ตัวผกผันนัยทั่วไปของเมทริกซ์คอมแพเนียนสองชั้นและเมทริกซ์เลสลีย์สองชั้นชนิดผืนผ้า

(The Generalized Inverse of a Rectangular Doubly Companion Matrix and a Doubly Leslie Matrix)

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ชื่อโครงการ ตัวผกผันนัยทั่วไปของเมทริกซ์คอมแพเนียนสองชั้นและเมทริกซ์เลสลีย์สองชั้นชนิดผืนผ้า
(The Generalized Inverse of a Rectangular Doubly Companion Matrix and a Doubly Leslie Matrix)

ชื่อผู้วิจัย รองศาสตราจารย์วิวรรธน์ วณิชาภิชาติ

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บทคัดย่อ

เมทริกซ์เลสลีย์สองชั้น (doubly Leslie matrix) เป็นเมทริกซ์ที่มีสมาชิกเป็นจำนวนจริงซึ่งมีรูปแบบดังนี้

$$L = \begin{bmatrix} -\mathrm{p}^T & -a_n - b_n \\ \Lambda & -\mathrm{q} \end{bmatrix}_{(n,n)}$$

เมื่อ $a_n, b_n \in \mathbb{R}$, $\mathrm{p}, \mathrm{q} \in \mathbb{R}^{n-1}$, และ $\Lambda = \operatorname{diag}(s_1, s_2, \ldots, s_{n-1})$ เป็นเมทริกซ์ทแยงมุมที่มีขนาด $n-1$ เมทริกซ์เลสลีย์สองชั้นเป็นนัยทั่วไปของเมทริกซ์คอมแพเนียนสองชั้น (doubly companion matrix) ของเมทริกซ์เลสลีย์ และของเมทริกซ์คอมแพเนียน งานวิจัยนี้ได้ศึกษาหารูปแบบชัดแจ้งของตัวผกผันนัยทั่วไปแบบมัวร์-พินโรสและแบบกรุป (group inverse) ของเมทริกซ์คอมแพเนียนสองชั้นและเมทริกซ์เลสลีย์สองชั้น และได้ศึกษารูปแบบตัวผกผันทั่วไปแบบมัวร์-พินโรสของเมทริกซ์เลสลีย์สองชั้นชนิดผืนผ้าด้วย (rectangle doubly Leslie matrix)

ชื่อโครงการ	ตัวผกผันนัยทั่วไปของเมทริกซ์คอมแพเนียนสองชั้นและเมทริกซ์เลสลีย์สองชั้นชนิดผืนผ้า (The Generalized Inverse of a Rectangular Doubly Companion Matrix and a Doubly Leslie Matrix)
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ABSTRACT

A doubly Leslie matrix is a bordered real matrix of the form

$$L = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}_{(n,n)}$$

where $a_n, b_n \in \mathbb{R}$, $\mathbf{p}, \mathbf{q} \in \mathbb{R}^{n-1}$, and $\Lambda = \text{diag}(s_1, s_2, \ldots, s_{n-1})$ is a diagonal matrix of order $n-1$. The matrix L is a closed form of a doubly companion matrix, a Leslie matrix and a companion matrix. This paper is discussed the explicit formula of the Moore-Penrose inverse and the group inverse of the doubly leslie matrix. In general the Moore-Penrose inverse of a rectangle doubly Leslie matrix is also discussed.

LIST OF CONTENTS

CHAPTER I

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

Let $\mathbb{R}$ and $\mathbb{C}$ be the field of real numbers and complex numbers respectively. The set of all polynomials in x over $\mathbb{C}$ is denoted by $\mathbb{C}[x]$. For a positive integer n, let M_n be the set of all $n \times n$ matrices over $\mathbb{C}$. The set of all complex vectors, or $n \times 1$ matrices over $\mathbb{C}$ is denoted by $\mathbb{C}^n$.

Let

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

be a polynomial with coefficients over an arbitrary field. As is well known, the matrix

$$C = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \ldots & -a_1 & -a_0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}$$

has the property that

$$\det(xI - C) = p(x).$$

The matrix A, or some of its modifications, is being called companion matrix of the polynomial $p(x)$ since its characteristic polynomial is $p(x)$.

Companion matrix appear in literature in several forms. To illustrate, consider the companion matrix

$$C_1 := \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & -a_n \\ 1 & 0 & \cdots & 0 & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & 0 & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_2 \\ 0 & 0 & \cdots & 0 & 1 & -a_1 \end{bmatrix} \in M_n$$

using the permutation matrix P of order n, the "backward identity" permutation. Since $P = P^T = P^{-1}$,

$$P^{-1}C_1P = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & -a_n \\ 1 & 0 & \cdots & 0 & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & 0 & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_2 \\ 0 & 0 & \cdots & 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{1.1}$$

The companion matrix C_1 is seen to be similar to the following matrix

$$C_2 := \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 & 0 \\ -a_2 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n-2} & 0 & 0 & \cdots & 1 & 0 \\ -a_{n-1} & 0 & 0 & \cdots & 0 & 1 \\ -a_n & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in M_n.$$

Moreover, any matrix is similar to it transpose (see, e.g. [6, pp.134-135]), thus the following companion matrices

$$C_3 := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix} \in M_n,$$

$$C_4 := \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \in M_n$$

similar to the matrix C_1, C_2 respectively. Therefore the companion matrices C_i, $i = 1, 2, 3, 4$ have the same characteristic polynomial:

$$\begin{aligned} p(x) &= \det(xI - C_i) \\ &= t^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n. \end{aligned}$$

Butcher and Wright [4, p.363] defined a doubly companion matrix for the pair of polynomials $\alpha(x) = x^n - a_1 x^{n-1} - a_2 x^{n-2} - \cdots - \alpha_n$ and $\beta(x) = x^n - b_1 x^{n-1} - b_2 x^{n-2} - \cdots - b_n$, as $C \in M_n$ given by

$$C = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \ldots & -a_{n-1} & -a_n - b_n \\ 1 & 0 & 0 & \ldots & 0 & -b_{n-1} \\ 0 & 1 & 0 & \ldots & 0 & -b_{n-2} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -b_2 \\ 0 & 0 & 0 & \ldots & 1 & -b_1 \end{bmatrix}, \qquad (1.2)$$

that is, a $n \times n$ matrix C with $n > 1$ is called a doubly companion matrix if its entries c_{ij} satisfy $c_{ij} = 1$ for all entries in the sub-maindiagonal of C and else $c_{ij} = 0$ for $i \neq 1$ and $j \neq n$, which is a special case of unreduced upper Hessenberg matrix. Butcher and Wright used the doubly companion matrices as a tool for analyzing various extension of classical methods with inherent Runge-Kutta stability. The doubly companion matrices is important for application in some certain matrix equations, numerical and linear methods.

It is well known that the companion matrices are nonderogatory. The matrix C is also nonderogatory, that is the characteristic polynomial $c_C(t)$ is equal to the minimal polynomial $m_C(t)$, see [10] for more details.

One of the most popular models of population growth is a matrix-based model, first introduced by P. H. Leslie. In 1945, he published his most famous article in Biometrika, a journal. The article was entitled, On the use of matrices in certain population mathematics [1, pp. 117—120]. The Leslie model describes the growth of the female portion of a population which is assumed to have a maximum lifespan. The females are divided into age classes all of which span an equal number of years. Using data about the average birthrates and survival probabilities of each class, the model is then able to determine the growth of the population over time, [9].

Chen and Li in [5] asserted that, Leslie matrix models are discrete models for the development of age-structured populations. It is known that eigenvalues of a Leslie matrix are important in describing the asymptotic behavior of the corresponding population model. It is also known that the ratio of the spectral radius and the second largest (subdominant) eigenvalue in modulus of a non-periodic Leslie matrix determines the rate of convergence of the corresponding population distributions to a stable age distribution.

A Leslie matrix arises in a discrete, age-dependent model for population growth. It is a matrix of the form

$$L = \begin{bmatrix} r_1 & r_2 & r_3 & \dots & r_{n-1} & r_n \\ s_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & s_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \dots & s_{n-1} & 0 \end{bmatrix}, \tag{1.3}$$

where $r_j \geq 0$, $0 < s_j \leq 1$, $j = 1, 2, \dots, n-1$.

We define a doubly Leslie matrix analogous as the doubly companion matrix by replacing the subdiagonal of the doubly companion matrix by $s_1, s_2, \dots, s_{n-1}$ where s_j, $j = 1, 2, \dots, n-1$, respectively, and denoted by $\mathbf{L}$, that is, a doubly Leslie matrix is defined to be a matrix as follows

$$\mathbf{L} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \dots & -a_{n-1} & -a_n - b_n \\ s_1 & 0 & 0 & \dots & 0 & -b_{n-1} \\ 0 & s_2 & 0 & \dots & 0 & -b_{n-2} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -b_2 \\ 0 & 0 & 0 & \dots & s_{n-1} & -b_1 \end{bmatrix}, \tag{1.4}$$

where $a_j, b_j \in \mathbb{R}$, the real numbers, $j = 1, 2, \dots, n$. As the Leslie matrix, we restriction only $s_j > 0$, $j = 1, 2, \dots, n-1$.

For convenience, we can be written the matrix $\mathbf{L}$ in a partitioned form as

$$\mathbf{L} = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}_{(n,n)} \quad where \quad \mathbf{p} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \end{bmatrix},$$

and $\Lambda = \operatorname{diag}(s_1, s_2, \dots, s_{n-1})$ is a diagonal matrix of order $n-1$.

Note: If we define the doubly Leslie matrix in an another form such as $\mathbf{L} = \begin{bmatrix} \mathbf{p}^T & a_n + b_n \\ \Lambda & \mathbf{q} \end{bmatrix}_{(n,n)}$, where all symbols are as above, then some consequence productions will be complicates forms.

We define a doubly Leslie matrix analogous as the doubly companion matrix by replacing the subdiagonal of the doubly companion matrix by $s_1, s_2, \ldots, s_{n-1}$ where s_j, $j = 1, 2, \ldots, n-1$, respectively, and denoted by **L**, that is, a doubly Leslie matrix is defined to be a matrix as follows

$$\mathbf{L} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \ldots & -a_{n-1} & -a_n - b_n \\ s_1 & 0 & 0 & \ldots & 0 & -b_{n-1} \\ 0 & s_2 & 0 & \ldots & 0 & -b_{n-2} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -b_2 \\ 0 & 0 & 0 & \ldots & s_{n-1} & -b_1 \end{bmatrix}, \tag{1.5}$$

where $a_j, b_j \in \mathbb{R}$, the real numbers, $j = 1, 2, \ldots, n$. As the Leslie matrix, we restriction only $s_j > 0$, $j = 1, 2, \ldots, n-1$.

For convenience, we can be written the matrix **L** in a partitioned form as

$$\mathbf{L} = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}_{(n,n)} \quad where \quad \mathbf{p} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \end{bmatrix},$$

and $\Lambda = \operatorname{diag}(s_1, s_2, \ldots, s_{n-1})$ is a diagonal matrix of order $n-1$.

