triangular inequality, we have

$$\begin{aligned}\xi(p(z, Tz)) &\leq \xi(p(z, x_{n+1}) + p(x_{n+1}, Tz)) \\ &\leq \xi(p(z, x_{n+1})) + \xi(p(x_{n+1}, Tz)) \\ &\leq \xi(p(z, x_{n+1})) + \xi(H_p(Tx_n, Tz)) \\ &\leq \xi(p(z, x_{n+1})) + \psi(\xi(M(x_n, z))) \\ &\leq \xi(p(z, x_{n+1})) + \psi(\xi(p(z, Tz))).\end{aligned}$$

Since ξ is upper semicontinuous from the right at 0 and by taking the limit superior in the above inequality, we have

$$\begin{aligned}\xi(p(z, Tz)) &\leq \limsup_{n\to\infty} \xi(p(z, x_{n+1})) + \psi(\xi(p(z, Tz))) \\ &\leq \xi(0) + \psi(\xi(p(z, Tz))) \\ &= \psi(\xi(p(z, Tz))) \\ &< \xi(p(z, Tz)),\end{aligned}$$

which is a contradiction. Then $z \in Tz$ and hence T has a fixed point. □

If we take $\eta(x, y) = 1$, we have the following result.

Corollary 2.12. *Let (X, p) be an α-complete partial metric space and $T : X \to CB^p(X)$ be a strictly (α, ψ, ξ)-contractive mapping. Assume that the following conditions hold:*

(i) T is an α-admissible mapping;
(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then T has a fixed point.

Using Corollary 2.12, we can extend the result proved by Kutbi and Sintunavarat (Theorem 2.6, [11]).

Corollary 2.13 ([11]). *Let (X, d) be an α-complete metric space and $T : X \to CB(X)$ be a strictly (α, ψ, ξ)-contractive mapping. Assume that the following conditions hold:*

(i) T is an α-admissible mapping;
(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to^d x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then T has a fixed point.

We give an example for supporting Theorem 2.11.

Example 2.14. Let $X = (-1, 5]$ and a partial metric $p : X \times X \to \mathbb{R}$ defined by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Define $T : X \to CB^p(X)$ by

$$Tx = \begin{cases} \{2x\}, & \text{if } x \in (-1, 0) \text{ ;} \\ \{\frac{x}{16}\}, & \text{if } x \in [0, 5]. \end{cases}$$

Also, we define mappings $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 2, & \text{if } x, y \in [0,5]\ ; \\ \frac{1}{6}, & \text{if otherwise,} \end{cases}$$

$$\eta(x,y) = \begin{cases} 1, & \text{if } x, y \in [0,5]\ ; \\ \frac{1}{4}, & \text{if otherwise .} \end{cases}$$

Define $\psi, \xi : [0,\infty) \to [0,\infty)$ by $\psi(t) = \frac{t}{4}$ and $\xi(t) = \sqrt{t}$. We see that $\psi \in \Psi$ and $\xi \in \Xi'$.

Firstly, we will show that T is a strictly $(\alpha, \eta, \psi, \xi)$-contractive mapping. For $x, y \in X$ and $\alpha(x,y) \geq \eta(x,y)$, we have $x, y \in [0,5]$ and then

$$\begin{aligned} \xi(H_p(Tx,Ty)) &= \sqrt{\max\{\frac{x}{16}, \frac{y}{16}\}} \\ &= \frac{1}{4}\sqrt{\max\{x,y\}} \\ &= \frac{1}{4}\sqrt{p(x,y)} \\ &\leq \frac{1}{4}\sqrt{M(x,y)} = \psi(\xi(M(x,y))). \end{aligned}$$

It is clear that ψ is a strictly increasing function. Therefore, T is strictly $(\alpha, \eta, \psi, \xi)$-contractive mapping. We next show that T is an α-admissible with respect to η. Let $x \in X$, $y \in Tx$ and $z \in Ty$ with $\alpha(x,y) \geq \eta(x,y)$, we have $x, y \in [0,5]$, it follows that $Ty \in [0,5]$. Since $z \in Ty$, we have $z \in [0,5]$. So $\alpha(y,z) \geq \eta(y,z)$. Therefore T is an α-admissible with respect to η. We will prove that (X,p) is an α-η-complete partial metric space. If $\{x_n\}$ is a Cauchy sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then $\{x_n\} \subseteq [0,5]$ for all $n \in \mathbb{N}$. Since $x \in \overline{[0,5]}$ iff $p(x,[0,5]) = p(x,x)$ iff $\inf_{y\in[0,5]} \max\{x,y\} = \inf_{y\in[0,5]} p(x,y) = p(x,x)$ iff $x \in [0,5]$, we obtain that $[0,5]$ is closed in (X,p). Now, since $([0,5],p)$ is a complete partial metric space, then the sequence $\{x_n\}$ converges in $[0,5] \subseteq X$. Next, there exist $x_0 = 1 \in X$ and $x_1 = 2 \in Tx_0$ such that

$$\alpha(x_0, x_1) = \alpha(1,2) = 2 > 1 = \eta(1,2) = \eta(x_0, x_1).$$

Then the condition (ii) of Theorem 2.11 is satisfied. Finally, for each sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ and $x_n \to x$ as $n \to \infty$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N}$. Thus the condition (iv) of Theorem 2.11 is satisfied. Then all the conditions of Theorem 2.11 are satisfied and so T has a fixed point which is $x = 0$.

3. Consequences

3.1. Fixed point results in partial metric spaces endowed with binary relations

Let (X,p) be a partial metric space and $\mathcal{R}$ be a binary relation over X. Denote $\mathcal{S} := \mathcal{R} \cup \mathcal{R}^{-1}$, that is

$$x, y \in X,\ \ x\mathcal{S}y \text{ if and only if } x\mathcal{R}y \text{ or } y\mathcal{R}x.$$

Definition 3.1 ([11]). Let X be a nonempty set and $\mathcal{R}$ be a binary relation over X. A multi-valued mapping $T : X \to N(X)$ is said to be weakly comparative if for each $x \in X$ and $y \in Tx$ with $x\mathcal{S}y$, we have $y\mathcal{S}z$ for all $z \in Ty$.

We now introduce the notions of $\mathcal{S}$-completeness, $\mathcal{S}$-continuity and $(\mathcal{S}, \psi, \xi)$-contractive mappings on partial metric spaces as follows.

Definition 3.2. Let (X, p) be a partial metric space and $\mathcal{R}$ be a binary relation over X. The partial metric space X is said to be $\mathcal{S}$-complete if and only if every Cauchy sequence $\{x_n\}$ in X with $x_n \mathcal{S} x_{n+1}$ for all $n \in \mathbb{N}$, converges in (X, p).

Definition 3.3. Let (X, p) be a partial metric space and $\mathcal{R}$ be a binary relation over X. $T : X \to CB^p(X)$ is an $\mathcal{S}$-continuous mapping if for given $x \in X$ and a sequence $\{x_n\}$ with $\lim_{n\to\infty} p(x_n, x) = p(x, x)$ and $x_n \mathcal{S} x_{n+1}$ for all $n \in \mathbb{N}$ imply $\lim_{n\to\infty} H_p(Tx_n, Tx) = H_p(Tx, Tx)$.

Definition 3.4. Let (X, p) be a partial metric space and $\mathcal{R}$ be a binary relation over X. A mapping $T : X \to CB^p(X)$ is called an $(\mathcal{S}, \psi, \xi)$-contractive mapping if there exist $\psi \in \Psi$ and $\xi \in \Xi'$ such that for all $x, y \in X$,

$$x\mathcal{S}y \text{ implies } \xi(H_p(Tx, Ty)) \le \psi(\xi(M(x, y))),$$

where

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Ty) + p(y, Tx)}{2}\}.$$

In the case when $\psi \in \Psi$ is strictly increasing, the $(\mathcal{S}, \psi, \xi)$-contractive mapping is called a strictly $(\mathcal{S}, \psi, \xi)$-contractive mapping.

We now assure the fixed point theorems for strictly $(\mathcal{S}, \psi, \xi)$-contractive mappings on partial metric spaces with binary relations.

