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Symmetry classes of tensors and non-abelian groups

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งบประมาณรายได้มหาวิทยาลัยนเรศวร

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ชื่อโครงการ คลาสเทนเซอร์เชิงสมมาตรและนอนอาบีเลียนกรุป

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หน่วยงานที่สังกัด ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยนเรศวร

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บทคัดย่อ(ภาษาไทย)

ในงานวิจัยนี้ ผู้วิจัยได้ค้นหาการมีอยู่ของฐานเชิงตั้งฉากปรกติชนิดพิเศษที่ประกอบไปด้วยเทนเซอร์ที่ลดทอนไม่ได้ของคลาสเทนเซอร์เชิงสมมาตรสำหรับนอนอาบีเลียนกรุปที่มีออร์เดอร์ pq ซึ่งผู้วิจัยได้ค้นพบเงื่อนไขที่จำเป็นและเพียงพอของการมีอยู่สำหรับฐานดังกล่าว

บทคัดย่อ(ภาษาอังกฤษ)

In this project, the researcher investigates the existence of orthogonal *-basis of the symmetry classes of tensors for non abelian groups of order pq. The necessary and sufficient condition for the existence has been provided.

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กิจติ รอดเทศ

สารบัญ

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1. Introduction

The study of symmetry classes of tensors is motivated by many branches of pure and applied mathematics: combinatorial theory, matrix theory, operator theory, group representation theory, differential geometry, partial differential equations, quantum mechanics and other areas, [3]. In the past decades, there are many papers devoted to study this concept, (e.g. [2-5], [8-13] and [16-18). In particular, finding examples of (higher) symmetry classes of tensors that possesses an orthogonal basis of decomposable symmetrized tensors (orthogonal $*$-basis or o-basis, for short) was of interest. In fact, this topic arose from the question by B.Y. Wang and M.P. Gong in [18].

The investigation for the existence of orthogonal $*$-basis of symmetry classes of tensors had been done for some classes of groups, for example, dihedral groups in [11], Dicyclic groups in [4], semi-dihedral groups in [13], some subgroups of full symmetric groups and some type of p-groups in [10].

Non-abelian groups of order pq has some applications in group theory and graph theory, e.g., [7]. The symmetry classes of tensors associated to these groups has been studied in some aspects. Precisely, Poursalavati computed some dimensions of the symmetry classes of tensors associated with certain Frobenius groups in [17]. However, he did not investigate the condition for the existence of the o-basis. This might be an effect from the complicated values in the character table. However, with the help of vanishing sums of roots of unity (a result of T.Y. Lam and K.H. Leung: On vanishing sums of roots of unity, [14]), the investigation is possible to complete.

2. Preliminary

Let V be an n-dimensional complex inner product space and G be a permutation group on m elements. Let Γ_n^m be the set of all sequences $\alpha = (\alpha_1, \dots, \alpha_m)$, with $1 \le \alpha_i \le n$. Define the action of G on Γ_n^m by

$$\alpha\sigma = (\alpha_{\sigma^{-1}(1)}, \dots, \alpha_{\sigma^{-1}(m)}).$$

Let $O(\alpha) = \{\alpha\sigma | \sigma \in G\}$ be the orbit of α. We write $\alpha \sim \beta$ if α and β belong to the same orbit in Γ_n^m. Let Δ be a system of distinct representatives of the orbits. Let G_α be the stabilizer subgroup of α, i.e., $G_\alpha = \{\sigma \in G | \alpha\sigma = \alpha\}$. Let χ be any irreducible character of G.

For any $\sigma \in G$, define the operator $P_\sigma : V^{\otimes m} \longrightarrow V^{\otimes m}$ on the m-folds tensor space $V^{\otimes m} := \bigotimes_1^m V$ by

$$P_\sigma(v_1 \otimes \cdots \otimes v_m) = (v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}).$$

The symmetry classes of tensors associated with G and χ is the image of the symmetry operator

$$T(G,\chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P_\sigma,$$

and it is denoted by $V_\chi^n(G)$. We say that the tensor $T(G,\chi)(v_1 \otimes \cdots \otimes v_m)$ is a decomposable symmetrized tensor, and we denote it by $v_1 * \cdots * v_m$. We call $V_\chi^n(G)$ the symmetry class of tensors associated with G and χ, and the dimension of $V_\chi^n(G)$ is given by

$$\dim(V_\chi^n(G)) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) n^{c(\sigma)},$$

where $c(\sigma)$ is the number of cycles, including cycles of length one, in the disjoint cycle factorization of σ, see [15].