We also define a rectangular doubly Leslie matrix of order $m \times n$, where

$m = n - k$ and $1 < k < n$ as follows:

$$R = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \dots & -a_{n-1} & -a_n - b_n \\ s_1 & 0 & 0 & \dots & 0 & -b_{n-1} \\ 0 & s_2 & 0 & \dots & 0 & -b_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & s_{n-k} & 0 & -b_k \end{bmatrix}_{(m,n)} . \quad (1.6)$$

For convenience, we can be written the matrix L in a partitioned form as

$$R = \begin{bmatrix} -\mathrm{p}^T & -a_n - b_n \\ \Lambda_k & -\mathrm{q}_k \end{bmatrix}_{(m,n)} \quad \text{where } \mathrm{p} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}, \mathrm{q}_k = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_k \end{bmatrix}, \quad (1.7)$$

and $\Lambda_k = [\mathrm{diag}(s_1, s_2, \dots, s_{n-k})|0]$ is a $(n-k) \times (n-1)$ block matrix which the first block is a diagonal matrix $\mathrm{diag}(s_1, s_2, \dots, s_{n-k})$ and the remainder block is a zero matrix of appropriated size.

We abbreviate doubly Leslie matrix to DLM and rectangular doubly Leslie matrix by RDLM.

Let M be a matrix partitioned into four blocks

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (1.8)$$

where the submatrix C is assumed to be square and nonsingular. Brezinski in [3, p.232] asserted that, the Schur complement of C in M, denoted by (M/C), is defined by

$$(M/C) = B - AC^{-1}D, \quad (1.9)$$

which is related to Gaussian elimination by

$$M = \begin{bmatrix} I & AC^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & (M/C) \\ C & D \end{bmatrix}. \quad (1.10)$$

Suppose that B and C are $k \times k$ and $(n-k) \times (n-k)$ matrices, respectively, $k < n$, and C is nonsingular, as in [7, p.39] we have the following theorem.

Theorem 1.1 (Schur's formula). Let M be a square matrix of order $n \times n$ partitioned as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where B and C are $k \times k$ and $(n-k) \times (n-k)$ matrices, respectively, $k < n$. If C is nonsingular, then

$$\det M = (-1)^{(n+1)k} \det C \det(M/C). \tag{1.11}$$

Proof. From the (1.10)

$$M = \begin{bmatrix} I & AC^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & (M/C) \\ C & D \end{bmatrix}.$$

The identity (1.11) follows by taking the determinant of both sides. Then,

$$\det M = \det \begin{bmatrix} I & AC^{-1} \\ 0 & I \end{bmatrix} \det \begin{bmatrix} 0 & (M/C) \\ C & D \end{bmatrix}.$$

Since $\det \begin{bmatrix} I & AC^{-1} \\ 0 & I \end{bmatrix} = 1$. Therefore

$$\det M = \det \begin{bmatrix} 0 & (M/C) \\ C & D \end{bmatrix}.$$

By Laplace's theorem, expansion of $\det \begin{bmatrix} 0 & (M/C) \\ C & D \end{bmatrix}$ by the first k rows i.e., rows $\{1, 2, \ldots, k\}$. We have

$$\det \begin{bmatrix} 0 & (M/C) \\ C & D \end{bmatrix} = (-1)^{(n+1)k} \det C \det(M/C).$$

Therefore

$$\det M = (-1)^{(n+1)k} \det C \det(M/C).$$

This completes the proof. □

The following useful formula, presents the inverse of a matrix in terms of Schur complements, analogous as in [12, p. 19], we obtain.

Theorem 1.2. Let M be partitioned as in (1.8) and suppose both M and C are nonsingular. Then (M/C) is nonsingular and

$$M^{-1} = \begin{bmatrix} -C^{-1}D\,(M/C)^{-1} & C^{-1} + \left(C^{-1}D\,(M/C)^{-1}\,AC^{-1}\right) \\ (M/C)^{-1} & -\,(M/C)^{-1}\,AC^{-1} \end{bmatrix}. \tag{1.12}$$

Proof. The Schur complements (M/C) is nonsingular by virtue of (1.11). Under the given hypotheses, from (1.10) one checks that

$$\begin{aligned} M &= \begin{bmatrix} I & AC^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & (M/C) \\ C & D \end{bmatrix} \\ &= \begin{bmatrix} I & AC^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & (M/C) \\ C & 0 \end{bmatrix} \begin{bmatrix} I & C^{-1}D \\ 0 & I \end{bmatrix}. \end{aligned}$$

Inverting both sides yields

$$\begin{aligned} M^{-1} &= \begin{bmatrix} I & C^{-1}D \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & (M/C) \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} I & AC^{-1} \\ 0 & I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} I & -C^{-1}D \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & C^{-1} \\ (M/C)^{-1} & 0 \end{bmatrix} \begin{bmatrix} I & -AC^{-1} \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} -C^{-1}D\,(M/C)^{-1} & C^{-1} \\ (M/C)^{-1} & 0 \end{bmatrix} \begin{bmatrix} I & -AC^{-1} \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} -C^{-1}D\,(M/C)^{-1} & C^{-1} + \left(C^{-1}D\,(M/C)^{-1}\,AC^{-1}\right) \\ (M/C)^{-1} & -\,(M/C)^{-1}\,AC^{-1} \end{bmatrix}, \end{aligned}$$

from which the identity (1.12) follows. □

Let M be a matrix partitioned into four blocks

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where the submatrix C is assumed to be square and nonsingular. Brezinski in [3, p.232] asserted that, the Schur complement of C in M, denoted by (M/C), is defined by

$$(M/C) = B - AC^{-1}D. \tag{1.13}$$

As in (1.13), the Schur complement of Λ in L, denoted by (L/Λ) , is a 1×1 matrix or a scalar

$$\begin{aligned}(L/\Lambda) &= (-a_n - b_n) - (-\mathrm{p}^T)\Lambda^{-1}(-\mathrm{q}) \\ &= -\left((a_n + b_n) + \sum_{i=1}^{n-1} \frac{a_i b_{n-i}}{s_i}\right).\end{aligned} \tag{1.14}$$

The author [11] asserted some basic properties of doubly Leslie matrix as in the following lemma.

Lemma 1.3. Let L be a doubly Leslie matrix as in (1.5) with partitioned as

$$L = \begin{bmatrix} -\mathrm{p}^T & -a_n - b_n \\ \Lambda & -\mathrm{q} \end{bmatrix}_{(n,n)},$$

where $\mathrm{p} = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}^T$, $\mathrm{q} = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_1 \end{bmatrix}^T$, and

$$\Lambda = \mathrm{diag}(s_1, s_2, \dots, s_{n-1}), \quad s_j > 0, \quad j = 1, 2, \dots, n-1$$

is a diagonal matrix of order $n-1$, then

$$\det L = (-1)^n \left((a_n + b_n) + \sum_{i=1}^{n-1} \frac{a_i b_{n-i}}{s_i}\right) \prod_{i=1}^{n-1} s_i. \tag{1.15}$$

and, if $\det L \neq 0$ then

$$L^{-1} = (L/\Lambda)^{-1} \begin{bmatrix} \Lambda^{-1}\mathrm{q} & (L/\Lambda)\Lambda^{-1} + (\Lambda^{-1}\mathrm{q}\mathrm{p}^T\Lambda^{-1}) \\ 1 & \mathrm{p}^T\Lambda^{-1} \end{bmatrix}_{(n,n)}, \tag{1.16}$$

where $(L/\Lambda) = -\left((a_n + b_n) + \sum_{i=1}^{n-1} \frac{a_i b_{n-i}}{s_i}\right)$, as in (1.14), and $\Lambda^{-1} = \mathrm{diag}(\frac{1}{s_1}, \frac{1}{s_2}, \dots, \frac{1}{s_{n-1}})$.

In the present paper we give explicit Moore-Penrose inverse and group inverse formulae for the doubly Leslie matrix and give some related topics.

1.2 Preliminaries

Let $\mathbb{R}^{m\times n}$ denote the set of all $m \times n$ matrices over the field of real numbers $\mathbb{R}$. The Moore-Penrose inverse of a matrix $A \in \mathbb{R}^{m\times n}$ is the unique matrix $X \in \mathbb{R}^{n\times m}$ satisfying the four Penrose conditions

$$A = AXA, \quad X = XAX, \quad (AX)^T = AX \quad \text{and} \quad (XA)^T = XA$$

and is denoted by $A^{\dagger}$. The group inverse of a matrix $A \in \mathbb{R}^{n\times n}$ is the unique matrix $X \in \mathbb{R}^{n\times n}$ satisfying

$$A = AXA, \quad X = XAX \quad \text{and} \quad AX = XA$$

and is denoted by $A^{\sharp}$. A well known characterization for the existence of $A^{\sharp}$ is that $\text{rank}(A) = \text{rank}(A^2)$, [2]. If A is nonsingular, then $A^{-1} = A^{\dagger} = A^{\sharp}$. Recall that $A \in \mathbb{R}^{n\times n}$ is called range-symmetric if $\text{range}(A) = \text{range}(A^T)$. If A is range-symmetric, then $A^{\dagger} = A^{\sharp}$.

A system of linear equation $A\mathrm{x} = \mathrm{b}$ need not possess a solution when $\text{rank}(A) \neq \text{rank}[A : \mathrm{b}]$. That is b is not in the range of A. The Moore-Penrose inverse is most often used to solve least squares systems. It is still desirable to to find a x_0 that is closest to a solution. The residual vector is a key component to solve these systems.

Theorem 1.4 ([2, ?]). Let $A \in \mathbb{R}^{m\times n}$ with $\text{rank}(A) = r \neq 0$, and suppose $A = FG$ is a full rank factorization of A. Then

1. $F^{\dagger} = (F^TF)^{-1}F^T$,
2. $F^{\dagger}F = I_r$, the $r \times r$ identity matrix,
3. $G^{\dagger} = G^T(GG^T)^{-1}$,
4. $GG^{\dagger} = I_r$,

5. $A^\dagger = G^\dagger F^\dagger$.

More generally, for any $m \times n$ matrix A of full row rank m, $A = I_m A$ is a full rank factorization of A. Then

$$A^\dagger = A^T(AA^T)^{-1}. \tag{1.17}$$

The group inverse is very useful and has applications in many fields such as singular differential and difference equations, Markov chains, and iterative methods, see for instance [2, ?, ?].

Theorem 1.5 ([2]). Let a square matrix A have the full rank factorization $A = FG$. Then A has a group inverse if and only if GF is nonsingular. In which case,

$$A^\# = F(GF)^{-2}G.$$

CHAPTER II

MAIN RESULTS

In this present report we give explicit Moore-Penrose inverse and group inverse formulae for the doubly Leslie matrix and give some related topics.

