สัญญาเลขที่ R2559C123

รายงานวิจัยฉบับสมบูรณ์

โครงการ : ขั้นตอนวิธีสำหรับการหาคำตอบของปัญหาการหาค่าเหมาะที่สุดเชิงคอนเวกซ์โดยปราศจากฟังก์ชันบังคับเชิงคอนเวกซ์

Algorithm for finding solutions of convex Optimization problem without convex constraint functions

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ชื่อโครงการ: ขั้นตอนวิธีสำหรับการหาคำตอบของปัญหาการหาค่าเหมาะที่สุดเชิงคอนเวกซ์โดยปราศจากฟังก์ชันบังคับเชิงคอนเวกซ์

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โครงการวิจัยนี้มุ่งเน้นไปที่คลาสของเซตและฟังก์ชันที่ไม่คอนเวกซ์ โดยได้นำเสนอเงื่อนไขและวิธีการสำหรับการหาคำตอบสำหรับปัญหาที่พิจารณาในสองรูปแบบคือ ในกรณีเมื่อเซตบังคับที่พิจารณาไม่เป็นคอนเวกซ์ และกรณีที่เซตบังคับที่พิจารณาเป็นคอนเวกซ์แต่ไม่สามารถนำเสนอให้อยู่ในรูปแบบฟังก์ชันตัวแทนที่เป็นคอนเวกซ์ฟังก์ชันได้ โดยพิจารณากรณีทั้งสองบนปัญหาอสมการการแปรผันและปัญหาค่าเหมาะสมที่สุด ตามลำดับ

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ABSTRACT

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This project is concentrated to the classes of nonconvex sets and nonconvex functions. We study and suggest coth the conditions and methods for the existence and finding a solution of the following two cases: first case is when the consered set is a nonconvex set, and the second case is when the considered set is a convex set but can not represented by a class of convex functions. We consider the two cases above equipped with the variational inequality problem and optimization problem, respectively

Keywords: convex set, convex function, variational inequality problem, optimization problem.

Executive Summary

In this project, we obtain a following manuscript:

1. **Jittiporn Tangkhawiwetkul and Narin Petrot, Existence and convergence theorems for the split quasi variational inequality problems on proximally smooth sets, J. Nonlinear Sci. Appl. 9 (2016), 2364-2375.**

   (ISI: Impact Factor= 0.949)

Here is the executive summary of this part.

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let 2^H be denoted for the class of all nonempty subsets of H, and K be a closed subset of H. For each $K \subseteq H$, we denote by $d(\cdot, K)$ for the usual distance function on H to K, that is, $d(u, K) = \inf_{v\in K} \|u - v\|$, for all $u \in H$.

For each $K \subseteq H$ and $u \in H$. A point $v \in K$ is called the closest point or the projection of u onto K if $d(u, K) = \|u - v\|$. The set of all such closest points is denoted by $Proj_K(u)$, that is , $Proj_K(u) = \{v \in K : d(u, K) = \|u - v\|\}$. The proximal normal cone to K at u is given by

$$N_K^P(u) = \{v \in H : \exists \rho > 0 \text{ such that } u \in Proj_K(u + \rho v)\}.$$

For a given $r \in (0, +\infty]$, a subset K of H is said to be *uniformly prox-regular with respect to r*, say, *uniformly r-prox-regular set*, if for all $\overline{x} \in K$ and for all $0 \neq z \in N_K^P(\overline{x})$, one has

$$\left\langle \frac{z}{\|z\|}, x - \overline{x} \right\rangle \leq \frac{1}{2r}\|x - \overline{x}\|^2, \text{ for all } x \in K.$$

Note that for the case of $r = \infty$, the uniform r-prox-regularity K is equivalent to the convexity of K. For the sake of simplicity, from now on, we will make use the following notation: for each $r \in (0, +\infty]$, we write

$$K_r := \{x \in H : d(x, K) < r\},$$

and $[Cl(H)]_r$ for the class of all uniformly r-prox regular subsets of H.

In this work, we will considering the following class of mappings.

- A mapping $T : H \to H$ is said to be a σ-strongly monotone if there exists $\sigma > 0$ such that for all $x, x^* \in K$,

$$\langle T(x) - T(x^*), x - x^* \rangle \geq \|x - x^*\|^2.$$

- A mapping $T : H \to H$ is said to be a β-Lipschitzian if there exists a real number $\beta > 0$ such that

$$\|T(x) - T(y)\| \leq \beta \|x - y\|, \quad \text{for all } x, y \in H.$$

- A multivalued mapping $C : H \to 2^H$ is said to be a κ-Lipschitz continuous if there exists a real number $\kappa > 0$ such that

$$\|d(y, C(x)) - d(y', C(x'))\| \leq \|y - y'\| + \kappa \|x - x'\|, \quad \text{for all } x, x', y, y' \in H.$$

Next is the main problem that we have interested.

Let H_1 and H_2 be real Hilbert spaces, $T_i : H_i \to H_i$ be nonlinear mappings, $C_i : H_i \to 2^{H_i}$ be nonlinear multivalued mappings for $i = 1, 2$ and $A : H_1 \to H_2$ be a bounded linear operator. In this paper, we are interesting in the following problem: find $x^* \in C_1(x^*)$ such that, $Ax^* \in C_2(Ax^*)$ and

$$\begin{aligned} -T_1(x^*) &\in N^P_{C_1(x^*)}(x^*), \\ -T_2(Ax^*) &\in N^P_{C_2(Ax^*)}(Ax^*). \end{aligned} \tag{1}$$

We introduce the following algorithm which will play an important role in our work.

Algorithm $(\mathcal{A})$: Let $T_i : H_i \to H_i, C_1 : H_1 \to [Cl(H_1)]_r$ and $C_2 : H_2 \to [Cl(H_2)]_s$ be nonlinear mappings where $r, s \in (0, +\infty)$ and $i = 1, 2$. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator, denoted by A^*. Given $x_0 \in H_1$, compute the algorithm sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ as the following projection method:

$$\begin{aligned} y_n &\in Proj_{C_1(x_n)}[x_n - \rho T_1(x_n)], \\ z_n &\in Proj_{C_2(Ay_n)}[Ay_n - \lambda T_2(Ay_n)], \\ x_{n+1} &\in Proj_{C_1(y_n)}[y_n + \gamma A^*(z_n - Ay_n)], \end{aligned} \tag{2}$$

where ρ, λ and γ are step size positive real numbers.

The following assumption is proposed, as the sufficient conditions.

Assumption $(\mathcal{C})$: Let $T_i : H_i \to H_i, C_1 : H_1 \to [Cl(H_1)]_r$ and $C_2 : H_2 \to [Cl(H_2)]_s$ be nonlinear mappings for $r, s \in (0, +\infty)$ and $i = 1, 2$ which are satisfied the following conditions:

(*i*) T_i is a β_i-Lipschitzian mapping and a σ_i-strongly monotone mapping for i = 1, 2;

(*ii*) C_i is a κ_i- Lipschitzian continuous mapping for some $\kappa_i \in [0, 1)$ and i = 1, 2;

(*iii*) for each $i = 1, 2$, there is $\omega_i \in [0, 1)$ such that

$$\|Proj_{C_i(x)}(z) - Proj_{C_i(y)}(z)\| \leq \omega_i \|x - y\|, \quad \text{for all } x, y, z \in H_i.$$

The following theorem shows that the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$, are all convergent sequences.

Theorem A Let H_1, H_2 be real Hilbert spaces. Assume that Assumption $(\mathcal{C})$ holds. If $\gamma < \min\{\frac{2}{\|A\|^2}, \frac{1-\omega_1-\varphi}{\varphi\theta_2\|A\|^2}\}$, where $\varphi = \frac{1+\mu\kappa_1}{\kappa_1(\mu-1)}, \theta_2 = t_{s^*}\sqrt{1 - 2\lambda\sigma_2 + \lambda^2\beta_2^2} + \omega_2$ and $t_{s^*} = \frac{s}{s-s^*}$, then $\{x_n\}, \{y_n\}$ and $\{z_n\}$, which are constructed in Algorithm $(\mathcal{A})$, are convergent sequences.

By using Theorem A, we are in position to present the sufficient condition for existence of solution of problem (1), which is our main theorem.

Theorem B Let H_1, H_2 be real Hilbert spaces. Let $T_i : H_i \to H_i$ be nonlinear mappings for $i = 1, 2$ and $C_1 : H_1 \to [Cl(H_1)]_r$ and $C_2 : H_2 \to [Cl(H_2)]_s$ be nonlinear set-valued mappings. Assume that all of the assumptions in Theorem A hold and if $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$ and $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} z_n$. Then, the problem (1) has a solution.

In conclusion, in this part, we introduce and study a type of split quasi variational inequality problem over a class of nonconvex sets. In order to proof the existence theorems, an algorithm is constructed as an important tool. This problem generalizes and extends the variational inequality problems and the split variational inequality problems from the setting of convex sets to nonconvex case. We desire that the results which presented here will be useful and valuable for researchers who study the branch of variational inequality and related applications.

2. **Porntip Promsinchai and Narin Petrot, Barrier method for convex optimization problem without regularity of constraint functions, (submitted).**

Here is the executive summary of this part.

We are interested in the following convex optimization problem:

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in C, \end{aligned} \tag{3}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function and set C is a feasible convex subset of $\mathbb{R}^n$ given by

$$C = \{x \in \mathbb{R}^n : g_i(x) \geq 0, i = 1, ..., m\},$$

for some constraint functions $g_i : \mathbb{R}^n \to \mathbb{R}, \forall i = 1, ..., m$. The considered set $\{g_i : i = 1, .., m\}$ is called a representation set of C. If each g_i is a concave function, we say that C has a convex representation, and in the case that each representation of C always has one or more g_i's which are not concave functions, we will call the problem of type (3) as a convex optimization without convex representation. Here we are interesting in the latter case, and we use the following so-called a barrier or a log-barrier function as a main tool for implementation this point. For each $\mu > 0$, a barrier or a log-barrier function $\varphi_\mu : C \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\varphi_\mu(x) = \begin{cases} f(x) - \mu \ \sum_{i=1}^{m} \ln(g_i(x)) & \text{if} \quad x \in S \\ +\infty & \text{otherwise,} \end{cases} \tag{4}$$

where $S := \bigcap_{i=1}^{m} \{x : g_i(x) > 0\}$.

In this work, we have concerned with the following concepts.