The inner product on V induces an inner product on $V_\chi(G)$ which satisfies

$$\langle v_1 * \cdots * v_m, u_1 * \cdots * u_m \rangle = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^m \langle v_i, u_{\sigma(i)} \rangle.$$

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of V. It is well known that $\{e_\alpha^\otimes | \alpha \in \Gamma_n^m\}$ forms an orthogonal basis for $V^{\otimes m}$ associated to the induced inner product. Here, we denote $e_\alpha^\otimes$ the m-folds tensor $e_{\alpha_1} \otimes e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_m}$ and also denote e_α^* the tensor $e_{\alpha_1} * \cdots * e_{\alpha_m}$. We have

$$\langle e_\alpha^*, e_\beta^* \rangle = \begin{cases} 0 & \text{if} \quad \alpha \nsim \beta \\ \frac{\chi(1)}{|G|} \sum_{\sigma \in G_\beta} \chi(\sigma h^{-1}) & \text{if} \quad \alpha = \beta h. \end{cases} \tag{2.1}$$

In particular, for $\sigma_1, \sigma_2 \in G$ and $\alpha \in \Gamma_n^m$, we obtain

$$\langle e_{\alpha\sigma_1}^*, e_{\alpha\sigma_2}^* \rangle = \frac{\chi(1)}{|G|} \sum_{x \in \sigma_2 G_\alpha \sigma_1^{-1}} \chi(x). \tag{2.2}$$

Thus,

$$\| e_\alpha^* \|^2 = \frac{\chi(1)}{|G|} \sum_{\sigma \in G_\alpha} \chi(\sigma).$$

Define

$$\Omega = \{\alpha \in \Gamma_n^m | \sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0\},$$

and put $\overline{\Delta} = \Delta \cap \Omega$. Then $e_\alpha^* \neq 0$ if and only if $\alpha \in \Omega$.

For $\alpha \in \overline{\Delta}$, $V_\alpha^* := \langle e_{\alpha\sigma}^* : \sigma \in G\rangle$ is called the orbital subspace of $V_\chi(G)$. By (2.1), it follows that

$$V_\chi(G) = \bigoplus_{\alpha \in \overline{\Delta}} V_\alpha^*$$

is an orthogonal direct sum. In [8], it is proved that

$$\dim(V_\alpha^*) = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma). \tag{2.3}$$

Thus we deduce that if χ is a linear character, then $\dim V_\alpha^* = 1$ and in this case, the set $\{e_\alpha^* | \alpha \in \overline{\Delta}\}$ is an orthogonal basis of $V_\chi(G)$. An orthogonal basis which consists of the decomposable symmetrized tensors e_α^* is called an *orthogonal $*$-basis* or *o-basis* for short. If χ is not linear, it is possible that $V_\chi(G)$ has no orthogonal $*$-basis. The following facts are also needed in this article.

Proposition 2.1. *Let $\tau \in G$. If $B = \{e_{\alpha\sigma_1}^*, e_{\alpha\sigma_2}^*, \ldots, e_{\alpha\sigma_t}^*\}$ is an o-basis for V_α^*, then so does $\tau B := \{e_{\alpha\sigma_1\tau^{-1}}^*, e_{\alpha\sigma_2\tau^{-1}}^*, \ldots, e_{\alpha\sigma_t\tau^{-1}}^*\}$.*

Proof. This is an immediate result of [12, Lemma 1.3]. □

For a given natural number m, if there exist mth roots of unity $\epsilon_1, \epsilon_2, \ldots, \epsilon_k \in \mathbb{C}$ such that $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_k = 0$, then the equation is said to be a vanishing sum of mth roots of unity of *weight* k. Let $W(m)$ be the set of weights k for which there exists a vanishing sum $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_k = 0$, where each ϵ_i is an mth roots of unity. The main result of Lam and Leung in [14], states that;

Theorem 2.2. ([14]) *For any $m = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$ (prime factorization, $a_i > 0$), the weight set $W(m)$ is exactly given by $\mathbb{N}_0 p_1 + \cdots + \mathbb{N}_0 p_t := \{k_1 p_1 + \cdots + k_t p_t \mid k_1, \ldots, k_t \in \mathbb{N}_0\}$, where $\mathbb{N}_0$ is the set of all non negative integers.*

3. Main Results

3.1. **Non-abelian groups of order pq.** Let q be prime and p be a positive integer such that $p \mid q-1$. It is well known that there is only one (up to isomorphism) non-abelian group G having order pq. Namely, G is a semidirect product of cyclic group $C_q = \langle a \rangle$ of order q and cyclic group $C_p = \langle b \rangle$ of order p ; i.e., $G = C_q \rtimes_\phi C_p$, where $\phi : C_p \longrightarrow \mathrm{Aut}(C_q)$ is a homomorphism with $\circ(\phi_b(a)) = p$. A presentation of G may be given by

$$G = \left\langle A, B | A^q = B^p = 1, BAB^{-1} = A^r \right\rangle,$$

where r is a primitive root of the congruence $z^p \equiv 1 (\mathrm{mod}\, q)$ and $A = (a,1), B = (1,b)$. In particular, if $p = 2$, then G is the dihedral group of order $2q$. An embedding of G into the symmetric group S_q is also well known. Explicitly, [1], as an element of S_q, $A = (1\ 2\ \cdots\ q)$ and B is the product of p disjoint cycles sending i to $1 + (i-1)r$.