2.1 Moore-Penrose Inverse of RDLM

Penrose [8, p.18]. It is possible to calculate $A^\dagger$ even when A^*A and AA^* are both singular by the following methods, where A^* is the conjugate transpose of the matrix A.

Any matrix M can be partitioned in the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $D = CA^{-1}B$, (using a suitable arrangement of rows and columns). A being any non-singular submatrix whose rank is equal to that of the whole matrix. It is then easily verified that

$$M^\dagger = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^\dagger = \begin{bmatrix} A^*KA^* & A^*KC^* \\ B^*KA^* & B^*KC^* \end{bmatrix}, \tag{2.18}$$

where $K = (AA^* + BB^*)^{-1}A(A^*A + C^*C)^{-1}$. The matrices $AA^* + BB^*$ and $A^*A + C^*C$ are positive definite, since A is non-singular. Thus the generalized inverse of any matrix can be expressed in terms of ordinary reciprocals of matrices.

We have the following main results.

Lemma 2.1. If $A = PB$, where P is a permutation matrix, then

$$A^{\dagger} = B^{\dagger}P^{T}. \tag{2.19}$$

Proof. It is straightforward to verify that $B^{\dagger}P^{T}$ satisfies the four Penrose conditions. Clearly,

1. $PB(B^{\dagger}P^{T})PB = PBB^{\dagger}B = PB,$
2. $(B^{\dagger}P^{T})PB(B^{\dagger}P^{T}) = B^{\dagger}BB^{\dagger}P^{T} = B^{\dagger}P^{T},$
3. $[PB(B^{\dagger}P^{T})]^{T} = [PBB^{\dagger}P^{T}]^{T} = P(BB^{\dagger})^{T}P^{T} = PBB^{\dagger}P^{T} = PB(B^{\dagger}P^{T}),$
4. $[(B^{\dagger}P^{T})PB]^{T} = [B^{\dagger}P^{T}PB]^{T} = [B^{\dagger}B]^{T} = B^{\dagger}B = B^{\dagger}(P^{T}P)B = (B^{\dagger}P^{T})PB.$

□

Lemma 2.2. For an $m \times n$ $\mathbb{R}$-matrix N of rank $r < \min(m, n)$, and N partitioned in the form

$$N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where C is $r \times r$ nonsingular. Then

$$N^{\dagger} = \begin{bmatrix} C^{T}KD^{T} & C^{T}KC^{T} \\ D^{T}KA^{T} & D^{T}KC^{T} \end{bmatrix}$$

where $K = (CC^{T} + DD^{T})^{-1}C(C^{T}C + A^{T}A)^{-1}$. The matrices $CC^{T} + DD^{T}$ and $C^{T}C + A^{T}A$ are positive definite, since C is non-singular.

Proof. Let $P = \begin{bmatrix} 0 & I_{r} \\ I_{m-r} & 0 \end{bmatrix}_{(m\times m)}$ be a permutation matrix. Premultiplying the matrix N by P.

$$PN = \begin{bmatrix} C & D \\ A & B \end{bmatrix}.$$

Since P is a unitary matrix and by (2.19). We have

$$(PN)^{\dagger} = N^{\dagger}P^{T}.$$

Therefore

$$(PN)^{\dagger} = \begin{bmatrix} C & D \\ A & B \end{bmatrix}^{\dagger} = N^{\dagger}P^{T}.$$

and

$$N^{\dagger} = \begin{bmatrix} C & D \\ A & B \end{bmatrix}^{\dagger} P.$$

As in (2.18), we have

$$\begin{bmatrix} C & D \\ A & B \end{bmatrix}^{\dagger} = \begin{bmatrix} C^TKC^T & C^TKA^T \\ D^TKC^T & D^TKA^T \end{bmatrix},$$

where $K = (CC^T + DD^T)^{-1}C(C^TC + A^TA)^{-1}$. The matrices $CC^T + DD^T$ and $C^TC + A^TA$ are positive definite, since C is non-singular. Therefore $CC^T + DD^T$ and $C^TC + A^TA$ are also non-singular matrices. We have

$$N^{\dagger} = \begin{bmatrix} C^TKC^T & C^TKA^T \\ D^TKC^T & D^TKA^T \end{bmatrix} P = \begin{bmatrix} C^TKA^T & C^TKC^T \\ D^TKA^T & D^TKC^T \end{bmatrix}.$$

The proof is complete. □

Theorem 2.3. Let L be a doubly Leslie matrix as in (1.5) with partitioned as

$$L = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}_{(n,n)},$$

where $\mathbf{p} = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}^T$, $\mathbf{q} = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_1 \end{bmatrix}^T$, and $\Lambda = \mathrm{diag}(s_1, s_2, \dots, s_{n-1})$, $s_j > 0$, $j = 1, 2, \dots, n-1$ is a diagonal matrix of order $n-1$, then

$$L^{\dagger} = \begin{bmatrix} -\Lambda K\mathbf{p} & \Lambda K \Lambda \\ \mathbf{q}^T K\mathbf{p} & -\mathbf{q}^T K\Lambda \end{bmatrix},$$

where $K = (\Lambda^2 + \mathbf{q}\mathbf{q}^T)^{-1}\Lambda(\Lambda^2 + \mathbf{p}\mathbf{p}^T)^{-1}$.

Proof. If $\det L \neq 0$ then $L^{\dagger} = L^{-1}$ which appeared in (1.16).

In general

$$\begin{aligned} L^{\dagger} &= \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{\dagger} = \begin{bmatrix} C^T K A^T & C^T K C^T \\ D^T K A^T & D^T K C^T \end{bmatrix} \\ &= \begin{bmatrix} \Lambda^T K(-\mathbf{p}^T)^T & \Lambda^T K \Lambda^T \\ (-\mathbf{q})^T K(-\mathbf{p}^T)^T & (-\mathbf{q})^T K \Lambda^T \end{bmatrix} \\ &= \begin{bmatrix} -\Lambda K \mathbf{p} & \Lambda K \Lambda \\ \mathbf{q}^T K \mathbf{p} & -\mathbf{q}^T K \Lambda \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} K &= (\Lambda\Lambda^T + (-\mathbf{q})(-\mathbf{q})^T)^{-1}\Lambda(\Lambda^T\Lambda + (-\mathbf{p}^T)^T(-\mathbf{p}^T))^{-1} \\ &= (\Lambda^2 + \mathbf{q}\mathbf{q}^T)^{-1}\Lambda(\Lambda^2 + \mathbf{p}\mathbf{p}^T)^{-1}. \end{aligned}$$

□

Corollary 2.4. Let R be a rectangle doubly Leslie matrix as in (1.7) with partitioned as

$$R = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda_k & -\mathbf{q}_k \end{bmatrix}_{(m,n)},$$

where $m = n - k$,

$$\mathbf{p} = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}^T, \quad \mathbf{q} = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_k \end{bmatrix}^T,$$

and $\Lambda_k = [\mathrm{diag}(s_1, s_2, \dots, s_{n-k})|0]$ is a $(n-k) \times (n-1)$ block matrix, then

$$L^{\dagger} = \begin{bmatrix} -\Lambda_k K \mathbf{p} & \Lambda_k K \Lambda_k \\ \mathbf{q}_k^T K \mathbf{p} & -\mathbf{q}_k^T K \Lambda_k \end{bmatrix}$$

where $K = (\Lambda_k\Lambda_k^T + \mathbf{q}_k\mathbf{q}_k^T)^{-1}\Lambda_k(\Lambda_k^T\Lambda_k + \mathbf{p}\mathbf{p}^T)^{-1}$.

Proof. The proof is an analogous as in Theorem 2.3. □

Let's consider some examples.

Example 1.

$$L = \begin{bmatrix} 1 & 2 & -1 & -3 \\ 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} =: \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}$$

where $\mathbf{p} = \begin{bmatrix} -1 & -2 & 1 \end{bmatrix}^T$, $\mathbf{q} = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix}^T$, and $\Lambda = \operatorname{diag}(1,2,1)$, is a diagonal matrix of order 3, then

$$L^{\dagger} = \begin{bmatrix} -\Lambda K\mathbf{p} & \Lambda K\Lambda \\ \mathbf{q}^T K\mathbf{p} & -\mathbf{q}^T K\Lambda \end{bmatrix},$$

where $K = (\Lambda^2 + \mathbf{q}\mathbf{q}^T)^{-1}\Lambda(\Lambda^2 + \mathbf{p}\mathbf{p}^T)^{-1}$.

First we calculate $\mathbf{q}\mathbf{q}^T$ and $\mathbf{p}\mathbf{p}^T$.

$$\mathbf{q}\mathbf{q}^T = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix},$$

$$\mathbf{p}\mathbf{p}^T = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix},$$

and

$$(\Lambda^2 + \mathbf{q}\mathbf{q}^T)^{-1} = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{5}{6} & 0 & \frac{1}{3} \\ 0 & \frac{1}{4} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix},$$

$$(\Lambda^2 + \mathbf{p}\mathbf{p}^T)^{-1} = \begin{bmatrix} 2 & 2 & -1 \\ 2 & 8 & -2 \\ -1 & -2 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{8} & \frac{1}{4} \\ -\frac{1}{8} & \frac{3}{16} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{8} & \frac{3}{4} \end{bmatrix},$$

we have

$$K = (\Lambda^2 + \mathbf{q}\mathbf{q}^T)^{-1}\Lambda(\Lambda^2 + \mathbf{p}\mathbf{p}^T)^{-1} = \begin{bmatrix} \frac{17}{24} & -\frac{1}{16} & \frac{11}{24} \\ -\frac{1}{16} & \frac{3}{32} & \frac{1}{16} \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}.$$

Finally,

$$-\Lambda K\mathbf{p} = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{17}{24} & -\frac{1}{16} & \frac{11}{24} \\ -\frac{1}{16} & \frac{3}{32} & \frac{1}{16} \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} \\ \frac{1}{8} \\ 0 \end{bmatrix},$$

$$\Lambda K\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{17}{24} & -\frac{1}{16} & \frac{11}{24} \\ -\frac{1}{16} & \frac{3}{32} & \frac{1}{16} \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{17}{24} & -\frac{1}{8} & \frac{11}{24} \\ -\frac{1}{8} & \frac{3}{8} & \frac{1}{8} \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix},$$

$$\mathrm{q}^T K\mathrm{p} = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} \frac{17}{24} & -\frac{1}{16} & \frac{11}{24} \\ -\frac{1}{16} & \frac{3}{32} & \frac{1}{16} \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} = -\tfrac{1}{8},$$

$$-\mathrm{q}^T K\Lambda = -\begin{bmatrix} 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} \frac{17}{24} & -\frac{1}{16} & \frac{11}{24} \\ -\frac{1}{16} & \frac{3}{32} & \frac{1}{16} \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{24} & \frac{1}{8} & \frac{5}{24} \end{bmatrix}.$$