Theorem 3.5. *Let (X, p) be a partial metric space, $\mathcal{R}$ be a binary relation over X and $T : X \to CB^p(X)$ be a strictly $(\mathcal{S}, \psi, \xi)$-contractive mapping. Assume that the following conditions hold:*

(i) (X, p) is an $\mathcal{S}$-complete partial metric space;
(ii) T is a weakly comparative mapping;
(iii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0 \mathcal{S} x_1$;
(iv) T is an $\mathcal{S}$-continuous multi-valued mapping;
(v) if $\{x_n\}$ is a sequence in X converging to a point x in (X, p), where $x_n \mathcal{S} x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, then we have $x\mathcal{S}x$.

Then T has a fixed point.

Proof. Define mappings $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in x\mathcal{S}y; \\ 0, & \text{if otherwise,} \end{cases}$$

$$\eta(x, y) = \begin{cases} \frac{1}{2}, & \text{if } x, y \in x\mathcal{S}y; \\ 2, & \text{if otherwise.} \end{cases}$$

Therefore we can obtain the result by using Theorem 2.9. □

By using Theorem 2.11, we immediately obtain the following result.

Theorem 3.6. *Let (X, p) be a partial metric space, $\mathcal{R}$ be a binary relation over X and $T : X \to CB^p(X)$ be a strictly $(\mathcal{S}, \psi, \xi)$-contractive mapping. Assume that the following conditions hold:*

(i) (X, p) is an $\mathcal{S}$-complete partial metric space;
(ii) T is a weakly comparative mapping;
(iii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0 \mathcal{S} x_1$;
(iv) if $\{x_n\}$ is a sequence in X with $x_n \to x \in X$ as $n \to \infty$ and $x_n \mathcal{S} x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, then we have $x_n \mathcal{S} x$.

Then T has a fixed point.

3.2. Fixed point results in partial metric spaces endowed with graph

Let (X,p) be a partial metric space. Let G be a graph such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops.

Definition 3.7 ([11]). Let (X,p) be a nonempty set endowed with a graph G and $T : X \to N(X)$ be a multi-valued mapping. We say that T weakly preserves edges if for each $x \in X$ and $y \in Tx$ with $(x,y) \in E(G)$, we have $(y,z) \in E(G)$ for all $z \in Ty$.

We now introduce the notions of $E(G)$-completeness, $E(G)$-continuity and $(E(G),\psi,\xi)$-contractive mappings on partial metric spaces as follows.

Definition 3.8. Let (X,p) be a partial metric space endowed with a graph G. The partial metric space X is said to be $E(G)$-complete if and only if every Cauchy sequence $\{x_n\}$ in X with $(x_n,x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, converges in (X,p).

Definition 3.9. Let (X,p) be a partial metric space endowed with a graph G. $T : X \to CB^p(X)$ is an $E(G)$-continuous mapping if for given $x \in X$ and a sequence $\{x_n\}$ with $\lim_{n\to\infty} p(x_n,x) = p(x,x)$ and $(x_n,x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ imply $\lim_{n\to\infty} H_p(Tx_n,Tx) = H_p(Tx,Tx)$.

Definition 3.10. Let (X,p) be a partial metric space endowed with a graph G. A mapping $T : X \to CB^p(X)$ is called an $(E(G),\psi,\xi)$-contractive mapping if there exist $\psi \in \Psi$ and $\xi \in \Xi'$ such that for all $x,y \in X$,

$$(x,y) \in E(G) \text{ implies } \xi(H_p(Tx,Ty)) \le \psi(\xi(M(x,y))),$$

where

$$M(x,y) = \max\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Ty)+p(y,Tx)}{2}\}.$$

In the case when $\psi \in \Psi$ is strictly increasing, the $(E(G),\psi,\xi)$-contractive mapping is called a strictly $(E(G),\psi,\xi)$-contractive mapping.

Theorem 3.11. *Let (X,p) be a partial metric space endowed with a graph G and $T : X \to CB^p(X)$ be a strictly $(E(G),\psi,\xi)$-contractive mapping. Assume that the following conditions hold:*

(i) (X,p) is an $E(G)$-complete partial metric space;
(ii) T weakly preserves edges;
(iii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0,x_1) \in E(G)$;
(iv) T is an $E(G)$-continuous mapping on (X,p);
(v) if $\{x_n\}$ is a sequence in X converging to a point x in (X,p), where $(x_n,x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}\cup\{0\}$, then we have $(x,x) \in E(G)$.

Then T has a fixed point.

Proof. Define mappings $\alpha,\eta : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } (x,y) \in E(G); \\ 0, & \text{if otherwise,} \end{cases}$$

$$\eta(x,y) = \begin{cases} \frac{1}{2}, & \text{if } (x,y) \in E(G); \\ 2, & \text{if otherwise }. \end{cases}$$

Therefore we can obtain the result by using Theorem 2.9. □

By using Theorem 2.11, we immediately obtain the the following result.

Theorem 3.12. *Let (X, p) be a partial metric space endowed with a graph G and $T : X \to CB^p(X)$ be a strictly $(E(G), \psi, \xi)$-contractive mapping. Assume that the following conditions hold:*

(i) (X, p) *is an* $E(G)$*-complete partial metric space;*

(ii) T *weakly preserves edges;*

(iii) there exist $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $(x_0, x_1) \in E(G)$*;*

(iv) if $\{x_n\}$ *is a sequence in* X *with* $x_n \to x \in X$ *as* $n \to \infty$ *and* $(x_n, x_{n+1}) \in E(G)$ *for all* $n \in \mathbb{N} \cup \{0\}$*, then we have* $(x_n, x) \in E(G)$*.*

Then T *has a fixed point.*

Remark 3.13. Our results extend and improve several results in the literature as the following:

(1) Theorem 2.11 extends Theorem 2.6 [2], Theorem 2.2 [5], Theorem 3.2 [8], Theorem 2.6 [11], Theorem 3.4 [13] and Theorem 2.2 [16].

(2) Theorem 3.6 extends Theorem 3.6 [11].

(3) Theorem 3.12 extends Theorem 3.12 [11].

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Fixed Point Theorems for Modified (α-ψ-φ-θ)-Rational Contractive Mappings in α-Complete b-Metric Spaces[1]

Preeyaluk Chuadchawna[†] and Anchalee Kaewcharoen[†,‡,2]

[†]Department of Mathematics, Faculty of Science, Naresuan University
Phitsanulok 65000, Thailand
e-mail : anchaleeka@nu.ac.th (A. Kaewcharoen)
e-mail : Chuadchawna@hotmail.com (P. Chuadchawan)
[‡] Centre of Excellence in Mathematics, CHE, Si Ayutthaya Rd.
Bangkok, 10400, Thailand

Abstract : In this paper, we introduce the notion of modified (α-ψ-φ-θ)-rational contractive mappings where some conditions of Bianchini-Grandolfi gauge function φ are omitted. We establish the existence of the unique fixed point theorems for such mappings which are triangular α-orbital admissible in α-complete b-metric spaces. Moreover, we also prove the unique common fixed point theorem for mappings T and g where T is a modified (α-ψ-φ-θ)-rational contractive mapping with respect to g. Our results extend the fixed point theorems in α-complete metric spaces proved by Hussian et al.[N. Hussain, M. A. Kutbi, P. Salimi, Fixed point theory in α-complete metric spaces with applications, Abstr. Appl. Anal. (2014) Article ID 280817] to α-complete b-metric spaces.

Keywords : triangular α-orbital admissible mappings; α-complete b-metric spaces; α-continuous mappings; modified (α-ψ-φ-θ)-rational contractive mappings; common fixed points.

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[2]Corresponding author.

1 Introduction

Fixed point theory in metric spaces is one of the most important tools for proving the existence and uniqueness of the solutions to various mathematical models. Later in 1993 Czerwik [1], generalized the notion of metric spaces by introducing the notion of b-metric spaces. On the other hand, Samet et al. [2] proved the fixed point theorems for α-admissible mappings which are α-φ-contractive mappings in complete metric spaces. Salimi et al. [3] and Hussain et al. [4] modified these notions and assured the fixed point theorems. Recently, Hussain et al. [5] established fixed point theorems for modified α-φ-rational contractive mappings in α-complete metric spaces and proved the existence of solutions of integral equations.