Since $G = C_q \rtimes_\phi C_p$, we can view C_q as a C_p - set with action given by $\phi_b(a) = a^r$. This action is induced to be an action of C_p on the set of irreducible representations $Irr(C_q) := C_q^\vee$ of C_q. Indeed,

$$b \cdot x = x\phi_b \text{ for each } x \in C_q^\vee.$$

Let O be an orbit of this action and $(C_p)_x$ be the stabilizer of x in C_p. For each $x \in O$ and $U \in Irr((C_p)_x)$, it can be shown that (see, [6], for example)

$$V_{(O,U)} = V_{O,x,U} = \mathrm{Ind}_{(C_p)_x}^{C_p} U = \{f : C_p \longrightarrow U | f(hg) = hf(g), h \in (C_p)_x\},$$

is an irreducible representation of G and $V_{O,x,U} \cong V_{O,y,U}$ for any $x, y \in O$. Furthermore, if $\{O_1, O_2, \ldots, O_k\}$ is the set of all disjoint orbits for the action of C_p on $C_q^\vee$, then ($x_i \in O_i$), [6],

$$\{V_{O_i,U} | U \in Irr((C_p)_{x_i}), i = 1, 2, \ldots, k\}$$

forms a complete set of irreducible representation of $G = C_q \rtimes_\phi C_p$. The character of $V = V_{(O,U)}$, such that $x \in O$, is given by the Mackey-type formula,

$$\chi_V(a,g) = \begin{cases} \frac{1}{|(C_p)_x|} \sum_{b \in C_p} x\phi_b(a)\chi_U(g), & \text{if } g \in (C_p)_x; \\ 0, & \text{if } g \notin (C_p)_x. \end{cases} \tag{3.1}$$

Now, we obtain

Proposition 3.1. *For the non-abelian group G of order pq with q is prime and $p \mid q-1$, there are $\frac{q-1}{p}$ irreducible characters of degree p and p irreducible characters of degree one.*

Proof. By the above discussion, it is sufficient to find the orbits and stabilizers for the action of C_p on $C_q^\vee$. Since p is the smallest positive integer such that $r^p \equiv 1 \mod q$,

$$[x] = \{b^t \cdot x | t = 0, 1, 2, \ldots, p-1\}$$

contains exactly p elements, for each $x \in C_q^\vee - \{1\}$. Thus, there are $\frac{q-1}{p}$ orbits of size p and one orbit of size 1. Let $\{1, x_1, x_2, \ldots, x_{\frac{q-1}{p}}\}$ be the set of representative of these orbits. By the orbit-stabilizer theorem, $(C_p)_{x_i} = \{1\}$, for each $i = 1, 2, \ldots, \frac{q-1}{p}$ and $(C_p)_1 = C_p$. Now, $Irr((C_p)_{x_i}) = \{\widetilde{0}\}$ and $Irr((C_p)_1) = \{\widetilde{0}, \widetilde{1}, \widetilde{2}, \ldots, \widetilde{p-1}\}$, where $\widetilde{k}(t) = e^{\frac{2\pi kti}{p}}$, for each $k, t = 0, 1, 2, \ldots, p-1$. Hence, $V_{([x_i],\widetilde{0})}$ and $V_{([1],\widetilde{k})}$, for each $i = 1, 2, \ldots, \frac{q-1}{p}$ and $k = 0, 1, \ldots, p-1$, form a completed list of irreducible representation of $G = C_q \rtimes_\phi C_p$.

By Mackey-type formula (3.1), we compute that $\dim(V_{([x_i],\tilde{0})}) = p$ and $\dim(V_{([1],\tilde{k})}) = 1$, which completes the proof. □

3.2. **Symmetry classes of tensors associated to G and non-linear characters.** Denote $Irr^p(G)$ the set of degree p irreducible characters of $G = C_q \rtimes_\phi C_p$. In the proof of the Proposition 3.1, we have seen that

$$Irr^p(G) = \{\chi_{V_{([x_i],\tilde{0})}} | i = 1, 2, \dots, \frac{q-1}{p}\}.$$

Let V be a finite inner product space, $\chi_V \in Irr^p(G)$ and $e^*_\alpha := T(G, \chi_V)(e^\otimes_\alpha)$, where $\alpha \in \Gamma^q_{\dim(V)}$.