Therefore

$$L^\dagger = \begin{bmatrix} -\Lambda K\mathrm{p} & \Lambda K\Lambda \\ \mathrm{q}^T K\mathrm{p} & -\mathrm{q}^T K\Lambda \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & \frac{17}{24} & -\frac{1}{8} & \frac{11}{24} \\ \frac{1}{8} & -\frac{1}{8} & \frac{3}{8} & \frac{1}{8} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ -\frac{1}{8} & -\frac{1}{24} & \frac{1}{8} & \frac{5}{24} \end{bmatrix}.$$

This matrix is satisfies the four Penrose conditions. □

Example 2. For a full row rank rectangle doubly Leslie matrix of order 3×4

$$R = \begin{bmatrix} 1 & 2 & -1 & -3 \\ 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \end{bmatrix}.$$

From (1.17),

$$\begin{aligned} R^{\dagger} &= R^T(RR^T)^{-1} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ -1 & 0 & 0 \\ -3 & -1 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 & -1 & -3 \\ 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ -1 & 0 & 0 \\ -3 & -1 & 0 \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ -1 & 0 & 0 \\ -3 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{11}{6} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{7}{12} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{3} & \frac{7}{6} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{3} \end{bmatrix}. \end{aligned}$$

This matrix is also satisfies the four Penrose conditions. □

In particular case, if $s_j = 1,\ j = 1, 2, \ldots, n-1$ the matrix L in Theorem 2.3 is become to a doubly companion matrix denoted by C, then we have the Moore-Penrose inverse of this matrix as the follows:

Corollary 2.5. Let C be a doubly companion matrix as in (1.2) with partitioned as

$$C = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ I_{n-1} & -\mathbf{q} \end{bmatrix}_{(n,n)},$$

where $\mathbf{p} = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}^T$, $\mathbf{q} = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_1 \end{bmatrix}^T$, and I_{n-1} is the identity matrix of order $n-1$, then

$$C^\dagger = \begin{bmatrix} -K\mathbf{p} & K \\ \mathbf{q}^T K\mathbf{p} & -\mathbf{q}^T K \end{bmatrix},$$

where $K = (I_{n-1} + \mathbf{q}\mathbf{q}^T)^{-1}(I_{n-1} + \mathbf{p}\mathbf{p}^T)^{-1}$.

2.2 Group Inverse of DLM

As in [2, p.167] we have the following useful result.

Theorem 2.6. Let A be a square singular matrix, $\operatorname{rank} A = \operatorname{rank} A^2$, and $R(A)$ be the range of A. If the system

$$A\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \in R(A)$$

has a solution, it is uniquely given by

$$\mathbf{x} = A^\sharp \mathbf{b}.$$

Proof. Suppose that $\mathbf{x} \in R(A)$ where $R(A)$ is the range of A. There is a vector $\mathbf{y}$ such that $A\mathbf{y} = \mathbf{x}$. Let a solution $\mathbf{x}$ be written as $\mathbf{x} = A\mathbf{y}_1$ for some $\mathbf{y}_1$. We have

$$A\mathbf{x} = AA\mathbf{y}_1 = A^2\mathbf{y}_1,$$

then $A^2\mathbf{y}_1 = \mathbf{b}$. Since $\operatorname{rank} A = \operatorname{rank} A^2$, there is a unique $A^\sharp$ such that

$$AA^\sharp A = A, \quad A^\sharp AA^\sharp = A^\sharp, \quad \text{and} \quad AA^\sharp = A^\sharp A.$$

Therefore

$$\begin{aligned} \mathbf{x} &= A\mathbf{y}_1 \\ &= AA^\sharp A\mathbf{y}_1 \\ &= A^2 A^\sharp \mathbf{y}_1 \\ &= A^\sharp A^2 \mathbf{y}_1 \\ &= A^\sharp A\mathbf{x} \\ &= A^\sharp \mathbf{b}. \end{aligned}$$

□

Let L be a doubly Leslie matrix as in (1.5) with partitioned as

$$L = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}_{(n,n)}.$$

If $\det L \neq 0$ then $L^\# = L^{-1}$ which was shown in (1.16). We interested in study the only case $\operatorname{rank}(L) \neq n$. By the definition of DLM the rank of L is at least $n-1$. Since equivalence matrix has the same rank, we reduce the matrix L to a reduced echelon form as follows:

$$\begin{bmatrix} 0 & \frac{1}{s_1} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{s_2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{s_{n-1}} \\ 1 & \frac{a_1}{s_1} & \frac{a_2}{s_2} & \cdots & \frac{a_{n-1}}{s_{n-1}} \end{bmatrix} \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n - b_n \\ s_1 & 0 & \cdots & 0 & -b_{n-1} \\ 0 & s_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -b_2 \\ 0 & \cdots & 0 & s_{n-1} & -b_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 & -\frac{b_{n-1}}{s_1} \\ 0 & 1 & 0 & \vdots & -\frac{b_{n-2}}{s_2} \\ 0 & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & 1 & -\frac{b_1}{s_{n-1}} \\ 0 & 0 & \cdots & 0 & -a_n - b_n - \frac{a_1}{s_1}b_{n-1} - \frac{a_2}{s_2}b_{n-2} - \cdots - \frac{a_{n-2}}{s_{n-2}}b_2 - \frac{a_{n-1}}{s_{n-1}}b_1 \end{bmatrix}. \tag{2.20}$$

We see that $\operatorname{rank}(L) = n-1$ if and only if

$$-a_n - b_n - \frac{a_1}{s_1}b_{n-1} - \frac{a_2}{s_2}b_{n-2} - \cdots - \frac{a_{n-2}}{s_{n-2}}b_2 - \frac{a_{n-1}}{s_{n-1}}b_1 = 0$$

if and only if

$$-a_n - b_n = \frac{a_1}{s_1}b_{n-1} + \frac{a_2}{s_2}b_{n-2} + \cdots + \frac{a_{n-2}}{s_{n-2}}b_2 + \frac{a_{n-1}}{s_{n-1}}b_1.$$

We factor L to full rank factorization as follows:

$$L = FG = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} \\ s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 & -\frac{b_{n-1}}{s_1} \\ 0 & 1 & \ddots & \vdots & -\frac{b_{n-2}}{s_2} \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -\frac{b_1}{s_{n-1}} \end{bmatrix}$$

$$= \begin{bmatrix} -\mathrm{p}^T \\ \Lambda \end{bmatrix} \begin{bmatrix} I_{n-1} & -\mathrm{q}_1 \end{bmatrix},$$

where $\mathrm{p} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$, $\mathrm{q}_1 = \begin{bmatrix} \frac{b_{n-1}}{s_1} \\ \frac{b_{n-2}}{s_2} \\ \vdots \\ \frac{b_1}{s_{n-1}} \end{bmatrix}$, and $\Lambda = \mathrm{diag}(s_1, s_2, \ldots, s_{n-1})$.

Also, by direct computation, we have

$$GF = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_{n-2} & -a_{n-1} - b_{n-1}\frac{s_{n-1}}{s_1} \\ s_1 & 0 & 0 & \cdots & 0 & -b_{n-2}\frac{s_{n-1}}{s_2} \\ 0 & s_2 & 0 & \cdots & 0 & -b_{n-3}\frac{s_{n-1}}{s_3} \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & -b_2\frac{s_{n-1}}{s_{n-2}} \\ 0 & 0 & \cdots & 0 & s_{n-2} & -b_1\frac{s_{n-1}}{s_{n-1}} \end{bmatrix} =: M.$$

The matrix $GF =: M$ is a doubly Leslie matrix of order $(n-1) \times (n-1)$.

$$M = \begin{bmatrix} -\mathrm{p}_1^T & -a_{n-1} - b_{n-1}\frac{s_{n-1}}{s_1} \\ \Lambda_1 & -\mathrm{q}_2 \end{bmatrix}_{(n-1,n-1)},$$

where $\mathrm{p}_1 = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-2} \end{bmatrix}$, $\mathrm{q}_2 = \begin{bmatrix} b_{n-2}\frac{s_{n-1}}{s_2} \\ b_{n-3}\frac{s_{n-1}}{s_3} \\ \vdots \\ b_1\frac{s_{n-1}}{s_{n-1}} \end{bmatrix}$, and $\Lambda_1 = \mathrm{diag}(s_1, s_2, \ldots, s_{n-2})$ is a diagonal matrix of order $n-1$.

By (1.16), we have

$$M^{-1} = (M/\Lambda_1)^{-1} \begin{bmatrix} \Lambda_1^{-1}\mathbf{q}_2 & (M/\Lambda_1)\Lambda_1^{-1} + \left(\Lambda_1^{-1}\mathbf{q}_2\mathbf{p}_1^T\Lambda_1^{-1}\right) \\ 1 & \mathbf{p}_1^T\Lambda_1^{-1} \end{bmatrix}, \tag{2.21}$$

where $(M/\Lambda_1) = -\left((a_{n-1} + b_{n-1}\frac{s_{n-1}}{s_1}) + s_{n-1}\sum_{i=1}^{n-2}\frac{a_i b_{n-i-1}}{s_i s_{i+1}}\right)$, as in (1.14), and $\Lambda_1^{-1} = \text{diag}(\frac{1}{s_1}, \frac{1}{s_2}, \ldots, \frac{1}{s_{n-2}})$.

From Theorem 1.5, we have

$$\begin{aligned} L^\sharp &= F(GF)^{-2}G \\ &= \begin{bmatrix} -\mathbf{p}^T \\ \Lambda \end{bmatrix} \left((M/\Lambda_1)^{-1} \begin{bmatrix} \Lambda_1^{-1}\mathbf{q}_2 & (M/\Lambda_1)\Lambda_1^{-1} + \left(\Lambda_1^{-1}\mathbf{q}_2\mathbf{p}_1^T\Lambda_1^{-1}\right) \\ 1 & \mathbf{p}_1^T\Lambda_1^{-1} \end{bmatrix} \right)^2 \times \\ &\quad \begin{bmatrix} I_{n-1} & -\mathbf{q}_1 \end{bmatrix} \\ &= (M/\Lambda_1)^{-2} \begin{bmatrix} -\mathbf{p}^T \\ \Lambda \end{bmatrix} \begin{bmatrix} \Lambda_1^{-1}\mathbf{q}_2 & (M/\Lambda_1)\Lambda_1^{-1} + \left(\Lambda_1^{-1}\mathbf{q}_2\mathbf{p}_1^T\Lambda_1^{-1}\right) \\ 1 & \mathbf{p}_1^T\Lambda_1^{-1} \end{bmatrix}^2 \times \\ &\quad \begin{bmatrix} I_{n-1} & -\mathbf{q}_1 \end{bmatrix}. \end{aligned}$$

We have proven the following important Theorem.