- A function $g : \mathbb{R}^n \to \mathbb{R}$ is called a *locally Lipschitz* at a point $x \in \mathbb{R}^n$ if there exist a real number $L_x > 0$ and $\epsilon_x > 0$ such that

$$|g(y) - g(z)| \leq L_x \|y - z\| \quad \text{for all} y, z \in B(x, \epsilon_x).$$

- Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. A *directional derivative* of f at x in the feasible direction h, denoted by $f'(x, h)$, is defined by

$$f'(x,h) = \lim_{t \downarrow 0} \frac{f(x+th) - f(x)}{t}.$$

We say that the directional derivative of f in the direction h at x exists if the above limit exists.

- If $g : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function at $x \in \mathbb{R}^n$, then the *generalized directional derivative* or *Clarke derivative* of g at x in the direction $h \in \mathbb{R}^n$ is defined by

$$g^{\circ}(x,h) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{g(y+th) - g(y)}{t}.$$

Note that under the locally Lipschitz assumption of g, the Clarke derivative always exists.

- For a locally Lipschitz function g and $x \in \mathbb{R}^n$, the *Clarke subdifferential* of g at x, denoted by $\partial^{\circ} g(x)$, is defined by

$$\partial^{\circ} g(x) = \{v \in \mathbb{R}^n : g^{\circ}(x,h) \geq \langle v, h \rangle, \forall h \in \mathbb{R}^n\}.$$

It is well known that, the Clarke subdifferential $\partial^{\circ} g(x)$ is a nonempty, convex and compact set.

- If g is a locally Lipschitz function which is also directionally differentiable at $x \in \mathbb{R}^n$ and $g'(x,h) = g^{\circ}(x,h)$ for all $h \in \mathbb{R}^n$, then g is said to be *regular* in the sense of Clarke at $x \in \mathbb{R}^n$.

- Condition NDC: Let C be a constraint set of problem (3). We say that the *non-degeneracy condition* holds if for all $i = 1, .., m$,

$$0 \notin \partial^{\circ} g_i(x), \text{whenever} x \in C \text{and} g_i(x) = 0.$$

In our work, we always assume that the constraint set C in the problem (3) is represented by a set of locally Lipschitz functions which are directionally differentiable. We begin by providing a sufficient condition for the existence of solutions of problem (3). This result will play an important role in the convergence analysis of our considered log-barrier method.

Theorem C Let C be defined as in the problem (3) and $x^* \in C$. Assume that there exist scalars $\lambda_i \geq 0$ such that for all $x \in C$ we have

(a) $f'(x^*, x - x^*) \geq \sum_{i=1}^{m} \lambda_i g_i^{\circ}(x^*, x - x^*)$

(b) $\lambda_i g_i(x^*) = 0,\ i = 1, ..., m.$

Then x^* is a global minimizer of the problem (3).

The following result is related to a property of the log-barrier function φ_μ, which was defined by (4).

Theorem D Let C be defined as in Problem (3). Assume that C is a compact set and the Slater condition holds. Then, for every $\mu > 0$ the log-barrier function φ_μ has a minimizer, which is an element of $intC$.

According to the Theorem D, under the Slater condition, we can construct a sequence $\{x_\mu\} \subset intC$ by

$$x_\mu = \arg\min_{x \in C} \varphi_\mu(x), \tag{5}$$

for each positive real number μ. In our next theorem, which is our main result, under the additional condition as NDC-condition, we show that every accumulation point of $\{x_\mu\}$ as $\mu \to 0$ is a global minimizer of the problem (3).

Theorem E Let C in Problem (3) be a compact set and $\{x_\mu\}$ defined as in (5). If the Slater condition and NDC-condition hold, then every accumulation point of $\{x_\mu\}$ with $\mu \to 0$ is a global minimizer of the problem (3).

In conclusion, in this work, we have considered the convex optimization problem when the objective function is convex and the constraint functions are just locally Lipschitz and directionally differentiable, but need not be continuously differentiable or regular. It is worth to pointing that, in fact, we are giving an affirmative answer to a problem that was proposed in [Dutta, J., Lalitha, C.S.: Optimality conditions in convex optimization revisited. Optim. Lett. 7(2),221-229 (2013)]. Moreover, after carefully considering, one may observe that the Slater condition is used only for guaranteeing that the log-barrier function φ_μ is a proper function and guarantee that its minimizer is an element of interior of constraint set (see Theorem C). This may rise an interesting following question: can we define a log-barrier function by using a condition that weaker than the Slater condition such that it is still a proper and its minimizer belongs to the set of all interior points of constraint set? In order to develop this research area, this question should be considered in the future works.

CONTENT OF RESEARCHES

BARRIER METHOD FOR CONVEX OPTIMIZATION PROBLEM WITHOUT REGULARITY OF CONSTRAINT FUNCTIONS

PORNTIP PROMSINCHAI AND NARIN PETROT†

ABSTRACT. We consider the convex optimization problem when the objective function may not smooth and the constraint set is represented by constraint functions that are locally Lipschitz and directionally differentiable, but neither necessarily concave nor continuously differentiable. The obtained results improve and extend those results that have been presented in [Dutta, J., Lalitha, C.S.: Optimality conditions in convex optimization revisited. Optim. Lett. 7(2),221-229 (2013)], and [Dutta, J.: Barrier method in nonsmooth convex optimization without convex representation. Optim. Lett. 9(6), 1177-1185 (2015)], by removing the regularity and continuously differentiable assumptions on the constraint functions from the considering.

Key Words and Phrases: Convex optimization; log-barrier function; locally Lipschitz function; directional derivative; Clarke derivative; regular function.

1. INTRODUCTION

In this paper we are interested in the following convex optimization problem:

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in C, \end{aligned} \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function and set C is a feasible convex subset of $\mathbb{R}^n$ given by

$$C = \{x \in \mathbb{R}^n : g_i(x) \geq 0, i = 1, ..., m\},$$

for some constraint functions $g_i : \mathbb{R}^n \to \mathbb{R}, \forall i = 1, ..., m$. The considered set $\{g_i : i = 1, .., m\}$ is called a representation set of C. If each g_i is a concave function, we say that C has a convex representation. In this case, we know that the **KKT** optimality

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condition[1] is both necessary and sufficient for a point to be a minimizer, under the Slater condition[2]. Further, let us notice that if C has a convex representation then the dual method for the problem (1.1) of type

$$\sup_{\lambda\in\mathbb{R}^m}\Big\{\inf_x f(x)-\sum_{i=1}^m \lambda_i g_i(x)\Big\},$$

is well defined, because $x\mapsto f(x)-\sum_{i=1}^m\lambda_i g_i(x)$ is a convex function. In particular, the Lagrangian $x\mapsto L_f(x):=f(x)-f^*-\sum_{i=1}^m\lambda_i g_i(x)$, where $f^*=\inf_x\{f(x): x\in C\}$, defined from an arbitary **KKT** point $(x^*,\lambda)\in K\times\mathbb{R}^m_+$, is convex and nonnegative on $\mathbb{R}^n$, with x^* being a global minimizer. However, if the g_i's are not concave this is not true in general, see [7].

In the case that each representation of C always has one or more g_i's which are not concave functions, we will call the problem of type (1.1) as a convex optimization without convex representation. In this situation, it is naturally interesting to ask that whether under the Slater condition, the **KKT** conditions still continue to be both necessary and sufficient. Motivated by this point, Lasserre [7] showed that if f and g_i's are differentiable functions and under an additional suitable condition, so-called a non-degeneracy condition, that is, for all $i=1,..,m$, $\nabla g_i(x)\neq 0, \forall x\in C$ with $g_i(x)=0$, then the **KKT** condition is both necessary and sufficient.

Later on, Lasserre [8] considered optimality condition of convex optimization problem without convex representation and focussed on the algorithmic issues of the considered convex optimization problems, when f and g_i's are continuously differentiable functions, by using the following so-called a barrier or a log-barrier function. For each $\mu>0$, a barrier or a log-barrier function $\varphi_\mu: C\to\mathbb{R}\cup\{+\infty\}$ is defined by

$$\varphi_\mu(x)=\begin{cases} f(x)-\mu\ \sum_{i=1}^m \ln(g_i(x)) & \text{if } x\in\mathcal{S}\\ +\infty & \text{otherwise,}\end{cases} \tag{1.2}$$

where $\mathcal{S}:=\bigcap_{i=1}^m\{x: g_i(x)>0\}$. Using this function, in that paper, he showed that even though the considered constraint set C does not have a convex representation,

[1]A point $x\in C$ is a **KKT** point if there exist $\lambda_i\geq 0$ for all $i=1,...,m$ such that

$$\lambda_i g_i(x)=0 \quad\text{and}\quad \nabla f(x)+\sum_{i=1}^m\lambda_i\nabla g_i(x)=0,$$

where $\nabla f(x)$ and $\nabla g_i(x)$ are denoted the *gradient vectors* of the function f and g_i at x, respectively.

[2]The Slater condition holds for C if there exists $x\in C$ such that $g_i(x)>0$ for all $i=1,..,m$.

the barrier method can be used for finding a solution of the problem (1.1), if the data of the considered problem is smooth and satisfies the Slater condition and non-degeneracy condition.

Recently, in [5], Dutta continuously focussed on the algorithmic issues of convex optimization problem without convex representation, by removing the continuously differentiability of the objective function and showed the following theorem.

Theorem 1.1. *Consider the problem (1.1), where the constraint set C satisfies the Slater condition and non-degeneracy condition, but may not have a convex representation. If C is a compact set and each g_i is continuously differentiable then for each $\mu > 0$ there exists a global minimizer x_μ of the φ_μ in the interior of C. Moreover, every accumulation point x^* of $\{x_\mu\}$ with $\mu \to 0$ is a global minimizer of f.*

Recall that a function $g : \mathbb{R}^n \to \mathbb{R}$ is called a *locally Lipschitz* at a point $x \in \mathbb{R}^n$ if there exist a real number $L_x > 0$ and $\epsilon_x > 0$ such that

$$|g(y) - g(z)| \leq L_x \|y - z\| \quad \text{for all} \quad y, z \in B(x, \epsilon_x).$$

It is well know that each continuously differentiable function is a locally Lipschitz function. Inspried by this relation, we are interested to improve the problem (1.1), by replacing the continuously differentiability of the representative functions by the locally Lipschitz assumption. The next example shows that there is a convex set such that one of its representative function may be a locally Lipschitz function, but such convex set cannot be represented by a set of smooth functions.