Proposition 3.2. *For $\alpha \in \Gamma^q_{\dim(V)}$, $e^*_\alpha = 0$ if and only if α is a constant sequence.*

Proof. It is clear that $(a^s, 1) \in G_\alpha$ if and only if $a^s \in (C_q)_\alpha$. Since $(C_q)_\alpha$ is a subgroup of C_q and $|C_q| = q$ (which is prime),

$$(C_q)_\alpha = \begin{cases} \{1\}, & \text{if } a \notin (C_q)_\alpha; \\ C_q, & \text{if } a \in (C_q)_\alpha, \end{cases} \tag{3.2}$$

$$= \begin{cases} \{1\}, & \text{if } \alpha \text{ is not a constant sequence;} \\ C_q, & \text{if } \alpha \text{ is a constant sequence.} \end{cases} \tag{3.3}$$

By (3.1) and the fact that $(C_p)_{x_i} = \{1\}$, we have, for each $i = 1, 2, \dots, \frac{q-1}{p}$,

$$\chi_{V_{([x_i],\tilde{0})}}(a^s, b^l) = \begin{cases} 0, & \text{if } l \neq 0; \\ \sum_{j=0}^{p-1} x_i\phi_{b^j}(a^s), & \text{if } l = 0. \end{cases} \tag{3.4}$$

Thus, for $\chi_V = \chi_{V_{([x_i],\tilde{0})}}$,

$$\begin{aligned} \sum_{\sigma \in G_\alpha} \chi_V(\sigma) &= \sum_{(a^s,1) \in G_\alpha} \sum_{j=0}^{p-1} x_i\phi_{b^j}(a^s) \\ &= \sum_{a^s \in (C_q)_\alpha} \sum_{j=0}^{p-1} x_i\phi_{b^j}(a^s) \\ &= \sum_{j=0}^{p-1} \sum_{a^s \in (C_q)_\alpha} x_i\phi_{b^j}(a^s). \end{aligned}$$

Note that $x_i\phi_{b^j}(1) = 1$ for each j, because $x_i\phi_{b^j} \in Irr(C_q)$. Thus, $e^*_\alpha = 0$ if and only if $\sum_{j=0}^{p-1} \sum_{a^s \in (C_q)_\alpha} x_i\phi_{b^j}(a^s) = 0$, which will be happened if and only if $(C_q)_\alpha \neq \{1\}$ (by (4.1) and the second orthogonality of irreducible characters). The proof is now completed by (4.2).

□

To obtain the condition for the existence of an o-basis, it is essential to calculate the dimension of the orbital subspace V^*_α. This is a direct consequence of the Freese's theorem.

Proposition 3.3. $\dim(V^*_\alpha) = \begin{cases} 0, & \textit{if } \alpha \textit{ is a constant sequence;} \\ \frac{p^2}{|G_\alpha|}, & \textit{otherwise.} \end{cases}$

Proof. By Proposition 3.2, if α is constant, then $\dim(V_\alpha^*) = 0$. Suppose that α is a non-constant sequence. So, $(C_q)_\alpha = \{1\}$. By (3.4) and the Freese's theorem, (2.3), we have

$$\begin{aligned}\dim(V_\alpha^*) &= \frac{\chi(1)}{|G_\alpha|}\sum_{\sigma\in G_\alpha}\chi(\sigma)\\ &= \frac{p}{|G_\alpha|}\sum_{\tilde{a}\in(C_q)_\alpha}\chi((\tilde{a},1))\\ &= \frac{p^2}{|G_\alpha|},\end{aligned}$$

which completes the proof. □

Theorem 3.4. *Let q be prime and G be the non-abelian group of order pq with $p \mid q-1$. Then, $V_\chi(G)$ does not admit o-basis if and only if χ is a non-linear irreducible character of G and $\dim(V) > 1$.*

Proof. It is well known that if χ is linear, then $V_\chi(G)$ always admits an o-basis. If $\dim(V) = 1$, then $\dim(V^{\otimes m}) = 1$, for any positive integer m. Thus, $\dim(V_\chi(G)) \le 1$ (for any irreducible characters χ) and hence $V_\chi(G)$ admits an o-basis. For the non-linear case with $\dim(V) > 1$, it is enough to consider the condition on each orbital subspace V_α^*. Let $\alpha = (1, 2, 1, 1, \ldots, 1)$. Since $\dim(V) > 1$, $\alpha \in \Gamma^q_{\dim V}$. By the embedding of G in S_q in which q is prime, it is easy to see that $G_\alpha = \{1\}$. Now, by Proposition 3.3, $\dim(V_\alpha^*) = p^2$. Assume that V_α^* has $B = \{e^*_{\alpha\sigma_1}, e^*_{\alpha\sigma_2}, \ldots, e^*_{\alpha\sigma_{p^2}}\}$ as an o-basis. Then, by the pigeon's hole principle, there must exist $1 \le l \le p$ and $i_1, i_2, \ldots, i_p \in \{1, 2, \ldots, p^2\}$ such that $\{\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_p}\} \subseteq \{(a, b^l), (a^2, b^l), \ldots, (a^q, b^l)\}$. Moreover, by Proposition 2.1, $B^o := \sigma_{i_1} B$ is an o-basis for V_α^* as well. Thus, B^o contains