Theorem 2.7. The doubly Leslie matrix L as in (1.5) have $\text{rank}(L) = n-1$ if and only if $-a_n - b_n = \frac{a_1}{s_1}b_{n-1} + \frac{a_2}{s_2}b_{n-2} + \cdots + \frac{a_{n-2}}{s_{n-2}}b_2 + \frac{a_{n-1}}{s_{n-1}}b_1$, and have the full rank factorization

$$L = FG = \begin{bmatrix} -\mathbf{p}^T \\ \Lambda \end{bmatrix} \begin{bmatrix} I_{n-1} & -\mathbf{q}_1 \end{bmatrix},$$

where $\mathbf{p} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$, $\mathbf{q}_1 = \begin{bmatrix} \frac{b_{n-1}}{s_1} \\ \frac{b_{n-2}}{s_2} \\ \vdots \\ \frac{b_1}{s_{n-1}} \end{bmatrix}$, and $\Lambda = \text{diag}(s_1, s_2, \ldots, s_{n-1})$. Then L has a group inverse and

$$L^\sharp = (M/\Lambda_1)^{-2}\begin{bmatrix} -\mathbf{p}^T \\ \Lambda \end{bmatrix}\begin{bmatrix} \Lambda_1^{-1}\mathbf{q}_2 & (M/\Lambda_1)\,\Lambda_1^{-1} + (\Lambda_1^{-1}\mathbf{q}_2\mathbf{p}_1^T\Lambda_1^{-1}) \\ 1 & \mathbf{p}_1^T\Lambda_1^{-1} \end{bmatrix}^2 \times$$

$$\begin{bmatrix} I_{n-1} & -\mathbf{q}_1 \end{bmatrix}.$$

where $M = GF$ and $\Lambda_1 = \mathrm{diag}(s_1, s_2, \ldots, s_{n-2})$.

Let's consider the same example.

Example 3.

$$L = \begin{bmatrix} 1 & 2 & -1 & -3 \\ 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

where $\mathbf{p} = \begin{bmatrix} -1 & -2 & 1 \end{bmatrix}^T$, $\mathbf{q} = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix}^T$, and $\Lambda = \mathrm{diag}(1, 2, 1)$, is a diagonal matrix of order 3.

Since $\det(L) = 0$, we have $\mathrm{rank}(L) = \mathrm{rank}(L^2)$, we know that the unique $L^\sharp$ exists. Now

$$L = FG = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

and

$$GF = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix},$$

$$(GF)^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix},$$

$$(GF)^{-2} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{8} & \frac{3}{8} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{8} \end{bmatrix}.$$

Finally

$$\begin{aligned} L^{\sharp} &= F(GF)^{-2}G \\ &= \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{8} & \frac{3}{8} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{3}{4} & \frac{1}{8} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{1}{8} & \frac{1}{4} \end{bmatrix}. \end{aligned}$$

This matrix is satisfies the three conditions for group inverse. □

In particular case, if $s_j = 1,\ j = 1, 2, \ldots, n-1$ the matrix L in Theorem 2.7 is become to a doubly companion matrix denoted by C, then we have the group inverse of the doubly companion matrix as the follows:

Corollary 2.8. The doubly companion matrix C as in (1.5) have rank$(C) = n-1$ if and only if $-a_n - b_n = a_1b_{n-1} + a_2b_{n-2} + \cdots + a_{n-2}b_2 + a_{n-1}b_1$, and have the full rank factorization

$$C = FG = \begin{bmatrix} -\mathbf{p}^T \\ I_{n-1} \end{bmatrix} \begin{bmatrix} I_{n-1} & -\mathbf{q}_1 \end{bmatrix},$$

where $\mathbf{p} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$, $\mathbf{q}_1 = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \end{bmatrix}$, and I_{n-1} is the identity matrix of order $n-1$.

Then C has a group inverse and

$$C^{\#} = (M/I_{n-2})^{-2} \begin{bmatrix} -\mathbf{p}^T \\ I_{n-1} \end{bmatrix} \begin{bmatrix} I_{n-2}^{-1}\mathbf{q}_2 & (M/I_{n-2})\,I_{n-2}^{-1} + (I_{n-2}^{-1}\mathbf{q}_2\mathbf{p}_1^T I_{n-2}^{-1}) \\ 1 & \mathbf{p}_1^T I_{n-2}^{-1} \end{bmatrix}^2 \times$$

$$\begin{bmatrix} I_{n-1} & -\mathbf{q}_1 \end{bmatrix}.$$

where $M = GF$ and I_{n-2} is the identity matrix of order $n-2$.

CHAPTER III

CONCLUSION

The following results are all main theorems of this thesis:

1. Let A be a matrix, if $A = PB$, where P is a permutation matrix, then $A^{\dagger} = B^{\dagger}P^{T}$.

2. For an $m \times n$ $\mathbb{R}$-matrix N of rank $r < \min(m, n)$, and N partitioned in the form

$$N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where C is $r \times r$ nonsingular. Then

$$N^{\dagger} = \begin{bmatrix} C^T K D^T & C^T K C^T \\ D^T K A^T & D^T K C^T \end{bmatrix}$$

where $K = (CC^T + DD^T)^{-1} C (C^T C + A^T A)^{-1}$. The matrices $CC^T + DD^T$ and $C^T C + A^T A$ are positive definite, since C is non-singular.

3. Let L be a doubly Leslie matrix as in (1.5) with partitioned as

$$L = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}_{(n,n)},$$

where $\mathbf{p} = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}^T$, $\mathbf{q} = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_1 \end{bmatrix}^T$, and $\Lambda = \operatorname{diag}(s_1, s_2, \dots, s_{n-1})$, $s_j > 0$, $j = 1, 2, \dots, n-1$ is a diagonal matrix of order $n-1$, then

$$L^{\dagger} = \begin{bmatrix} -\Lambda K \mathbf{p} & \Lambda K \Lambda \\ \mathbf{q}^T K \mathbf{p} & -\mathbf{q}^T K \Lambda \end{bmatrix},$$

where $K = (\Lambda^2 + \mathbf{q}\mathbf{q}^T)^{-1} \Lambda (\Lambda^2 + \mathbf{p}\mathbf{p}^T)^{-1}$.

4. Let R be a rectangle doubly Leslie matrix as in (1.7) with partitioned as

$$R = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda_k & -\mathbf{q}_k \end{bmatrix}_{(m,n)},$$

where $m = n - k$,

$$\mathbf{p} = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}^T, \quad \mathbf{q} = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_k \end{bmatrix}^T,$$

and $\Lambda_k = [\mathrm{diag}(s_1, s_2, \dots, s_{n-k})|0]$ is a $(n-k) \times (n-1)$ block matrix, then

$$L^\dagger = \begin{bmatrix} -\Lambda_k K \mathbf{p} & \Lambda_k K \Lambda_k \\ \mathbf{q}_k^T K \mathbf{p} & -\mathbf{q}_k^T K \Lambda_k \end{bmatrix}$$

where $K = (\Lambda_k \Lambda_k^T + \mathbf{q}_k \mathbf{q}_k^T)^{-1} \Lambda_k (\Lambda_k^T \Lambda_k + \mathbf{p}\mathbf{p}^T)^{-1}$.

5. Let C be a doubly companion matrix as in (1.2) with partitioned as

$$C = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ I_{n-1} & -\mathbf{q} \end{bmatrix}_{(n,n)},$$

where $\mathbf{p} = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}^T$, $\mathbf{q} = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_1 \end{bmatrix}^T$, and I_{n-1} is the identity matrix of order $n-1$, then

$$C^\dagger = \begin{bmatrix} -K\mathbf{p} & K \\ \mathbf{q}^T K \mathbf{p} & -\mathbf{q}^T K \end{bmatrix},$$

where $K = (I_{n-1} + \mathbf{q}\mathbf{q}^T)^{-1}(I_{n-1} + \mathbf{p}\mathbf{p}^T)^{-1}$.

6. Let A be a square singular matrix, $\operatorname{rank} A = \operatorname{rank} A^2$, and $R(A)$ be the range of A. If the system

$$A\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \in R(A)$$

has a solution, it is uniquely given by

$$\mathbf{x} = A^\# \mathbf{b}.$$

7. The doubly Leslie matrix L as in (1.5) have $\text{rank}(L) = n-1$ if and only if $-a_n - b_n = \frac{a_1}{s_1}b_{n-1} + \frac{a_2}{s_2}b_{n-2} + \cdots + \frac{a_{n-2}}{s_{n-2}}b_2 + \frac{a_{n-1}}{s_{n-1}}b_1$, and have the full rank factorization

$$L = FG = \begin{bmatrix} -\mathrm{p}^T \\ \Lambda \end{bmatrix} \begin{bmatrix} I_{n-1} & -\mathrm{q}_1 \end{bmatrix},$$

where $\mathrm{p} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$, $\mathrm{q}_1 = \begin{bmatrix} \frac{b_{n-1}}{s_1} \\ \frac{b_{n-2}}{s_2} \\ \vdots \\ \frac{b_1}{s_{n-1}} \end{bmatrix}$, and $\Lambda = \text{diag}(s_1, s_2, \ldots, s_{n-1})$. Then L has a group inverse and

$$L^\sharp = (M/\Lambda_1)^{-2} \begin{bmatrix} -\mathrm{p}^T \\ \Lambda \end{bmatrix} \begin{bmatrix} \Lambda_1^{-1}\mathrm{q}_2 & (M/\Lambda_1)\,\Lambda_1^{-1} + \left(\Lambda_1^{-1}\mathrm{q}_2\mathrm{p}_1^T\Lambda_1^{-1}\right) \\ 1 & \mathrm{p}_1^T\Lambda_1^{-1} \end{bmatrix}^2 \times \begin{bmatrix} I_{n-1} & -\mathrm{q}_1 \end{bmatrix}.$$

where $M = GF$ and $\Lambda_1 = \text{diag}(s_1, s_2, \ldots, s_{n-2})$.

8. The doubly companion matrix C as in (1.5) have $\text{rank}(C) = n-1$ if and only if $-a_n - b_n = a_1b_{n-1} + a_2b_{n-2} + \cdots + a_{n-2}b_2 + a_{n-1}b_1$, and have the full rank factorization

$$C = FG = \begin{bmatrix} -\mathrm{p}^T \\ I_{n-1} \end{bmatrix} \begin{bmatrix} I_{n-1} & -\mathrm{q}_1 \end{bmatrix},$$

where $\mathrm{p} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$, $\mathrm{q}_1 = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \end{bmatrix}$, and I_{n-1} is the identity matrix of order $n-1$. Then C has a group inverse and

$$C^\sharp = (M/I_{n-2})^{-2} \begin{bmatrix} -\mathrm{p}^T \\ I_{n-1} \end{bmatrix} \begin{bmatrix} I_{n-2}^{-1}\mathrm{q}_2 & (M/I_{n-2})\,I_{n-2}^{-1} + \left(I_{n-2}^{-1}\mathrm{q}_2\mathrm{p}_1^T I_{n-2}^{-1}\right) \\ 1 & \mathrm{p}_1^T I_{n-2}^{-1} \end{bmatrix}^2 \times \begin{bmatrix} I_{n-1} & -\mathrm{q}_1 \end{bmatrix}.$$

where $M = GF$ and I_{n-2} is the identity matrix of order $n-2$.