Example 1.2. *Let C be a convex subset of $\mathbb{R}^2$, which is represented by*

$$C = \{(x, y) \in \mathbb{R}^2 : g_1(x, y) \geq 0, \quad g_2(x, y) \geq 0\},$$

where

$$g_1(x, y) = \begin{cases} x^2 \sin(1/x) - y^2 & \text{if} \quad x \neq 0 \\ -y^2 & \text{if} \quad x = 0, \end{cases}$$

and

$$g_2(x, y) = xy - 1,$$

for all $(x, y) \in \mathbb{R}^2$.

We can check that g_1 is a locally Lipschitz function which is not smooth and the convex set C cannot be represented by a set of smooth functions.

Next, we will recall some important concepts that we will use in this literature.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. A *directional derivative* of f at x in the feasible direction h, denoted by $f'(x,h)$, is defined by

$$f'(x,h) = \lim_{t \downarrow 0} \frac{f(x+th) - f(x)}{t}.$$

We say that the directional derivative of f in the direction h at x exists if the above limit exists. It is well known that if f is a convex function, then the directional derivative $f'(x,h)$ exists in every direction $h \in \mathbb{R}^n$.

Now, let us focus on the class of locally Lipschitz functions, which will be the main target of this paper. We start by recalling the concepts of generalized directional derivative and generalized subdifferential of a locally Lipschitz function in the sense of Clarke [3].

If $g : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function at $x \in \mathbb{R}^n$, then the *generalized directional derivative* or *Clarke derivative* of g at x in the direction $h \in \mathbb{R}^n$ is defined by

$$g^\circ(x,h) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{g(y+th) - g(y)}{t}.$$

Note that under the locally Lipschitz assumption of g, the Clarke derivative always exists. For a locally Lipschitz function g and $x \in \mathbb{R}^n$, the *Clarke subdifferential* of g at x, denoted by $\partial^\circ g(x)$, is defined by

$$\partial^\circ g(x) = \{v \in \mathbb{R}^n : g^\circ(x,h) \geq \langle v, h \rangle, \forall h \in \mathbb{R}^n\}.$$

It is well known that, the Clarke subdifferential $\partial^\circ g(x)$ is a nonempty, convex and compact set. Furthermore, for each $h \in \mathbb{R}^n$ we know that $g^\circ(x,h) = \max\{\langle v, h\rangle : v \in \partial^\circ g(x)\}$. For more information on these concepts, the readers may consult [1–3, 9].

By the definition of Clarke derivative, it is worth noting that $g^\circ(x,h) \geq g'(x,h)$, for all $h \in \mathbb{R}^n$, and under locally Lipschitz setting we also know that $g^\circ(x,h)$ is a positively homogeneous, subadditive and upper semicontinuous with respect to h. Further, if g is a locally Lipschitz function which is also directionally differentiable at $x \in \mathbb{R}^n$ and $g'(x,h) = g^\circ(x,h)$ for all $h \in \mathbb{R}^n$, then g is said to be *regular* in the sense of Clarke at $x \in \mathbb{R}^n$. Note that if g is a convex function, then g is a locally Lipschitz function and regular in the sense of Clarke.

Under the framework of locally Lipschitz constraint functions, Dutta and Lalitha [4] considered the optimality conditions for the convex optimization problem (1.1), by assuming that each constraint function is regular in the sense of Clarke. They also introduced the following nonsmooth degeneracy type condition:

Condition NDC: Let C be a constraint set of problem (1.1). We say that the *non-degeneracy condition* holds if for all $i = 1, .., m$,

$$0 \notin \partial^{\circ} g_i(x), \quad \text{whenever} \quad x \in C \quad \text{and} \quad g_i(x) = 0.$$

In that work, Dutta and Lalitha [4] showed that if both the Slater condition and **NDC**-conditions hold then the **KKT** condition is both necessary and sufficient. Further, the authors gave a noticeable question that whether the assumption of regularity of the constraint functions can be removed.

Remark 1.3. *From the Example 1.2, we can check that the function g_1 is a locally Lipschitz function which is directionally differentiable but not regular in the sense of Clarke. Indeed, one can see that $(-1, 0) \in \partial^{\circ} g_1((0, 0))$ and this gives $g^{\circ}((0,0),(h_1,h_2)) \geq \langle(-1,0),(h_1,h_2)\rangle = -h_1$, for all direction $(h_1, h_2) \in \mathbb{R}^2$. This implies that there is a direction such that the Clarke derivative of g at $(0,0)$ is a positive real number. However, since g_1 is differentiable at $(0,0)$ and its derivative is $(0,0)$, we have $g_1'((0,0),(h_1,h_2)) = \langle(0,0),(h_1,h_2)\rangle = 0$, for all direction $(h_1, h_2) \in \mathbb{R}^2$. This shows that g_1 is not a regular in the sense of Clarke at $(0,0)$.*

Motivated by above literature, in this paper, we will give an affirmative answer to the question which was asked in [4], by removing the regularity assumption of the constraints functions. Meanwhile, our obtained result shows that Dutta's framework [5] still works when the convex constraint set is described by locally Lipschitz functions which is directionally differentiable, but not necessarily continuously differentiable functions. To do this, in order to considering the case that function is not regular, we have the following important tool which can be found in [6].

Lemma 1.4. *[6] Let $g : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function which is directionally differentiable over the open set $U \subset \mathbb{R}^n$. Then at each point $x \in U$ and in each direction $v \in \mathbb{R}^n$ we have*

$$g^{\circ}(x, v) = \limsup_{y \to x} g'(y, v).$$

2. Main Result

From now on, we will always assume that the constraint set C in the problem (1.1) is represented by a set of locally Lipschitz functions which are directionally differentiable. We begin by providing a sufficient condition for the existence of solutions of problem (1.1). This result will play an important role in the convergence analysis of our considered log-barrier method.

Lemma 2.1. *Let C be defined as in the problem (1.1) and $x^* \in C$. Assume that there exist scalars $\lambda_i \geq 0$ such that for all $x \in C$ we have*

(a) $f'(x^*, x - x^*) \geq \sum_{i=1}^{m} \lambda_i g_i^\circ(x^*, x - x^*)$

(b) $\lambda_i g_i(x^*) = 0, \ i = 1, ..., m.$

Then x^* is a global minimizer of the problem (1.1).

Proof. Firstly, by the convexity of C, we have

$$x^* + \alpha(x - x^*) = \alpha x + (1-\alpha)x^* \in C,$$

for all $x \in C$ and for all $\alpha \in (0, 1)$. Thus,

$$g_i(x^* + \alpha(x - x^*)) = g_i(\alpha x + (1-\alpha)x^*) \geq 0,$$

for each $i \in \{1, ..., m\}$. Consequently, by the assumption (b), we can deduce that

$$\lambda_i \left[\frac{g_i(x^* + \alpha(x - x^*)) - g_i(x^*)}{\alpha} \right] \geq 0,$$

for each $i \in \{1, ..., m\}$. Letting $\alpha \to 0$, in view of a relationship between directional derivative and Clarke derivative of g_i at x^* in direction $x - x^*$, we obtain

$$\lambda_i g_i^\circ(x^*, x - x^*) \geq 0,$$

for each $i \in \{1, ..., m\}$. Subsequently, by the assumption (a), we have

$$f'(x^*, x - x^*) \geq \sum_{i=1}^{m} \lambda_i g_i^\circ(x^*, x - x^*) \geq 0.$$

Hence, since f is a convex function, we can conclude that x^* is a global minimizer of f over C. This completes the proof.

□

The next result is an implication of NDC-condition.

Lemma 2.2. *Let C be defined as in Problem (1.1). If NDC-condition is satisfied then for each $x \in C \setminus S$ and $y \in intC$, we have*

$$g_i^\circ(x, y - x) > 0, \quad \forall i \in I(x),$$

where $I(x) = \{i \in \{1, .., m\} : g_i(x) = 0\}$, and $intC$ is denoted for the set of all interior points of C.

Proof. Suppose, on the contrary, that there exists $i \in I(x)$ such that $g_i^\circ(x, y-x) \leq 0$.

Note that, since $y \in intC$, we can find a $\delta > 0$ such that $y + u \in C$ for all $u \in B(0, \delta)$. Let us set $w = y + u$. Subsequently, by convexity of C, we have

$$x + \alpha(w - x) = \alpha w + (1 - \alpha)x \in C,$$

for all $\alpha \in (0, 1)$. This gives,

$$\frac{g_i(x + \alpha(w - x)) - g_i(x)}{\alpha} \geq 0,$$

for all $\alpha \in (0, 1)$. So, by a relation between the directional derivative and Clarke derivative, we have

$$g_i^\circ(x, w - x) \geq 0.$$

Then, by the subadditivity property of $g_i^\circ(x, \cdot)$, we have

$$0 \leq g_i^\circ(x, y + u - x) \leq g_i^\circ(x, y - x) + g_i^\circ(x, u) \leq g_i^\circ(x, u),$$

for all $u \in B(0, \delta)$. Using this one together with the positive homogeneity of $u \mapsto g_i(x, u)$, we would have $0 \in \partial^\circ g_i(x)$. This contradicts to the NDC-condition, and the proof is completed. □

The following result is related to a property of the log-barrier function φ_μ, which was defined by (1.2). Note that, in fact, the following result has been presented in [8], when each g_i is a continuously differentiable function. While, here, we are pointing that the continuity condition of each g_i is sufficient.

Lemma 2.3. *Let C be defined as in Problem (1.1). Assume that C is a compact set and the Slater condition holds. Then, for every $\mu > 0$ the log-barrier function φ_μ has a minimizer, which is an element of $intC$.*

Proof. Let $\mu > 0$ be given and φ_μ be the corresponding log-barrier function, which was defined in (1.2). Firstly, let us notice that by the Slater condition, we can guarantee that the function φ_μ is a proper function, that is $\mathrm{Dom}(\varphi_\mu) \neq \emptyset$. Next, we will show that φ_μ is a continuous extended real valued function on C. Indeed, we have to prove that for each sequence $\{x_k\}$ in $\mathcal{S}$ such that $x_k \to x$ for some $x \in C \setminus \mathcal{S}$, it holds $\varphi_\mu(x_k) \to \infty$.