$$S = \{e^*_\alpha, e^*_{\alpha(a^{t_1},1)}, e^*_{\alpha(a^{t_2},1)}, \ldots, e^*_{\alpha(a^{t_{p-1}},1)}\},$$

for some $t_1, t_2, \ldots, t_{p-1} \in \{1, 2, \ldots, q\}$. Since elements in S are pairwise orthogonal, by (2.2), for each $k = 1, 2, \ldots, p-1$,

$$0 = \langle e^*_\alpha, e^*_{\alpha(a^{t_k},1)}\rangle = \frac{\chi(1)}{|G|}\sum_{\sigma\in(a^{t_k},1)G_\gamma}\chi(\sigma) = \frac{1}{q}\chi((a^{t_k},1)) = \frac{1}{q}\sum_{j=0}^{p-1} x\phi_{b^j}(a^{t_k}).$$

Hence, $\sum_{j=0}^{p-1} x\phi_{b^j}(a^{t_k}) = 0$, which is a p-terms sum of the qth roots of unity (because $x\phi_{b^j}$ is an irreducible character of C_q and q is prime). In other words, this is a vanishing sum of qth roots of unity of weight p. This yields a contradiction to the Theorem 2.2 which asserts that $W(q) = \mathbb{N}_0 q$. □

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Appendix

ตารางเปรียบเทียบ

วัตถุประสงค์ที่วางไว้ และ สิ่งที่ได้

วัตถุประสงค์ที่วางไว้	สิ่งที่ได้
To study symmetry classes of tensors associated with some non-abelian groups.	Understand symmetry classes of tensors associated with some non-abelian groups.
To provide a necessary condition for the existence of the o-basis of symmetry classes of tensors associated with some non-abelian groups.	Obtain a necessary condition for the existence of the o-basis of symmetry classes of tensors associated with some non-abelian groups.
To provide a sufficient condition for the existence of the o-basis of symmetry classes of tensors associated with some non-abelian groups.	Obtain a sufficient condition for the existence of the o-basis of symmetry classes of tensors associated with some non-abelian groups.

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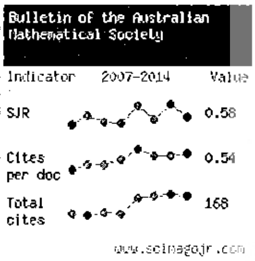

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SYMMETRY CLASSES OF TENSORS ASSOCIATED TO NONABELIAN GROUPS OF ORDER pq

KIJTI RODTES

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Abstract

Necessary and sufficient conditions for the existence of an orthogonal $*$-basis of symmetry classes of tensors associated to nonabelian groups of order pq are provided by using vanishing sums of roots of unity.

2010 *Mathematics subject classification*: primary 20C30; secondary 15A69.

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1. Introduction

The study of symmetry classes of tensors is motivated by many branches of pure and applied mathematics: combinatorial theory, matrix theory, operator theory, group representation theory, differential geometry, partial differential equations, quantum mechanics and other areas (see, for example, [11] and the references cited below). In particular, finding examples of (higher) symmetry classes of tensors that possess an orthogonal basis of decomposable symmetrised tensors (orthogonal $*$-basis or o-basis, for short) is of considerable interest. This topic arose from the question by Wang and Gong in [14], and the existence of an orthogonal $*$-basis of symmetry classes of tensors has been studied for several classes of groups: for example, dihedral groups in [8], dicyclic groups in [2], semi-dihedral groups in [9], some subgroups of full symmetric groups and some types of p-groups in [6].

Nonabelian groups of order pq have applications in group theory and graph theory (see, for example, [4]). Aspects of the symmetry classes of tensors associated to these groups have been considered. In particular, Poursalavati computed some dimensions of the symmetry classes of tensors associated with certain Frobenius groups in [13]. However, he did not investigate the condition for the existence of an o-basis. To do so, we need to handle the complicated values in the character table. We carry this through with the help of a result of Lam and Leung [10] on vanishing sums of roots of unity.

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2. Preliminaries

Let V be an n-dimensional complex inner product space and G a permutation group on m elements. Let Γ_n^m be the set of all sequences $\alpha = (\alpha_1, \dots, \alpha_m)$, with $1 \le \alpha_i \le n$. Define the action of G on Γ_n^m by

$$\alpha\sigma = (\alpha_{\sigma^{-1}(1)}, \dots, \alpha_{\sigma^{-1}(m)}).$$

Let $O(\alpha) = \{\alpha\sigma \mid \sigma \in G\}$ be the orbit of α. We write $\alpha \sim \beta$ if α and β belong to the same orbit in Γ_n^m. Let Δ be a system of distinct representatives of the orbits and let G_α be the stabiliser subgroup of α, that is, $G_\alpha = \{\sigma \in G \mid \alpha\sigma = \alpha\}$. Let χ be any irreducible character of G.