In this paper, we extend the fixed point results in α-complete metric spaces proved by Hussian et al. [5] to α-complete b-metric spaces by introducing the notion of modified $(\alpha$-ψ-φ-$\theta)$-rational contractive mappings where some conditions of Bianchini-Grandolfi gauge function φ are omitted. We establish the existence of the unique fixed point theorems for such mappings which are triangular α-orbital admissible. Moreover, we also prove the unique common fixed point theorem for mappings T and g where T is a modified $(\alpha$-ψ-φ-$\theta)$-rational contractive mapping with respect to g and is triangular g-α-admissible in the setting of α-complete b-metric spaces.

2 Preliminaries

We now recall some definitions and lemmas that will be used in the sequel.

In 2012, Samet et al. [2] introduced the notion of α-admissible mappings.

Definition 2.1 ([2]). Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then T is said to be *α-admissible* if for all $x, y \in X$,

$$\alpha(x, y) \geq 1 \quad \text{implies} \quad \alpha(Tx, Ty) \geq 1.$$

Recently Hussain et al. [5] introduced the concept of modified α-φ-rational contractive mappings and proved the fixed point theorems for such mappings in α-complete metric spaces.

Definition 2.2. A function $\varphi : [0, \infty) \to [0, \infty)$ is called a *Bianchini-Grandolfi gauge function* [6] if the following conditions hold:

(i) φ is nondecreasing;

(ii) $\sum_{k=1}^{\infty} \varphi^k(t)$ converges for all $t > 0$.

We denote by Φ the set of all Bianchini-Grandolfi gauge functions.

Lemma 2.3 ([7]). *If $\varphi \in \Phi$, then the following statements hold:*

(i) $\varphi(t) < t$ for all $t > 0$;

(ii) φ *is continuous at 0;*

(iii) $\varphi(0) = 0$.

Definition 2.4 ([5]). Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is a *modified* α-ψ-*rational contractive mapping* if for all $x, y \in X$,

$$\alpha(x,y) \geq 1 \text{ implies } d(Tx, Ty) \leq \varphi(M(x, y)), \tag{2.1}$$

where

$$M(x,y) = \max\{d(x,y), \frac{d(x,Tx)}{1+d(x,Tx)}, \frac{d(y,Ty)}{1+d(y,Ty)}, \frac{d(x,Ty)+d(y,Tx)}{2}\},$$

and $\varphi \in \Phi$.

Theorem 2.5 ([5]). *Let* (X,d) *be a metric space,* $\alpha : X \times X \to [0,\infty)$ *and* $T : X \to X$*. Assume that the following conditions are satisfied:*

(i) X *is an* α*-complete metric space;*

(ii) T *is a modified* α-φ*-rational contractive mapping;*

(iii) T *is an* α*-admissible mapping;*

(iv) there exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$*;*

(v) T *is an* α*-continuous mapping.*

Then T *has a fixed point.*

Recently, Popescu [8] studied the definitions of α-orbital admissible mappings and triangular α-orbital admissible mappings.

Definition 2.6 ([8]). Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then T is said to be α-*orbital admissible* if

$$\alpha(x, Tx) \geq 1 \quad \text{implies} \quad \alpha(Tx, T^2x) \geq 1.$$

Definition 2.7 ([8]). Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then T is said to be *triangular* α-*orbital admissible* if

(a) T is α-orbital admissible;

(b) $\alpha(x,y) \geq 1$ and $\alpha(y,Ty) \geq 1$ imply $\alpha(x,Ty) \geq 1$.

Lemma 2.8 ([8]). *Let* $T : X \to X$ *be a triangular* α*-orbital admissible mapping. Assume that there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$*. Define a sequence* $\{x_n\}$ *by* $x_{n+1} = Tx_n$ *for all* $n \in \mathbb{N}$*. Then* $\alpha(x_m, x_n) \geq 1$ *for all* $m, n \in \mathbb{N}$ *with* $m < n$*.*

Definition 2.9 ([1]). Let X be a nonempty set and let $s \geq 1$ a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a *b-metric* if for all $x, y, z \in X$,

(i) $d(x,y) = 0$, if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

Then the pair (X,d) is called a *b-metric space.*

Note that a metric space is evidently a b-metric space but the converse is not generally true. For more details see [9].

In this paper, we use the following concepts in b-metric spaces.

Definition 2.10. Let (X,d) be a b-metric space and $\alpha : X \times X \to [0,+\infty)$. Then X is said to be an *α-complete b-metric space* if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ converges in X.

Definition 2.11. Let (X,d) be a b-metric space, $\alpha : X \times X \to [0,+\infty)$ and $T : X \to X$. Then T is said to be an *α-continuous mapping on (X,d)* if for every sequence $\{x_n\}$ with $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ implies $Tx_n \to Tx$ as $n \to \infty$.

In 2014, Rosa and Vetro [10] introduced the notion of triangular g-α-admissible mappings.

Definition 2.12. Let $T, g : X \to X$ and $\alpha : X \times X \to [0,\infty)$. Then T is said to be *triangular g-α-admissible* if

1. $\alpha(gx, gy) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$;
2. $\alpha(gx, gy) \geq 1$ and $\alpha(gy, gz) \geq 1$ imply $\alpha(gx, gz) \geq 1$.

Lemma 2.13 ([5]). *Let $T : X \to X$ be a triangular g-α-admissible. Assume that that there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$. Define a sequence $\{gx_n\}$ by $gx_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Then $\alpha(gx_m, gx_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$.*

Definition 2.14. Let $T, g : X \to X$. If $w = Tx = gx$ for some $x \in X$, then x is called a *coincidence point of T and g*, and w is called a point of coincidence of T and g.

Definition 2.15. Let $T, g : X \to X$. The pair $\{T, g\}$ is said to be *weakly compatible* if $Tgx = gTx$, whenever $Tx = gx$ for some x in X.

Abbas and Rhoades [11] proved the existence of the unique common fixed points of a pair of weakly compatible mappings by using the following proposition as a main tool.

Proposition 2.16 ([11]). *Let $T, g : X \to X$ and $\{T, g\}$ is weakly compatible. If T and g have a unique point of coincidence $w = Tx = gx$, then w is the unique common fixed point of T and g.*

3 Main results

In this section, unique fixed point theorems and unique common fixed point theorems in α-complete b-metric spaces and applications to integral equations are presented.

3.1 The unique fixed point theorems

We first introduce the concept of modified $(\alpha$-ψ-φ-$\theta)$-rational contractive mappings and prove the existence of fixed point theorems for such mappings.

Definition 3.1. Let (X, d) be a b-metric space and $\alpha : X \times X \to [0, \infty)$. A mapping $T : X \to X$ is a *modified $(\alpha$-ψ-φ-$\theta)$-rational contractive mapping* if there exists $L \geq 0$ such that for all $x, y \in X$,

$$\alpha(x,y) \geq 1 \text{ implies } \psi(s^3 d(Tx,Ty)) \leq \varphi(\psi(M_b(x,y))) + L\theta(N_b(x,y)), \tag{3.1}$$

where

$$M_b(x,y) = \max\{d(x,y), \frac{d(x,Tx)}{1+d(x,Tx)}, \frac{d(y,Ty)}{1+d(y,Ty)}, \frac{d(x,Ty)+d(y,Tx)}{2s}\},$$

$$N_b(x,y) = \min\{d(x,Tx), d(x,Ty), d(y,Tx)\}$$

and $\psi, \varphi, \theta : [0,\infty) \to [0,\infty)$ are continuous functions with $\varphi(t) < t$, $\theta(t) > 0$ for each $t > 0$, φ is nondecreasing, $\theta(0) = 0$, $\psi(t) = 0$ if and only if $t = 0$ and ψ is increasing.

Theorem 3.2. *Let (X, d) be an α-complete b-metric space, $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$ is a modified $(\alpha$-ψ-φ-$\theta)$-rational contractive mapping. Assume that the following conditions hold:*

(i) T is triangular α-orbital admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is α-continuous.