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APPENDIX

Explicit Moore-Penrose inverse and group inverse of doubly Leslie matrix

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Abstract

A doubly Leslie matrix is a bordered real matrix of the form $L = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}_{(n,n)}$ where $a_n, b_n \in \mathbb{R}$, $\mathbf{p}, \mathbf{q} \in \mathbb{R}^{n-1}$, and $\Lambda = \operatorname{diag}(s_1, s_2, \ldots, s_{n-1})$ is a diagonal matrix of order $n-1$. The matrix L is a closed form of a doubly companion matrix, a Leslie matrix and a companion matrix. This paper is discussed the explicit formula of the Moore-Penrose inverse and the group inverse of the doubly leslie matrix. In general the Moore-Penrose inverse of a rectangle doubly Leslie matrix is also discussed.

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Key Words and Phrases: Companion matrix, doubly companion matrix, Leslie matrix, doubly Leslie matrix, Moore-Penrose inverse, group inverse.

1 Introduction

One of the most popular models of population growth is a matrix-based model, first introduced by P. H. Leslie. In 1945, he published his most famous article in *Biometrika*, a journal. The article was entitled, *On the use of matrices in certain population mathematics* [2, pp. 117–120]. The Leslie model describes the growth of the female portion of a population which is assumed to have a maximum lifespan. The females are divided into age classes all of which span an equal number of years. Using data about the average birthrates and survival probabilities of each class, the model is then able to determine the growth of the population over time, [6].

A *Leslie matrix* arises in a discrete, age-dependent model for population growth. It is a matrix of the form

$$\mathrm{L} = \begin{bmatrix} r_1 & r_2 & r_3 & \dots & r_{n-1} & r_n \\ s_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & s_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \dots & s_{n-1} & 0 \end{bmatrix}, \tag{1}$$

where $r_j \geq 0$, $0 < s_j \leq 1$, $j = 1, 2, \dots, n-1$.

Doubly companion matrices $C \in M_n$ were first introduced by Butcher and Chartier in [4, pp.274–276], given by

$$C = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & \dots & -\alpha_{n-1} & -\alpha_n - \beta_n \\ 1 & 0 & 0 & \dots & 0 & -\beta_{n-1} \\ 0 & 1 & 0 & \dots & 0 & -\beta_{n-2} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -\beta_2 \\ 0 & 0 & 0 & \dots & 1 & -\beta_1 \end{bmatrix}, \tag{2}$$

that is, a $n \times n$ matrix C with $n > 1$ is called a *doubly companion matrix* if its entries c_{ij} satisfy $c_{ij} = 1$ for all entries in the sub-maindiagonal of C and else $c_{ij} = 0$ for $i \neq 1$ and $j \neq n$.

We define a *doubly Leslie matrix* analogous as the doubly companion matrix by replacing the subdiagonal of the doubly companion matrix by $s_1, s_2, \dots, s_{n-1}$ where s_j, $j = 1, 2, \dots, n-1$, respectively, and denoted by L, that is, a doubly Leslie matrix is defined to be a matrix as follows:

$$L = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \dots & -a_{n-1} & -a_n - b_n \\ s_1 & 0 & 0 & \dots & 0 & -b_{n-1} \\ 0 & s_2 & 0 & \dots & 0 & -b_{n-2} \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & -b_2 \\ 0 & 0 & \dots & 0 & s_{n-1} & -b_1 \end{bmatrix}, \qquad (3)$$

where $a_j, b_j \in \mathbb{R}$, the real numbers, $j = 1, 2, \dots, n$. As the Leslie matrix, we restriction only $s_j > 0$, $j = 1, 2, \dots, n-1$.

For convenience, we can be written the matrix L in a partitioned form as

$$L = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}_{(n,n)} \text{ where } \mathbf{p} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}, \mathbf{q} = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \end{bmatrix},$$

and $\Lambda = \operatorname{diag}(s_1, s_2, \dots, s_{n-1})$ is a diagonal matrix of order $n-1$.

We also define a rectangular doubly Leslie matrix of order $m \times n$, where $m = n - k$ and $1 < k < n$ as follows:

$$R = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \dots & -a_{n-1} & -a_n - b_n \\ s_1 & 0 & 0 & \dots & 0 & -b_{n-1} \\ 0 & s_2 & 0 & \dots & 0 & -b_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & s_{n-k} & 0 & -b_k \end{bmatrix}_{(m,n)}. \qquad (4)$$

For convenience, we can be written the matrix L in a partitioned form as

$$R = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda_k & -\mathbf{q}_k \end{bmatrix}_{(m,n)} \text{ where } \mathbf{p} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}, \mathbf{q}_k = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_k \end{bmatrix}, \qquad (5)$$

and $\Lambda_k = [\text{diag}(s_1, s_2, \ldots, s_{n-k})|0]$ is a $(n-k)\times(n-1)$ block matrix which the first block is a diagonal matrix $\text{diag}(s_1, s_2, \ldots, s_{n-k})$ and the remainder block is a zero matrix of appropriated size.

We abbreviate doubly Leslie matrix to DLM and rectangular doubly Leslie matrix by RDLM.

Let M be a matrix partitioned into four blocks

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where the submatrix C is assumed to be square and nonsingular. Brezinski in [3, p.232] asserted that, the Schur complement of C in M, denoted by (M/C), is defined by

$$(M/C) = B - AC^{-1}D. \tag{6}$$

As in (6), the Schur complement of Λ in L, denoted by (L/Λ) , is a 1×1 matrix or a scalar

$$\begin{aligned}(L/\Lambda) &= (-a_n - b_n) - (-\mathbf{p}^T)\Lambda^{-1}(-\mathbf{q}) \\ &= -\left((a_n + b_n) + \sum_{i=1}^{n-1} \frac{a_i b_{n-i}}{s_i}\right).\end{aligned} \tag{7}$$

The author [7] asserted some basic properties of doubly Leslie matrix as in the following lemma.

Lemma 1. *Let L be a doubly Leslie matrix as in* (3) *with partitioned as*

$$L = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}_{(n,n)},$$

where $\mathbf{p} = \begin{bmatrix} a_1 & a_2 & \ldots & a_{n-1} \end{bmatrix}^T$, $\mathbf{q} = \begin{bmatrix} b_{n-1} & b_{n-2} & \ldots & b_1 \end{bmatrix}^T$, *and* $\Lambda = \text{diag}(s_1, s_2, \ldots, s_{n-1})$, $s_j > 0$, $j = 1, 2, \ldots, n-1$ *is a diagonal matrix of order* $n-1$, *then*

$$\det L = (-1)^n \left((a_n + b_n) + \sum_{i=1}^{n-1} \frac{a_i b_{n-i}}{s_i}\right) \prod_{i=1}^{n-1} s_i. \tag{8}$$

and, if $\det L \neq 0$ *then*

$$L^{-1} = (L/\Lambda)^{-1} \begin{bmatrix} \Lambda^{-1}\mathbf{q} & (L/\Lambda)\Lambda^{-1} + (\Lambda^{-1}\mathbf{q}\mathbf{p}^T\Lambda^{-1}) \\ 1 & \mathbf{p}^T\Lambda^{-1} \end{bmatrix}_{(n,n)}, \tag{9}$$

where $(L/\Lambda) = -\left((a_n + b_n) + \sum_{i=1}^{n-1} \frac{a_i b_{n-i}}{s_i}\right)$, *as in* (7), *and* $\Lambda^{-1} = \mathrm{diag}(\frac{1}{s_1}, \frac{1}{s_2}, \ldots, \frac{1}{s_{n-1}})$.

In the present paper we give explicit Moore-Penrose inverse and group inverse formulae for the doubly Leslie matrix and give some related topics.

2 Preliminaries

Let $\mathbb{R}^{m\times n}$ denote the set of all $m \times n$ matrices over the field of real numbers $\mathbb{R}$. The Moore-Penrose inverse of a matrix $A \in \mathbb{R}^{m\times n}$ is the unique matrix $X \in \mathbb{R}^{n\times m}$ satisfying the four Penrose conditions

$$A = AXA, \quad X = XAX, \quad (AX)^T = AX \quad \text{and} \quad (XA)^T = XA$$

and is denoted by $A^\dagger$. The group inverse of a matrix $A \in \mathbb{R}^{n\times n}$ is the unique matrix $X \in \mathbb{R}^{n\times n}$ satisfying

$$A = AXA, \quad X = XAX \quad \text{and} \quad AX = XA$$

and is denoted by $A^\sharp$. A well known characterization for the existence of $A^\sharp$ is that $\mathrm{rank}(A) = \mathrm{rank}(A^2)$, [1]. If A is nonsingular, then $A^{-1} = A^\dagger = A^\sharp$. Recall that $A \in \mathbb{R}^{n\times n}$ is called range-symmetric if $\mathrm{range}(A) = \mathrm{range}(A^T)$. If A is range-symmetric, then $A^\dagger = A^\sharp$.

A system of linear equation $A\mathbf{x} = \mathbf{b}$ need not possess a solution when $\mathrm{rank}(A) \neq \mathrm{rank}[A : \mathbf{b}]$. That is $\mathbf{b}$ is not in the range of A. The Moore-Penrose inverse is most often used to solve least squares systems. It is still desirable to to find a $\mathbf{x}_0$ that is closest to a solution. The residual vector is a key component to solve these systems.

Theorem 2 ([1, ?]). *Let* $A \in \mathbb{R}^{m\times n}$ *with* $\mathrm{rank}(A) = r \neq 0$, *and suppose* $A = FG$ *is a full rank factorization of* A. *Then*

1. $F^\dagger = (F^T F)^{-1} F^T$,
2. $F^\dagger F = I_r$, *the* $r \times r$ *identity matrix,*
3. $G^\dagger = G^T (G G^T)^{-1}$,

4. $GG^{\dagger} = I_r$,

5. $A^{\dagger} = G^{\dagger}F^{\dagger}$.

More generally, for any $m \times n$ matrix A of full row rank m, $A = I_m A$ is a full rank factorization of A. Then

$$A^{\dagger} = A^T(AA^T)^{-1}. \tag{10}$$

The group inverse is very useful and has applications in many fields such as singular differential and difference equations, Markov chains, and iterative methods, see for instance [1, ?, ?].

Theorem 3 ([1]). *Let a square matrix A have the full rank factorization $A = FG$. Then A has a group inverse if and only if GF is nonsingular. In which case,*

$$A^{\#} = F(GF)^{-2}G.$$

3 Moore-Penrose Inverse of RDLM

Penrose [5, p.18]. It is possible to calculate $A^{\dagger}$ even when A^*A and AA^* are both singular by the following methods, where A^* is the conjugate transpose of the matrix A.