For any $\sigma \in G$, define the operator $P_\sigma : V^{\otimes m} \longrightarrow V^{\otimes m}$ on the m-fold tensor space $V^{\otimes m} := \bigotimes_1^m V$ by

$$P_\sigma(v_1 \otimes \cdots \otimes v_m) = (v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}).$$

The symmetry class of tensors associated with G and χ is the image in $V^{\otimes m}$ of the symmetry operator

$$T(G,\chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P_\sigma,$$

and it is denoted by $V_\chi^n(G)$. We say that the tensor $T(G,\chi)(v_1 \otimes \cdots \otimes v_m)$ is a decomposable symmetrised tensor, and we denote it by $v_1 * \cdots * v_m$. The dimension of $V_\chi^n(G)$ is given by

$$\dim(V_\chi^n(G)) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) n^{c(\sigma)},$$

where $c(\sigma)$ is the number of cycles, including cycles of length one, in the disjoint cycle factorisation of σ (see [12]).

The inner product on V induces an inner product on $V_\chi(G)$ which satisfies

$$\langle v_1 * \cdots * v_m, u_1 * \cdots * u_m \rangle = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{m} \langle v_i, u_{\sigma(i)} \rangle.$$

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of V. It is well known that $\{e_\alpha^\otimes \mid \alpha \in \Gamma_n^m\}$ forms an orthogonal basis for $V^{\otimes m}$ associated to the induced inner product. Here, $e_\alpha^\otimes$ denotes the m-fold tensor $e_{\alpha_1} \otimes e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_m}$ and we also write $e_\alpha^* = e_{\alpha_1} * \cdots * e_{\alpha_m}$. We have

$$\langle e_\alpha^*, e_\beta^* \rangle = \begin{cases} 0 & \text{if } \alpha \nsim \beta, \\ \dfrac{\chi(1)}{|G|} \displaystyle\sum_{\sigma \in G_\beta} \chi(\sigma h^{-1}) & \text{if } \alpha = \beta h. \end{cases} \tag{2.1}$$

In particular, for $\sigma_1, \sigma_2 \in G$ and $\alpha \in \Gamma_n^m$,

$$\langle e_{\alpha\sigma_1}^*, e_{\alpha\sigma_2}^* \rangle = \frac{\chi(1)}{|G|} \sum_{x \in \sigma_2 G_\alpha \sigma_1^{-1}} \chi(x). \tag{2.2}$$

where r is a primitive root of the congruence $z^p \equiv 1 \pmod q$ and $A = (a, 1), B = (1, b)$. In particular, if $p = 2$, then G is the dihedral group of order $2q$. An embedding of G into the symmetric group S_q is also well known. Explicitly, from [1], $A = (1\ 2\ \cdots\ q)$ as an element of S_q, and B is the product of p disjoint cycles, where the cycle containing i sends i to $1 + (i-1)r$.

Since $G = C_q \rtimes_\phi C_p$, we can view C_q as a C_p-set with action given by $\phi_b(a) = a^r$. This action is induced as an action of C_p on the set of irreducible representations $\mathrm{Irr}(C_q) := C_q^\vee$ of C_q. Indeed,

$$b \cdot x = x\phi_b \quad \text{for each } x \in C_q^\vee.$$

Let O be an orbit of this action and $(C_p)_x$ the stabiliser of x in C_p. For each $x \in O$ and $U \in \mathrm{Irr}((C_p)_x)$, it can be shown (see, for example, [3]) that

$$V_{(O,U)} = V_{O,x,U} = \mathrm{Ind}_{(C_p)_x}^{C_p} U = \{f : C_p \longrightarrow U \mid f(hg) = hf(g),\ h \in (C_p)_x\},$$

is an irreducible representation of G and $V_{O,x,U} \cong V_{O,y,U}$ for any $x, y \in O$. Furthermore, if $\{O_1, O_2, \ldots, O_k\}$ is the set of all disjoint orbits for the action of C_p on $C_q^\vee$, then

$$\{V_{O_i,U} \mid U \in \mathrm{Irr}((C_p)_{x_i}), i = 1, 2, \ldots, k\},$$

with $x_i \in O_i$, forms a complete set of irreducible representations of $G = C_q \rtimes_\phi C_p$ [3]. The character of $V = V_{(O,U)}$, such that $x \in O$, is given by the Mackey-type formula

$$\chi_V(a, g) = \begin{cases} \dfrac{1}{|(C_p)_x|} \displaystyle\sum_{b \in C_p} x\phi_b(a)\chi_U(g) & \text{if } g \in (C_p)_x, \\ 0 & \text{if } g \notin (C_p)_x. \end{cases} \tag{3.1}$$