Then T has a fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ in X by

$$x_{n+1} = Tx_n \text{ for all } n \in \mathbb{N}.$$

By Lemma 2.8, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N}. \tag{3.2}$$

If $x_N = x_{N+1}$ for some $N \in \mathbb{N}$, then T has a fixed point. Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since T is a modified $(\alpha$-ψ-φ-$\theta)$-rational contraction and by (3.2), we obtain that

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &\leq \psi(s^3 d(x_n, x_{n+1}))\\ &= \psi(s^3 d(Tx_{n-1}, Tx_n))\\ &\leq \varphi(\psi(M_b(x_{n-1}, x_n))) + L\theta(N_b(x_{n-1}, x_n)) \qquad (3.3)\end{aligned}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned}N_b(x_{n-1}, x_n) &= \min\{d(x_{n-1}, Tx_{n-1}), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\}\\ &= \min\{d(x_{n-1}, x_n), d(x_{n-1}, x_{n+1}), d(x_n, x_n)\}\\ &= 0\end{aligned}$$

and

$$\begin{aligned}M_b(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, Tx_{n-1})}{1 + d(x_{n-1}, Tx_{n-1})}, \frac{d(x_n, Tx_n)}{1 + d(x_n, Tx_n)},\\ &\quad \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2s}\}\\ &= \max\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n)}{1 + d(x_{n-1}, x_n)}, \frac{d(x_n, x_{n+1})}{1 + d(x_n, x_{n+1})},\\ &\quad \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2s}\}\\ &\leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2s}\}.\end{aligned}$$

Since

$$\frac{d(x_{n-1}, x_{n+1})}{2s} \leq \frac{s[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]}{2s},$$

it follows that

$$M_b(x_{n-1}, x_n) \leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \qquad (3.4)$$

By (3.3) and (3.4), we obtain that

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &\leq \varphi(\psi(M_b(x_{n-1}, x_n))) + L\theta(N_b(x_{n-1}, x_n))\\ &\leq \varphi(\psi(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\})).\end{aligned}$$

If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, we have

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &\leq \varphi(\psi(d(x_n, x_{n+1})))\\ &< \psi(d(x_n, x_{n+1})),\end{aligned}$$

which is a contradiction. This implies that

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &\leq \varphi(\psi(d(x_{n-1}, x_n))) \\ &< \psi(d(x_{n-1}, x_n)), \end{aligned} \tag{3.5}$$

for each $n \in \mathbb{N}$. Since ψ is increasing, we get $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$ for each $n \in \mathbb{N}$. Therefore $\{d(x_n, x_{n+1})\}$ is a nonincreasing sequence. Consequently, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. We claim that $r = 0$. Assume that $r > 0$. Since ψ and φ are continuous, from (3.5), we have

$$\psi(r) \leq \varphi(\psi(r)) \leq \psi(r).$$

This implies that $\psi(r) = \varphi(\psi(r))$. Since $\varphi(t) < t$, for each $t > 0$, we obtain that

$$\psi(r) = \varphi(\psi(r)) < \psi(r),$$

which is a contradiction and therefore $r = 0$. It follows that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.6}$$

Next we will prove that the sequence $\{x_n\}$ is a Cauchy sequence. Suppose on the contrary, that there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, there exist two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) \geq k$ such that

$$d(x_{m(k)}, x_{n(k)}) \geq \varepsilon. \tag{3.7}$$

Let $n(k)$ be the smallest number satisfying (3.7). Thus

$$d(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{3.8}$$

By triangle inequality, (3.7) and (3.8), we obtain that

$$\begin{aligned}\varepsilon \leq d(x_{n(k)}, x_{m(k)}) &\leq sd(x_{n(k)}, x_{n(k)-1}) + sd(x_{n(k)-1}, x_{m(k)}) \\ &< sd(x_{n(k)}, x_{n(k)-1}) + s\varepsilon.\end{aligned}$$

By taking the upper limit as $k \to \infty$ and (3.6), we have

$$\varepsilon \leq \limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)}) \leq s\varepsilon. \tag{3.9}$$

Using triangle inequality again, we obtain that

$$\begin{aligned}\varepsilon \leq d(x_{m(k)}, x_{n(k)}) &\leq sd(x_{m(k)}, x_{n(k)+1}) + sd(x_{n(k)+1}, x_{n(k)}) \\ &\leq s^2 d(x_{m(k)}, x_{n(k)}) + s^2 d(x_{n(k)}, x_{n(k)+1}) + sd(x_{n(k)+1}, x_{n(k)}) \\ &\leq s^2 d(x_{m(k)}, x_{n(k)}) + (s^2 + s) d(x_{n(k)}, x_{n(k)+1}).\end{aligned}$$

From above inequality, we obtain that

$$\varepsilon \leq sd(x_{m(k)}, x_{n(k)+1}) + sd(x_{n(k)+1}, x_{n(k)}) \leq s^2 d(x_{m(k)}, x_{n(k)}) + (s^2+s) d(x_{n(k)}, x_{n(k)+1}).$$

Taking the upper limit as $k \to \infty$, by (3.6) and (3.9), we have

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) \leq s^2\varepsilon. \tag{3.10}$$

Similarly, we obtain that

$$\begin{aligned}\varepsilon \leq d(x_{n(k)}, x_{m(k)}) &\leq sd(x_{n(k)}, x_{m(k)+1}) + sd(x_{m(k)+1}, x_{m(k)})\\ &\leq s^2 d(x_{n(k)}, x_{m(k)}) + s^2 d(x_{m(k)}, x_{m(k)+1}) + sd(x_{m(k)+1}, x_{m(k)})\\ &\leq s^2 d(x_{n(k)}, x_{m(k)}) + (s^2+s) d(x_{m(k)}, x_{m(k)+1}).\end{aligned}$$

So from (3.6) and (3.9), we have

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)+1}) \leq s^2\varepsilon. \tag{3.11}$$

Since

$$d(x_{m(k)+1}, x_{n(k)}) \leq sd(x_{m(k)+1}, x_{n(k)+1}) + sd(x_{n(k)+1}, x_{n(k)}),$$

and by using (3.6) and (3.11), we get that

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}). \tag{3.12}$$

Using (3.6), (3.9), (3.10) and (3.11), we have

$$\begin{aligned}\limsup_{k\to\infty} M_b(x_{n(k)}, x_{m(k)}) &= \max\{\limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)}), \limsup_{k\to\infty} \frac{d(x_{n(k)}, Tx_{n(k)})}{1+d(x_{n(k)}, Tx_{n(k)})},\\ &\quad \limsup_{k\to\infty} \frac{d(x_{m(k)}, Tx_{m(k)})}{1+d(x_{m(k)}, Tx_{m(k)})},\\ &\quad \frac{\limsup_{k\to\infty} d(x_{n(k)}, Tx_{m(k)}) + \limsup_{k\to\infty} d(x_{m(k)}, Tx_{n(k)})}{2s}\}\\ &= \max\{\limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)}), \limsup_{k\to\infty} \frac{d(x_{n(k)}, x_{n(k)+1})}{1+d(x_{n(k)}, x_{n(k)+1})},\\ &\quad \limsup_{k\to\infty} \frac{d(x_{m(k)}, x_{m(k)+1})}{1+d(x_{m(k)}, x_{m(k)+1})},\\ &\quad \frac{\limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)+1}) + \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1})}{2s}\}\\ &\leq \max\{s\varepsilon, 0, 0, \frac{s^2\varepsilon + s^2\varepsilon}{2s}\} = s\varepsilon.\end{aligned}$$

This implies that

$$\limsup_{k\to\infty} M_b(x_{n(k)}, x_{m(k)}) \leq s\varepsilon. \tag{3.13}$$

By using the same argument as above, we have

$$\limsup_{k\to\infty} N_b(x_{n(k)}, x_{m(k)}) = 0. \tag{3.14}$$

Since T is a modified (α-ψ-φ-θ)-rational contraction, by using Lemma 2.8 and (3.12), we have

$$\begin{aligned}
\psi(s\varepsilon) = \psi(s^3 \cdot \frac{\varepsilon}{s^2}) &\leq \psi(s^3 \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1})) \\
&= \limsup_{k\to\infty} \psi(s^3 d(x_{m(k)+1}, x_{n(k)+1})) \\
&= \limsup_{k\to\infty} \psi(s^3 d(Tx_{m(k)}, Tx_{n(k)})) \\
&\leq \limsup_{k\to\infty} [\varphi(\psi(M_b(x_{m(k)}, x_{n(k)}))) + L\theta(N_b(x_{m(k)}, x_{n(k)}))] \\
&= \varphi(\psi(\limsup_{k\to\infty} M_b(x_{m(k)}, x_{n(k)}))) + L\theta(\limsup_{k\to\infty} N_b(x_{m(k)}, x_{n(k)})) \\
&\leq \varphi(\psi(s\varepsilon)) \\
&< \psi(s\varepsilon),
\end{aligned}$$

which is a contradiction. Then we can conclude that $\{x_n\}$ is a Cauchy sequence. From (3.2) and since X is an α-complete b-metric space, we have $\lim_{n\to\infty} x_n = x$ for some $x \in X$. Since T is α-continuous, we obtain that $\lim_{n\to\infty} Tx_n = Tx$. This implies that $\lim_{n\to\infty} d(x_{n+1}, Tx) = \lim_{n\to\infty} d(Tx_n, Tx) = 0$. Then T has a fixed point. □