Note that, since $x \in C \setminus \mathcal{S}$ and each g_i is a continuous function, there is an index j such that $g_j(x_k) \to 0$ as $k \to \infty$. Focusing on the such index j, we now consider

$$\begin{aligned}\varphi_\mu(x_k) =& f(x_k) - \mu \sum_{i=1}^{m} \ln(g_i(x_k)) \\ \geq& f^* - \mu(m-1)\ln K - \mu \ln(g_j(x_k)),\end{aligned} \tag{2.1}$$

where f^* is the minimum of f on C, and all the g_i's are bounded above on C by K. Subsequently, the inequality (2.1) implies that $\varphi_\mu(x_k) \to \infty$ as $k \to \infty$. This asserts that φ_μ is a continuous extended real valued function on C. Then, since C is a compact set, we know that φ_μ must has a minimizer on $\mathcal{S}$. Finally, since $\mathcal{S} \subseteq intC$, the minimizer must be an element of $intC$. □

According to the Lemma 2.3, under the Slater condition, we can construct a sequence $\{x_\mu\} \subset intC$ by

$$x_\mu = \arg\min_{x \in C} \varphi_\mu(x), \tag{2.2}$$

for each positive real number μ. In our next theorem, which is our main result, under the additional condition as NDC-condition, we show that every accumulation point of $\{x_\mu\}$ as $\mu \to 0$ is a global minimizer of the problem (1.1).

Theorem 2.4. *Let C in Problem (1.1) be a compact set and $\{x_\mu\}$ defined as in (2.2). If the Slater condition and NDC-condition hold, then every accumulation point of $\{x_\mu\}$ with $\mu \to 0$ is a global minimizer of the problem (1.1).*

Proof. Notice that, by carefully reading the proof of Lemma 2.3, one can see that $\{x_\mu\}$ is a sequence in $\mathcal{S}$. Moreover, let us observe that φ_μ is directionally differentiable at each $x \in \mathcal{S}$. Thus, from the basic necessary optimality conditions, we must have

$$\varphi'_\mu(x_\mu, v) \geq 0,$$

for all $v \in \mathbb{R}^n$. Subsequently, by applying the chain rule for the directional differentiability, we have

$$f'(x_\mu, v) \geq \sum_{i=1}^{m} \frac{\mu}{g_i(x_\mu)} g'_i(x_\mu, v), \tag{2.3}$$

for all $v \in \mathbb{R}^n$.

Now, let x^* be an accumulation point of $\{x_\mu\}$ as $\mu \to 0$. It follows that there is a null sequence $\{\mu_k\} \subset (0,1)$ such that $x_{\mu_k} \to x^*$ as $k \to \infty$. To complete the proof, we shall show that x^* is a global minimizer of f. We will now consider the following two possible cases.

Case(I) $g_i(x^*) > 0$ for all $i = 1, ...m$.
Case(II) $g_i(x^*) = 0$ for some $i \in \{1, ...m\}$.

Let us discuss Case(I). Since $x_{\mu_k} \to x^*$ as $k \to \infty$, by the continuity of each g_i, for all $i \in \{1, 2, ..., m\}$, we have $g_i(x_{\mu_k}) \to g_i(x^*) > 0$. Then, for all $i \in \{1, 2, ..., m\}$, we have

$$\lim_{k\to\infty} \frac{\mu_k}{g_i(x_{\mu_k})} = 0,$$

since $\mu_k \to 0$ as $k \to \infty$. Moreover, by Lemma 1.4, we know that $\{g_i'(x_{\mu_k}, x - x^*)\}_{k=1}^{\infty}$ is a bounded sequence, for each $i \in \{1, 2, ..., m\}$, and these imply that

$$\lim_{k\to\infty} \sum_{i=1}^{m} \frac{\mu_k}{g_i(x_{\mu_k})} g_i'(x_{\mu_k}, x - x^*) = 0. \tag{2.4}$$

Again, by using Lemma 1.4, we also have

$$\limsup_{k\to\infty} f'(x_{\mu_k}, v) = f^\circ(x^*, v) = f'(x^*, v), \tag{2.5}$$

since f is a convex function. Thus, by considering $v = x - x^*$ in (2.3), we see that (2.4) and (2.5) give

$$f'(x^*, x - x^*) \geq 0,$$

for all $x \in C$. This means that x^* is a global minimizer of the problem (1.1).

Next let us consider Case(II). Let us pick an element $x_0 \in intC$. Then, by Lemma 2.2, we know that

$$g_i^\circ(x^*, x_0 - x^*) > 0, \tag{2.6}$$

for all $i \in I(x^*)$. Subsequently, since $g_i^\circ(x^*, x_0 - x^*)$ is the superior limit of $\{g_i'(x_{\mu_k}, x_0 - x^*)\}_{k=1}^{\infty}$, we may assume without loss of generality (passing to a subsequence if necessary) that $g_i'(x_{\mu_k}, x_0 - x^*) \to g_i^\circ(x^*, x_0 - x^*)$ and $g_i'(x_{\mu_k}, x_0 - x^*) > 0$, for all $k \in \mathbb{N}$ and $i \in I(x^*)$.

Now, by considering $v = x_0 - x^*$, we rewritten (2.3) as

$$f'(x_{\mu_k}, x_0 - x^*) \geq \sum_{i \notin I(x^*)} \frac{\mu_k}{g_i(x_{\mu_k})} g_i'(x_{\mu_k}, x_0 - x^*) + \sum_{i \in I(x^*)} \frac{\mu_k}{g_i(x_{\mu_k})} g_i'(x_{\mu_k}, x_0 - x^*), \tag{2.7}$$

for each $k \in \mathbb{N}$. For the sake of simplicity, for each $k \in \mathbb{N}$, let us put

$$B_k := \sum_{i \notin I(x^*)} \frac{\mu_k}{g_i(x_{\mu_k})} g_i'(x_{\mu_k}, x_0 - x^*)$$

Notice that, by following the lines as proving the Case(I), we know that $B_k \to 0$ as $k \to \infty$. Subsequently, in view of (2.7), we can choose a positive real number B and its corresponding natural number $k_0 \in \mathbb{N}$ such that

$$f'(x_{\mu_k}, x_0 - x^*) + B \geq \sum_{i \in I(x^*)} \frac{\mu_k}{g_i(x_{\mu_k})} g_i'(x_{\mu_k}, x_0 - x^*), \tag{2.8}$$

for all $k \geq k_0$. Thus, by using Lemma 1.4, we have

$$\begin{aligned} f'(x^*, x_0 - x^*) + B &\geq \limsup_{k \to \infty} \sum_{i \in I(x^*)} \frac{\mu_k}{g_i(x_{\mu_k})} g_i'(x_{\mu_k}, x_0 - x^*) \\ &\geq \limsup_{k \to \infty} \frac{\mu_k}{g_j(x_{\mu_k})} g_j'(x_{\mu_k}, x_0 - x^*), \end{aligned}$$

for all $j \in I(x^*)$. This implies that the sequence $\left\{ \frac{\mu_k}{g_j(x_{\mu_k})} \right\}_{k=1}^{\infty}$ must be a bounded sequence, for each $j \in I(x^*)$. Invoking this fact, it allows us to define a function $\lambda : \{1, 2, ..., m\} \to [0, \infty)$ by

$$\lambda_i = \begin{cases} \limsup_{k \to \infty} \frac{\mu_k}{g_i(x_{\mu_k})}, & \text{if } i \in I(x^*); \\ 0, & \text{otherwise.} \end{cases}$$

Then it immediately follows that $\lambda_i g_i(x^*) = 0$ for all $i \in \{1, 2, ..., m\}$.

Next, let $x \in C$ be arbitrarity given. In view of (2.3), with $v = x - x^*$, and by using Lemma 1.4 we obtain

$$f'(x^*, x - x^*) \geq \limsup_{k \to \infty} \sum_{i=1}^{m} \frac{\mu_k}{g_i(x_{\mu_k})} g_i'(x_{\mu_k}, x - x^*) \tag{2.9}$$

$$= \sum_{i=1}^{m} \lambda_i g_i^{\circ}(x^*, x - x^*) \tag{2.10}$$

Hence, by using Lemma 2.1, we can conclude that x^* is a global minimizer of the problem (1.1). This completes the proof.

□

Remark 2.5. (a) *Theorem 2.4 improves a presented result in [4], by removing the regularity assumption from the considered constraint functions.*

(b) *Since Clarke subdifferential of a smooth function will consist only the gradient vector, so in this situation the NDC-condition is coincided with the non-degeneracy condition in the sense of Lasserre [7]. Further, since every continuously differentiable function is a locally Lipschitz and regular function, we can deduce that Theorem 2.4 contains Theorem 1.1 as a special case.*

3. Conclusion

In this work, we have considered the convex optimization problem when the objective function is convex and the constraint functions are just locally Lipschitz and directionally differentiable, but need not be continuously differentiable or regular. It is worth to pointing that, in fact, we are giving an affirmative answer to a problem that was proposed in [4]. Moreover, after carefully considering, one may observe that the Slater condition is used only for guaranteeing that the log-barrier function φ_μ is a proper function and guarantee that its minimizer is an element of interior of constraint set (see Lemma 2.3). This may rise an interesting following question: can we define a log-barrier function by using a condition that weaker than the Slater condition such that it is still a proper and its minimizer belongs to the set of all interior points of constraint set? In order to develop this research area, this question should be considered in the future works.

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EXISTENCE AND CONVERGENCE THEOREMS FOR THE SPLIT QUASI VARIATIONAL INEQUALITY PROBLEMS ON PROXIMALLY SMOOTH SETS

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ABSTRACT. In this paper, we consider the split quasi variational inequality problems over a class of nonconvex sets, as uniformly prox-regular sets. The sufficient conditions for the existence of solutions of such a problem are provided. Furthermore, an iterative algorithm for finding a solution is constructed and its convergence analysis are considered. The results in this paper improve and extend the variational inequality problems which have been appeared in literature.

Key Words and Phrases: Split quasi variational inequality; proximally smooth set; uniformlly prox-regular set; Lipschitzian mapping; strongly monotone mapping.

1. INTRODUCTION

A well known problem, which was studied and interested for many researchers, is the variational inequality problem. The variational inequality problem is a problem of finding $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle \geq 0, \quad \text{for all } x \in K, \tag{1.1}$$

where T is a nonlinear operator on H, K is a nonempty closed and convex subset of a Hilbert space H. This problem was introduced by Stampacchai [32] in 1960s, and it is a power tool which has been used in branches of both pure and applied sciences. Subsequently, the most nature, direct, simple and efficient framework for general treatment of wide range of problems are provided for the variational inequalities. Roughly speaking, many researchers interest to develop several numerical methods for solving variational inequalities and relaxed optimization problems(see [1, 8, 10, 11, 16, 33–36] and the references therein).