Proposition 3.1. *For the nonabelian group G of order pq, where p is a positive integer and q is a prime such that $p \mid q - 1$, there are $(q-1)/p$ irreducible characters of degree p and p irreducible characters of degree one.*

Proof. By the above discussion, it is sufficient to find the orbits and stabilisers for the action of C_p on $C_q^\vee$. Since p is the smallest positive integer such that $r^p \equiv 1 \pmod q$,

$$[x] = \{b^t \cdot x \mid t = 0, 1, 2, \ldots, p-1\}$$

contains exactly p elements for each $x \in C_q^\vee - \{1\}$. Thus, there are $(q-1)/p$ orbits of size p and one orbit of size 1. Let $\{1, x_1, x_2, \ldots, x_{(q-1)/p}\}$ be the set of representatives of these orbits. By the orbit-stabiliser theorem, $(C_p)_{x_i} = \{1\}$ for each $i = 1, 2, \ldots, (q-1)/p$ and $(C_p)_1 = C_p$. Now, $\mathrm{Irr}((C_p)_{x_i}) = \{\widetilde{0}\}$ and $\mathrm{Irr}((C_p)_1) = \{\widetilde{0}, \widetilde{1}, \widetilde{2}, \ldots, \widetilde{p-1}\}$, where $\widetilde{k}(t) = e^{2\pi kti/p}$ for each $k, t = 0, 1, 2, \ldots, p-1$. Hence, $V_{([x_i],\widetilde{0})}$ and $V_{([1],\widetilde{k})}$, for each $i = 1, 2, \ldots, (q-1)/p$ and $k = 0, 1, \ldots, p-1$, form a complete list of irreducible representations of $G = C_q \rtimes_\phi C_p$. By the Mackey-type formula (3.1), we compute $\dim(V_{([x_i],\widetilde{0})}) = p$ and $\dim(V_{([1],\widetilde{k})}) = 1$, which completes the proof. □

4. Symmetry classes of tensors associated to G and nonlinear characters

Denote by $\mathrm{Irr}^p(G)$ the set of degree-p irreducible characters of $G = C_q \rtimes_\phi C_p$. In the proof of the Proposition 3.1, we have seen that

$$\mathrm{Irr}^p(G) = \{\chi_{V_{((x_i),\bar{0})}} \mid i = 1, 2, \ldots, (q-1)/p\}.$$

Suppose that V is a finite-dimensional inner product space, $\chi_V \in \mathrm{Irr}^p(G)$ and $\alpha \in \Gamma^q_{\dim(V)}$. Let $e^*_\alpha := T(G, \chi_V)(e^\otimes_\alpha)$.

Proposition 4.1. *For $\alpha \in \Gamma^q_{\dim(V)}$, $e^*_\alpha = 0$ if and only if α is a constant sequence.*

Proof. It is clear that $(a^s, 1) \in G_\alpha$ if and only if $a^s \in (C_q)_\alpha$. Since $(C_q)_\alpha$ is a subgroup of C_q and $|C_q| = q$ (which is prime),

$$(C_q)_\alpha = \begin{cases} \{1\} & \text{if } a \notin (C_q)_\alpha, \\ C_q & \text{if } a \in (C_q)_\alpha \end{cases} \tag{4.1}$$

$$= \begin{cases} \{1\} & \text{if } \alpha \text{ is not a constant sequence,} \\ C_q & \text{if } \alpha \text{ is a constant sequence.} \end{cases} \tag{4.2}$$

By (3.1) and the fact that $(C_p)_{x_i} = \{1\}$, for each $i = 1, 2, \ldots, (q-1)/p$,

$$\chi_{V_{((x_i),\bar{0})}}(a^s, b^l) = \begin{cases} 0 & \text{if } l \neq 0, \\ \displaystyle\sum_{j=0}^{p-1} x_i\phi_{b^j}(a^s) & \text{if } l = 0. \end{cases} \tag{4.3}$$

Thus, for $\chi_V = \chi_{V_{((x_i),\bar{0})}}$,

$$\begin{aligned} \sum_{\sigma \in G_\alpha} \chi_V(\sigma) &= \sum_{(a^s,1)\in G_\alpha} \sum_{j=0}^{p-1} x_i\phi_{b^j}(a^s) \\ &= \sum_{a^s \in (C_q)_\alpha} \sum_{j=0}^{p-1} x_i\phi_{b^j}(a^s) \\ &= \sum_{j=0}^{p-1} \sum_{a^s \in (C_q)_\alpha} x_i\phi_{b^j}(a^s). \end{aligned}$$

Note that $x_i\phi_{b^j}(1) = 1$ for each j, because $x_i\phi_{b^j} \in \mathrm{Irr}(C_q)$. Thus, $e^*_\alpha = 0$ if and only if $\sum_{j=0}^{p-1} \sum_{a^s \in (C_q)_\alpha} x_i\phi_{b^j}(a^s) = 0$, which happens if and only if $(C_q)_\alpha \neq \{1\}$ (by (4.1) and the second orthogonality relation for irreducible characters). The proof is now completed by (4.2). □

To obtain the condition for the existence of an o-basis, it is necessary to calculate the dimension of the orbital subspace V^*_α. This is a direct consequence of Freese's theorem, which we stated at (2.3).