Example 3.3. Let $X = [0,6)$ and $d : X \times X \to \mathbb{R}$ defined by $d(x,y) = |x-y|^2$. Then d is a b-metric on X with $s = 2$. Define $T : X \to X$ by

$$T(x) = \begin{cases} \frac{\sqrt{2}}{6}x, & \text{if } x \in [0,1] ; \\ \frac{1}{2}x, & \text{if } x \in (1,6), \end{cases}$$

and define $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x, y \in [0,1] ; \\ 0, & \text{if otherwise.} \end{cases}$$

Define $\psi, \varphi : [0,\infty) \to [0,\infty)$ by $\psi(t) = \frac{t}{2}$ and $\varphi(t) = \frac{4}{9}t$. For all $x, y \in X$ and $\alpha(x,y) \geq 1$, we have $x, y \in [0,1]$ and then

$$\begin{aligned}
\psi(s^3 d(Tx,Ty)) &= \frac{s^3 d(Tx,Ty)}{2} \\
&= \frac{2^3|\frac{\sqrt{2}}{6}x - \frac{\sqrt{2}}{6}y|^2}{2} \\
&= 4\Big|\frac{\sqrt{2}}{6}x - \frac{\sqrt{2}}{6}y\Big|^2 \\
&= 4 \cdot \frac{2}{36}|x-y|^2 \\
&= \frac{4}{9}\frac{|x-y|^2}{2}
\end{aligned}$$

$$\begin{aligned} &= \frac{4}{9}\frac{d(x,y)}{2} \qquad (3.15)\\ &= \frac{4}{9}\psi(d(x,y)) \\ &= \varphi(\psi(d(x,y))) \leq \varphi(\psi(M_b(x,y))). \end{aligned}$$

Then T is a modified (α-ψ-φ-θ)-rational contractive mapping. We next show that (X,d) is an α-complete b-metric. If $\{x_n\}$ is a Cauchy sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\{x_n\} \subseteq [0,1]$. Now, since $([0,1],d)$ is a complete b-metric space, then the sequence $\{x_n\}$ converges in $[0,1]$. We will show that T is α-continuous. If $x_n \to x$ as $n \to \infty$ and $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $x_n \in [0,1]$ for all $n \in \mathbb{N}$ and so

$$d(Tx_n,Tx) = |\frac{\sqrt{2}}{6}x_n - \frac{\sqrt{2}}{6}x|^2 = \frac{1}{18}|x_n - x|^2 = \frac{1}{18}d(x_n,x) \to 0 \text{ as } n \to \infty.$$

Let $\alpha(x,Tx) \geq 1$. Thus $x \in [0,1]$ and $Tx \in [0,1]$ and so $T^2x = T(Tx) \in [0,1]$. Then $\alpha(Tx,T^2x) \geq 1$. Thus T is α-orbital admissible. Let $\alpha(x,y) \geq 1$ and $\alpha(y,Ty) \geq 1$. We have $x,y,Ty \in [0,1]$. This implies that $\alpha(x,Ty) \geq 1$. Hence T is triangular α-orbital admissible. It is clear that condition(ii) of Theorem 3.2 is satisfied with $x_0 = 0$ since $\alpha(x_0,Tx_0) = \alpha(0,T(0)) = \alpha(0,0) = 1$. Thus all assumptions of Theorem 3.2 are satisfied and so T has a fixed point which is $x = 0$.

We next replace the α-continuity of the mapping T by some appropriate conditions.

Theorem 3.4. *Let (X,d) be an α-complete b-metric space and $\alpha : X \times X \to [0,\infty)$. Suppose that $T : X \to X$ is a modified (α-ψ-φ-θ)-rational contractive mapping. Assume that the following conditions hold:*

(i) T is triangular α-orbital admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x) \geq 1$ for all $k \in \mathbb{N}$.

Then T has a fixed point.

Proof. As in Theorem 3.2, we can construct the sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$, $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$. From condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\alpha(x_{n(k)},x) \geq 1 \text{ for all } k \in \mathbb{N}. \qquad (3.16)$$

We claim that x is a fixed point of T. Assume that $d(x,Tx) > 0$. By triangle inequality, we obtain that

$$\begin{aligned} d(x,Tx) &\leq sd(x,x_{n(k)+1}) + sd(x_{n(k)+1},Tx) \\ &= sd(x,x_{n(k)+1}) + sd(Tx_{n(k)},Tx). \end{aligned}$$

Taking limit $k \to \infty$ in above inequality, we have

$$d(x,Tx) \leq \lim_{k\to\infty} sd(Tx_{n(k)},Tx). \tag{3.17}$$

Since T is a modified (α-ψ-φ-θ)-rational contractive mapping, using (3.16) and (3.17), we have

$$\begin{aligned} \psi(s^2 d(x,Tx)) &\leq \lim_{k\to\infty} \psi(s^3 d(Tx_{n(k)},Tx)) \\ &\leq \lim_{k\to\infty} [\varphi(\psi(M_b(x_{n(k)},x))) + L\theta(N_b(x_{n(k)},x))] \\ &\leq \varphi(\psi(\lim_{k\to\infty} M_b(x_{n(k)},x))) + L\theta(\lim_{k\to\infty} N_b(x_{n(k)},x)), \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} M_b(x_{n(k)},x) &= \max\{d(x_{n(k)},x), \frac{d(x_{n(k)},Tx_{n(k)})}{1+d(x_{n(k)},Tx_{n(k)})}, \frac{d(x,Tx)}{1+d(x,Tx)}, \\ &\quad \frac{d(x_{n(k)},Tx)+d(x,Tx_{n(k)})}{2s}\} \\ &= \max\{d(x_{n(k)},x), \frac{d(x_{n(k)},x_{n(k)+1})}{1+d(x_{n(k)},x_{n(k)+1})}, \frac{d(x,Tx)}{1+d(x,Tx)}, \\ &\quad \frac{d(x_{n(k)},Tx)+d(x,x_{n(k)+1})}{2s}\} \\ &\leq \max\{d(x_{n(k)},x), d(x_{n(k)},x_{n(k)+1}), d(x,Tx) \\ &\quad \frac{d(x_{n(k)},Tx)+d(x,x_{n(k)+1})}{2s}\} \end{aligned}$$

and

$$\begin{aligned} N_b(x_{n(k)},x) &= \min\{d(x_{n(k)},Tx_{n(k)}), d(x_{n(k)},Tx), d(x,Tx_{n(k)}\} \\ &= \min d(x_{n(k)},x_{n(k)+1}), d(x_{n(k)},Tx), d(x,x_{n(k)+1})\}. \end{aligned}$$

Taking limit as $k \to \infty$, we obtain that

$$\lim_{k\to\infty} M_b(x_{n(k)},x) \leq \max\{d(x,Tx), \frac{d(x,Tx}{2}\} = d(x,Tx)$$

and

$$\lim_{k\to\infty} N_b(x_{n(k)},x) = 0.$$

By (3.18), we have

$$\begin{aligned}\psi(s^2d(x,Tx)) &\leq \varphi(\psi(\lim_{k\to\infty} M_b(x_{n(k)},x))) + L\theta(\lim_{k\to\infty} N_b(x_{n(k)},x))\\ &\leq \varphi(\psi(d(x,Tx)))\\ &< \psi(d(x,Tx)),\end{aligned}$$

which is a contradiction because $s \geq 1$. Then $d(x,Tx) = 0$ and hence x is a fixed point of T. □

For the uniqueness of a fixed point of a modified (α-ψ-φ-θ)-rational contractive mapping, we investigate some conditions introduced in [5].