Any matrix M can be partitioned in the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $D = CA^{-1}B$, (using a suitable arrangement of rows and columns). A being any non-singular submatrix whose rank is equal to that of the whole matrix. It is then easily verified that

$$M^{\dagger} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{\dagger} = \begin{bmatrix} A^*KA^* & A^*KC^* \\ B^*KA^* & B^*KC^* \end{bmatrix}, \tag{11}$$

where $K = (AA^* + BB^*)^{-1}A(A^*A + C^*C)^{-1}$. The matrices $AA^* + BB^*$ and $A^*A + C^*C$ are positive definite, since A is non-singular. Thus the generalized inverse of any matrix can be expressed in terms of ordinary reciprocals of matrices.

We have the following main results.

Lemma 4. *If $A = PB$, where P is a permutation matrix, then*

$$A^{\dagger} = B^{\dagger}P^{T}. \tag{12}$$

Proof. It is straightforward to verify that $B^{\dagger}P^{T}$ satisfies the four Penrose conditions. Clearly,

1. $PB(B^{\dagger}P^{T})PB = PBB^{\dagger}B = PB$,
2. $(B^{\dagger}P^{T})PB(B^{\dagger}P^{T}) = B^{\dagger}BB^{\dagger}P^{T} = B^{\dagger}P^{T}$,
3. $[PB(B^{\dagger}P^{T})]^{T} = [PBB^{\dagger}P^{T}]^{T} = P(BB^{\dagger})^{T}P^{T} = PBB^{\dagger}P^{T} = PB(B^{\dagger}P^{T})$,
4. $[(B^{\dagger}P^{T})PB]^{T} = [B^{\dagger}P^{T}PB]^{T} = [B^{\dagger}B]^{T} = B^{\dagger}B = B^{\dagger}(P^{T}P)B = (B^{\dagger}P^{T})PB.$

□

Lemma 5. *For an $m \times n$ $\mathbb{R}$-matrix N of rank $r < \min(m, n)$, and N partitioned in the form*

$$N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where C is $r \times r$ nonsingular. Then

$$N^{\dagger} = \begin{bmatrix} C^{T}KD^{T} & C^{T}KC^{T} \\ D^{T}KA^{T} & D^{T}KC^{T} \end{bmatrix}$$

where $K = (CC^{T}+DD^{T})^{-1}C(C^{T}C+A^{T}A)^{-1}$. The matrices $CC^{T}+DD^{T}$ and $C^{T}C+A^{T}A$ are positive definite, since C is non-singular.

Proof. Let $P = \begin{bmatrix} 0 & I_r \\ I_{m-r} & 0 \end{bmatrix}_{(m \times m)}$ be a permutation matrix. Premultiplying the matrix N by P.

$$PN = \begin{bmatrix} C & D \\ A & B \end{bmatrix}.$$

Since P is a unitary matrix and by (12). We have

$$(PN)^{\dagger} = N^{\dagger}P^{T}.$$

Therefore

$$(PN)^\dagger = \begin{bmatrix} C & D \\ A & B \end{bmatrix}^\dagger = N^\dagger P^T.$$

and

$$N^\dagger = \begin{bmatrix} C & D \\ A & B \end{bmatrix}^\dagger P.$$

As in (11), we have

$$\begin{bmatrix} C & D \\ A & B \end{bmatrix}^\dagger = \begin{bmatrix} C^TKC^T & C^TKA^T \\ D^TKC^T & D^TKA^T \end{bmatrix},$$

where $K = (CC^T + DD^T)^{-1}C(C^TC + A^TA)^{-1}$. The matrices $CC^T + DD^T$ and $C^TC + A^TA$ are positive definite, since C is non-singular. Therefore $CC^T + DD^T$ and $C^TC + A^TA$ are also non-singular matrices. We have

$$N^\dagger = \begin{bmatrix} C^TKC^T & C^TKA^T \\ D^TKC^T & D^TKA^T \end{bmatrix} P = \begin{bmatrix} C^TKA^T & C^TKC^T \\ D^TKA^T & D^TKC^T \end{bmatrix}.$$

The proof is complete. □

Theorem 6. *Let L be a doubly Leslie matrix as in (3) with partitioned as*

$$L = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}_{(n,n)},$$

where $\mathbf{p} = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}^T$, $\mathbf{q} = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_1 \end{bmatrix}^T$, *and* $\Lambda = \mathrm{diag}(s_1, s_2, \dots, s_{n-1})$, $s_j > 0$, $j = 1, 2, \dots, n-1$ *is a diagonal matrix of order* $n-1$, *then*

$$L^\dagger = \begin{bmatrix} -\Lambda K\mathbf{p} & \Lambda K\Lambda \\ \mathbf{q}^TK\mathbf{p} & -\mathbf{q}^TK\Lambda \end{bmatrix},$$

where $K = (\Lambda^2 + \mathbf{q}\mathbf{q}^T)^{-1}\Lambda(\Lambda^2 + \mathbf{p}\mathbf{p}^T)^{-1}$.

Proof. If $\det L \neq 0$ then $L^\dagger = L^{-1}$ which appeared in (9).
In general

$$\begin{aligned} L^\dagger &= \begin{bmatrix} A & B \\ C & D \end{bmatrix}^\dagger = \begin{bmatrix} C^TKA^T & C^TKC^T \\ D^TKA^T & D^TKC^T \end{bmatrix} \\ &= \begin{bmatrix} \Lambda^TK(-\mathbf{p}^T)^T & \Lambda^TK\Lambda^T \\ (-\mathbf{q})^TK(-\mathbf{p}^T)^T & (-\mathbf{q})^TK\Lambda^T \end{bmatrix} \\ &= \begin{bmatrix} -\Lambda K\mathbf{p} & \Lambda K\Lambda \\ \mathbf{q}^TK\mathbf{p} & -\mathbf{q}^TK\Lambda \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} K &= (\Lambda\Lambda^T + (-\mathbf{q})(-\mathbf{q})^T)^{-1}\Lambda(\Lambda^T\Lambda + (-\mathbf{p}^T)^T(-\mathbf{p}^T))^{-1} \\ &= (\Lambda^2 + \mathbf{q}\mathbf{q}^T)^{-1}\Lambda(\Lambda^2 + \mathbf{p}\mathbf{p}^T)^{-1}. \end{aligned}$$

□

Corollary 7. *Let R be a rectangle doubly Leslie matrix as in (5) with partitioned as*

$$R = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda_k & -\mathbf{q}_k \end{bmatrix}_{(m,n)},$$

where $m = n - k$,

$$\mathbf{p} = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}^T, \quad \mathbf{q} = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_k \end{bmatrix}^T,$$

and $\Lambda_k = [\mathrm{diag}(s_1, s_2, \dots, s_{n-k})|0]$ is a $(n-k)\times(n-1)$ block matrix, then

$$L^\dagger = \begin{bmatrix} -\Lambda_k K\mathbf{p} & \Lambda_k K \Lambda_k \\ \mathbf{q}_k^T K\mathbf{p} & -\mathbf{q}_k^T K \Lambda_k \end{bmatrix}$$

where $K = (\Lambda_k\Lambda_k^T + \mathbf{q}_k\mathbf{q}_k^T)^{-1}\Lambda_k(\Lambda_k^T\Lambda_k + \mathbf{p}\mathbf{p}^T)^{-1}$.

Proof. The proof is an analogous as in Theorem 6. □

Let's consider some examples.

EXAMPLE

$$L = \begin{bmatrix} 1 & 2 & -1 & -3 \\ 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} =: \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}$$

where $\mathbf{p} = \begin{bmatrix} -1 & -2 & 1 \end{bmatrix}^T$, $\mathbf{q} = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix}^T$, and $\Lambda = \mathrm{diag}(1,2,1)$, is a diagonal matrix of order 3, then

$$L^\dagger = \begin{bmatrix} -\Lambda K\mathbf{p} & \Lambda K\Lambda \\ \mathbf{q}^T K\mathbf{p} & -\mathbf{q}^T K\Lambda \end{bmatrix},$$

where $K = (\Lambda^2 + \mathbf{q}\mathbf{q}^T)^{-1}\Lambda(\Lambda^2 + \mathbf{p}\mathbf{p}^T)^{-1}$.

First we calculate $\mathbf{q}\mathbf{q}^T$ and $\mathbf{p}\mathbf{p}^T$.

$$\begin{aligned}
\mathbf{q}\mathbf{q}^T &= \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}, \\
\mathbf{p}\mathbf{p}^T &= \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix},
\end{aligned}$$

and

$$\begin{aligned}
(\Lambda^2 + \mathbf{q}\mathbf{q}^T)^{-1} &= \begin{bmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{5}{6} & 0 & \frac{1}{3} \\ 0 & \frac{1}{4} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}, \\
(\Lambda^2 + \mathbf{p}\mathbf{p}^T)^{-1} &= \begin{bmatrix} 2 & 2 & -1 \\ 2 & 8 & -2 \\ -1 & -2 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{8} & \frac{1}{4} \\ -\frac{1}{8} & \frac{3}{16} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{8} & \frac{3}{4} \end{bmatrix},
\end{aligned}$$

we have

$$K = (\Lambda^2 + \mathbf{q}\mathbf{q}^T)^{-1}\Lambda(\Lambda^2 + \mathbf{p}\mathbf{p}^T)^{-1} = \begin{bmatrix} \frac{17}{24} & -\frac{1}{16} & \frac{11}{24} \\ -\frac{1}{16} & \frac{3}{32} & \frac{1}{16} \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}.$$

Finally,

$$\begin{aligned}
-\Lambda K\mathbf{p} &= -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{17}{24} & -\frac{1}{16} & \frac{11}{24} \\ -\frac{1}{16} & \frac{3}{32} & \frac{1}{16} \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{8} \\ \frac{1}{8} \\ 0 \end{bmatrix}, \\
\Lambda K \Lambda &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{17}{24} & -\frac{1}{16} & \frac{11}{24} \\ -\frac{1}{16} & \frac{3}{32} & \frac{1}{16} \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{17}{24} & -\frac{1}{8} & \frac{11}{24} \\ -\frac{1}{8} & \frac{3}{8} & \frac{1}{8} \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix},
\end{aligned}$$

$$\mathbf{q}^T K\mathbf{p} = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{17}{24} & -\frac{1}{16} & \frac{11}{24} \\ -\frac{1}{16} & \frac{3}{32} & \frac{1}{16} \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} = -\frac{1}{8},$$

$$\begin{aligned} -\mathbf{q}^T K\Lambda &= -\begin{bmatrix} 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{17}{24} & -\frac{1}{16} & \frac{11}{24} \\ -\frac{1}{16} & \frac{3}{32} & \frac{1}{16} \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{24} & \frac{1}{8} & \frac{5}{24} \end{bmatrix}. \end{aligned}$$

Therefore

$$L^\dagger = \begin{bmatrix} -\Lambda K\mathbf{p} & \Lambda K\Lambda \\ \mathbf{q}^T K\mathbf{p} & -\mathbf{q}^T K\Lambda \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & \frac{17}{24} & -\frac{1}{8} & \frac{11}{24} \\ \frac{1}{8} & -\frac{1}{8} & \frac{3}{8} & \frac{1}{8} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ -\frac{1}{8} & -\frac{1}{24} & \frac{1}{8} & \frac{5}{24} \end{bmatrix}.$$

This matrix is satisfies the four Penrose conditions. □

EXAMPLE. For a full row rank rectangle doubly Leslie matrix of order 3×4

$$R = \begin{bmatrix} 1 & 2 & -1 & -3 \\ 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \end{bmatrix}.$$

From (10),

$$\begin{aligned} R^\dagger &= R^T(RR^T)^{-1} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ -1 & 0 & 0 \\ -3 & -1 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 & -1 & -3 \\ 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ -1 & 0 & 0 \\ -3 & -1 & 0 \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ -1 & 0 & 0 \\ -3 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{11}{6} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{7}{12} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{3} & \frac{7}{6} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{3} \end{bmatrix}. \end{aligned}$$

This matrix is also satisfies the four Penrose conditions. □

4 Group Inverse of DLM

As in [1, p.167] we have the following useful result.