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In the early 1970s, Bensoussan et al. [3] developed the concept of variational inequality, by introducing the following concept of quasi-variational inequality problem: find $x^* \in C(x^*)$ such that

$$\langle T(x^*), x - x^* \rangle \geq 0, \quad \text{for all} \quad x \in C(x^*), \tag{1.2}$$

where C is a set-valued mapping on H. We see that if $C(x) = K$ for all $x \in H$, then the problem (1.2) is reduced to the problem (1.1). Notice that, since in many important problems the considered set also depend upon the solutions explicitly or implicitly, evidently, the problem (1.2) is of interesting to study, see [15, 22–24]

On the other hand, in 2012, Cencer et al. [9] introduced the following concept of split variational inequality problem: let H_1, H_2 be real Hilbert spaces and K, Q be nonempty closed and convex subsets of H_1 and H_2, respectively, $T : H_1 \to H_1, S : H_2 \to H_2$ be nonlinear mappings and $A : H_1 \to H_2$ be a bounded linear operator then they are interesting in finding $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle \geq 0, \quad \text{for all} \quad x \in K,$$

and such that $Ax^* \in Q$ solves

$$\langle S(Ax^*), y - Ax^* \rangle \geq 0, \quad \text{for all} \quad y \in Q. \tag{1.3}$$

This problem extends and permits the split minimization between two spaces so the image of minimizer of a given function, under a bounded linear operator, is a minimizer at another function. Furthermore, the split zero problem and split feasibility problem which was studied and used in a model of intensity-modulated radiation therapy treatment planning are contained as special cases of this problem, see [6, 7, 14]. This formulation is also at core of the modeling of many inverse problems arising for phase retrieval and other real-world problems; for instance, in sensor networks in computerized tomography and data compression; see [5, 14, 21].

By the way, in the early period of these development, it should be pointed out that almost all the results regarding the existence and iterative schemes for solving those variational inequality problems are being considered in the convexity setting. This is because, perhaps, they need the convexity assumption for guaranteeing the well definedness of the proposed iterative algorithm, which almost depends on the projection properties. However, in fact, the convexity assumption may not be required, because the algorithm may be well defined even if the considered set is nonconvexs (e.g., when the considered set is a closed subset of a finite dimensional space or a compact subset of Hilbert space, etc.) see [2,4,20,25,26,30]. While, it may be from the practical point of view, one may see that the nonconvex problems are more useful and general than convex case, subsequently, now many researchers are convinced and paid attention to many nonconvex cases. Here, we are focusing the following case, which was presented in 2013 by K. R. Kazmi [19]: let $T_i : H_i \to H_i, A : H_1 \to H_2$ be nonlinear mappings for $i = 1, 2$ and K_r, Q_s are uniformly prox-regular subsets of H_1 and H_2, respectively, with $r, s \in (0, \infty)$ for finding $(x^*, y^*) \in K_r \times Q_s$, where $y^* = Ax^*$ such that

$$\begin{aligned} 0 &\in \rho T_1(x^*) + N^P_{K_r}(x^*), \\ 0 &\in \lambda T_2(y^*) + N^P_{Q_s}(y^*), \end{aligned} \tag{1.4}$$

where ρ, λ are parameters with positive values and $N^P_K(x)$ is the proximal normal cone of K at x.

In this paper, base on above literatures, we are interested to study split quasi variational inequality of nonconvex type problem. The existence theorems and an algorithm for finding such solution will be considered and introduced, respectively. Our results represent an improvement and refinement of the literature results for the variational inequality problem.

2. Preliminaries

In this section, we will recall some basic concepts and useful results which will be used in this paper.

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let 2^H be denoted for the class of all nonempty subsets of H, and K be a closed subset of H. For each $K \subseteq H$, we denote by $d(\cdot, K)$ for the usual distance function on H to K, that is, $d(u, K) = \inf_{v \in K} \|u - v\|$, for all $u \in H$.

For each $K \subseteq H$ and $u \in H$. A point $v \in K$ is called the closest point or the projection of u onto K if $d(u, K) = \|u - v\|$. The set of all such closest points is denoted by $Proj_K(u)$, that is , $Proj_K(u) = \{v \in K : d(u, K) = \|u - v\|\}$. The proximal normal cone to K at u is given by

$$N^P_K(u) = \{v \in H : \exists \rho > 0 \text{ such that } u \in Proj_K(u + \rho v)\}.$$

The following characterization of $N_K^P(u)$ can be found in [13].

Lemma 2.1. *Let K be a closed subset of a Hilbert space H. Then*

$$v \in N_K^P(u) \Leftrightarrow \exists \sigma > 0 \ \text{ such that } \ \langle v, z-u\rangle \leq \sigma\|z-u\|^2, \ \text{ for all } \ z \in K. \tag{2.1}$$

The inequality (2.1) is called the proximal normal inequality.

We recall also that the Clarke normal cone is given by

$$N(K, x) = \overline{co}[N_K^P(x)],$$

where $\overline{co}[S]$ means the closure of the convex hull of S (see [12]). It is clear that one always has $N_K^P(x) \subset N(K,x)$, but the converse is not true in general. Note that $N(K,x)$ is always a closed and convex cone and that $N_K^P(x)$ is always a convex cone but may be nonclosed (see [12,13]). Also, in 1995, Clarke et al. [17] introduced a new class of nonconvex sets, which is called proximally smooth sets, and it has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions. Subsequently, in recent years, Bounkhel et al. [4], Cho et al. [11], Noor [25,26], Petrot [27] and J. Suwannait and N. Petrot [26,29,31] have considered both variational inequalities and equilibrium problems in the context of proximally smooth sets. They suggested and analyzed some projection type iterative algorithms by using the prox-regular technique and auxiliary principle technique. Note that the original definition of proximally smooth set was given in terms of the differentiability of the distance function (see [17, 28]), while here, we will take the following characterization, which was proved in [13], as the definition of proximally smooth sets.

Definition 2.2. For a given $r \in (0, +\infty]$, a subset K of H is said to be *uniformly prox-regular with respect to* r, say, *uniformly r-prox-regular set*, if for all $\overline{x} \in K$ and for all $0 \neq z \in N_K^P(\overline{x})$, one has

$$\left\langle \frac{z}{\|z\|}, x - \overline{x} \right\rangle \leq \frac{1}{2r}\|x - \overline{x}\|^2, \quad \text{for all} \quad x \in K.$$

For the case of $r = \infty$, the uniform r-prox-regularity K is equivalent to the convexity of K (see [17]). Moreover, it is known that the class of uniformly prox-regular sets is sufficiently large to include the class p-convex sets, $C^{1,1}$submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets, see [13,28].

For the sake of simplicity, from now on, we will make use the following notation: for each $r \in (0, +\infty]$, we write

$$K_r := \{x \in H : d(x, K) < r\},$$

and $[Cl(H)]_r$ for the class of all uniformly r-prox regular subsets of H.

Lemma 2.3. *Let $r \in (0, +\infty]$ and K be a nonempty closed subset of H. If K is a uniformly r-prox-regular set, then the following holds*

(i) *For all $x \in K_r$, $Proj_K(x) \neq \emptyset$;*
(ii) *For all $s \in (0, r)$, $Proj_K$ is a $\frac{r}{r-s}$-Lipschitz on K_s;*
(iii) *The proximal normal cone is closed as a set-valued mapping.*

Remark 2.4. If K is a uniformly r-prox-regular set, as a direct consequence of Lemma 2.3(iii), we know that $N(K, x) = N_K^P(x)$.

In this work, we will considering the following class of mappings.

Definition 2.5. A mapping $T : H \to H$ is said to be a σ-strongly monotone if there exists $\sigma > 0$ such that for all $x, x^* \in K$,

$$\langle T(x) - T(x^*), x - x^* \rangle \geq \|x - x^*\|^2.$$

Definition 2.6. A mapping $T : H \to H$ is said to be a β-Lipschitzian if there exists a real number $\beta > 0$ such that

$$\|T(x) - T(y))\| \leq \beta\|x - y\|, \quad \text{for all} \quad x, y \in H.$$

Definition 2.7. A multivalued mapping $C : H \to 2^H$ is said to be a κ-Lipschitz continuous if there exists a real number $\kappa > 0$ such that

$$\|d(y, C(x)) - d(y', C(x'))\| \leq \|y - y'\| + \kappa\|x - x'\|, \quad \text{for all} \quad x, x', y, y' \in H.$$

The following lemma is a very important tool, in order to prove our main results.

Lemma 2.8. [4] Let $r \in (0, +\infty]$ and let $C : H \to 2^H$ be a κ- Lipschitz continuous multivalued mapping with uniformly r-prox regular valued then the following closedness property holds: for any $x_n \to x^*, y_n \to y^*$ and $u_n \to u^*$ with $y_n \in C(x_n)$ and $u_n \in N_{C(x_n)}^P(y_n)$, one has $u^* \in N_{C(x^*)}^P(y^*)$.

3. Main results

Let H_1 and H_2 be real Hilbert spaces, $T_i : H_i \to H_i$ be nonlinear mappings, $C_i : H_i \to 2^{H_i}$ be nonlinear multivalued mappings for $i = 1,2$ and $A : H_1 \to H_2$ be a bounded linear operator. In this paper, we are interesting in the following problem: find $x^* \in C_1(x^*)$ such that, $Ax^* \in C_2(Ax^*)$ and

$$\begin{aligned} -T_1(x^*) &\in N^P_{C_1(x^*)}(x^*), \\ -T_2(Ax^*) &\in N^P_{C_2(Ax^*)}(Ax^*). \end{aligned} \tag{3.1}$$

Notice that, the problem (3.1) can be reformulated as the following: find $(x^*, z^*) \in C_1(x^*) \times C_2(z^*)$ with $z^* = Ax^*$ such that

$$\begin{aligned} x^* &= Proj_{C_1(x^*)}(x^* - \rho T_1(x^*)), \\ z^* &= Proj_{C_2(z^*)}(z^* - \lambda T_2(z^*)), \end{aligned} \tag{3.2}$$

for some $\rho, \lambda > 0$ are constants.