Theorem 3.5. *Suppose that all hypotheses of Theorem 3.2 (respectively Theorem 3.4) hold. Assume that either $\alpha(u,v) \geq 1$ or $\alpha(v,u) \geq 1$ whenever $Tu = u$ and $Tv = v$. Then T has a unique fixed point.*

Proof. Assume that w and z are fixed points of T with $w \neq z$. By assumption, we have

$$\alpha(w,z) \geq 1 \text{ or } \alpha(z,w) \geq 1.$$

Suppose that $\alpha(w,z) \geq 1$. Since T is a modified (α-ψ-φ-θ)-rational contractive mapping, we have

$$\begin{aligned}\psi(s^3(d(w,z))) &= \psi(s^3(d(Tw,Tz)))\\ &\leq \varphi(\psi(M_b(w,z))) + L\theta(N_b(w,z))),\end{aligned}$$

where

$$\begin{aligned}M_b(w,z) &= \max\{d(w,z), \frac{d(w,Tw)}{1+d(w,Tw)}, \frac{d(z,Tz)}{1+d(z,Tz)}, \frac{d(w,Tz)+d(z,Tw)}{2s}\}\\ &= \max\{d(w,z), \frac{d(w,w)}{1+d(w,w)}, \frac{d(z,z)}{1+d(z,z)}, \frac{d(w,z)+d(z,w)}{2s}\}\\ &= d(w,z)\end{aligned}$$

and

$$N_b(w,z) = \min\{d(w,Tw), d(w,Tz), d(z,Tw)\} = 0.$$

Then

$$\begin{aligned}\psi(s^3(d(w,z))) &= \varphi(\psi(d(w,z)))\\ &< \psi(d(w,z)\end{aligned}$$

which is a contradiction because $s \geq 1$. Thus $w = z$. Similarly, if $\alpha(z,w) \geq 1$, then we can prove that $w = z$. Hence T has a unique fixed point. □

In Theorem 3.5, if we take $\psi(t) = t$ for all $t \in [0,\infty)$, then we immediately obtain the following result.

Corollary 3.6. *Let (X,d) be an α-complete b-metric space where $\alpha : X \times X \to [0,\infty)$ and $T : X \to X$. Assume that there exists $L \geq 0$ such that for all $x, y \in X$,*

$$\alpha(x,y) \geq 1 \text{ implies } s^3 d(Tx,Ty) \leq \varphi(M_b(x,y)) + L\theta(N_b(x,y)), \tag{3.19}$$

where

$$M_b(x,y) = \max\{d(x,y), \frac{d(x,Tx)}{1+d(x,Tx)}, \frac{d(y,Ty)}{1+d(y,Ty)}, \frac{d(x,Ty)+d(y,Tx)}{2s}\},$$

$$N_b(x,y) = \min\{d(x,Tx), d(x,Ty), d(y,Tx)\}$$

and $\varphi, \theta : [0,\infty) \to [0,\infty)$ are continuous functions such that $\theta(0) = 0$, $\varphi(t) < t$, $\theta(t) > 0$ for each $t > 0$ and φ is nondecreasing. Assume that the following conditions hold:

(i) T is triangular α-orbital admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is α-continuous or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all $k \in \mathbb{N}$.

Then T has a fixed point. Moreover, either $\alpha(u,v) \geq 1$ or $\alpha(v,u) \geq 1$ whenever $Tu = u$ and $Tv = v$. Then T has a unique fixed point.

In Corollary 3.6, if $\varphi(t) = t - \varphi'(t)$ for all $t \in [0,\infty)$ where $\varphi' : [0,\infty) \to [0,\infty)$ is continuous such tha $\varphi'(t) < t$ for each $t > 0$ and φ' is nonincreasing and $L = 0$, then we obtain the following corollary.

Corollary 3.7. *Let (X,d) be an α-complete b-metric space where $\alpha : X \times X \to [0,\infty)$. Suppose that $T : X \to X$ is a mapping such that for all $x, y \in X$,*

$$\alpha(x,y) \geq 1 \text{ implies } s^3 d(Tx,Ty) \leq M_b(x,y) - \varphi'(M_b(x,y)), \tag{3.20}$$

where

$$M_b(x,y) = \max\{d(x,y), \frac{d(x,Tx)}{1+d(x,Tx)}, \frac{d(y,Ty)}{1+d(y,Ty)}, \frac{d(x,Ty)+d(y,Tx)}{2s}\},$$

and $\varphi' : [0,\infty) \to [0,\infty)$ is continuous such that $\varphi'(0) = 0$, $\varphi'(t) < t$ for each $t > 0$ and φ' is nonincreasing Assume that the following conditions hold:

(i) T is triangular α-orbital admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is α-continuous or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all $k \in \mathbb{N}$.

Then T has a fixed point. Moreover, either $\alpha(u,v) \geq 1$ or $\alpha(v,u) \geq 1$ whenever $Tu = u$ and $Tv = v$. Then T has a unique fixed point.

3.2 The unique of common fixed point theorems

In this section, we introduce the concept of modified (α-ψ-φ-θ)-rational contractive mappings with respect to g and prove the the existence of unique common fixed point theorems in α-complete b-metric spaces.

Definition 3.8. Let (X,d) be a b-metric space, $\alpha : X \times X \to [0,\infty)$, and $T, g : X \to X$. We say that $T : X \to X$ is a *modified (α-ψ-φ-θ)-rational contractive mapping with respect to g* if there exists $L \geq 0$ such that for all $x, y \in X$,

$$\alpha(x,y) \geq 1 \text{ implies } \psi(s^3 d(Tx,Ty)) \leq \varphi(\psi(M_b(x,y))) + L\theta(N_b(x,y)), \quad (3.21)$$

where

$$M_b(x,y) = \max\{d(gx,gy), \frac{d(gx,Tx)}{1+d(gx,Tx)}, \frac{d(gy,Ty)}{1+d(gy,Ty)}, \frac{d(gx,Ty)+d(gy,Tx)}{2s}\},$$

$$N_b(x,y) = \min\{d(gx,Tx), d(gx,Ty), d(gy,Tx)\}$$

and $\psi, \varphi, \theta : [0,\infty) \to [0,\infty)$ are continuous functions with $\varphi(t) < t$, $\theta(t) > 0$ for each $t > 0$, φ is nondecreasing, $\theta(0) = 0$, $\psi(t) = 0$ if and only if $t = 0$ and ψ is increasing.

Definition 3.9. Let (X,d) be a b-metric space and $\alpha : X \times X \to [0,+\infty)$ and $T, g : X \to X$. Then T is said to be *α-continuous with respect to g*, if for each sequence $\{gx_n\}$ with $gx_n \to gx$ as $n \to \infty$, $\alpha(gx_n, gx_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, we have $Tx_n \to Tx$ as $n \to \infty$.

Theorem 3.10. *Let (X,d) be an α-complete b-metric space and $T, g : X \to X$ be such that $TX \subseteq gX$ and suppose that gX is closed. Let $\alpha : X \times X \to [0,\infty)$ and T is a modified (α-ψ-φ-θ)-rational contractive mapping with respect to g. Assume that the following conditions hold:*

(i) T is triangular g-α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$;

(iii) T is α-continuous with respect to g.

Then T and g have a coincidence point.

Proof. Let $x_0 \in X$ be such that $\alpha(gx_0, Tx_0) \geq 1$. Since $TX \subseteq gX$, we can construct a sequence $\{gx_n\}$ such that

$$gx_{n+1} = Tx_n \text{ for all } n \in \mathbb{N}.$$

By using Lemma 2.13, we have

$$\alpha(gx_n, gx_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N}. \quad (3.22)$$

By the analogous proof as in Theorem 3.2, we can prove that $\{gx_n\}$ is a Cauchy sequence. Since $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and X is an α-complete b-metric space, we have $\{gx_n\}$ converges to $z \in gX$. Thus there exists $x \in X$ such that $\lim_{n\to\infty} gx_n = gx$. Since T is α-continuous with respect to g, so $Tx = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_{n+1} = gx$. Then x is a coincidence point of T and g. □

We replace the α-continuity of the mapping T with respect to g by some appropriate conditions.