PROPOSITION 4.2. *For* $\alpha \in \Gamma^q_{\dim(V)}$,

$$\dim(V^*_\alpha) = \begin{cases} 0 & \text{if } \alpha \text{ is a constant sequence,} \\ \dfrac{p^2}{|G_\alpha|} & \text{otherwise.} \end{cases}$$

PROOF. By Proposition 4.1, if α is constant, then $\dim(V^*_\alpha) = 0$. Suppose that α is not a constant sequence. Then $(C_q)_\alpha = \{1\}$. By (4.3) and Freese's theorem, (2.3),

$$\begin{aligned} \dim(V^*_\alpha) &= \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma) \\ &= \frac{p}{|G_\alpha|} \sum_{\tilde{a} \in (C_q)_\alpha} \chi((\tilde{a}, 1)) \\ &= \frac{p^2}{|G_\alpha|}, \end{aligned}$$

which completes the proof. □

THEOREM 4.3. *Suppose that p is a positive integer and q is a prime with* $p \mid q - 1$, G *is a nonabelian group of order pq and* χ *is an irreducible character of G.*

(1) *If* $\dim V = 1$ *or* χ *is linear, then* $V_\chi(G)$ *always admits an o-basis.*
(2) *If* $\dim V > 1$ *and* χ *is nonlinear, then* $V_\chi(G)$ *does not admit an o-basis.*

PROOF. It is well known that if χ is linear, then $V_\chi(G)$ always admits an o-basis. If $\dim V = 1$, then $\dim(V^{\otimes m}) = 1$ for any positive integer m. Thus, $\dim(V_\chi(G)) \leq 1$ (for any irreducible character χ) and hence $V_\chi(G)$ admits an o-basis.

For the nonlinear case with $\dim V > 1$, it is enough to consider the condition on each orbital subspace V^*_α. Let $\alpha = (1, 2, 1, 1, \ldots, 1)$. Since $\dim V > 1$, $\alpha \in \Gamma^q_{\dim V}$. By the embedding of G in S_q, where q is prime, it is easy to see that $G_\alpha = \{1\}$. Now, by Proposition 4.2, $\dim(V^*_\alpha) = p^2$. Assume that V^*_α has $B = \{e^*_{\alpha\sigma_1}, e^*_{\alpha\sigma_2}, \ldots, e^*_{\alpha\sigma_{p^2}}\}$ as an o-basis. Then, by the pigeonhole principle, there must exist $1 \leq l \leq p$ and $i_1, i_2, \ldots, i_p \in \{1, 2, \ldots, p^2\}$ such that $\{\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_p}\} \subseteq \{(a, b^l), (a^2, b^l), \ldots, (a^q, b^l)\}$. Moreover, by Proposition 2.1, $B^\circ := \sigma_{i_1} B$ is an o-basis for V^*_α as well. Thus, B° contains

$$S = \{e^*_\alpha, e^*_{\alpha(a^{t_1}, 1)}, e^*_{\alpha(a^{t_2}, 1)}, \ldots, e^*_{\alpha(a^{t_{p-1}}, 1)}\}$$

for some $t_1, t_2, \ldots, t_{p-1} \in \{1, 2, \ldots, q\}$. Since elements in S are pairwise orthogonal, by (2.2), for each $k = 1, 2, \ldots, p-1$,

$$0 = \langle e^*_\alpha, e^*_{\alpha(a^{t_k}, 1)} \rangle = \frac{\chi(1)}{|G|} \sum_{\sigma \in (a^{t_k}, 1)G_\alpha} \chi(\sigma) = \frac{1}{q}\chi((a^{t_k}, 1)) = \frac{1}{q} \sum_{j=0}^{p-1} x\phi_{b^j}(a^{t_k}).$$

Hence, $\sum_{j=0}^{p-1} x\phi_{b^j}(a^{t_k}) = 0$, which is a vanishing sum of qth roots of unity (because $x\phi_{b^j}$ is an irreducible character of C_q and q is prime). The weight of the sum, that is, the number of terms, is p. This contradicts Theorem 2.2, which asserts that the weight is in $\mathbb{N}_0 q$. □

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