Moreover, by using the definition of uniformly prox-regular set, we also see that the problem (3.1) is of finding $x^* \in C_1(x^*)$, and $z^* = Ax^* \in C_2(z^*)$ such that

$$\begin{aligned} \langle T_1(x^*), \widehat{x} - x^* \rangle + \frac{\|T_1(x^*)\|}{2r}\|\widehat{x} - x^*\|^2 &\geq 0, \forall \widehat{x} \in C_1(x^*), \\ \langle T_2(z^*), \widehat{z} - z^* \rangle + \frac{\|T_2(z^*)\|}{2s}\|\widehat{z} - z^*\|^2 &\geq 0, \forall \widehat{z} \in C_2(z^*). \end{aligned} \tag{3.3}$$

In a special case, when K and Q are closed subsets of H_1 and H_2, respectively, and $C_i : H_i \to 2^{H_i}$, for $i = 1,2$ are defined by

$$\begin{aligned} C_1(x) &= K, \quad \text{for all } x \in H_1, \\ C_2(y) &= Q, \quad \text{for all } y \in H_2, \end{aligned} \tag{3.4}$$

then the problem (3.1) is reduced to the problem of finding $(x^*, z^*) \in K \times Q$ with $z^* = Ax^*$ such that

$$\begin{aligned} -T(x^*) &\in N^P_K(x^*), \\ -S(z^*) &\in N^P_Q(z^*), \end{aligned} \tag{3.5}$$

which was studied by K. R. Kazmi [18].

Now, we introduce an algorithm which will play an important role in our prove.

Algorithm ($\mathcal{A}$): Let $T_i : H_i \to H_i, C_1 : H_1 \to [Cl(H_1)]_r$ and $C_2 : H_2 \to [Cl(H_2)]_s$ be nonlinear mappings where $r, s \in (0, +\infty)$ and $i = 1, 2$. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator, denoted by A^*. Given $x_0 \in H_1$, compute the algorithm sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ as the following projection method:

$$\begin{aligned} y_n &\in Proj_{C_1(x_n)}[x_n - \rho T_1(x_n)], \\ z_n &\in Proj_{C_2(Ay_n)}[Ay_n - \lambda T_2(Ay_n)], \\ x_{n+1} &\in Proj_{C_1(y_n)}[y_n + \gamma A^*(z_n - Ay_n)], \end{aligned} \tag{3.6}$$

where ρ, λ and γ are step size positive real numbers.

The following assumption will be proposed, as the sufficient conditions.

Assumption ($\mathcal{C}$) : Let $T_i : H_i \to H_i, C_1 : H_1 \to [Cl(H_1)]_r$ and $C_2 : H_2 \to [Cl(H_2)]_s$ be nonlinear mappings for $r, s \in (0, +\infty)$ and $i = 1, 2$ which are satisfied the following conditions:

- (*i*) T_i is a β_i-Lipschitzian mapping and a σ_i-strongly monotone mapping for i = 1, 2;
- (*ii*) C_i is a κ_i- Lipschitzian continuous mapping for some $\kappa_i \in [0, 1)$ and i = 1, 2;
- (*iii*) for each $i = 1, 2$, there is $\omega_i \in [0, 1)$ such that

$$\|Proj_{C_i(x)}(z) - Proj_{C_i(y)}(z)\| \leq \omega_i \|x - y\|, \quad \text{for all } x, y, z \in H_i.$$

Firstly, based on the assumption ($\mathcal{C}$), we notice the following key remark.

Remark 3.1. For a real Hilbert space H and $r \in (0, \infty)$. If $T : H \to H$ and $C : H \to [Cl(H)]_r$ are nonlinear mappings. Then, for each $x_0 \in H$ with $d(x_0, C(x_0)) \leq r^* - \rho\|Tx_0\|$, where $r^* \in (0, r)$ and ρ is a positive real number, $Proj_{C(x_0)}[x_0 - \rho Tx_0] \neq \emptyset$. Indeed, Since C is a κ-Lipschitz continuous mapping, we have

$$\begin{aligned} d(x_0 - \rho Tx_0, C(x_0)) &\leq d(x_0, C(x_0)) + \rho\|Tx_0\| \\ &\leq r^* - \rho\|Tx_0\| + \rho\|Tx_0\| \\ &< r. \end{aligned}$$

By Lemma 2.3(i), we obtain that $Proj_{C(x_0)}[x_0 - \rho Tx_0] \neq \emptyset$.

The following lemma asserts that, under our setting, Algorithm $(\mathcal{A})$ is well-define.

Lemma 3.2. *Let H_1, H_2 be real Hilbert spaces. Assume that Assumption $(\mathcal{C})(ii)$ and (iii) hold and there are $\mu > 1$ and $x_0 \in H_1$ such that*

(*i*) $d(x_0, C_1(x_0)) \leq r^* - \rho\|T_1x_0\|$,

(*ii*) $0 < \rho < \frac{r^*}{\delta_{T_1}}, 0 < \lambda < \frac{s^*-\Phi}{\delta_{T_2}}$ *and* $0 < \gamma < \frac{r^*}{\delta_{A^*}}$,

where $\delta_{T_1} = \sup\{\|T_1x_n\| : x_n \in H_1\}, \delta_{T_2} = \sup\{\|T_2(Ay_n)\| : Ay_n \in H_2\}$, $\delta_{A^*} = \sup\{\|A^*(z_n)\| : z_n \in H_2\}, \Phi = \sup\{d(Ay_n, C_2(Ay_n)) \mid y_n \in H_1\}, r^* = \frac{r(1-\kappa)}{1+\mu\kappa}$, $s^* = \frac{s(1-\kappa)}{1+\mu\kappa}$, *$A : H_1 \to H_2$ is a bounded linear operator and $\{x_n\}, \{y_n\}$ are constructed as in Algorithm $(\mathcal{A})$ with the initial vector x_0. Then, the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ which are constructed by Algorithm $(\mathcal{A})$ are well-defined.*

Proof. By condition (i) and Remark 3.1, we know that $Proj_{C_1(x_0)}[x_0 - \rho T_1x_0] \neq \emptyset$. Subsequently, we put $y_0 \in Proj_{C_1(x_0)}[x_0 - \rho T_1x_0]$. Next, by the condition (ii), we see that

$$
\begin{aligned}
d(Ay_0 - \lambda T_2(Ay_0), C_2(Ay_0)) &\leq d(Ay_0, C_2(Ay_0)) + \lambda\|T_2(Ay_0)\| \\
&\leq d(Ay_0, C_2(Ay_0)) + \lambda\|T_2(Ay_0)\| \\
&< d(Ay_0, C_2(Ay_0)) + \left(\frac{s^* - \Phi}{\delta_{T_2}}\right)\|T_2(Ay_0)\| \\
&< d(Ay_0, C_2(Ay_0)) + s^* - \Phi \\
&< s^*.
\end{aligned}
$$

Thus, $Proj_{C_2(Ay_0)}[Ay_0 - \lambda T_2(Ay_0)] \neq \emptyset$. Let $z_0 \in Proj_{C_2(Ay_0)}[Ay_0 - \lambda T_2(Ay_0)]$. Notice that, by using the κ_1-Lipschitz continuous mapping of C_1, we see that

$$
\begin{aligned}
d(y_0 + \gamma A^*(z_0 - Ay_0), C_1(y_0)) &\leq d(y_0, C_1(y_0)) + \gamma\|A^*(z_0 - Ay_0)\| \\
&= d(y_0, C_1(y_0)) - d(y_0, C_1(x_0)) + \gamma\|A^*(z_0 - Ay_0)\| \\
&\leq \kappa_1\|y_0 - x_0\| + r^*. \qquad (3.7)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\|y_0 - x_0\| &\leq \|y_0 - (x_0 - \rho T_1(x_0)\| + \|x_0 - \rho T_1(x_0) - x_0\| \\
&= d(x_0 - \rho T_1(x_0), C_1(x_0)) + \rho\|T_1(x_0)\| \\
&\leq r^* + r^*
\end{aligned}
$$

$$= \quad 2r^*. \tag{3.8}$$

Thus, (3.7) and (3.8), give

$$\begin{aligned} d(y_0 + \gamma A^*(z_0 - Ay_0), C_1(y_0)) &\leq 2\kappa_1 r^* + r^* \\ &= r^*(2\kappa_1 + 1) \\ &= r^*\left(\frac{1+\kappa_1 - 2\kappa_1^2}{1-\kappa_1}\right) \\ &< r. \end{aligned} \tag{3.9}$$

This implies that, $Proj_{C_1(y_0)}[y_0 + \gamma A^*(z_0 - Ay_0)] \neq \emptyset$. Let $x_1 \in Proj_{C_1(y_0)}[y_0 + \gamma A^*(z_0 - Ay_0)]$, and consider

$$\begin{aligned} d(x_1 - \rho T_1 x_1, C_1(x_1)) &\leq d(x_1, C_1(x_1)) + \rho\|T_1 x_1\| \\ &= d(x_1, C_1(x_1)) - d(x_1, C_1(y_0)) + \rho\|T_1 x_1\| \\ &\leq \kappa_1\|x_1 - y_0\| + r^*. \end{aligned} \tag{3.10}$$

And, since

$$\begin{aligned} \|x_1 - y_0\| &\leq \|x_1 - (y_0 + \gamma A^*(z_0 - Ay_0))\| + \|y_0 + \gamma A^*(z_0 - Ay_0) - y_0\| \\ &= d(y_0 + \gamma A^*(z_0 - Ay_0), C_1(y_0)) + \gamma\|A^*(z_0 - Ay_0)\| \\ &< r^*\left(\frac{1+\kappa_1 - 2\kappa_1^2}{1-\kappa_1}\right) + r^*, \end{aligned} \tag{3.11}$$

we obtain

$$\begin{aligned} d(x_1 - \rho T_1 x_1, C_1(x_1)) &\leq r^*\kappa_1\left(\frac{1+\kappa_1 - 2\kappa_1^2}{1-\kappa_1}\right) + r^*\kappa_1 + r^* \\ &= r^*\left(\frac{1+\kappa_1 - 2\kappa_1^3}{1-\kappa_1}\right) \\ &< r. \end{aligned} \tag{3.12}$$

This implies that, $Proj_{C_1(x_1)}[x_1 - \rho T_1 x_1] \neq \emptyset$. Let $y_1 \in Proj_{C_1(x_1)}[x_1 - \rho T_1(x_1)]$, and we see that

$$\begin{aligned} d(Ay_1 - \lambda T_2(Ay_1), C_2(Ay_1)) &\leq d(Ay_1, C_2(Ay_1)) + \lambda\|T_2(Ay_1)\| \\ &< d(Ay_1, C_2(Ay_1)) + \left(\frac{s^* - \Phi}{\delta_{T_2}}\right)\|T_2(Ay_1)\| \\ &\leq d(Ay_1, C_2(Ay_1)) + s^* - \Phi \\ &\leq s^* \\ &< s. \end{aligned} \tag{3.13}$$