Theorem 3.11. *Let (X, d) be an α-complete b-metric space and $T, g : X \to X$ be such that $TX \subseteq gX$ and suppose that gX is closed. Let $\alpha : X \times X \to [0, \infty)$ and T is a modified (α-ψ-φ-θ)-rational contractive mapping with respect to g. Assume that the following conditions hold:*

(i) T is triangular g-α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$;

(iii) if $\{gx_n\}$ is a sequence in X such that $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $gx_n \to gx$ as $n \to \infty$, then there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $\alpha(gx_{n(k)}, gx) \geq 1$ for all $k \in \mathbb{N}$.

Then T and g have a coincidence point.

Proof. As in the proof of Theorem 3.10, we can construct the sequence $\{gx_n\}$ with $Tx_n = gx_{n+1}$ for all $n \in \mathbb{N}$, $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} gx_n = gx$. By (iii), there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $\alpha(gx_{n(k)}, gx) \geq 1$, for all $k \in \mathbb{N}$. By the analogous proof as in Theorem 3.4, we obtain that T and g have a coincidence point. □

For the uniqueness of a common fixed point, we add some appropriate conditions to the hypotheses.

Theorem 3.12. *Suppose that all hypotheses of Theorem 3.10 (respectively Theorem 3.11) hold. Assume that the following conditions hold:*

(i) the pair $\{T, g\}$ is weakly compatible;

(ii) either $\alpha(u, v) \geq 1$ or $\alpha(v, u) \geq 1$ whenever $Tu = gu$ and $Tv = gv$.

Then T and g have a unique common fixed point.

Proof. Assume that $Tu = gu$ and $Tv = gv$. We will show that $gu = gv$. Suppose that $gu \neq gv$. Therefore $\alpha(u, v) \geq 1$ or $\alpha(v, u) \geq 1$. Suppose that $\alpha(u, v) \geq 1$. It follows that

$$\psi(s^3 d(gu, gv)) = \psi(s^3 d(Tu, Tv)) \leq \varphi(\psi(M_b(u, v))) + L\theta(N_b(u, v)),$$

where

$$\begin{aligned} M_b(u,v) &= \max\{d(gu,gv), \frac{d(gu,Tu)}{1+d(gu,Tu)}, \frac{d(v,Tv)}{1+d(v,Tv)}, \frac{d(gu,Tv)+d(gv,Tu)}{2s}\} \\ &= \max\{d(gu,gv), \frac{d(gu,gu)}{1+d(gu,gu)}, \frac{d(gv,gv)}{1+d(gv,gv)}, \frac{d(gu,gv)+d(gv,gu)}{2s}\} \\ &= d(gu,gv) \end{aligned}$$

and

$$N_b(u,v) = \min\{d(gu,Tu), d(gu,Tv), d(gv,Tu)\} = 0.$$

This implies that

$$\begin{aligned} \psi(s^3(d(gu,gv))) &\leq \varphi(\psi(d(gu,gv))) \\ &< \psi(d(gu,gv)) \end{aligned}$$

which is a contradiction because $s \geq 1$. Thus $gu = gv$. Similarly, if $\alpha(v,u) \geq 1$, we can prove that $gu = gv$. This implies that T and g have a unique point of coincidence. Since the pair $\{T, g\}$ is weakly compatible and by Theorem 2.16, we can conclude that T and g have a unique common fixed point. □

Corollary 3.13. *Let (X,d) be an α-complete b-metric space with respect to g and $T, g : X \to X$ be such that $TX \subseteq gX$. Assume that gX is closed and there exist $\alpha : X \times X \to \mathbb{R}$ and $L \geq 0$ such that for all $x, y \in X$,*

$$\alpha(x,y) \geq 1 \text{ implies } s^3 d(Tx,Ty) \leq \varphi(M_b(x,y)) + L\theta(N_b(x,y)), \tag{3.23}$$

where

$$M_b(x,y) = \max\{d(gx,gy), \frac{d(gx,Tx)}{1+d(gx,Tx)}, \frac{d(gy,Ty)}{1+d(gy,Ty)}, \frac{d(gx,Ty)+d(gy,Tx)}{2s}\},$$

$$N_b(x,y) = \min\{\frac{d(gx,Tx)}{1+d(g,Tx)}, \frac{d(gx,Ty)}{1+d(gx,Ty)}, \frac{d(gy,Tx)}{1+d(gy,Tx)}\}$$

and $\varphi, \theta : [0,\infty) \to [0,\infty)$ are continuous functions such that $\theta(0) = 0$, $\varphi(t) < t$, $\theta(t) > 0$ for each $t > 0$. Assume that the following conditions hold:

(i) T is triangular g-α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$;

(iii) T is α-continuous with respect to g or if $\{gx_n\}$ is a sequence in X such that $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $gx_n \to gx$ as $n \to \infty$, then there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $\alpha(gx_{n(k)}, gx) \geq 1$ for all $k \in \mathbb{N}$.

Then T and g have a coincidence point. Moreover, assume that the following conditions hold:

(iv) the pair $\{T, g\}$ is weakly compatible;

(v) either $\alpha(u,v) \geq 1$ or $\alpha(v,u) \geq 1$ whenever $Tu = gu$ and $Tv = gv$.

Then T and g have a unique common fixed point.

3.3 Applications to integral equations

In this section, we prove the existence of a solution of a nonlinear quadratic integral equation taken from Allahari et al. [12].

Let $C(I)$ be the set of all continuous functions defined on $I = [0,1]$ and $\rho : C(I) \times C(I) \to \mathbb{R}$ defined by

$$\rho(x,y) = \sup_{t \in I} |x(t) - y(t)| \text{ for } x, y \in C(I).$$

Let $p \geq 1$. We define $d : C(I) \times C(I) \to \mathbb{R}$ defined by

$$d(x,y) = (\rho(x,y))^p = (\sup_{t \in I} |x(t) - y(t)|)^p = \sup_{t \in I} |x(t) - y(t)|^p \text{ for all } x, y \in C(I).$$

It is well known that (X,d) is a complete b-metric space with $s = 2^{p-1}$ (see [13]). Let Γ be the set of functions $\gamma : [0,+\infty) \to [0,+\infty)$ which satisfy the following conditions:

(i) γ is nondecreasing and $(\gamma(t))^p \leq \gamma(t^p)$ for all $p \geq 1$;

(ii) There exists $\varphi : [0,+\infty) \to [0,+\infty)$ which is nonincreasing and continuous, $\varphi(t) < t$ for all $t > 0$ such that $\gamma(t) = t - \varphi(t)$ for all $t \in [0,+\infty)$.

Consider the nonlinear quadratic equation as follows:

$$x(t) = h(t) + \lambda \int_0^1 k(t,s) f(s, x(s)) ds, \; t \in I, \; \lambda \geq 0. \tag{3.24}$$

Suppose that the following conditions hold:

(A1) $h : I \to \mathbb{R}$ is continuous;

(A2) $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(t,x) \geq 0$ and there exist $L \geq 0$, $\gamma \in \Gamma$ and a function $\xi : \mathbb{R}^2 \to \mathbb{R}$ such that for all $t \in I$, for all $a, b \in \mathbb{R}$ with $\xi(a,b) \geq 0$,

$$|f(t,a) - f(t,b)| \leq L\gamma(|a-b|);$$

(A3) $k : I \times I \to \mathbb{R}$ is continuous at $t \in I$ for every $s \in I$ and measurable at $s \in I$ for all $t \in I$ such that $k(t,s) \geq 0$ and $\int_0^1 k(t,s) ds \leq K$;

(A4) $\lambda^p K^p L^p \leq \frac{1}{2^{3p-3}}$;

(A5) there exists $x_0 \in C(I)$ such that for all $t \in I$,

$$\xi(x_0(t), h(t) + \lambda \int_0^1 k(t,s) f(s, x_0(s)) ds) \geq 0;$$

(A6) for all $t \in I$ and for all $x, y, z \in C(I)$,

$$\xi(x(t), y(t)) \geq 0 \text{ and } \xi(y(t), z(t)) \geq 0 \text{ imply } \xi(x(t), z(t)) \geq 0;$$

(A7) for all $t \in I$ and for all $x, y \in C(I)$,

$$\xi(x(t), y(t)) \geq 0 \text{ implies } \xi(h(t)+\lambda \int_0^1 k(t,s)f(s,x(s))ds, h(t)+\lambda \int_0^1 k(t,s)f(s,y(s))ds) \geq 0;$$

(A8) if $\{x_n\}$ is a sequence in $C(I)$ such that $x_n \to x \in C(I)$ and $\xi(x_n(t), x_{n+1}(t)) \geq 0$ for all $n \in \mathbb{N}$ and for all $t \in I$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\xi(x_{n(k)}(t), x(t)) \geq 0$ for all $k \in \mathbb{N}$ and for all $t \in I$.