Theorem 8. *Let A be a square singular matrix,* $\operatorname{rank} A = \operatorname{rank} A^2$*, and $R(A)$ be the range of A. If the system*

$$A\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \in R(A)$$

has a solution, it is uniquely given by

$$\mathbf{x} = A^{\sharp}\mathbf{b}.$$

Proof. Suppose that $\mathbf{x} \in R(A)$ where $R(A)$ is the range of A. There is a vector $\mathbf{y}$ such that $A\mathbf{y} = \mathbf{x}$. Let a solution $\mathbf{x}$ be written as $\mathbf{x} = A\mathbf{y}_1$ for some $\mathbf{y}_1$. We have

$$A\mathbf{x} = AA\mathbf{y}_1 = A^2\mathbf{y}_1,$$

then $A^2\mathbf{y}_1 = \mathbf{b}$. Since $\operatorname{rank} A = \operatorname{rank} A^2$, there is a unique $A^{\sharp}$ such that

$$AA^{\sharp}A = A, \quad A^{\sharp}AA^{\sharp} = A^{\sharp}, \quad \text{and} \quad AA^{\sharp} = A^{\sharp}A.$$

Therefore

$$\begin{aligned} \mathbf{x} &= A\mathbf{y}_1 \\ &= AA^{\sharp}A\mathbf{y}_1 \\ &= A^2A^{\sharp}\mathbf{y}_1 \\ &= A^{\sharp}A^2\mathbf{y}_1 \\ &= A^{\sharp}A\mathbf{x} \\ &= A^{\sharp}\mathbf{b}. \end{aligned}$$

□

Let L be a doubly Leslie matrix as in (3) with partitioned as

$$L = \begin{bmatrix} -\mathbf{p}^T & -a_n - b_n \\ \Lambda & -\mathbf{q} \end{bmatrix}_{(n,n)}.$$

If $\det L \neq 0$ then $L^{\sharp} = L^{-1}$ which was shown in (9). We interested in study the only case $\operatorname{rank}(L) \neq n$. By the definition of DLM the rank of L is at least $n-1$. Since equivalence matrix has the same rank, we reduce the matrix L to a reduced echelon form as follows:

$$\begin{bmatrix} 0 & \frac{1}{s_1} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{s_2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{s_{n-1}} \\ 1 & \frac{a_1}{s_1} & \frac{a_2}{s_2} & \cdots & \frac{a_{n-1}}{s_{n-1}} \end{bmatrix} \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n - b_n \\ s_1 & 0 & \cdots & 0 & -b_{n-1} \\ 0 & s_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -b_2 \\ 0 & \cdots & 0 & s_{n-1} & -b_1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & -\frac{b_{n-1}}{s_1} \\ 0 & 1 & 0 & \vdots & -\frac{b_{n-2}}{s_2} \\ 0 & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & 1 & -\frac{b_1}{s_{n-1}} \\ 0 & 0 & \cdots & 0 & -a_n - b_n - \frac{a_1}{s_1}b_{n-1} - \frac{a_2}{s_2}b_{n-2} - \cdots - \frac{a_{n-2}}{s_{n-2}}b_2 - \frac{a_{n-1}}{s_{n-1}}b_1 \end{bmatrix}.$$

We see that $\operatorname{rank}(L) = n - 1$ if and only if

$$-a_n - b_n - \frac{a_1}{s_1}b_{n-1} - \frac{a_2}{s_2}b_{n-2} - \cdots - \frac{a_{n-2}}{s_{n-2}}b_2 - \frac{a_{n-1}}{s_{n-1}}b_1 = 0$$

if and only if

$$-a_n - b_n = \frac{a_1}{s_1}b_{n-1} + \frac{a_2}{s_2}b_{n-2} + \cdots + \frac{a_{n-2}}{s_{n-2}}b_2 + \frac{a_{n-1}}{s_{n-1}}b_1.$$

We factor L to full rank factorization as follows:

$$\begin{aligned} L = FG &= \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} \\ s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 & -\frac{b_{n-1}}{s_1} \\ 0 & 1 & \ddots & \vdots & -\frac{b_{n-2}}{s_2} \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -\frac{b_1}{s_{n-1}} \end{bmatrix} \\ &= \begin{bmatrix} -\mathbf{p}^T \\ \Lambda \end{bmatrix} \begin{bmatrix} I_{n-1} & -\mathbf{q}_1 \end{bmatrix}, \end{aligned}$$

where $\mathbf{p} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$, $\mathbf{q}_1 = \begin{bmatrix} \frac{b_{n-1}}{s_1} \\ \frac{b_{n-2}}{s_2} \\ \vdots \\ \frac{b_1}{s_{n-1}} \end{bmatrix}$, and $\Lambda = \operatorname{diag}(s_1, s_2, \ldots, s_{n-1})$.

Also, by direct computation, we have

$$GF = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \dots & -a_{n-2} & -a_{n-1} - b_{n-1}\frac{s_{n-1}}{s_1} \\ s_1 & 0 & 0 & \dots & 0 & -b_{n-2}\frac{s_{n-1}}{s_2} \\ 0 & s_2 & 0 & \dots & 0 & -b_{n-3}\frac{s_{n-1}}{s_3} \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & -b_2\frac{s_{n-1}}{s_{n-2}} \\ 0 & 0 & \cdots & 0 & s_{n-2} & -b_1\frac{s_{n-1}}{s_{n-1}} \end{bmatrix} =: M.$$

The matrix $GF =: M$ is a doubly Leslie matrix of order $(n-1) \times (n-1)$.

$$M = \begin{bmatrix} -\mathbf{p}_1^T & -a_{n-1} - b_{n-1}\frac{s_{n-1}}{s_1} \\ \Lambda_1 & -\mathbf{q}_2 \end{bmatrix}_{(n-1,n-1)},$$

where $\mathbf{p}_1 = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-2} \end{bmatrix}$, $\mathbf{q}_2 = \begin{bmatrix} b_{n-2}\frac{s_{n-1}}{s_2} \\ b_{n-3}\frac{s_{n-1}}{s_3} \\ \vdots \\ b_1\frac{s_{n-1}}{s_{n-1}} \end{bmatrix}$, and

$$\Lambda_1 = \text{diag}(s_1, s_2, \dots, s_{n-2})$$

is a diagonal matrix of order $n-1$.

By (9), we have

$$M^{-1} = (M/\Lambda_1)^{-1} \begin{bmatrix} \Lambda_1^{-1}\mathbf{q}_2 & (M/\Lambda_1)\,\Lambda_1^{-1} + (\Lambda_1^{-1}\mathbf{q}_2\mathbf{p}_1^T\Lambda_1^{-1}) \\ 1 & \mathbf{p}_1^T\Lambda_1^{-1} \end{bmatrix}, \quad (13)$$

where $(M/\Lambda_1) = -\left((a_{n-1} + b_{n-1}\frac{s_{n-1}}{s_1}) + s_{n-1}\sum_{i=1}^{n-2}\frac{a_i b_{n-i-1}}{s_i s_{i+1}}\right)$, as in (7), and $\Lambda_1^{-1} = \text{diag}(\frac{1}{s_1}, \frac{1}{s_2}, \dots, \frac{1}{s_{n-2}})$.

From Theorem 3, we have

$$
\begin{aligned}
L^\sharp &= F(GF)^{-2}G \\
&= \begin{bmatrix} -\mathbf{p}^T \\ \Lambda \end{bmatrix} \times \\
&\quad \left((M/\Lambda_1)^{-1} \begin{bmatrix} \Lambda_1^{-1}\mathbf{q}_2 & (M/\Lambda_1)\,\Lambda_1^{-1} + \left(\Lambda_1^{-1}\mathbf{q}_2\mathbf{p}_1^T\Lambda_1^{-1}\right) \\ 1 & \mathbf{p}_1^T\Lambda_1^{-1} \end{bmatrix} \right)^2 \times \\
&\quad \begin{bmatrix} I_{n-1} & -\mathbf{q}_1 \end{bmatrix} \\
&= (M/\Lambda_1)^{-2} \begin{bmatrix} -\mathbf{p}^T \\ \Lambda \end{bmatrix} \times \\
&\quad \begin{bmatrix} \Lambda_1^{-1}\mathbf{q}_2 & (M/\Lambda_1)\,\Lambda_1^{-1} + \left(\Lambda_1^{-1}\mathbf{q}_2\mathbf{p}_1^T\Lambda_1^{-1}\right) \\ 1 & \mathbf{p}_1^T\Lambda_1^{-1} \end{bmatrix}^2 \begin{bmatrix} I_{n-1} & -\mathbf{q}_1 \end{bmatrix}.
\end{aligned}
$$

Let's consider the same example.

EXAMPLE

$$
L = \begin{bmatrix} 1 & 2 & -1 & -3 \\ 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}
$$

where $\mathbf{p} = \begin{bmatrix} -1 & -2 & 1 \end{bmatrix}^T$, $\mathbf{q} = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix}^T$, and $\Lambda = \operatorname{diag}(1,2,1)$, is a diagonal matrix of order 3.

Since $\det(L) = 0$, we have $\operatorname{rank}(L) = \operatorname{rank}(L^2)$, we know that the unique $L^\sharp$ exists. Now

$$
L = FG = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}
$$

and

$$
GF = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix},
$$

$$
\begin{aligned}
(GF)^{-1} &= \begin{bmatrix} 1 & 2 & -2 \\ 1 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}, \\
(GF)^{-2} &= \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{8} & \frac{3}{8} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{8} \end{bmatrix}.
\end{aligned}
$$

Finally

$$
\begin{aligned}
L^{\#} &= F(GF)^{-2}G \\
&= \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{8} & \frac{3}{8} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \frac{3}{4} & \frac{1}{8} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{1}{8} & \frac{1}{4} \end{bmatrix}.
\end{aligned}
$$

This matrix is satisfies the three conditions for group inverse. □

5 Conclusion

In this paper, we mainly study about the explicit formula of Moore-Penrose inverse and group inverse of doubly Leslie matrix.

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