Thus, $Proj_{C_2(Ay_1)}[Ay_1 - \lambda T_2(Ay_1)] \neq \emptyset$. Let $z_1 \in Proj_{C_2(Ay_1)}[Ay_1 - \lambda T_2(Ay_1)]$, and computes

$$\begin{aligned} d(y_1 + \gamma A^*(z_1 - Ay_1), C_1(y_1)) &\leq d(y_1, C_1(y_1)) + \gamma\|A^*(z_1 - Ay_1)\| \\ &\leq d(y_1, C_1(y_1)) - d(y_1, C_1(x_1)) + r^* \\ &\leq \kappa_1\|y_1 - x_1\| + r^*. \end{aligned} \tag{3.14}$$

Since

$$\begin{aligned} \|y_1 - x_1\| &\leq \|y_1 - (x_1 - \rho T_1 x_1)\| + \|x_1 - \rho T_1 x_1 - x_1\| \\ &= d(x_1 - \rho T_1 x_1, C_1(x_1)) + \rho\|T_1 x_1\| \\ &< r^*\left(\frac{1+\kappa_1 - 2\kappa_1^3}{1-\kappa_1}\right) + r^*, \end{aligned} \tag{3.15}$$

we have

$$\begin{aligned} d(y_1 + \gamma A^*(z_1 - Ay_1), C_1(y_1)) &< r^*\kappa_1\left(\frac{1+\kappa_1 - 2\kappa_1^3}{1-\kappa_1}\right) + \kappa_1 r^* + r^* \\ &= r^*\left(\frac{1+\kappa_1 - 2\kappa_1^4}{1-\kappa_1}\right) \\ &< r. \end{aligned} \tag{3.16}$$

Thus, $Proj_{C_1(y_1)}[y_1 + \gamma A^*(z_1 - Ay_1)] \neq \emptyset$. Let $x_2 \in Proj_{C_1}(y_1)[y_1 + \gamma A^*(z_1 - Ay_1)]$. In the same way of (3.10), (3.11) and (3.12), we have

$$\begin{aligned} d(x_2 - \rho T_1(x_2), C_1(x_2)) &\leq \kappa_1\|x_2 - y_1\| + r^* \\ \|x_2 - y_1\| &< r^*\left(\frac{1+\kappa_1 - 2\kappa_1^4}{1-\kappa_1}\right) + r^* \end{aligned}$$

and

$$d(x_2 - \rho T_1 x_2, C_1(x_2)) \leq r^*\left(\frac{1+\kappa_1 - 2\kappa_1^5}{1-\kappa_1}\right).$$

Thus, $Proj_{C_1(x_2)}[x_2 - \rho T_1 x_2] \neq \emptyset$. Let $y_2 \in Proj_{C_1(x_2)}[x_2 - \rho T_1 x_2]$. In similarly way (3.13), we obtain

$$d(Ay_2 - \lambda T_2(Ay_2), C_2(Ay_2)) \leq s^*.$$

Thus, $Proj_{C_2(Ay_2)}[Ay_2 - \lambda T_2(Ay_2)] \neq \emptyset$. Let $z_2 \in Proj_{C_2(Ay_2)}[Ay_2 - \lambda T_2(Ay_2)]$. In the same way as obtaining (3.14), (3.15) and (3.16), we have

$$d(y_2 + \gamma A^*(z_2 - Ay_2), C_1(y_2)) \leq \kappa_1\|y_2 - x_2\| + r^*$$

$$\|y_2 - x_2\| < r^* \left(\frac{1+\kappa_1 - 2\kappa_1^5}{1-\kappa_1}\right) + r^*,$$

and

$$d(y_2 + \gamma A^*(z_2 - Ay_2), C_1(y_2)) < r^* \left(\frac{1+\kappa_1 - 2\kappa_1^6}{1-\kappa_1}\right).$$

Thus, $Proj_{C_1(y_2)}[y_2 + \gamma A^*(z_2 - Ay_2)] \neq \emptyset$, and we then put $x_3 \in Proj_{C_1(y_2)}[y_2 + \gamma A^*(z_2 - Ay_2)]$.

By using this process, we can construct the sequences $\{x_n\}, \{y_n\}$ in H_1 and $\{z_n\}$ in H_2 such that

$$\begin{aligned} x_n - \rho T_1(x_n) &\in [C_1(x_n)]_{\frac{r(1+\kappa_1)}{1+\mu\kappa_1}} \\ y_n &\in Proj_{C_1(x_n)}[x_n - \rho T_1(x_n)] \\ Ay_n - \lambda S(Ay_n) &\in [C_2(Ay_n)]_{s^*} \\ z_n &\in Proj_{C_2(Ay_n)}[Ay_n - \lambda T_2(Ay_n)] \\ y_n + \gamma A^*(z_n - Ay_n) &\in [C_1(y_n)]_{\frac{r(1+\kappa_1)}{1+\mu\kappa_1}} \\ x_{n+1} &\in Proj_{C_1(y_n)}[y_n + \gamma A^*(z_n - Ay_n)], \end{aligned} \tag{3.17}$$

which is, in fact, the Algorithm ($\mathcal{A}$). □

Remark 3.3. Let us consider the proposed assumptions of Lemma 3.2. In the application point of view, one may ask for the best choice of the real number μ, and hence r^* and s^*. We would like to notice here that, the real number $\mu = \frac{\kappa\Delta - 1}{\kappa(1-\Delta)}$, where $\Delta = \frac{\beta(1-\omega)}{\sqrt{\beta^2-\sigma^2}}$, should provide the answer. This is because, by the following observation:

- the domain of function f is $\frac{\beta(1-\omega)}{\sqrt{\beta^2-\sigma^2}}$,
- $r^* = \frac{r(1-\kappa)}{1+\mu\kappa} \Leftrightarrow t_r = \frac{1+\mu\kappa}{\kappa(1-\mu)}$ and $s^* = \frac{s(1-\kappa)}{1+\mu\kappa} \Leftrightarrow t_s = \frac{1+\mu\kappa}{\kappa(1-\mu)}$,
- the function $\mu \mapsto \frac{1+\mu\kappa}{\kappa(1+\mu)}$ is an increasing function on its domain,
- $\frac{1+\mu\kappa}{\kappa(1+\mu)} = \Delta \Leftrightarrow \mu = \frac{\kappa\Delta-1}{\kappa(1-\Delta)}$, where $\Delta = \frac{\beta(1-\omega)}{\sqrt{\beta^2-\sigma^2}}$.

The following theorem shows that the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$, which are considered in Lemma 3.2 are all convergent sequences.

Theorem 3.4. *Let H_1, H_2 be real Hilbert spaces. Assume that Assumption* ($\mathcal{C}$) *and all of assumptions in Lemma 3.2 hold. If* $\gamma < \min\{\frac{2}{\|A\|^2}, \frac{1-\omega_1-\varphi}{\varphi\theta_2\|A\|^2}\}$, *where* $\varphi =$

$\frac{1+\mu\kappa_1}{\kappa_1(\mu-1)}, \theta_2 = t_{s^*}\sqrt{1-2\lambda\sigma_2+\lambda^2\beta_2^2}+\omega_2$ *and* $t_{s^*} = \frac{s}{s-s^*}$, *then* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$, *which are constructed in Algorithm* $(\mathcal{A})$, *are convergent sequences.*

Proof. Using the definition of sequence $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in Algorithm $(\mathcal{A})$, we have

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|Proj_{C_1}(x_{n+1})[x_{n+1}-\rho T_1(x_{n+1})] - Proj_{C_1}(x_n)[x_n-\rho T_1(x_n)]\| \\ &\leq \|Proj_{C_1}(x_{n+1})[x_{n+1}-\rho T_1(x_{n+1})] - Proj_{C_1(x_{n+1})}[x_n-\rho T_1(x_n)]\| \\ &\quad +\|Proj_{C_1(x_{n+1})}[x_n-\rho T_1x_n] - Proj_{C_1(x_n)}[x_n-\rho T_1(x_n)]\| \\ &\leq \frac{1+\mu\kappa_1}{\kappa_1(\mu-1)}\|x_{n+1}-\rho T_1(x_{n+1})-x_n+\rho T_1(x_n)\|+\omega_1\|x_{n+1}-x_n\| \end{aligned} \tag{3.18}$$

On the other hand, we have

$$\begin{aligned} &\|x_{n+1}-\rho T_1(x_{n+1})-x_n+\rho T_1(x_n)\|^2 \\ \leq\ & \|x_{n+1}-x_n\|^2 - 2\rho\langle T_1(x_{n+1})-T_1(x_n), x_{n+1}-x_n\rangle + \rho^2\|T_1(x_{n+1})-T_1(x_n)\|^2 \\ \leq\ & \|x_{n+1}-x_n\|^2 - 2\rho\sigma_1\|x_{n+1}-x_n\|^2+\rho^2\beta_1^2\|x_{n+1}-x_n\|^2 \\ =\ & (1-2\rho\sigma_1+\rho^2\beta_1^2)\|x_{n+1}-x_n\|^2. \end{aligned} \tag{3.19}$$

From (3.18) and (3.19), we get

$$\begin{aligned}\|y_{n+1}-y_n\| &\leq \frac{1+\mu\kappa_1}{\kappa_1(\mu-1)}\sqrt{1-2\rho\sigma_1+\rho^2\beta_1^2}\|x_{n+1}-x_n\|+\omega_1\|x_{n+1}-x_n\| \\ &= \left(\frac{1+\mu\kappa_1}{\kappa_1(\mu-1)}\sqrt{1-2\rho\sigma_1+\rho^2\beta_1^2}+\omega_1\right)\|x_{n+1}-x_n\| \\ &= \theta_1\|x_{n+1}-x_n\|, \end{aligned} \tag{3.20}$$

where $\theta_1 = \frac{1+\mu\kappa_1}{\kappa_1(\mu-1)}\sqrt{1-2\rho\sigma_1+\rho^2\beta_1^2}+\omega_1$. Observe that, by the choice of ρ, μ and κ_1, we have $\theta_1 < 1$.