Theorem 3.14. *Under assumptions (A1)-(A8), the integral equation (3.24) has a solution in $C(I)$.*

Proof. Let $T : C(I) \to C(I)$ be defined by

$$T(x)(t) = h(t) + \lambda \int_0^1 k(t,s)f(s,x(s))ds \text{ for } t \in I.$$

Let $x, y \in C(I)$ such that $\xi(x(t), y(t)) \geq 0$ for all $t \in I$. Therefore

$$\begin{aligned}
|T(x)(t) - T(y)(t)| &= |h(t) + \lambda \int_0^1 k(t,s)f(s,x(s))ds - h(t) - \lambda \int_0^1 k(t,s)f(s,y(s))ds| \\
&\leq \lambda \int_0^1 k(t,s)|f(s,x(s)) - f(s,y(s))|ds \\
&\leq \lambda \int_0^1 k(t,s)L\gamma(|x(s) - y(s)|)ds.
\end{aligned}$$

Since γ is nondecreasing, we obtain that

$$\gamma(|x(s) - y(s)|) \leq \gamma(\sup_{s \in I}|x(s) - y(s)|) = \gamma(\rho(x,y)).$$

This implies that

$$|T(x)(t) - T(y)(t)| \leq \lambda KL\gamma(\rho(x,y)).$$

Therefore

$$\begin{aligned}
d(Tx, Ty) &= \sup_{t \in I}|T(x)(t) - T(y)(t)|^p \\
&\leq [\lambda KL\gamma(\rho(x,y))]^p \\
&\leq \lambda^p K^p L^p \gamma(d(x,y)) \\
&\leq \lambda^p K^p L^p \gamma(M(x,y)) \\
&\leq \lambda^p K^p L^p [M(x,y) - \varphi(M(x,y))] \\
&\leq \frac{1}{2^{3p-3}}[M(x,y) - \varphi(M(x,y))],
\end{aligned}$$

for all $x, y \in C(I)$ such that $\xi(x(t), y(t)) \geq 0$ for all $t \in I$. We next define $\alpha : C(I) \times C(I) \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } \xi(x(t), y(t)) \geq 0,\ t \in I \\ 0, & \text{otherwise.} \end{cases}$$

Let $x, y \in C(I)$ be such that $\alpha(x,y) \geq 1$. It follows that $\xi(x(t), y(t)) \geq 0$ for all $t \in I$. This yields

$$s^3 d(Tx, Ty) \leq M(x,y) - \varphi(M(x,y).$$

This implies that T satisfies the contractive condition in Corollary 3.7. Using (A7), for each $x \in C(I)$ such that $\alpha(x, Tx) \geq 1$ we obtain that $\xi(Tx(t), T^2x(t)) \geq 0$. This implies that $\alpha(Tx, T^2x) \geq 1$. Let $x, y \in C(I)$ be such that $\alpha(x,y) \geq 1$ and $\alpha(y, Ty) \geq 1$. Thus

$$\xi(x(t), y(t)) \geq 0 \text{ and } \xi(y(t), Ty(t)) \geq 0 \text{ for all } t \in I.$$

By applying (A6), we obtain that $\xi(x(t), Ty(t)) \geq 0$ and so $\alpha(x, Ty) \geq 1$. It follows that T is triangular α-orbital admissible. Using (A5), there exists $x_0 \in C(I)$ such that $\alpha(x_0, Tx_0) \geq 1$. Let $\{x_n\}$ be a sequence in $C(I)$ such that $x_n \to x \in C(I)$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. By (A8), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\xi(x_{n(k)}(t), x(t)) \geq 0$. This implies that $\alpha(x_{n(k)}, x) \geq 1$. Therefore all assumptions in Corollary 3.7 are satisfied. Hence T has a fixed point in $C(I)$ that is a solution of the integral equation (3.24). □

Corollary 3.15. *Assume that the following conditions hold:*

(i) $h : I \to \mathbb{R}$ *is a continuous;*

(ii) $f : I \times \mathbb{R} \to [0, \infty)$ *is continuous and nondecreasing and* $f(t,s) \geq 0$.

(iii) there exist $L \geq 0$ *and* $\gamma \in \Gamma$ *such that for all* $t \in I$, *for all* $a, b \in \mathbb{R}$ *with* $a \leq b$, *we have*

$$|f(t,a) - f(t,b)| \leq L\gamma(|a-b|);$$

(iv) $k : I \times I \to \mathbb{R}$ *is continuous at* $t \in I$ *for every* $s \in I$ *and measurable at* $s \in I$ *for all* $t \in I$ *such that* $k(t,s) \geq 0$ *and* $\int_0^1 k(t,s)ds \leq K$;

(v) $\gamma^p K^p L^p \leq \frac{1}{2^{3p-3}}$;

(vi) there exists $x_0 \in C([0,1])$ *such that for all* $t \in I$, *we have*

$$x_0(t) \leq h(t) + \lambda \int_0^1 k(t,s) f(s, x_1(s))ds.$$

Then (3.24) has a solution in $C(I)$.

Proof. Define a mapping $\xi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\xi(a,b) = b - a \quad \text{for all } a, b \in \mathbb{R}.$$

By the analogous proof as in Theorem 3.14, we obtain that (3.24) has a solution in $C(I)$. □

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Thai Journal of Mathematics

Country	Thailand
Subject Area	Mathematics
Subject Category	Mathematics (miscellaneous)
Publisher	Chiang Mai University

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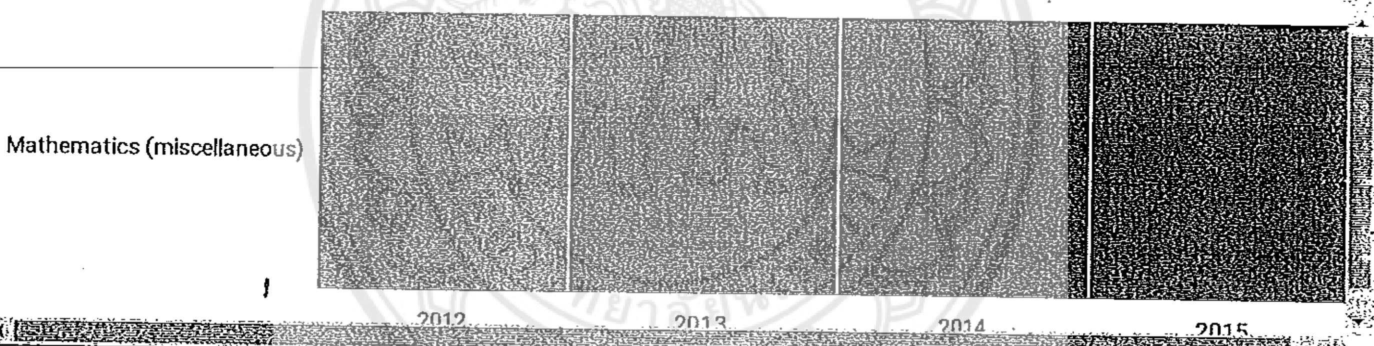
H Index

Publication type	Journals
ISSN	16860209
Coverage	2011-ongoing

Quartiles



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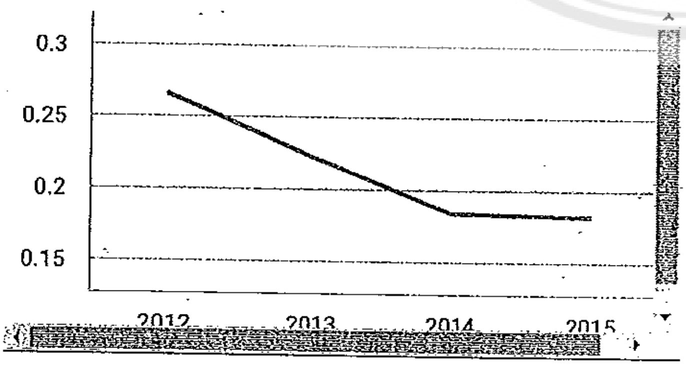

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