Next, by the definition of $\{z_n\}$, we have

$$\begin{aligned}\|z_{n+1}-z_n\| &= \|Proj_{C_2(Ay_{n+1})}[Ay_{n+1}-\lambda T_2(Ay_{n+1})] - Proj_{C_2(Ay_n)}[Ay_n-\lambda T_2(Ay_n)]\| \\ &\leq \|Proj_{C_2(Ay_{n+1})}[Ay_{n+1}-\lambda T_2(Ay_{n+1})] - Proj_{C_2(Ay_{n+1})}[Ay_n-\lambda T_2(Ay_n)]\| \\ &\quad +\|Proj_{C_2(Ay_{n+1})}[Ay_n-\lambda T_2(Ay_n)] - Proj_{C_2(Ay_n)}[Ay_n-\lambda T_2(Ay_n)]\| \\ &\leq t_{s*}\|(Ay_{n+1}-Ay_n)-\lambda(T_2(Ay_{n+1})-T_2(Ay_n))\|+\omega_2\|Ay_{n+1}-Ay_n\|. \end{aligned}$$

And, since

$$\|(Ay_{n+1}-Ay_n)-\lambda(T_2(Ay_{n+1})-T_2(Ay_n))\|^2$$

$$
\begin{aligned}
&\leq \|Ay_{n+1} - Ay_n\|^2 - 2\lambda\langle T_2(Ay_{n+1}) - T_2(Ay_n)), Ay_{n+1} - Ay_n\rangle + \lambda\|T_2(Ay_{n+1}) - T_2(Ay_n)\|^2 \\
&\leq \|A\|^2\|y_{n+1} - y_n\|^2 - 2\lambda\sigma_2\|Ay_{n+1} - Ay_n\|^2 + \lambda^2\beta_2^2\|Ay_{n+1} - Ay_n\|^2 \\
&= (1 - 2\lambda\sigma_2 + \lambda^2\beta_2^2)\|A\|^2\|y_{n+1} - y_n\|^2, \qquad (3.21)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\|z_{n+1} - z_n\| &\leq t_{s^*}\sqrt{1 - 2\lambda\sigma_2 + \lambda^2\beta_2^2}\|A\|\|y_{n+1} - y_n\| + \omega_2\|A\|\|y_{n+1} - y_n\| \\
&= \left(t_{s^*}\sqrt{1 - 2\lambda\sigma_2 + \lambda^2\beta_2^2} + \omega_2\right)\|A\|\|y_{n+1} - y_n\| \\
&= \theta_2\|A\|\|y_{n+1} - y_n\|. \qquad (3.22)
\end{aligned}
$$

Note that, by the choice of λ, we have $\theta_2 < 1$. Next, we consider

$$
\begin{aligned}
&\|x_{n+1} - x_n\| \\
&= \|Proj_{C_1(y_n)}[y_n + \gamma A^*(z_n - Ay_n)] - Proj_{C_1(y_{n+1})}[y_{n+1} + \gamma A^*(z_{n-1} - Ay_{n-1})]\| \\
&\leq \|Proj_{C_1(y_n)}[y_n + \gamma A^*(z_n - Ay_n)] - Proj_{C_1(y_{n-1})}[y_n + \gamma A^*(z_n - Ay_n)]\| \\
&\quad + \|Proj_{C_1(y_{n-1})}[y_n + \gamma A^*(z_n - Ay_n)] - Proj_{C_1(y_{n-1})}[y_{n-1} + \gamma A^*(z_{n-1} - Ay_{n-1})]\| \\
&\leq \omega_1\|y_n - y_{n-1}\| + \varphi\|y_n - y_{n-1} - \gamma(A^*(z_{n-1} - Ay_{n-1}) - A^*(z_n - Ay_n))\| \\
&\leq \omega_1\|y_n - y_{n-1}\| + \varphi\|y_n - y_{n-1} - \gamma(A^*(Ay_n - Ay_{n-1}))\| + \varphi\gamma\|A^*(z_n) - A^*(z_{n-1})\| \qquad (3.23)
\end{aligned}
$$

Since,

$$
\begin{aligned}
&\|y_n - y_{n-1} - \gamma(A^*(Ay_n - Ay_{n-1}))\|^2 \\
&\leq \|y_n - y_{n-1}\|^2 - 2\gamma\langle y_n - y_{n-1}, A^*(Ay_n - Ay_{n-1})\rangle + \gamma^2\|A^*(Ay_n - Ay_{n-1})\|^2 \\
&= \|y_n - y_{n-1}\|^2 - 2\gamma\langle Ay_n - Ay_{n-1}, Ay_n - Ay_{n-1}\rangle + \gamma^2\|A^*(Ay_n - Ay_{n-1})\|^2 \\
&\leq \|y_n - y_{n-1}\|^2 - 2\gamma\|Ay_n - Ay_{n-1}\|^2 + \gamma^2\|A\|^2\|Ay_n - Ay_{n-1}\|^2 \\
&= \|y_n - y_{n-1}\|^2 - (2\gamma - \gamma^2\|A\|^2)\|Ay_n - Ay_{n-1}\|^2 \\
&= \|y_n - y_{n-1}\|^2 - \gamma(2 - \gamma\|A\|^2)\|Ay_n - Ay_{n-1}\|^2 \\
&\leq \|y_n - y_{n-1}\|^2 \qquad (3.24)
\end{aligned}
$$

and $\|A^*(z_n) - A^*(z_{n-1})\| \leq \|A\|\|z_n - z_{n-1}\|$, we get

$$
\begin{aligned}
\|x_{n+1} - x_n\| &\leq \omega_1\|y_n - y_{n-1}\| + \varphi\|y_n - y_{n-1}\| + \varphi\gamma\|A\|\|z_n - z_{n-1}\| \qquad (3.25) \\
&\leq \omega_1\|y_n - y_{n-1}\| + \varphi\|y_n - y_{n-1}\| + \varphi\gamma\theta_2\|A\|^2\|y_n - y_{n-1}\| \\
&= (\gamma\theta_2\varphi\|A\|^2 + \varphi + \omega_1)\|y_n - y_{n-1}\|
\end{aligned}
$$

$$\begin{aligned} &\leq \theta_1(\gamma\theta_2\varphi\|A\|^2+\varphi+\omega_1)\|x_n-x_{n-1}\| \\ &= \theta_3\|x_n-x_{n-1}\|, \end{aligned} \tag{3.26}$$

where $\theta_3 = \theta_1(\gamma\theta_2\varphi\|A\|^2+\varphi+\omega_1)$. Also, by the choice of γ, we know that $\theta_3 < 1$. Hence, for any $m \geq n > 1$, we see that

$$\|x_m - x_n\| \leq \Sigma_{i=n}^{m-1}\|x_{i+1}-x_i\| \tag{3.27}$$

$$\begin{aligned} &\leq \Sigma_{i=n}^{m-1}\theta_3^i\|x_1-x_0\| \\ &\leq \|x_1-x_0\|\Sigma_{i=n}^{\infty}\theta_3^i \\ &\leq \frac{\theta_3^n}{1-\theta_3}\|x_1-x_0\|. \end{aligned} \tag{3.28}$$

Since $\theta_3 < 1$, we can conclude that $\{x_n\}$ is a Cauchy sequence in H_1. By the completeness of H_1,we know that $\{x_n\}$ is a convergent sequence. Also, by (3.20) and the convergence of the sequence $\{x_n\}$, we see that $\{y_n\}$ is a convergent sequence. In similarly way, by (3.22) and the convergence of the sequence $\{y_n\}$, we obtain that $\{z_n\}$ is a convergent sequence. This completes the proof. □

Now, we are in position to present the sufficient condition for existence of solution of problem (3.1) our main theorem.

Theorem 3.5. *Let H_1, H_2 be real Hilbert spaces. Let $T_i : H_i \to H_i$ be nonlinear mappings for $i = 1,2$ and $C_1 : H_1 \to [Cl(H_1)]_r$ and $C_2 : H_2 \to [Cl(H_2)]_s$ be nonlinear set-valued mappings. Assume that all of the assumptions in Theorem 3.4 hold and if $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$ and $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} z_n$. Then, the problem (3.1) has a solution.*

Proof. Let $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x^*$ and $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} z_n = z^*$. Firstly, we will show that $-T_1(x^*) \in N^P_{C_1(x^*)}(x^*)$. Since $y_n \in Proj_{C_1(x_n)}[x_n - \rho T_1(x_n)]$, we see that $x_n - y_n - \rho T_1(x_n) \in N^P_{C_1(x_n)}(y_n)$. Using this one together with the closedness property of the proximal cone, we obtain that $-\rho T_1 x^* \in N^P_{C_1(x^*)}(x^*)$. This means, $-T_1(x^*) \in N^P_{C_1(x^*)}(x^*)$.

Next, we want to show that $-T_2(z^*) \in N^P_{C_2(z^*)}(z^*)$. Since $z_n \in Proj_{C_2(Ay_n)}[Ay_n - \lambda T_2(Ay_n)]$, we have $Ay_n - z_n - \lambda T_2(Ay_n) \in N^P_{C_2(Ay_n)}(z_n)$. Again, by using the closedness property of the proximal cone, we have $Ax^* - z^* - \lambda T_2(Ax^*) \in N^P_{C_2(Ax^*)}(z^*)$. This is $-\lambda T_2(Ax^*) \in N^P_{C_2(z^*)}(z^*)$. Hence, $-T_2(z^*) \in N^P_{C_2}(z^*)$. The proof is completed. □

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Existence and convergence theorems for the split quasi variational inequality problems on proximally smooth sets

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Abstract

In this paper, we consider the split quasi variational inequality problems over a class of nonconvex sets, as uniformly prox-regular sets. The sufficient conditions for the existence of solutions of such a problem are provided. Furthermore, an iterative algorithm for finding a solution is constructed and its convergence analysis are considered. The results in this paper improve and extend the variational inequality problems which have been appeared in literature. ©2016 All rights reserved.

Keywords: Split quasi variational inequality, proximally smooth set, uniformly prox-regular set, Lipschitzian mapping, strongly monotone mapping.
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1. Introduction

A well known problem, which was studied and interesting for many researchers, is the variational inequality problem. The variational inequality problem is a problem of finding $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle \geq 0, \quad \text{for all} \quad x \in K, \tag{1.1}$$

where T is a nonlinear operator on H, K is a nonempty closed and convex subset of a Hilbert space H. This problem was introduced by Stampacchai[31] in 1960s, and it is a power tool which has been used in branches

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